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A note on pairwise fuzzy Volterra spaces

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Abstract. In this paper we investigate several characterizations of pairwise fuzzy Volterra spaces and study the conditions under which a fuzzy bitopological space is a pairwise fuzzy Volterra space.

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Keywords: Pairwise fuzzy dense set, Pairwise fuzzy G_δ-set, Pairwise fuzzy F_σ-set, Pairwise fuzzy first category set, Pairwise fuzzy Baire space, Pairwise fuzzy σ-Baire space, Pairwise fuzzy P-space, Pairwise fuzzy hyperconnected space, Pairwise fuzzy strongly irresolvable space, Pairwise fuzzy Volterra space.

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1. Introduction

In order to deal with uncertainties, the idea of fuzzy sets and fuzzy set operations was introduced by L. A. Zadeh [16] in his classical paper in the year 1965, describing fuzziness mathematically for the first time. Among the first field of Mathematics to be considered in the context of fuzzy sets was general topology. The concept of fuzzy topology was defined by C. L. Chang [3] in the year 1968. The paper of Chang paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Since then much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed. Today fuzzy topology has been firmly established as one of the basic disciplines of fuzzy mathematics. In 1989, A. Kandil [9] introduced the concept of fuzzy bitopological spaces. The concepts of Volterra spaces have been studied extensively in classical topology in [4], [5], [6], [7] and [8]. The concept of Volterra spaces in fuzzy setting was introduced and studied by G. Thangaraj and S. Soundararajan in [12]. The concept of pairwise Volterra spaces in fuzzy setting was introduced and studied by G. Thangaraj and V. Chandiran in [15]. In this paper, several characterizations of pairwise fuzzy Volterra spaces are studied.

2. Preliminaries

Now we introduce some basic notions and results used in the sequel. By a fuzzy bitopological space (Kandil, 1989) we mean an ordered triple (X, T_1, T_2), where T_1 and T_2 are fuzzy topologies on the non-empty set X. The complement $\lambda^{'}$ of a fuzzy set λ is defined by $\lambda^{'}(x) = 1 - \lambda(x),\ x \in X$.

Definition 2.1. Let λ and μ be any two fuzzy sets in (X, T). Then we define $\lambda \vee \mu : X \to [0,1]$ as follows : $(\lambda \vee \mu)(x) = Max\{\lambda(x), \mu(x)\}$. Also we define $\lambda \wedge \mu : X \to [0,1]$ as follows : $(\lambda \wedge \mu)(x) = Min\{\lambda(x), \mu(x)\}$.

For a family $\{\lambda_i / i \in I\}$ of fuzzy sets in (X, T), the union $\psi = \vee_i \lambda_i$ and intersection $\delta = \wedge_i \lambda_i$ are defined respectively as $\psi(x) = sup_i\{\lambda_i(x), x \in X\}$ and $\delta(x) = inf_i\{\lambda_i(x), x \in X\}$.

Definition 2.2 ([1])**.** Let (X, T) be a fuzzy topological space. For a fuzzy set λ of X, the interior $int(\lambda)$ and the closure $cl(\lambda)$ of (X, T) are defined respectively as $int(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \in T\}$ and $cl(\lambda) = \wedge\{\mu/\lambda \leq \mu, 1 - \mu \in T\}$.

Lemma 2.3 ([1])**.** *Let λ be any fuzzy set in a fuzzy topological space (X, T). Then $1 - cl(\lambda) = int(1 - \lambda)$ and $1 - int(\lambda) = cl(1 - \lambda)$.*

Definition 2.4 ([2])**.** Let (X, T) be a fuzzy topological space and λ be a fuzzy set in X. Then λ is called a fuzzy G_δ-set if $\lambda = \wedge_{i=1}^{\infty} \lambda_i$ for each $\lambda_i \in T$.

Definition 2.5 ([2])**.** Let (X, T) be a fuzzy topological space and λ be a fuzzy set in X. Then λ is called a fuzzy F_σ-set if $\lambda = \vee_{i=1}^{\infty} \lambda_i$ for each $1 - \lambda_i \in T$.

