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Commutative fuzzy languages and their generalizations

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Abstract. In this paper we describe the commutative fuzzy languages and their generalizations. Also we give the Eilenberg variety theorem for these classes of regular fuzzy languages.

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1. Introduction

The theory of fuzzy languages was developed as a generalization of the classical notion of (crisp) languages. There are several methods for studying fuzzy languages. One method is by associating a monoid (syntactic monoid) to every fuzzy language and then study properties of the fuzzy language using the algebraic properties of the syntactic monoid. This method has a strong basis because every monoid is a syntactic monoid of some fuzzy language and most of our work is based on this concept [1], [2] and [3]. The aim of this paper is to provide a variety structure of commutative fuzzy languages and their generalizations.

2. Preliminaries

Here we recall the basic definitions and notations that will be used in the sequel. All undefined terms are as in [4, 5]. A nonempty set S with an associative binary operation is called a semigroup. If there is an element $1 \in S$ with $1s = s = s1$ for all $s \in S$, then S is called a monoid (semigroup with identity). If $xy = yx$ for all $x, y \in S$, then the semigroup S is called a commutative semigroup.

Let A be a nonempty finite set called an alphabet. Elements of A are called letters. A word is a finite sequence of letters of A. The length of a word is the number of letters in it. A word of length zero is called the empty word, it is denoted by 1. A^+ denote the set of all nonempty words. Then $A^* = A^+ \cup \{1\}(A^+)$ together with the binary operation concatenation is called a free monoid(semigroup) on A. A language L' is a subset of $A^*(A^+)$.

A fuzzy language λ in $A^*(A^+)$ is a fuzzy subset of $A^*(A^+)$. To each fuzzy language λ we associate a congruence P_λ called syntactic congruence, as follows. For $u, v \in A^*(A^+)$, $uP_\lambda v$ if and only if $\lambda(xuy) = \lambda(xvy)$ for all $x, y \in A^*(A^+)$. The quotient monoid(semigroup), $\text{Syn}(\lambda) = A^*/P_\lambda(Syn\lambda = A^+/P_\lambda)$ is called the syntactic monoid (semigroup) of λ.

A fuzzy language λ over an alphabet A is recognizable by a monoid(semigroup) S if there is a homomorphism $\phi : A^* \to S(\phi : A^+ \to S)$ and a fuzzy subset π of S such that $\lambda = \pi\phi^{-1}$, where $\pi\phi^{-1}(u) = \pi(\phi(u))$.

For fuzzy languages $\lambda, \lambda_1, \lambda_2$ over an alphabet A, complement, union and intersection are defined respectively by $\overline{\lambda}(u) = 1 - \lambda(u)$, $(\lambda_1 \vee \lambda_2)(u) = \lambda_1(u) \vee \lambda_2(u)$, $(\lambda_1 \wedge \lambda_2)(u) = \lambda_1(u) \wedge \lambda_2(u)$.

Further left and right quotients are defined respectively by;

$$(\lambda_1^{-1}\lambda_2)(u) = \bigvee_{v \in A^*} (\lambda_2(vu) \wedge \lambda_1(v)), (\lambda_2\lambda_1^{-1})(u) = \bigvee_{v \in A^*} (\lambda_2(uv) \wedge \lambda_1(v)).$$

Let $c \in [0, 1]$ be arbitrary. Then the fuzzy language $c\lambda$ defined by $(c\lambda)(u) = c \cdot \lambda(u)$ is called multiplication by constant c.

Let A, B be finite alphabets, $\phi : A^* \to B^*(\phi : A^+ \to B^+)$ be a homomorphism and ψ a fuzzy language in $B^*(B^+)$, then the inverse image of ψ (under ϕ) is a fuzzy language $\psi\phi^{-1}$ over A defined by $(\psi\phi^{-1})(u) = \psi(\phi u)$.

For a fuzzy language λ by a c-cut, $c \in [0, 1]$, we mean the crisp language λ_c defined by $\lambda_c = \{u \in A^* | \lambda(u) \geq c\}$.

The following theorem gives a characterization for regular fuzzy languages.

Theorem 2.1 ([5]). *A fuzzy language λ is regular if and only if $Im(\lambda)$ is finite and language λ_c is regular for every $c \in [0, 1]$, where $Im(\lambda) = \{c | c \in [0, 1]$ and there exists $u \in A^*$ such that $\lambda(u) = c\}$.*

Unless otherwise specified all the fuzzy languages considered here are regular.

