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Solving fuzzy linear differential equations by a new method

S. Melliani, E. El Jaoui, L. S. Chadli

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Abstract. In this work, we develop an operator method for solving first order fuzzy linear differential equations, which was introduced by T. Allahviranloo et all in [2], it was limited to solve only fuzzy linear differential equations with crisp constant coefficients, and its main result was formal and lacks proof. We extend this method for some equations with variable coefficients and we give the general formula's solution with necessary proofs.

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Corresponding Author: Said Melliani (melliani@fstbm.ac.ma)

1. Introduction

A natural way to model dynamic systems under uncertainty is to use fuzzy differential equations (FDEs). So, the topic of FDEs has been rapidly growing in recent years. The theory of FDEs was treated by several authors (see [9], [10], [11] and [12]) and others researchers (see [1], [2], [3], [4] and [5]) studied numerical algorithms for solving this kind of equations.
In [2] Allahviranloo et al. proposed a novel method for solving fuzzy linear differential equations which its construction based on the equivalent integral forms of original problems under the assumption of strongly generalized differentiability. By using the lower and upper functions of obtained integral equations, the lower and upper functions of solutions are determined. More precisely, they studied the following FIVPs

$$\begin{cases} y'(x) = y(x) \\ y(0) = y_0 \in E \end{cases} \quad \text{and} \quad \begin{cases} y'(x) = -y(x) + x + 1 \\ y(0) = y_0 \in E \end{cases},$$

using the operator J defined by

$$J\underline{y}(x,\alpha)=\int_0^x \underline{y}(t,\alpha)dt;\quad J\overline{y}(x,\alpha)=\int_0^x \overline{y}(t,\alpha)dt.$$

They solved the first equation only under the condition of (1)-differentiability of the solution y and the second problem only under the assumption of (2)-differentiability. They used the bijectivity of the operators $I-J, I+J$ and they claimed that

$$\begin{cases}(I-J)^{-1}=I+J+J^2+J^3+J^4+\cdots\\(I+J)^{-1}=I-J+J^2-J^3+J^4-\cdots\end{cases}$$

But these results which represent the basis of their algorithm, were not proved.
The aim of this paper, is to modify and develop their method using new operators denoted by J and K to solve the following first order fuzzy linear differential equations, with variable coefficients in both cases: under (1) or (2)-differentiability

$$\begin{cases}y'(x)=f(x)y(x)\\ay(0)=y_0\in E\end{cases}\quad\text{and}\quad\begin{cases}y'(x)=-f(x)y(x)\\y(0)=y_0\in E\end{cases}$$

where f is a crisp function verifying some assumptions to be determined later.
Moreover, we prove that each of the operators $I-J$, $I+J$, $I-K$, $I+K$ are bijective and we give the inverse operator's formulas.
The remainder of this work is organized as follows:
Section 2 is reserved for some preliminaries. Section 3 is devoted to notations and terminology. Then in section 4, we present our operator method to solve first order fuzzy linear differential equations. Section 5 deals with some numerical examples. In the last section, we present conclusion and a further research topic.

2. Preliminaries

By $P_K(\mathbb{R})$ we denote the family of all nonempty compact convex subsets of $\mathbb{R}$ and define the addition and scalar multiplication in $P_K(\mathbb{R})$ as usual. Denote

$$E=\Big\{u:\mathbb{R}\longrightarrow[0,1]\mid \text{u satisfies (i)}-\text{(iv) below}\Big\}$$

where

(i) u is normal, i.e. $\exists x_0\in\mathbb{R}$ for which $u(x_0)=1$,
(ii) u is fuzzy convex, i.e.

$$u(\lambda x+(1-\lambda y))\geq\min(u(x),u(y))\quad\text{for any } x,\ y\in\mathbb{R},\quad\text{and } \lambda\in[0,1],$$

(iii) u is upper semi-continuous,
(iv) supp$u=\{x\in\mathbb{R}|u(x)>0\}$ is the support of the u, and its closure cl (supp u) is compact.

For $0<\alpha\leq1$, denote

$$[u]^\alpha=\{x\in\mathbb{R}\mid u(x)\geq\alpha\}$$

Then, from (i)-(iv), it follows that the α-level set $[u]^\alpha\in P_K(\mathbb{R})$ for all $0\leq\alpha\leq1$.
According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy number space E as usual.

Let $D : E \times E \longrightarrow [0, \infty)$ be a function which is defined by the equation

$$D(u, v) = \sup_{0 \le \alpha \le 1} d\Big([u]^\alpha \,,\, [v]^\alpha\Big)$$

where d is the Hausdorff metric defined in $P_K(\mathbb{R})$. Then, it is easy to see that D is a metric in E and has the following properties [13]:

(1) (E, D) is a complete metric space;
(2) $D(u + w, v + w) = D(u, v)$ for all $u,\ v,\ w \in E$;
(3) $D(k\,u, k\,v) = |k|\ D(u, v)$ for all $u,\ v \in E$ and $k \in \mathbb{R}$;
(4) $D(u + w, v + t) \le D(u, v) + D(w, t)$ for all $u,\ v,\ w,\ t \in E$.