Lemma 2.6 ([1])**.** *For a family $\mathscr{A} = \{\lambda_\alpha\}$ of fuzzy sets of a fuzzy space X, $\vee\big(cl(\lambda_\alpha)\big) \leq cl\big(\vee(\lambda_\alpha)\big)$. In case $\mathscr{A}$ is a finite set, $\vee\big(cl(\lambda_\alpha)\big) = cl\big(\vee(\lambda_\alpha)\big)$. Also $\vee\big(int(\lambda_\alpha)\big) \leq int\big(\vee(\lambda_\alpha)\big)$.*

Definition 2.7 ([14])**.** A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy open set if $\lambda \in T_i,\ (i = 1, 2)$. The complement of pairwise fuzzy open set in (X, T_1, T_2) is called a *pairwise fuzzy closed set*.

Definition 2.8 ([15])**.** A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy G_δ-set if $\lambda = \wedge_{i=1}^{\infty} \lambda_i$, where λ_i's are pairwise fuzzy open sets in (X, T_1, T_2).

Definition 2.9 ([15])**.** A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy F_σ-set if $\lambda = \vee_{i=1}^{\infty} \lambda_i$, where λ_i's are pairwise fuzzy closed sets in (X, T_1, T_2).

Definition 2.10 ([2])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy dense if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$.

Definition 2.11 ([10])**.** A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy dense set if $cl_{T_1} cl_{T_2}(\lambda) = 1 = cl_{T_2} cl_{T_1}(\lambda)$.

Definition 2.12 ([11])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy nowhere dense if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is, $intcl(\lambda) = 0$.

Definition 2.13 ([10]). A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy nowhere dense set if $int_{T_1} cl_{T_2}(\lambda) = 0 = int_{T_2} cl_{T_1}(\lambda)$.

Definition 2.14 ([13]). Let (X, T_1, T_2) be a fuzzy bitopological space. A fuzzy set λ in (X, T_1, T_2) is called a pairwise fuzzy first category set if $\lambda = \vee_{k=1}^{\infty}(\lambda_k)$, where λ_k's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Any other fuzzy set in (X, T_1, T_2) is said to be a pairwise fuzzy second category set in (X, T_1, T_2).

Definition 2.15 ([13]). If λ is a pairwise fuzzy first category set in a fuzzy bitopological space (X, T_1, T_2), then the fuzzy set $1 - \lambda$ is called a pairwise fuzzy residual set in (X, T_1, T_2).

3. Pairwise fuzzy Volterra spaces

Motivated by the classical concept introduced in [4] we shall now define :

Definition 3.1 ([15]). A fuzzy bitopological space (X, T_1, T_2) is said to be a pairwise fuzzy Volterra space if $cl_{T_i}\Big(\wedge_{k=1}^{N}(\lambda_k)\Big) = 1$, $(i = 1, 2)$, where λ_k's are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2).

Proposition 3.2. *If λ is a pairwise fuzzy G_δ-set such that $cl_{T_i}(\lambda) = 1,\ (i = 1, 2)$, in a fuzzy bitopological space (X, T_1, T_2), then $1 - \lambda$ is a pairwise fuzzy first category set in (X, T_1, T_2).*

Proof. Let λ be a pairwise fuzzy G_δ-set such that $cl_{T_i}(\lambda) = 1,\ (i = 1, 2)$, in (X, T_1, T_2). Then $\lambda = \wedge_{k=1}^{\infty}(\lambda_k)$, where λ_k's are pairwise fuzzy open sets in (X, T_1, T_2). Since $cl_{T_i}(\lambda) = cl_{T_i}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) \leq \wedge_{k=1}^{\infty}\big(cl_{T_i}(\lambda_k)\big)$, we have $1 \leq \wedge_{k=1}^{\infty} cl_{T_i}(\lambda_k)$. That is., $\wedge_{k=1}^{\infty} cl_{T_i}(\lambda_k) = 1$. This implies that $cl_{T_i}(\lambda_k) = 1\ (i = 1, 2)$. Now $1 - \lambda = 1 - \wedge_{k=1}^{\infty}(\lambda_k) = \vee_{k=1}^{\infty}(1 - \lambda_k) \longrightarrow (A)$. Since λ_k's are pairwise fuzzy open sets in (X, T_1, T_2), $(1 - \lambda_k)$'s are pairwise fuzzy closed sets in (X, T_1, T_2). Then $cl_{T_i}(1 - \lambda_k) = 1 - \lambda_k,\ (i = 1, 2)$. Now $cl_{T_i}(\lambda_k) = 1$ implies that $1 - cl_{T_i}(\lambda_k) = 0$ and hence $int_{T_i}(1 - \lambda_k) = 0$. Now $int_{T_2} cl_{T_1}(1 - \lambda_k) = int_{T_2}(1 - \lambda_k) = 0$ and $int_{T_1} cl_{T_2}(1 - \lambda_k) = int_{T_1}(1 - \lambda_k) = 0$. Hence we have $int_{T_2} cl_{T_1}(1 - \lambda_k) = 0$ and $int_{T_1} cl_{T_2}(1 - \lambda_k) = 0$. This implies that $(1 - \lambda_k)$'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Therefore, from (A), $1 - \lambda$ is a pairwise fuzzy first category set in (X, T_1, T_2). □