Definition 2.2. A family $\mathscr{F} = \mathscr{F}(A)$ of regular fuzzy languages is a variety of fuzzy languages in $A^*(A^+)$ if it is closed under unions, intersections, complements, multiplication by constants, quotients, inverse homomorphic images and cuts.

For a variety of fuzzy languages $\mathscr{F}$, let $\mathscr{F}^s$ be the family of finite monoids defined by $\mathscr{F}^s = \{\text{Syn}(\lambda) | \lambda \in \mathscr{F}(A), \text{ for some } A\}$. For a variety of finite monoids $\mathscr{S}$, let $\mathscr{S}^f = \mathscr{S}^f(A)$ be the family of fuzzy languages defined by $\mathscr{S}^f(A) = \{\lambda \text{ is a fuzzy language over } A | \text{Syn}(\lambda) \in \mathscr{S}\}$.

Theorem 2.3 (cf. [6], Theorem 7). *The mappings $\mathscr{F} \to \mathscr{F}^s$ and $\mathscr{S} \to \mathscr{S}^f$ are mutually inverse lattice isomorphisms between the lattices of all varieties of fuzzy languages and all varieties of finite monoids.*

3. Commutative fuzzy languages

Definition 3.1. Let λ be a fuzzy language on A^*, λ is called a commutative fuzzy language if it satisfies the condition $\lambda(xuvy) = \lambda(xvuy)$ for all $x, y, u, v \in A^*$.

Example 3.2. Let $L \subseteq A^*, M$ be a commutative monoid and $\varphi : A^* \to M$ be a homomorphism such that $L = \varphi^{-1}(P)$ for some $P \subseteq M$. Then $\lambda = \chi_P \varphi = \chi_L$ is a commutative fuzzy language, since

$$\begin{aligned} \lambda(xuvy) &= \chi_P\varphi(xuvy) = \chi_P(\varphi(xuvy)) \\ &= \chi_P(\varphi(x)\varphi(u)\varphi(v)\varphi(y)) \\ &= \chi_P(\varphi(x)[\varphi(u)\varphi(v)]\varphi(y)) \\ &= \chi_P(\varphi(x)[\varphi(v)\varphi(u)]\varphi(y)) \\ &= \chi_P(\varphi(xvuy)) = \chi_P\varphi(xvuy) \\ &= \lambda(xvuy) \end{aligned}$$

for all $x, y, u, v \in A^*$.

Example 3.3. Let $\lambda : A^* \to [0,1]$ be defined by

$$\lambda(u) = \begin{cases} 1 & \text{if } |u| \text{ is prime} \\ \frac{1}{2} & \text{if } |u| \text{ is composite} \\ \frac{1}{3} & \text{if } |u| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then λ is a commutative fuzzy language.

The class of commutative fuzzy languages on A^* is denoted by $\mathbf{CFL}(A^*)$ ($\mathbf{CFL}$). By Example 3.2, we have $\mathbf{CFL}(A^*) \neq \phi$.

3.1. **Variety of commutative fuzzy languages.** The following result shows that **CFL** is closed under the boolean operations.

Lemma 3.4. *Let $\lambda, \lambda_1, \lambda_2 \in$ **CFL**. Then $\overline{\lambda}, \lambda_1 \vee \lambda_2$ and $\lambda_1 \wedge \lambda_2$ are in **CFL**.*

Proof. Let $\lambda \in \mathbf{CFL}$. Then $\lambda(xuvy) = \lambda(xvuy)$ for all $x, y, u, v \in A^*$. So for all $x, y, u, v \in A^*$, we have

$$\begin{aligned} \overline{\lambda}(xuvy) &= 1 - \lambda(xuvy) \\ &= 1 - \lambda(xvuy) = \overline{\lambda}(xvuy). \end{aligned}$$

Thus $\overline{\lambda} \in \mathbf{CFL}$. Since $\lambda_1, \lambda_2 \in \mathbf{CFL}$, we have $\lambda_1(xuvy) = \lambda_1(xvuy)$ and $\lambda_2(xuvy) = \lambda_2(xvuy)$ for all $x, y, u, v \in A^*$. So

$$\begin{aligned} (\lambda_1 \vee \lambda_2)(xuvy) &= \lambda_1(xuvy) \vee \lambda_2(xuvy) \\ &= \lambda_1(xvuy) \vee \lambda_2(xvuy) = (\lambda_1 \vee \lambda_2)(xvuy) \end{aligned}$$

for all $x, y, u, v \in A^*$. Thus $(\lambda_1 \vee \lambda_2) \in \mathbf{CFL}$. Since $\lambda_1 \wedge \lambda_2 = \overline{(\overline{\lambda_1} \vee \overline{\lambda_2})}$, we have $\lambda_1 \wedge \lambda_2 \in \mathbf{CFL}$. □