We recall some measurability, integrability properties for fuzzy set-valued mappings (see [9]). Let $T = [c, d] \subset \mathbb{R}$ be a compact interval.

Definition 2.1. A mapping $F : T \to E$ is strongly measurable if for all $\alpha \in [0, 1]$ the set-valued function $F_\alpha : T \to \mathcal{P}_K(\mathbb{R})$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable.
A mapping $F : T \to E$ is called integrably bounded if there exists an integrable function k such that $\|x\| \le k(t)$ for all $x \in F_0(t)$.

Definition 2.2. Let $F : T \to E$, then the integral of F over T denoted by $\displaystyle\int_T F(t)dt$ or $\displaystyle\int_c^d F(t)dt$, is defined by the equation

$$\begin{aligned}\left[\int_T F(t)dt\right]^\alpha &= \int_T F_\alpha(t)dt \\ &= \left\{\int_T f(t)dt / f : T \to \mathbb{R} \textbf{ is a measurable selection for } F_\alpha\right\}\end{aligned}$$

$\alpha \in]0, 1]$.
Also, a strongly measurable and integrably bounded mapping $F : T \to E$ is said to be integrable over T if $\displaystyle\int_T F(t)dt \in E$.

Proposition 2.3. *(Aumann [6]). If $F : T \to E$ is strongly measurable and integrably bounded, then F is intergrable.*

The following definitions and theorems are given in [9] , [2] and [7].

Proposition 2.4. *Let $F, G : T \to E$ be integrable and $\lambda \in \mathbb{R}$. Then*

(i) $\displaystyle\int_T (F(t) + G(t))dt = \int_T F(t)dt + \int_T G(t)dt$,
(ii) $\displaystyle\int_T \lambda F(t)dt = \lambda \int_T F(t)dt,$
(iii) $D(F, G)$ *is integrable,*
(iv) $\displaystyle D\left(\int_T F(t)dt, \int_T G(t)dt\right) \le \int_T D(F, G)(t)dt.$

For $u, v \in E$, if there exists $w \in E$ such that $u = v + w$, then w is the Hukuhara difference of u and v denoted by $u \ominus v$.

Definition 2.5. We say that a mapping $f:(a,b)\to E$ is strongly generalized differentiable at $x_0\in(a,b)$; if there exists an element $f'(x_0)\in E$; such that

(i) for all $h>0$ sufficiently small, there exist $f(x_0+h)\ominus f(x_0)$; $f(x_0)\ominus f(x_0-h)$ and the limits

$$\lim_{h\to 0^+}\frac{f(x_0+h)\ominus f(x_0)}{h}=\lim_{h\to 0^+}\frac{f(x_0)\ominus f(x_0-h)}{h}=f'(x_0)$$

or

(ii) for all $h>0$ sufficiently small, there exist $f(x_0)\ominus f(x_0+h)$; $f(x_0-h)\ominus f(x_0)$ and the limits

$$\lim_{h\to 0^+}\frac{f(x_0)\ominus f(x_0+h)}{(-h)}=\lim_{h\to 0^+}\frac{f(x_0-h)\ominus f(x_0)}{(-h)}=f'(x_0)$$

or

(iii) for all $h>0$ sufficiently small, there exist $f(x_0+h)\ominus f(x_0)$; $f(x_0-h)\ominus f(x_0)$ and the limits

$$\lim_{h\to 0^+}\frac{f(x_0+h)\ominus f(x_0)}{h}=\lim_{h\to 0^+}\frac{f(x_0-h)\ominus f(x_0)}{(-h)}=f'(x_0)$$

or

(iv) for all $h>0$ sufficiently small, there exist $f(x_0)\ominus f(x_0+h)$; $f(x_0)\ominus f(x_0-h)$ and the limits

$$\lim_{h\to 0^+}\frac{f(x_0)\ominus f(x_0+h)}{(-h)}=\lim_{h\to 0^+}\frac{f(x_0)\ominus f(x_0-h)}{h}=f'(x_0)$$

Here the limits are taken in the metric space $(E\,,\,D)$ at the end points of (a,b) we consider only one-sided derivatives.

The following theorem (see [8]) allows us to consider case (i) or (ii) of the previous definition almost everywhere in the domain of the functions under discussion.