Proposition 3.3. *If the pairwise fuzzy first category sets μ_k are formed from the pairwise fuzzy G_δ-sets λ_k such that $cl_{T_i}(\lambda_k) = 1\ (i = 1, 2)$ in a pairwise fuzzy Volterra space (X, T_1, T_2), then $int_{T_i}\big(\vee_{k=1}^{N}(\mu_k)\big) = 0$.*

Proof. Let λ_k's ($k = 1$ *to* N) be pairwise fuzzy G_δ-sets such that $cl_{T_i}(\lambda_k) = 1$ $(i = 1, 2)$ in a pairwise fuzzy Volterra space (X, T_1, T_2). Now $cl_{T_1} cl_{T_2}(\lambda_k) = 1$ and $cl_{T_2} cl_{T_1}(\lambda_k) = 1$. Thus λ_k's are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy Volterra space, $cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big) = 1,\ (i = 1, 2)$. Now $1 - cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big) = 0$, implies that $int_{T_i}\big(\vee_{k=1}^{N}(1 - \lambda_k)\big) = 0,\ (i = 1, 2)$. Since the fuzzy sets λ_k's ($k = 1$ *to* N) are pairwise fuzzy G_δ-sets such that $cl_{T_i}(\lambda_k) = 1\ (i = 1, 2)$ in (X, T_1, T_2), by proposition 3.2, $(1 - \lambda_k)$'s are pairwise fuzzy first category sets in (X, T_1, T_2). Let $\mu_k = 1 - \lambda_k$. Hence $int_{T_i}\big(\vee_{k=1}^{N}(\mu_k)\big) = 0,\ (i = 1, 2)$, where μ_k's are pairwise fuzzy first category sets in (X, T_1, T_2). □

Definition 3.4 ([13]). A fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy Baire space if $int_{T_i}(\vee_{k=1}^{\infty}(\lambda_k)) = 0$, $(i = 1, 2)$, where λ_k's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2).

Theorem 3.5 ([13]). *Let (X, T_1, T_2) be a fuzzy bitopological space. Then the following are equivalent :*

(1). *(X, T_1, T_2) is a pairwise fuzzy Baire space.*
(2). *$int_{T_i}(\lambda) = 0$, $(i = 1, 2)$, for every pairwise fuzzy first category set λ in (X, T_1, T_2).*
(3). *$cl_{T_i}(\mu) = 1$, $(i = 1, 2)$, for every pairwise fuzzy residual set μ in (X, T_1, T_2).*

Proposition 3.6. *If the pairwise fuzzy first category sets μ_k are formed from the pairwise fuzzy G_δ-sets λ_k such that $cl_{T_i}(\lambda_k) = 1$ $(i = 1, 2)$ in a pairwise fuzzy Volterra space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy Baire space.*

Proof. Now $\vee_{k=1}^{N}\big(int_{T_i}(\mu_k)\big) \leq int_{T_i}\big(\vee_{k=1}^{N}(\mu_k)\big)$. By Proposition 3.3, $int_{T_i}\big(\vee_{k=1}^{N}(\mu_k)\big) = 0$. Then $\vee_{k=1}^{N}\big(int_{T_i}(\mu_k)\big) = 0$ implies that $int_{T_i}(\mu_k) = 0$, $(i = 1, 2)$, where μ_k's are pairwise fuzzy first category set in (X, T_1, T_2). Hence, by theorem 3.5, (X, T_1, T_2) is a pairwise fuzzy Baire space. □

Definition 3.7 ([15]). A fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy P-space if countable intersection of pairwise fuzzy open sets in (X, T_1, T_2) is pairwise fuzzy open. That is, every non-zero pairwise fuzzy G_δ-set in (X, T_1, T_2) is pairwise fuzzy open in (X, T_1, T_2).