Lemma 3.5. *Let λ be a commutative fuzzy language on A^*, X be a finite alphabet and $\varphi : X^* \to A^*$ be a homomorphism. Then $\lambda\varphi^{-1}$ is a commutative fuzzy language over X where $\lambda\varphi^{-1}(u) = \lambda(\varphi(u))$ for all $u \in X^*$.*

Proof. Since $\lambda \in \mathbf{CFL}$, we have $\lambda(xuvy) = \lambda(xvuy)$ for all $x, y, u, v \in A^*$. So

$$\begin{aligned}(\lambda\varphi^{-1})(xuvy) &= \lambda(\varphi(xuvy)) \\ &= \lambda(\varphi(x)\varphi(u)\varphi(v)\varphi(y)) \\ &= \lambda(\varphi(x)\varphi(v)\varphi(u)\varphi(y)) \\ &= \lambda(\varphi(xvuy)) = \lambda\varphi^{-1}(xvuy)\end{aligned}$$

for all $x, y, u, v \in X^*$. Thus $\lambda\varphi^{-1}$ is a commutative fuzzy language. □

From the above lemma, it follows that **CFL** is closed under the inverse homomorphic images.

Lemma 3.6. *Let* $\lambda_1, \lambda_2 \in \boldsymbol{CFL}$. *Then*
(i) $\lambda_1^{-1}\lambda_2 \in \boldsymbol{CFL}$ *and*
(ii) $\lambda_2\lambda_1^{-1} \in \boldsymbol{CFL}$.

Proof. **(i)** Since $\lambda_1, \lambda_2 \in \mathbf{CFL}$, we have $\lambda_1(xuvy) = \lambda_1(xvuy)$ and $\lambda_2(xuvy) = \lambda_2(xvuy)$ for all $x, y, u, v \in A^*$. So

$$\begin{aligned}(\lambda_1^{-1}\lambda_2)(xuvy) &= \bigvee_{v_1 \in A^*} \{\lambda_2(v_1xuvy) \wedge \lambda_1(v_1)\} \\ &= \bigvee_{v_1 \in A^*} \{\lambda_2((v_1x)uvy) \wedge \lambda_1(v_1)\} \\ &= \bigvee_{v_1 \in A^*} \{\lambda_2((v_1x)vuy) \wedge \lambda_1(v_1)\} \\ &= \bigvee_{v_1 \in A^*} \{\lambda_2(v_1xvuy) \wedge \lambda_1(v_1)\} = (\lambda_1^{-1}\lambda_2)(xvuy)\end{aligned}$$

for all $x, y, u, v \in A^*$. Thus $\lambda_1^{-1}\lambda_2 \in \mathbf{CFL}$.

(ii) Similarly, $\lambda_1, \lambda_2 \in \mathbf{CFL}$, then $\lambda_2\lambda_1^{-1} \in \mathbf{CFL}$. □

Lemma 3.7. *Let* λ *be a commutative fuzzy language and* $c \in [0, 1]$. *Then* $c\lambda$ *is a commutative fuzzy language.*

Proof. Since $\lambda \in \mathbf{CFL}$, we have $\lambda(xuvy) = \lambda(xvuy)$ for all $x, y, u, v \in A^*$ and $c \in [0, 1]$. So

$$\begin{aligned}(c\lambda)(xuvy) &= c \cdot \lambda(xuvy) \\ &= c \cdot \lambda(xvuy) = (c\lambda)(xvuy)\end{aligned}$$

for all $x, y, u, v \in A^*$. Thus $c\lambda$ is a commutative fuzzy language. Hence **CFL** is closed under the multiplication by constants. □

Lemma 3.8. *Let* $\lambda \in \boldsymbol{CFL}$ *and* $\lambda_c = \{u \in A^* : \lambda(u) \geq c\}$ *for* $c \in [0, 1]$. *Then the syntactic monoid of* λ_c *is a commutative monoid.*

Proof. Since $\lambda \in \mathbf{CFL}$, we have $\lambda(xuvy) = \lambda(xvuy)$ for all $x, y, u, v \in A^*$. So

$$\begin{aligned}xuvy \in \lambda_c &\Leftrightarrow c \leq \lambda(xuvy) = \lambda(xvuy) \ \ (\text{since } \lambda \in \mathbf{CFL}) \\ &\Leftrightarrow xvuy \in \lambda_c\end{aligned}$$

for all $x, y, u, v \in A^*$. So $xuvyP_{\lambda_c}xvuy$. Hence $M(\lambda_c)$ is a commutative monoid. □