Theorem 2.6. *Let $f:(a,b)\to E$ be strongly generalized differentiable on each point $x\in(a,b)$ in the sense of Definition 2.3, (iii) or (iv). Then $f'(x)\in\mathbb{R}$ for all $x\in(a,b)$.*

Another result concerned the derivation of a fuzzy constant multiplied by a crisp function (see [8]):

Theorem 2.7. *If $g:(a,b)\to\mathbb{R}$ is differentiable on (a,b) such that g' has at most a finite number of roots in (a,b) and $c\in E$, then $f(x)=g(x).c$ is strongly generalized differentiable on (a,b) and $f'(x)=g'(x).c$, for all $x\in(a,b)$.*

3. Definition and properties of operators J and K

Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be a fixed continuous nonnegative crisp function defined on $\mathbb{R}_+$.
We consider $F:\mathbb{R}_+\to\mathbb{R}$ the unique primitive of f which vanishing at $t=0$ i.e $F(0)=0$.
It is well known that F is nondecreasing and is nonnegative on $\mathbb{R}_+$.
Denote $\mathcal{C}^1(\mathbb{R}_+)$ the (vectorial) space of all functions of $\mathcal{C}^1$ class on $\mathbb{R}_+$ into $\mathbb{R}$ and $\mathcal{C}^1_0(\mathbb{R}_+)$ its subspace defined by

$$\mathcal{C}^1_0(\mathbb{R}_+)=\{g:\mathbb{R}_+\to\mathbb{R}/g \text{ is of } \mathcal{C}^1 \text{ class } \textit{and } g(0)=0\}$$

We define two operators J and K as follows
$J,\, K : \mathcal{C}^1(\mathbb{R}_+) \times \mathcal{C}^1(\mathbb{R}_+) \to \mathcal{C}^1(\mathbb{R}_+) \times \mathcal{C}^1(\mathbb{R}_+)$,

$$J\,(g,h)\,(x) = (J_1 g(x), J_1 h(x)) = \left(\int_0^x f(t)g(t)dt, \int_0^x f(t)h(t)dt\right),$$

and

$$K\,(g,h)\,(x) = (J_1 h(x), J_1 g(x)) = \left(\int_0^x f(t)h(t)dt, \int_0^x f(t)g(t)dt\right),$$

where $J_1 g(x) = \displaystyle\int_0^x f(t)g(t)dt$ denote the operator J used in [2], for all $x \in \mathbb{R}$. For short, we can write $J\,(g(x), h(x))$ instead of $J\,(g,h)\,(x)$ and $K\,(g(x), h(x))$ instead of $K\,(g,h)\,(x)$.
We recall that, if $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function, then its primitive $\phi : x \mapsto \displaystyle\int_0^x \varphi(t)dt$ is the unique element of $\mathcal{C}_0^1(\mathbb{R}_+)$ verifying $\phi' = \varphi$.

Lemma 3.1. (a) *The linear operator* $I - J : \mathcal{C}^1(\mathbb{R}_+) \times \mathcal{C}^1(\mathbb{R}_+) \to \mathcal{R}(I-J)$, *with* $\mathcal{R}(I-J) = \mathcal{C}_0^1(\mathbb{R}_+) \times \mathcal{C}_0^1(\mathbb{R}_+)$, *is bijective and its inverse is given by*

$$(I-J)^{-1}(g,h) = (g,h) + J(g,h) + J^2(g,h) + \cdots + J^n(g,h) + R_n(g,h)$$

where I *is the identity operator,* $J^2 = J \circ J$, $J^n = J \circ J^{n-1}$, *for* $n \geq 2$ *and the remainder term* $R_n(g,h)$ *is defined by*

$$R_n(g,h)(x) = e^{F(x)}\left(\int_0^x f(t)(J_1^n g)(t)e^{-F(t)}dt, \int_0^x f(t)(J_1^n h)(t)e^{-F(t)}dt\right),$$

for $x \geq 0$. *Moreover, the sequence* $R_n(g,h)$ *converges to 0 uniformly with respect to* x *in each compact subset* $[0,a]$ *of* $\mathbb{R}_+$. *Therefore, we have*

$$(I-J)^{-1} = I + J + J^2 + J^3 + J^4 + \cdots = \sum_{n=0}^{+\infty} J^n.$$

(b) *The linear operator* $I + J : \mathcal{C}^1(\mathbb{R}_+) \times \mathcal{C}^1(\mathbb{R}_+) \to \mathcal{R}(I+J)$ *is bijective, with* $\mathcal{R}(I+J) = \mathcal{C}_0^1(\mathbb{R}_+) \times \mathcal{C}_0^1(\mathbb{R}_+)$, *and its inverse is given by*

$$(I+J)^{-1}(g,h) = (g,h) - J(g,h) + J^2(g,h) + \cdots + (-1)^n J^n(g,h) + (-1)^{n+1} R_n(g,h)$$