Definition 3.8. A fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy hyperconnected space if λ is a pairwise fuzzy open set, then $cl_{T_i}(\lambda) = 1$, $(i = 1, 2)$.

Proposition 3.9. *If the fuzzy bitopological P-space (X, T_1, T_2) is a pairwise fuzzy hyperconnected space, then (X, T_1, T_2) is a pairwise fuzzy Volterra space.*

Proof. Let λ_k's $(k = 1 \text{ to } N)$ be pairwise fuzzy G_δ-sets in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy P-space and λ_k's are pairwise fuzzy G_δ-sets, λ_k's are pairwise fuzzy open sets in (X, T_1, T_2). This implies that $\lambda_k \in T_i$. Then $\wedge_{k=1}^{N}(\lambda_k) \in T_i$, $(i = 1, 2)$. Thus $\wedge_{k=1}^{N}(\lambda_k)$ is a pairwise fuzzy open set in (X, T_1, T_2). Also, since (X, T_1, T_2) is a pairwise fuzzy hyperconnected space and λ_k's are pairwise fuzzy open sets, λ_k's are pairwise fuzzy dense sets in (X, T_1, T_2). Hence λ_k's are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Now $\wedge_{k=1}^{N}(\lambda_k)$ is a pairwise fuzzy open set in a pairwise fuzzy hyperconnected space (X, T_1, T_2). Then $cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big) = 1$, $(i = 1, 2)$. Therefore (X, T_1, T_2) is a pairwise fuzzy Volterra space. □

Theorem 3.10 ([13]). *If λ is a pairwise fuzzy nowhere dense set in a fuzzy bitopological space (X, T_1, T_2), then $1 - \lambda$ is a pairwise fuzzy dense set in (X, T_1, T_2).*

Theorem 3.11 ([14]). *If the pairwise fuzzy first category set λ is a pairwise fuzzy closed set in a pairwise fuzzy Baire space (X, T_1, T_2), then λ is a pairwise fuzzy nowhere dense set in (X, T_1, T_2).*

Proposition 3.12. *If the pairwise fuzzy first category sets λ_k $(k = 1 \text{ to } N)$ are pairwise fuzzy closed and pairwise fuzzy F_σ-sets in a pairwise fuzzy Baire space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy Volterra space.*

Proof. Let the pairwise fuzzy first category sets λ_k ($k = 1$ *to* N) be pairwise fuzzy closed and pairwise fuzzy F_σ-sets in a pairwise fuzzy Baire space (X, T_1, T_2). Then by theorem 3.11, λ_k's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Then by theorem 3.10, $(1 - \lambda_k)$'s are pairwise fuzzy dense sets in (X, T_1, T_2). Since the λ_k's are pairwise fuzzy F_σ-sets, $(1 - \lambda_k)$'s are pairwise fuzzy G_δ-sets in (X, T_1, T_2). Hence $(1 - \lambda_k)$'s ($k = 1$ *to* N) are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2).

Now $cl_{T_i}\big(\wedge_{k=1}^{N}(1 - \lambda_k)\big) = cl_{T_i}\big(1 - \vee_{k=1}^{N}(\lambda_k)\big) = 1 - int_{T_i}\big(\vee_{k=1}^{N}(\lambda_k)\big)$, $(i = 1, 2) \longrightarrow (1)$. If μ_k's are the pairwise fuzzy nowhere dense sets in (X, T_1, T_2) in which the first N pairwise fuzzy nowhere dense sets be λ_k, then $\vee_{k=1}^{N}(\lambda_k) \leq \vee_{k=1}^{\infty}(\mu_k)$. This implies that $int_{T_i}\big(\vee_{k=1}^{N}(\lambda_k)\big) \leq int_{T_i}\big(\vee_{k=1}^{\infty}(\mu_k)\big)$, $(i = 1, 2) \longrightarrow (2)$. Since (X, T_1, T_2) is a pairwise fuzzy Baire space, $int_{T_i}\big(\vee_{k=1}^{\infty}(\mu_k)\big) = 0$, $(i = 1, 2)$ where the μ_k's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Then from (2), $int_{T_i}\big(\vee_{k=1}^{N}(\lambda_k)\big) = 0$. Now $1 - int_{T_i}\big(\vee_{k=1}^{N}(\lambda_k)\big) = 1$. Hence from (1), $cl_{T_i}\big(\wedge_{k=1}^{N}(1 - \lambda_k)\big) = 1$, $(i = 1, 2)$ where $(1 - \lambda_k)$'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a pairwise fuzzy Volterra space. □