Theorem 3.9. ***CFL*** *is a variety of fuzzy languages.*

Proof. By Lemma 3.4, **CFL** is closed under the boolean operations. By Lemma 3.5, **CFL** is closed under the inverse homomorphic images. By Lemmas 3.6, 3.7 and 3.8, **CFL** is closed under quotients, multiplication by constants and c-cuts. Thus **CFL** is a variety of fuzzy languages. □

The following theorem shows that the pseudovariety associated with the variety of fuzzy languages **CFL** is that of commutative monoids.

Theorem 3.10. *Let M be a finite commutative monoid recognizing the fuzzy language λ. Then $\lambda \in$* ***CFL***.

Proof. Since M is a finite commutative monoid recognizing the fuzzy language λ over an alphabet A, if there is a homomorphism $\varphi : A^* \to M$ and a fuzzy subset $\pi : M \to [0,1]$ such that $\lambda = \pi\varphi^{-1}$, where $\lambda(u) = \pi\varphi^{-1}(u) = \pi(\varphi(u))$ for all $u \in A^*$. Since M is a commutative monoid, we have

$$
\begin{aligned}
\lambda(xuvy) &= \pi\varphi^{-1}(xuvy) \\
&= \pi(\varphi(xuvy)) \\
&= \pi(\varphi(x)\varphi(u)\varphi(v)\varphi(y)) \\
&= \pi([\varphi(xu)\varphi(v)]\varphi(y)) \\
&= \pi([\varphi(v)\varphi(xu)]\varphi(y)) \\
&= \pi([\varphi(v)\varphi(x)\varphi(u)]\varphi(y)) \\
&= \pi([\varphi(v)\varphi(x)]\varphi(u)\varphi(y)) \\
&= \pi([\varphi(x)\varphi(v)]\varphi(u)\varphi(y)) \\
&= \pi(\varphi(x)\varphi(v)\varphi(u)\varphi(y)) \\
&= \pi(\varphi(xvuy)) = \pi\varphi^{-1}(xvuy) = \lambda(xvuy)
\end{aligned}
$$

for all $x, y, u, v \in A^*$. Thus $\lambda \in$ **CFL**. □

Theorem 3.11. *Let λ be a fuzzy language. Then $\lambda \in$* ***CFL*** *if and only if Syn(λ) is a commutative monoid.*

Proof. Since Syn(λ) is a commutative monoid, we have $[u]_\lambda \cdot [v]_\lambda = [v]_\lambda \cdot [u]_\lambda$ for all $u, v \in A^*$. So $(uv)P_\lambda(vu)$. Thus $\lambda(x_1uvx_2) = \lambda(x_1vux_2)$ for all $x_1, x_2 \in A^*$. Hence $\lambda \in$ **CFL**.

Conversely, since $\lambda \in$ **CFL**, we have $\lambda(xuvy) = \lambda(xvuy)$ for all $x, y, u, v \in A^*$. So $(uv)P_\lambda(vu)$. That is $[u]_\lambda \cdot [v]_\lambda = [v]_\lambda \cdot [u]_\lambda$ for all $u, v \in A^*$. Thus Syn(λ) is a commutative monoid. □

Theorem 3.12. *There exists a one-one correspondence between* ***CFL*** *and pseudovariety* ***Com***$^f = \{\lambda \mid Syn(\lambda) \in$ ***Com***$\}$ *of commutative monoid.*

Proof. Let $M \in$ **Com**^{f}. Then by Theorem 3.10, $\lambda \in$ **CFL**.

Conversely, if $\lambda \in$ **CFL**, then by Theorem 3.11 and Theorem 2.3, Syn(λ) $\in$ **Com**^{f}. □

4. Fuzzy languages whose syntactic monoids satisfies permutation identities

4.1. Different types of permutation identity fuzzy languages.

Definition 4.1. A fuzzy language λ in A^* is called a permutation identity fuzzy language of type 1, if it satisfies the condition $\lambda(uvw) = \lambda(vuw)$ for all $u, v, w \in A^*$.

Example 4.2. Let $\lambda : A^* \to [0,1]$ be defined by

$$\lambda(u) = \begin{cases} \frac{1}{25} & \text{if } |u| \geq 2 \\ \frac{1}{30} & \text{if } |u| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then λ is a permutation identity fuzzy language of type 1.