Moreover, the sequence $R_n(g,h)$ *converges to 0 uniformly with respect to* x *in each compact subset* $[0,a]$ *of* $\mathbb{R}_+$. *Therefore, we have*

$$(I+J)^{-1} = I - J + J^2 - J^3 + J^4 + \cdots = \sum_{n=0}^{+\infty} (-1)^n J^n.$$

Proof. (a) Step 1 : It is clear that J is linear, then let $(y,z) \in \ker(I-J)$ i.e $(I-J)(y,z) = (0,0)$. Thus

$$\begin{cases} y(x) = \int_0^x f(t)y(t)dt \\ z(x) = \int_0^x f(t)z(t)dt \end{cases}$$

Since y and z are of $\mathcal{C}^1$ class, then by derivation

$$\begin{cases} y'(x) - f(x)y(x) = 0; \quad y(0) = 0 \\ z'(x) - f(x)z(x) = 0; \quad z(0) = 0 \end{cases}$$

So, y and z are solutions of the same Cauchy problem, which has as unique solution $y = z = 0$. Then, $I - J$ is injective.

Step 2: Let $(g,h) \in \mathcal{C}_0^1(\mathbb{R}_+) \times \mathcal{C}_0^1(\mathbb{R}_+)$, we prove that the following equation, has at least one solution $(y,z) \in \mathcal{C}^1(\mathbb{R}_+) \times \mathcal{C}^1(\mathbb{R}_+) : (I-J)(y,z) = (g,h)$ i.e.,

$$\begin{cases} y(x) = \int_0^x f(t)y(t)dt + g(x) \\ z(x) = \int_0^x f(t)z(t)dt + h(x) \end{cases} \tag{3.1}$$

y and z are of $\mathcal{C}^1$ class, then we deduce by derivation

$$\begin{cases} y'(x) - f(x)y(x) = g'(x); \quad y(0) = g(0) = 0 \\ z'(x) - f(x)z(x) = h'(x); \quad z(0) = h(0) = 0 \end{cases} \tag{3.2}$$

Using the variation of constants formula for differential equations, we get

$$\begin{cases} y(x) = e^{F(x)} \int_0^x g'(t)e^{F(t)}dt \\ z(x) = e^{F(x)} \int_0^x h'(t)e^{F(t)}dt \end{cases} \tag{3.3}$$

Conversely, we suppose that equations (3.3) hold. Since $F'(t) = f(t)$, then $(I-J)(y(x), z(x))$ has as components $(\varphi(x), \psi(x))$ given by

$$\begin{cases} \varphi(x) = e^{F(x)} \int_0^x g'(t)e^{-F(t)}dt - \int_0^x F'(t)e^{F(t)} \int_0^t g'(s)e^{-F(s)}dsdt \\ \psi(x) = e^{F(x)} \int_0^x h'(t)e^{-F(t)}dt - \int_0^x F'(t)e^{F(t)} \int_0^t h'(s)e^{-F(s)}dsdt \end{cases} \tag{3.4}$$

Using integration by parts, we obtain

$$\begin{cases} \varphi(x) = \int_0^x e^{F(t)}g'(t)e^{-F(t)}dt = \int_0^x g'(t)dt = g(x) \\ \psi(x) = \int_0^x e^{F(t)}h'(t)e^{-F(t)}dt = \int_0^x h'(t)dt = h(x) \end{cases}$$

Therefore $(I-J)(y(x), z(x)) = (g(x), h(x))$.
Consequently, the operator $(I-J)$ is bijective and we have

$$(I-J)^{-1}(g(x), h(x)) = \left(e^{F(x)} \int_0^x g'(t)e^{-F(t)}dt, e^{F(x)} \int_0^x h'(t)e^{-F(t)}dt \right)$$

Step 3 : Note that $(J_1g)'(x) = f(x)g(x)$ and by induction, we have

$$(J_1^n g)'(x) = f(x)J_1^{n-1}g(x); \quad n \geq 1; x \geq 0.$$

We denote $(I-J)^{-1}(g(x), h(x)) = (\varphi(x), \psi(x))$. Since all the functions g, J_1g, $\dots$, $J_1^n g$ are in $\mathcal{C}_0^1(\mathbb{R}_+)$, then $g(0) = J_1g(0) = \dots = J_1^n g(0) = 0$.