Definition 3.13. A fuzzy set λ in a fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy σ-nowhere dense set if λ is a pairwise fuzzy F_σ-set in (X, T_1, T_2) such that $int_{T_1}int_{T_2}(\lambda) = int_{T_2}int_{T_1}(\lambda) = 0$.

Proposition 3.14. *If λ is a pairwise fuzzy dense set in a fuzzy bitopological space (X, T_1, T_2) such that $\mu \leq 1 - \lambda$ where μ is a pairwise fuzzy F_σ-set in (X, T_1, T_2), then μ is a pairwise fuzzy σ-nowhere dense set in (X, T_1, T_2).*

Proof. Let λ be a pairwise fuzzy dense set in (X, T_1, T_2) such that $\mu \leq 1 - \lambda$. Then $cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda)$ and $int_{T_1}int_{T_2}(\mu) \leq int_{T_1}int_{T_2}(1 - \lambda) = 1 - cl_{T_1}cl_{T_2}(\lambda) = 1 - 1 = 0$ implies that $int_{T_1}int_{T_2}(\mu) = 0$. Also, $int_{T_2}int_{T_1}(\mu) = 0$. Hence μ is a pairwise fuzzy F_σ-set and $int_{T_1}int_{T_2}(\mu) = 0 = int_{T_2}int_{T_1}(\mu)$. Therefore μ is a pairwise fuzzy σ-nowhere dense set in (X, T_1, T_2). □

Proposition 3.15. *In a fuzzy bitopological space (X, T_1, T_2), a fuzzy set λ is pairwise fuzzy σ-nowhere dense in (X, T_1, T_2) if and only if $1 - \lambda$ is a pairwise fuzzy dense and pairwise fuzzy G_δ-set in (X, T_1, T_2).*

Proof. Let λ be a pairwise fuzzy σ-nowhere dense set in (X, T_1, T_2). Then λ is a pairwise fuzzy F_σ-set and $int_{T_1}int_{T_2}(\lambda) = 0 = int_{T_2}int_{T_1}(\lambda)$. Clearly, $1 - \lambda$ is a pairwise fuzzy G_δ-set in (X, T_1, T_2) and $1 - int_{T_1}int_{T_2}(\lambda) = 1$. That is, $cl_{T_1}cl_{T_2}(1 - \lambda) = 1$. Similarly, $cl_{T_2}cl_{T_1}(1 - \lambda) = 1$. Hence $1 - \lambda$ is a pairwise fuzzy dense set in (X, T_1, T_2). Therefore $1 - \lambda$ is a pairwise fuzzy dense and pairwise fuzzy G_δ-set in (X, T_1, T_2).

Conversely, let λ be a pairwise fuzzy dense and pairwise fuzzy G_δ-set in (X, T_1, T_2). Then $cl_{T_1}cl_{T_2}(\lambda) = 1 = cl_{T_2}cl_{T_1}(\lambda)$ and $1-\lambda$ is a pairwise fuzzy F_σ-set in (X, T_1, T_2). Now $int_{T_1}int_{T_2}(1-\lambda) = 1-cl_{T_1}cl_{T_2}(\lambda) = 1-1 = 0$. Similarly, $int_{T_2}int_{T_1}(1-\lambda) = 0$. Hence $1-\lambda$ is a pairwise fuzzy F_σ-set and $int_{T_1}int_{T_2}(1-\lambda) = 0 = int_{T_2}int_{T_1}(1-\lambda)$. Therefore $1 - \lambda$ is a pairwise fuzzy σ-nowhere dense set in (X, T_1, T_2). □

Proposition 3.16. *If λ is a pairwise fuzzy F_σ-set and pairwise fuzzy nowhere dense set in a fuzzy bitopological space (X, T_1, T_2), then λ is a pairwise fuzzy σ-nowhere dense set in (X, T_1, T_2).*