The class of permutation identity fuzzy languages of type 1 in A^* is denoted by $\mathbf{P_1IF}$.

Definition 4.3. A fuzzy language λ in A^* is called a permutation identity fuzzy language of type 2, if it satisfies the condition $\lambda(uvw) = \lambda(uwv)$ for all $u, v, w \in A^*$.

Example 4.4. Let $\lambda : A^* \to [0,1]$ be defined by

$$\lambda(u) = \begin{cases} \frac{1}{100} & \text{if } |u| \text{ is even} \\ \frac{1}{110} & \text{if } |u| \text{ is odd} \end{cases}$$

Then λ is a permutation identity fuzzy language of type 2.

The class of permutation identity fuzzy languages of type 2 in A^* is denoted by $\mathbf{P_2IF}$.

Definition 4.5. A fuzzy language λ in A^* is called a permutation identity fuzzy language of type 3, if it satisfies the condition $\lambda(uvw) = \lambda(wvu)$ for all $u, v, w \in A^*$.

Example 4.6. Let $\lambda : A^* \to [0,1]$ be defined by

$$\lambda(u) = \begin{cases} 1 & \text{if } |u| \text{ is prime} \\ \frac{1}{2} & \text{if } |u| \text{ is composite} \\ \frac{1}{3} & \text{if } |u| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then λ is a permutation identity fuzzy language of type 3.

The class of permutation identity fuzzy languages of type 3 in A^* is denoted by $\mathbf{P_3IF}$.

Definition 4.7. A fuzzy language λ in A^* is called a permutation identity fuzzy language of type 4, if it satisfies the condition $\lambda(uvw) = \lambda(vwu)$ for all $u, v, w \in A^*$.

Example 4.8. Let $\lambda_n : A^* \to [0,1]$ be defined by

$$\lambda_n(u) = \begin{cases} 1 & \text{if } |u| \geq 3^n \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Then λ_n is a permutation identity fuzzy language of type 4.

The class of permutation identity fuzzy languages of type 4 in A^* is denoted by $\mathbf{P}_4\mathbf{IF}$.

Definition 4.9. A fuzzy language λ in A^* is called a permutation identity fuzzy language of type 5, if it satisfies the condition $\lambda(uvw) = \lambda(wuv)$ for all $u, v, w \in A^*$.

Example 4.10. Let $\lambda_n : A^* \to [0,1]$ be defined by

$$\lambda_n(u) = \begin{cases} \frac{1}{12} & \text{if } |u| \geq n+1 \\ \frac{1}{13} & \text{otherwise.} \end{cases}$$

Then λ_n is a permutation identity fuzzy language of type 5.

The class of permutation identity fuzzy languages of type 5 in A^* is denoted by $\mathbf{P}_5\mathbf{IF}$.

4.2. **Variety of permutation identity fuzzy language of type 1.**

Lemma 4.11. *Let* λ, λ_1, $\lambda_2 \in \boldsymbol{P}_1\boldsymbol{IF}$. *Then*

(i) $\overline{\lambda} \in \boldsymbol{P}_1\boldsymbol{IF}$
(ii) $\lambda_1 \vee \lambda_2 \in \boldsymbol{P}_1\boldsymbol{IF}$ *and*
(iii) $\lambda_1 \wedge \lambda_2 \in \boldsymbol{P}_1\boldsymbol{IF}$.

Proof. **(i)** Since $\lambda \in \mathbf{P}_1\mathbf{IF}$, we have $\lambda(uvw) = \lambda(vuw)$ for all $u, v, w \in A^*$. Then

$$\begin{aligned} \overline{\lambda}(uvw) &= 1 - \lambda(uvw) \\ &= 1 - \lambda(vuw) = \overline{\lambda}(vuw). \end{aligned}$$

So $\overline{\lambda} \in \mathbf{P}_1\mathbf{IF}$.

(ii) Since $\lambda_1, \lambda_2 \in \mathbf{P}_1\mathbf{IF}$, we have $\lambda_1(uvw) = \lambda_1(vuw)$ and $\lambda_2(uvw) = \lambda_2(vuw)$ for all $u, v, w \in A^*$. Then

$$\begin{aligned} (\lambda_1 \vee \lambda_2)(uvw) &= \lambda_1(uvw) \vee \lambda_2(uvw) \\ &= \lambda_1(vuw) \vee \lambda_2(vuw) = (\lambda_1 \vee \lambda_2)(vuw) \end{aligned}$$

for all $u, v, w \in A^*$. Thus $(\lambda_1 \vee \lambda_2) \in \mathbf{P}_1\mathbf{IF}$.