We conclude using successive integrations by parts that

$$\begin{aligned}
\varphi(x) &= e^{F(x)}\int_0^x g'(t)e^{-F(t)}dt\\
&= g(x)+e^{F(x)}\int_0^x g(t)f(t)e^{-F(t)}dt\\
&= g(x)+e^{F(x)}\int_0^x (J_1g)'(t)e^{-F(t)}dt\\
&= g(x)+J_1g(x)+e^{F(x)}\int_0^x J_1g(t)f(t)e^{-F(t)}dt\\
&= g(x)+J_1g(x)+e^{F(x)}\int_0^x (J_1^2g)'(t)e^{-F(t)}dt\\
&= g(x)+J_1g(x)+J_1^2g(x)+e^{F(x)}\int_0^x J_1^2g(t)f(t)e^{-F(t)}dt
\end{aligned}$$

And by induction, we deduce that

$$\varphi(x)=g(x)+J_1g(x)+J_1^2g(x)+\cdots+J_1^ng(x)+e^{F(x)}\int_0^x J_1^ng(t)f(t)e^{-F(t)}dt$$

Similarly, we can prove that

$$\begin{aligned}
\psi(x) &= e^{F(x)}\int_0^x h'(t)e^{-F(t)}dt\\
\psi(x) &= h(x)+J_1h(x)+J_1^2h(x)+\cdots+J_1^nh(x)+e^{F(x)}\int_0^x J_1^nh(t)f(t)e^{-F(t)}dt
\end{aligned}$$

Hence

$$\begin{aligned}
(I-J)^{-1}(g,h)(x) &= (g(x)+J_1g(x)+\cdots+J_1^ng(x),h(x)+J_1h(x)+\cdots+J_1^nh(x))\\
&\quad+e^{F(x)}\left(\int_0^x J_1^ng(t)f(t)e^{-F(t)}dt,\int_0^x J_1^nh(t)f(t)e^{-F(t)}dt\right)\\
&= (g,h)(x)+J(g,h)(x)+J^2(g,h)(x)+\cdots\\
&\qquad+J^n(g,h)(x)+R_n(g,h)(x)
\end{aligned}$$

$$(I-J)^{-1}(g,h)=(g,h)+J(g,h)+J^2(g,h)+\cdots+J^n(g,h)+R_n(g,h)$$

and the remainder term $R_n(g,h)$ is given for $x\geq 0$ by

$$R_n(g,h)(x)=\left(e^{F(x)}\int_0^x f(t)(J_1^ng)(t)e^{-F(t)}dt,e^{F(x)}\int_0^x f(t)(J_1^nh)(t)e^{-F(t)}dt\right)$$

Let $a>0$ and denote $M=\sup\limits_{t\in[0,a]}|g(t)|$. Therefore

$$|J_1g(x)|=\left|\int_0^x f(t)g(t)dt\right|\leq M\int_0^x f(t)dt=MF(x)$$

Similarly, we have

$$\left|J_1^2g(x)\right|=\left|\int_0^x f(t)J_1g(t)dt\right|\leq M\int_0^x f(t)F(t)dt=M\frac{(F(x))^2}{2!}$$

Suppose that for $n \geq 1$, we have

$$\left|J_1^{n-1} g(x)\right| \leq M \frac{(F(x))^{n-1}}{(n-1)!}$$

Then by induction, we get

$$|J_1^n g(x)| = \left|\int_0^x f(t) J_1^{n-1} g(t) dt\right| \leq M \int_0^x f(t) \frac{(F(t))^{n-1}}{(n-1)!} dt = M \frac{(F(x))^n}{n!}$$

Since the function F is nondecreasing, then for all $x \in [0, a]$

$$\begin{aligned} e^{F(x)} \left|\int_0^x J_1^n g(t) f(t) e^{-F(t)} dt\right| &\leq M e^{F(a)} \int_0^x f(t) \frac{(F(t))^n}{n!} e^{-F(t)} dt \\ &\leq M e^{F(a)} \frac{(F(a))^n}{n!} \int_0^a f(t) e^{-F(t)} dt \\ &= M e^{F(a)} \frac{(F(a))^n}{n!} \left[1 - e^{-F(a)}\right] \end{aligned}$$

Then

$$e^{F(x)} \left|\int_0^x J_1^n g(t) f(t) e^{-F(t)} dt\right| \leq M e^{F(a)} \frac{(F(a))^n}{n!}$$

The convergence of the series $\sum_{n \geq 0} \frac{(F(a))^n}{n!}$ implies that $\frac{(F(a))^n}{n!} \to 0$ as $n \to \infty$. So

$$e^{F(x)} \left(\int_0^x J_1^n g(t) f(t) e^{-F(t)} dt\right) \to 0 \quad \text{as}\, n \to \infty, \text{uniformly with respect to}\, x$$

in $[0, a]$. And analogously, we obtain

$$e^{F(x)} \left(\int_0^x J_1^n h(t) f(t) e^{-F(t)} dt\right) \to 0 \quad \text{as}\, n \to \infty, \text{uniformly with respect to}\, x$$

in $[0, a]$. Therefore $R_n(g, h)$ converges to 0 uniformly with respect to x in each compact subset $[0, a]$ of $\mathbb{R}_+$. Consequently

$$(I - J)^{-1} = I + J + J^2 + J^3 + J^4 + \ldots = \sum_{n=0}^{+\infty} J^n$$

(b) The second part of the lemma can be proved similarly. □

Analogously, we can prove the following result:

Lemma 3.2. *(a) The linear operator $I - K : \mathcal{C}^1(\mathbb{R}_+) \times \mathcal{C}^1(\mathbb{R}_+) \to \mathcal{R}(I - K)$ is bijective, with $\mathcal{R}(I - K) = \mathcal{C}_0^1(\mathbb{R}_+) \times \mathcal{C}_0^1(\mathbb{R}_+)$, and its inverse is given by*

$$(I - K)^{-1}(g, h) = (g, h) + K(g, h) + K^2(g, h) + \cdots + K^n(g, h) + R'_n(g, h)$$

where the remainder term $R'_n(g, h)$ is defined by

$$R'_n(g, h)(x) = e^{F(x)} \left(\int_0^x f(t)(J_1^n h)(t) e^{-F(t)} dt, \int_0^x f(t)(J_1^n g)(t) e^{-F(t)} dt\right),$$

for $x \geq 0$. Moreover, the sequence $R'_n(g,h)$ converges to 0 uniformly with respect to x in each compact subset $[0,a]$ of $\mathbb{R}_+$. Therefore, we have

$$(I-K)^{-1} = I + K + K^2 + K^3 + K^4 + \cdots = \sum_{n=0}^{+\infty} K^n.$$

(b) The linear operator $I+K : \mathcal{C}^1(\mathbb{R}_+) \times \mathcal{C}^1(\mathbb{R}_+) \to \mathcal{R}(I+K)$ is bijective, with $\mathcal{R}(I+K) = \mathcal{C}^1_0(\mathbb{R}_+) \times \mathcal{C}^1_0(\mathbb{R}_+)$, and its inverse is given by

$$(I+K)^{-1}(g,h) = (g,h) - K(g,h) + K^2(g,h) + \cdots + (-1)^n K^n(g,h) + (-1)^{n+1} R'_n(g,h)$$

Moreover, the sequence $R'_n(g,h)$ converges to 0 uniformly with respect to x in each compact subset $[0,a]$ of $\mathbb{R}_+$. Therefore, we have

$$(I+K)^{-1} = I - K + K^2 - K^3 + K^4 + \cdots = \sum_{n=0}^{+\infty} (-1)^n K^n.$$

4. Method for first order fuzzy linear differential equations

Our aim is to present an operator method to solve the following first order fuzzy homogeneous linear differential equations, under strongly generalized differentiability:

$$(P_1) \qquad \begin{cases} y'(x) = f(x)y(x) \\ y(0) = y_0 \in E \end{cases}$$

and

$$(P'_1) \qquad \begin{cases} y'(x) = -f(x)y(x) \\ y(0) = y_0 \in E \end{cases}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nonnegative crisp function defined on $\mathbb{R}_+$.
We need to use the operators J and K defined above.
Let $F : \mathbb{R}_+ \to \mathbb{R}$ the unique primitive of f which vanishes at $t = 0$ i.e $F(0) = 0$.

4.1. **Resolution of equation (P_1).** (a) If $y(x)$ is (i)-strongly generalized differentiable : then (P_1) is equivalent to its following integral form

$$\begin{cases} \underline{y}(x,\alpha) = \underline{y}(0,\alpha) + \int_0^x f(t)\underline{y}(t,\alpha)dt \\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha) + \int_0^x f(t)\overline{y}(t,\alpha)dt \end{cases}$$

which can be written as follows

$$\left(\underline{y}(x,\alpha), \overline{y}(x,\alpha)\right) = \left(\underline{y}(0,\alpha), \overline{y}(0,\alpha)\right) + \left(\int_0^x f(t)\underline{y}(t,\alpha)dt, \int_0^x f(t)\overline{y}(t,\alpha)dt\right)$$

Hence, (P_1) is equivalent to

$$\left(\underline{y}(x,\alpha), \overline{y}(x,\alpha)\right) = \left(\underline{y}(0,\alpha), \overline{y}(0,\alpha)\right) + J\left(\underline{y}(x,\alpha), \overline{y}(x,\alpha)\right)$$

This identity can be expressed in the following form

$$(I-J)\left(\underline{y}(x,\alpha), \overline{y}(x,\alpha)\right) = \left(\underline{y}(0,\alpha), \overline{y}(0,\alpha)\right)$$

Since $I - J$ is bijective, then

$$\left(\underline{y}(x,\alpha), \overline{y}(x,\alpha)\right) = (I-J)^{-1}\left(\underline{y}(0,\alpha), \overline{y}(0,\alpha)\right)$$

Consequently (P_1) is equivalent to

$$\left(\underline{y}(x,\alpha),\overline{y}(x,\alpha)\right) = \left(I + J + J^2 + J^3 + \cdots + J^n + \ldots\right)\left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right)$$