Proof. Let λ be a pairwise fuzzy nowhere dense set in (X, T_1, T_2). Then $int_{T_1} cl_{T_2}(\lambda) = 0 = int_{T_2} cl_{T_1}(\lambda)$. Now $\lambda \leq cl_{T_1}(\lambda)$ implies that $int_{T_2}(\lambda) \leq int_{T_2} cl_{T_1}(\lambda) = 0$. It follows that $int_{T_1} int_{T_2}(\lambda) \leq int_{T_1}(0) = 0$. That is, $int_{T_1} int_{T_2}(\lambda) = 0$. Also, $int_{T_1}(\lambda) \leq int_{T_1} cl_{T_2}(\lambda) = 0$ and hence $int_{T_2} int_{T_1}(\lambda) = int_{T_2}(0) = 0$. Hence λ is a F_σ-set and $int_{T_1} int_{T_2}(\lambda) = 0 = int_{T_2} int_{T_1}(\lambda)$. Therefore λ is a pairwise fuzzy σ-nowhere dense set in (X, T_1, T_2). □

Definition 3.17. A fuzzy bitopological space (X, T_1, T_2) is called a pairwise fuzzy σ-Baire space if $int_{T_i}\Big(\vee_{k=1}^{\infty}(\lambda_k)\Big) = 0,\ (i = 1, 2)$ where λ_k's are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2).

Proposition 3.18. *If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy σ-Baire space, then (X, T_1, T_2) is a pairwise fuzzy Volterra space.*

Proof. Let (X, T_1, T_2) be a pairwise fuzzy σ-Baire space. Then $int_{T_i}\Big(\vee_{k=1}^{\infty}(\lambda_k)\Big) = 0,\ (i = 1, 2)$ where λ_k's are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Now $int_{T_i}\Big(\vee_{k=1}^{N}(\lambda_k)\Big) \leq int_{T_i}\Big(\vee_{k=1}^{\infty}(\lambda_k)\Big) = 0$ implies that $int_{T_i}\Big(\vee_{k=1}^{N}(\lambda_k)\Big) = 0,\ (i = 1, 2)$. Then $1 - int_{T_i}\Big(\vee_{k=1}^{N}(\lambda_k)\Big) = 1$. Then we have $cl_{T_i}\Big(1 - \vee_{k=1}^{N}(\lambda_k)\Big) = 1$. That is, $cl_{T_i}\Big(\wedge_{k=1}^{N}(1 - \lambda_k)\Big) = 1$. By proposition 3.15, $(1 - \lambda_k)$'s are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Hence (X, T_1, T_2) is a pairwise fuzzy Volterra space. □

Proposition 3.19. *If the fuzzy bitopological space (X, T_1, T_2) is a pairwise fuzzy Baire space and if the pairwise fuzzy nowhere dense sets in (X, T_1, T_2) are pairwise fuzzy F_σ-sets in (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy σ-Baire space.*

Proof. Let (X, T_1, T_2) be a pairwise fuzzy Baire space such that every pairwise fuzzy nowhere dense set λ_k is a pairwise fuzzy F_σ-set in (X, T_1, T_2). Then $int_{T_i}\big(\vee_{k=1}^{\infty}(\lambda_k)\big) = 0$ where λ_k's are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). By proposition 3.16, λ_k's are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Hence $int_{T_i}\big(\vee_{k=1}^{\infty}(\lambda_k)\big) = 0$, where λ_k's are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Therefore (X, T_1, T_2) is a pairwise fuzzy σ-Baire space. □

Proposition 3.20. *If the pairwise fuzzy nowhere dense sets are pairwise fuzzy F_σ-sets in a pairwise fuzzy Baire space (X, T_1, T_2), then (X, T_1, T_2) is a pairwise fuzzy Volterra space.*

Proof. Suppose that every pairwise fuzzy nowhere dense set is a pairwise fuzzy F_σ-set in a pairwise fuzzy Baire space (X, T_1, T_2). Then by proposition 3.19, (X, T_1, T_2) is a pairwise fuzzy σ-Baire space. Also by proposition 3.18, (X, T_1, T_2) is a pairwise fuzzy Volterra space. □