(iii) Since $\lambda_1 \wedge \lambda_2 = \overline{(\overline{\lambda_1} \vee \overline{\lambda_2})}$, we have $\lambda_1 \wedge \lambda_2 \in \mathbf{P}_1\mathbf{IF}$. Hence $\mathbf{P}_1\mathbf{IF}$ is closed under the boolean operations. □

Lemma 4.12. *Let* λ *be a fuzzy language over* A, $\lambda \in \boldsymbol{P}_1\boldsymbol{IF}$, X *be a finite alphabet and* $\varphi : X^* \to A^*$ *be a homomorphism. Then* $\lambda\varphi^{-1} \in \boldsymbol{P}_1\boldsymbol{IF}$, *where* $\lambda\varphi^{-1}(u) = \lambda(\varphi(u))$ *for all* $u \in X^*$.

Proof. From the definition of $\lambda\varphi^{-1}$, we have $\lambda\varphi^{-1}$ is a fuzzy language in X^*. Then for all $u, v, w \in X^*$, we have

$$\begin{aligned} (\lambda\varphi^{-1})(uvw) &= \lambda(\varphi(uvw)) \\ &= \lambda(\varphi(u)\varphi(v)\varphi(w)) \\ &= \lambda(\varphi(v)\varphi(u)\varphi(w)) \\ &= \lambda(\varphi(vuw)) = \lambda\varphi^{-1}(vuw). \end{aligned}$$

So $\lambda\varphi^{-1} \in \mathbf{P}_1\mathbf{IF}$. □

Lemma 4.13. *Let* λ_1, $\lambda_2 \in \boldsymbol{P}_1\boldsymbol{IF}$. *Then*

(i) $(\lambda_1^{-1}\lambda_2) \in \boldsymbol{P}_1\boldsymbol{IF}$ *and*

(ii) $(\lambda_2\lambda_1^{-1}) \in \boldsymbol{P}_1\boldsymbol{IF}$.

Proof. **(i)** Since $\lambda_1, \lambda_2 \in \mathbf{P}_1\mathbf{IF}$, we have $\lambda_1(uvw) = \lambda_1(vuw)$ and $\lambda_2(uvw) = \lambda_2(vuw)$ for all $u, v, w \in A^*$. Then for all $u, v, w \in A^*$, we have

$$\begin{aligned} (\lambda_1^{-1}\lambda_2)(uvw) &= \bigvee_{v_1 \in A^*} \{\lambda_2(v_1uvw) \wedge \lambda_1(v_1)\} \\ &= \bigvee_{v_1 \in A^*} \{\lambda_2(v_1(uv)w) \wedge \lambda_1(v_1)\} \\ &= \bigvee_{v_1 \in A^*} \{\lambda_2((uv)v_1w) \wedge \lambda_1(v_1)\} \\ &= \bigvee_{v_1 \in A^*} \{\lambda_2(u(vv_1)w) \wedge \lambda_1(v_1)\} \\ &= \bigvee_{v_1 \in A^*} \{\lambda_2((vv_1)uw) \wedge \lambda_1(v_1)\} \\ &= \bigvee_{v_1 \in A^*} \{\lambda_2(vv_1(uw)) \wedge \lambda_1(v_1)\} \\ &= \bigvee_{v_1 \in A^*} \{\lambda_2(v_1v(uw)) \wedge \lambda_1(v_1)\} \\ &= \bigvee_{v_1 \in A^*} \{\lambda_2(v_1vuw) \wedge \lambda_1(v_1)\} = (\lambda_1^{-1}\lambda_2)(vuw). \end{aligned}$$

Thus $\lambda_1^{-1}\lambda_2 \in \mathbf{P}_1\mathbf{IF}$.

(ii) Similarly, if $\lambda_1, \lambda_2 \in \mathbf{P}_1\mathbf{IF}$, then $\lambda_2\lambda_1^{-1} \in \mathbf{P}_1\mathbf{IF}$. □

Lemma 4.14. *Let* $\lambda \in \boldsymbol{P}_1\boldsymbol{IF}$ *and* $c \in [0, 1]$. *Then* $(c\lambda) \in \boldsymbol{P}_1\boldsymbol{IF}$.