In the other hand, we have

$$\begin{aligned} J\left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right) &= \left(\int_0^x f(t)\underline{y}(0,\alpha)dt, \int_0^x f(t)\overline{y}(0,\alpha)dt\right) \\ &= \left(F(x)\underline{y}(0,\alpha), F(x)\overline{y}(0,\alpha)\right) = F(x)\left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right) \end{aligned}$$

Hence

$$\begin{aligned} J^2\left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right) &= \left(\int_0^x f(t)F(t)\underline{y}(0,\alpha)dt, \int_0^x f(t)F(t)\overline{y}(0,\alpha)dt\right) \\ &= \left(\int_0^x F'(t)F(t)\underline{y}(0,\alpha)dt, \int_0^x F'(t)F(t)\overline{y}(0,\alpha)dt\right) \\ &= \left(\frac{(F(x))^2}{2}\underline{y}(0,\alpha), \frac{(F(x))^2}{2}\overline{y}(0,\alpha)\right) \end{aligned}$$

Then, by induction we can establish that for all $n \in \mathbb{N}$, we have

$$\begin{aligned} J^n\left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right) &= \left(\frac{(F(x))^n}{n!}\underline{y}(0,\alpha), \frac{(F(x))^n}{n!}\overline{y}(0,\alpha)\right) \\ &= \frac{(F(x))^n}{n!}\left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right). \end{aligned}$$

Therefore

$$\begin{aligned} \left(\underline{y}(x,\alpha),\overline{y}(x,\alpha)\right) &= \sum_{n=0}^{+\infty}\frac{(F(x))^n}{n!}\left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right) \\ &= \exp(F(x))\left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right). \end{aligned}$$

Then finally , we deduce that

$$\begin{cases} \underline{y}(x,\alpha) = \underline{y}(0,\alpha)\exp(F(x)) \\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha)\exp(F(x)) \end{cases}$$

(b) If $y(x)$ is (ii)-strongly generalized differentiable : then (P_1) is equivalent to its following integral form

$$\begin{cases} \underline{y}(x,\alpha) = \underline{y}(0,\alpha) + \int_0^x f(t)\overline{y}(t,\alpha)dt \\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha) + \int_0^x f(t)\underline{y}(t,\alpha)dt \end{cases}$$

which can be written as follows

$$\left(\underline{y}(x,\alpha),\overline{y}(x,\alpha)\right) = \left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right) + \left(\int_0^x f(t)\overline{y}(t,\alpha)dt, \int_0^x f(t)\underline{y}(t,\alpha)dt\right)$$

Hence, (P_1) is equivalent to

$$\left(\underline{y}(x,\alpha),\overline{y}(x,\alpha)\right) = \left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right) + K\left(\underline{y}(x,\alpha),\overline{y}(x,\alpha)\right)$$

This identity can be expressed in the following form

$$(I - K)\left(\underline{y}(x,\alpha),\overline{y}(x,\alpha)\right) = \left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right)$$

Since $I - K$ is bijective, then

$$\left(\underline{y}(x,\alpha), \overline{y}(x,\alpha)\right) = (I-K)^{-1}\left(\underline{y}(0,\alpha), \overline{y}(0,\alpha)\right)$$

Consequently (P_1) is equivalent to following equation

$$\left(\underline{y}(x,\alpha), \overline{y}(x,\alpha)\right) = (I + K + K^2 + K^3 + K^4 + \ldots)\left(\underline{y}(0,\alpha), \overline{y}(0,\alpha)\right)$$

which we can write

$$\begin{aligned}\left(\underline{y}(x,\alpha), \overline{y}(x,\alpha)\right) = {} & (I + K^2 + K^4 + \ldots)\left(\underline{y}(0,\alpha), \overline{y}(0,\alpha)\right) \\ & + (K + K^3 + K^5 + \ldots)\left(\underline{y}(0,\alpha), \overline{y}(0,\alpha)\right)\end{aligned}$$

In the other hand, we have

$$\begin{aligned} K\left(\underline{y}(0,\alpha), \overline{y}(0,\alpha)\right) &= \left(\int_0^x f(t)\overline{y}(0,\alpha)dt, \int_0^x f(t)\underline{y}(0,\alpha)dt\right) \\ &= \left(F(x)\overline{y}(0,\alpha), F(x)\underline{y}(0,\alpha)\right)\end{aligned}$$

Hence

$$\begin{aligned} K^2\left(\underline{y}(0,\alpha), \overline{y}(0,\alpha)\right) &= \left(\int_0^x f(t)F(t)\underline{y}(0,\alpha)dt, \int_0^x f(t)F(t)\overline{y}(0,\alpha)dt\right) \\ &= \left(\int_0^x F'(t)F(t)\underline{y}(0,\alpha)dt, \int_0^x F'(t)F(t)\overline{y}(0,\alpha)dt\right) \\ &= \frac{(F(x))^2}{2}\left(\underline{y}(0,\alpha), \overline{y}(0,\alpha)\right)\end{aligned}$$