The inter relations between pairwise fuzzy Baire spaces, pairwise fuzzy σ-Baire spaces and pairwise fuzzy Volterra spaces can be summarized as follows :

$$\begin{tikzcd}
\text{pairwise fuzzy Baire spaces} \arrow[d, "\begin{array}{l}\text{pairwise fuzzy nowhere dense sets}\\ \text{are pairwise fuzzy } F_\sigma\text{-sets}\end{array}"] \\
\text{pairwise fuzzy } \sigma\text{-Baire spaces} \arrow[d] \\
\text{pairwise fuzzy Volterra spaces}
\end{tikzcd}$$

Proposition 3.21. *If* $cl_{T_i}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) = 1,\ (i = 1, 2)$ *where* λ_k*'s are pairwise fuzzy dense and pairwise fuzzy* G_δ*-sets in a fuzzy bitoplogical space* (X, T_1, T_2)*, then* (X, T_1, T_2) *is a pairwise fuzzy* σ*-Baire space.*

Proof. Now $cl_{T_i}\big(\wedge_{k=1}^{\infty}(\lambda_k)\big) = 1,\ (i = 1, 2)$ implies that $int_{T_i}\big(\vee_{k=1}^{\infty}(1-\lambda_k)\big) = 0$. Since λ_k's are pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2), by proposition 3.15, $(1-\lambda_k)$'s are pairwise fuzzy σ-nowhere dense sets in (X, T_1, T_2). Hence (X, T_1, T_2) is a pairwise fuzzy σ-Baire space. □

Definition 3.22 ([14])**.** A fuzzy bitopological space (X, T_1, T_2) is said to be a pairwise fuzzy strongly irresolvable space if $cl_{T_1} int_{T_2}(\lambda) = 1 = cl_{T_2} int_{T_1}(\lambda)$ for each pairwise fuzzy dense set λ in (X, T_1, T_2).

Proposition 3.23. *If the fuzzy bitopological space* (X, T_1, T_2) *is a pairwise fuzzy strongly irresolvable Baire space, then* (X, T_1, T_2) *is a pairwise fuzzy Volterra space.*

Proof. Let λ_k's ($k = 1$ *to* N) be pairwise fuzzy dense and pairwise fuzzy G_δ-sets in (X, T_1, T_2). Since (X, T_1, T_2) is a pairwise fuzzy strongly irresolvable space, $cl_{T_1} int_{T_2}(\lambda_k) = 1 = cl_{T_2} int_{T_1}(\lambda_k)$. This implies that $int_{T_1} cl_{T_2}(1-\lambda_k) = 0$ and $int_{T_2} cl_{T_1}(1-\lambda_k) = 0$. Hence $(1-\lambda_k)$'s are pairwise fuzzy nowhere dense sets in (X, T_1, T_2). Now $\vee_{k=1}^{N}(1-\lambda_k) \leq \vee_{k=1}^{\infty}(1-\lambda_k)$ implies that $int_{T_1}\big(\vee_{k=1}^{N}(1-\lambda_k)\big) \leq int_{T_1}\big(\vee_{k=1}^{\infty}(1-\lambda_k)\big)$. Also, $int_{T_2}\big(\vee_{k=1}^{N}(1-\lambda_k)\big) \leq int_{T_2}\big(\vee_{k=1}^{\infty}(1-\lambda_k)\big)$. That is, $int_{T_i}\big(\vee_{k=1}^{N}(1-\lambda_k)\big) \leq int_{T_i}\big(\vee_{k=1}^{\infty}(1-\lambda_k)\big),\ (i = 1, 2)$. Since (X, T_1, T_2) is a pairwise fuzzy Baire space, $int_{T_i}\big(\vee_{k=1}^{\infty}(1-\lambda_k)\big) = 0$. Hence $int_{T_i}\big(\vee_{k=1}^{N}(1-\lambda_k)\big) = 0$. Then $1 - cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big) = 0$. This implies that $cl_{T_i}\big(\wedge_{k=1}^{N}(\lambda_k)\big) = 1,\ (i = 1, 2)$. Therefore (X, T_1, T_2) is a pairwise fuzzy Volterra space. □

4. Conclusions

In this paper several characterizations of pairwise fuzzy Volterra are studied. The inter relations between pairwise fuzzy Baire space, pairwise fuzzy σ-Baire space and pairwise fuzzy Volterra space are investigated.

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