Proof. Since $\lambda \in \mathbf{P}_1\mathbf{IF}$, we have $\lambda(uvw) = \lambda(vuw)$ for all $u, v, w \in A^*$. Then

$$\begin{aligned} (c\lambda)(uvw) &= c \cdot \lambda(uvw) \\ &= c \cdot \lambda(vuw) = (c\lambda)(vuw) \end{aligned}$$

for all $u, v, w \in A^*$. Thus $c\lambda \in \mathbf{P}_1\mathbf{IF}$. □

Lemma 4.15. $\boldsymbol{P}_1\boldsymbol{IF}$ *is closed under the c-cuts.*

Proof. Let $\lambda \in \mathbf{P}_1\mathbf{IF}$ and $c \in [0, 1]$. Then

$$\begin{aligned} uvw \in \lambda_c &\Leftrightarrow c \leq \lambda(uvw) = \lambda(vuw) \\ &\Leftrightarrow vuw \in \lambda_c \end{aligned}$$

for all $u, v, w \in A^*$. Thus $\chi_{\lambda_c}(uvw) = \chi_{\lambda_c}(vuw)$ for all $u, v, w \in A^*$. That is $\chi_{\lambda_c} \in \mathbf{P}_1\mathbf{IF}$. Hence $\mathbf{P}_1\mathbf{IF}$ is closed under c-cuts. □

Theorem 4.16. $\boldsymbol{P}_1\boldsymbol{IF}$ *is a variety of fuzzy languages.*

Proof. By Lemma 4.11, $\mathbf{P}_1\mathbf{IF}$ is closed under the boolean operations. By Lemmas 4.12 and 4.13, $\mathbf{P}_1\mathbf{IF}$ is closed under the inverse homomorphic images and quotients. By Lemmas 4.14 and 4.15, $\mathbf{P}_1\mathbf{IF}$ is closed under the multiplication by constants and c-cuts. Hence $\mathbf{P}_1\mathbf{IF}$ is a variety of fuzzy languages. □

The following theorem shows that the pseudovariety associated with the variety of fuzzy languages $\mathbf{P}_1\mathbf{IF}$ is that of monoids satisfying the identity $s_1s_2s_3 = s_2s_1s_3$.

Theorem 4.17. *Let M be a finite monoid satisfying the identity $s_1s_2s_3 = s_2s_1s_3$ and recognizing the fuzzy language λ. Then $\lambda \in \boldsymbol{P}_1\boldsymbol{IF}$.*

Proof. Since M recognizes the fuzzy language λ, there is a homomorphism $\varphi : A^* \to M$ and a mapping $\pi : M \to [0,1]$ such that $\lambda(u) = \pi\varphi^{-1}(u) = \pi(\varphi(u))$ for all $u \in A^*$. So we have

$$\begin{array}{rcl}\lambda(uvw) & = & \pi\varphi^{-1}(uvw)\\ & = & \pi(\varphi(uvw))\\ & = & \pi(\varphi(u)\varphi(v)\varphi(w))\\ & = & \pi(\varphi(v)\varphi(u)\varphi(w))\\ & = & \pi(\varphi(vuw))\\ & = & \pi\varphi^{-1}(vuw) = \lambda(vuw)\end{array}$$

for all $u, v, w \in A^*$. Thus $\lambda \in \mathbf{P}_1\mathbf{IF}$. □

Similarly we have the following results.

Theorem 4.18. *Let M be a finite monoid satisfying the identity $s_1s_2s_3 = s_1s_3s_2$ and recognizing the fuzzy language λ. Then $\lambda \in \boldsymbol{P}_2\boldsymbol{IF}$.* □

Theorem 4.19. *Let M be a finite monoid satisfying the identity $s_1s_2s_3 = s_3s_2s_1$ and recognizing the fuzzy language λ. Then $\lambda \in \boldsymbol{P}_3\boldsymbol{IF}$.* □

Theorem 4.20. *Let M be a finite monoid satisfying the identity $s_1s_2s_3 = s_2s_3s_1$ and recognizing the fuzzy language λ. Then $\lambda \in \boldsymbol{P}_4\boldsymbol{IF}$.* □

Theorem 4.21. *Let M be a finite monoid satisfying the identity $s_1s_2s_3 = s_3s_1s_2$ and recognizing the fuzzy language λ. Then $\lambda \in \boldsymbol{P}_5\boldsymbol{IF}$.* □

Theorem 4.22. *Let λ be a fuzzy language in A^*. Then $\lambda \in \boldsymbol{P}_1\boldsymbol{IF}$ if and only if $Syn(\lambda)$ is a monoid satisfying the condition $s_1s_2s_3 = s_2s_1s_3$ for all $s_1, s_2, s_3 \in Syn(\lambda)$.*

Proof. Assume that $\mathrm{Syn}(\lambda)$ is a monoid satisfying the the condition $s_1s_2s_3 = s_2s_1s_3$ for all $s_1, s_2, s_3 \in Syn(\lambda)$. Then for all $u, v, w \in A^*$ we have $[u]_\lambda \cdot [v]_\lambda \cdot [w]_\lambda = [v]_\lambda \cdot [u]_\lambda \cdot [w]_\lambda$. That is, $(uvw)P_\lambda(vuw)$. So $\lambda(x(uvw)y) = \lambda(x(vuw)y)$ for all $x, y \in A^*$.

Conversely, let $\lambda \in \mathbf{P}_1\mathbf{IF}$, we have $\lambda(uvw) = \lambda(vuw)$ for all $u, v, w \in A^*$. Then

$$\begin{array}{rcl}\lambda(x(uvw)y) & = & \lambda((xu)v(wy))\\ & = & \lambda(v(xu)(wy))\\ & = & \lambda(vxu(wy))\\ & = & \lambda(vx(uwy))\\ & = & \lambda(xvuwy)\end{array}$$

for all $x, y \in A^*$. So $(uvw)P_\lambda(vuw)$. Thus $[u]_\lambda \cdot [v]_\lambda \cdot [w]_\lambda = [v]_\lambda \cdot [u]_\lambda \cdot [w]_\lambda$ for all $u, v, w \in A^*$. Then $\mathrm{Syn}(\lambda)$ satisfies the condition $s_1s_2s_3 = s_2s_1s_3$ for all $s_1, s_2, s_3 \in Syn(\lambda)$. □

Theorem 4.23. *There exists a one-one correspondence between* $\boldsymbol{P}_1\boldsymbol{IF}$ *and the pseudovariety of monoids satisfies the permutation identity* $s_1s_2s_3 = s_2s_1s_3$.

Proof. This follows from Theorem 4.17 and Theorem 4.22 □

Remark 4.24. By the similar arguments as in Lemmas 4.11, 4.12, 4.13, 4.14, 4.15 and Theorem 4.16, we can show that $\mathbf{P}_2\mathbf{IF}$, $\mathbf{P}_3\mathbf{IF}$, $\mathbf{P}_4\mathbf{IF}$ and $\mathbf{P}_5\mathbf{IF}$ are varieties of fuzzy languages. Also we have the following.

(i) There exists a one-one correspondence between $\mathbf{P}_2\mathbf{IF}$ and the pseudovariety of monoids satisfying $s_1s_2s_3 = s_1s_3s_2$.
(ii) There exists a one-one correspondence between $\mathbf{P}_3\mathbf{IF}$ and the pseudovariety of monoids satisfying $s_1s_2s_3 = s_3s_2s_1$.
(iii) There exists a one-one correspondence between $\mathbf{P}_4\mathbf{IF}$ and the pseudovariety of monoids satisfying $s_1s_2s_3 = s_2s_3s_1$.
(iv) There exists a one-one correspondence between $\mathbf{P}_5\mathbf{IF}$ and the pseudovariety of monoids satisfying $s_1s_2s_3 = s_3s_1s_2$.

Remark 4.25. If M is a commutative monoid, then M satisfies the identity $s_1s_2s_3 = s_1s_3s_2 = s_3s_2s_1 = s_3s_1s_2$. So **CFL** is a subclass of $\mathbf{P}_1\mathbf{IF}$, $\mathbf{P}_2\mathbf{IF}$, $\mathbf{P}_3\mathbf{IF}$, $\mathbf{P}_4\mathbf{IF}$ and $\mathbf{P}_5\mathbf{IF}$.

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References

[1] V. P. Archana, On varieties of fuzzy languages whose syntactic monoids has commuting idempotents, Adv. Fuzzy Sets Syst. 13(2) (2012) 115–125.
[2] V. P. Archana and P. Ramesh Kumar, Full expletive fuzzy languages, Int. J. Algebra 6(5-8) (2012) 393–397.
[3] V. P. Archana and P. Ramesh Kumar, Varieties of some fuzzy languages, Advances in Algebra 5(2) (2012) 119–124.
[4] J. M. Howie, An Introduction to Semigroup Theory, Academic Press, London, 1976.
[5] J. N. Mordeson and D. S. Malik, Fuzzy Automata and Languages; Theory and Applications, Chapman & Hall CRC, 2002.
[6] T. Petkovic, Varieties of fuzzy languages, Proc. 1st Inernational Conference on Algebraic Informatics, Aristotle University of Thessaloniki, Thessaloniki, 2005.

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