Then, by induction, we have

$$K^{2n}\left(\underline{y}(0,\alpha), \overline{y}(0,\alpha)\right) = \frac{(F(x))^{2n}}{(2n)!}\left(\underline{y}(0,\alpha), \overline{y}(0,\alpha)\right)$$

and

$$K^{2n+1}\left(\underline{y}(0,\alpha), \overline{y}(0,\alpha)\right) = \frac{(F(x))^{2n+1}}{(2n+1)!}\left(\overline{y}(0,\alpha), \underline{y}(0,\alpha)\right)$$

Therefore

$$\begin{aligned}\left(\underline{y}(x,\alpha), \overline{y}(x,\alpha)\right) = {} & \sum_{n=0}^{+\infty}\frac{(F(x))^{2n}}{(2n)!}\left(\underline{y}(0,\alpha), \overline{y}(0,\alpha)\right) \\ & + \sum_{n=0}^{+\infty}\frac{(F(x))^{2n+1}}{(2n+1)!}\left(\overline{y}(0,\alpha), \underline{y}(0,\alpha)\right)\end{aligned}$$

Thus

$$\left(\underline{y}(x,\alpha), \overline{y}(x,\alpha)\right) = \cosh\left(F(x)\right)\left(\underline{y}(0,\alpha), \overline{y}(0,\alpha)\right) + \sinh\left(F(x)\right)\left(\overline{y}(0,\alpha), \underline{y}(0,\alpha)\right)$$

Then, we deduce that

$$\begin{cases} \underline{y}(x,\alpha) = \underline{y}(0,\alpha)\cosh\left(F(x)\right) + \overline{y}(0,\alpha)\sinh\left(F(x)\right) \\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha)\cosh\left(F(x)\right) + \underline{y}(0,\alpha)\sinh\left(F(x)\right)\end{cases}$$

So, we can resume the results above in the following proposition.

Proposition 4.1. *(a) If $y(x)$ is (i)-strongly generalized differentiable, then the solution of (P_1) is given by*

$$\begin{cases} \underline{y}(x,\alpha) = \underline{y}(0,\alpha)\exp(F(x)) \\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha)\exp(F(x)) \end{cases}$$

(b) If $y(x)$ is (ii)-strongly generalized differentiable, then the solution of (P_1) is given by

$$\begin{cases} \underline{y}(x,\alpha) = \underline{y}(0,\alpha)\cosh(F(x)) + \overline{y}(0,\alpha)\sinh(F(x)) \\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha)\cosh(F(x)) + \underline{y}(0,\alpha)\sinh(F(x)) \end{cases}$$

Remark 4.2. We recall that the length of the solution $y(x,\alpha)$ is given by

$$len\,(y(x,\alpha)) = \overline{y}(x,\alpha) - \underline{y}(x,\alpha)$$

Assume that $\lim\limits_{x\to\infty} F(x) = \infty$ and $len\,(y(0,\alpha)) > 0$ i.e $y_0 \in E \setminus \mathbb{R}$.

(1) Under (i)-strong generalized differentiability, then

$$len\,(y(x,\alpha)) = e^{F(x)}\left[\overline{y}(0,\alpha) - \underline{y}(0,\alpha)\right] = e^{F(x)} len\,(y(0,\alpha))$$

Therefore

$$len\,(y(x,\alpha)) \to \infty \text{ as } x \to \infty.$$

So, this solution is asymptotically unstable.

(2) Under (ii)-strong generalized differentiability, then

$$len\,(y(x,\alpha)) \quad = \quad e^{-F(x)}\left[\overline{y}(0,\alpha) - \underline{y}(0,\alpha)\right] = e^{-F(x)} len\,(y(0,\alpha))$$

Therefore

$$len\,(y(x,\alpha)) \to 0 \text{ as } x \to \infty.$$

So, this solution is asymptotically stable.

If f is negative and $\lim\limits_{x\to\infty} F(x) = -\infty$ then, the solution became asymptotically stable in the first case, and asymptotically unstable in the second case.
To solve (eq.1) under the conditions $\lim\limits_{x\to\infty} F(x) = \infty$ (respectively $\lim\limits_{x\to\infty} F(x) = -\infty$)and $len\,(y(0,\alpha)) > 0$, we can choose the (ii)-differentiability (respectively (i)-differentiability)as the appropriate kind of differentiability, because in this case the behavior of the fuzzy solution is the same as the deterministic solution (for more details see [2] and [8]).

4.2. **Resolution of equation (P_1').** (a) If $y(x)$ is (i)-strongly generalized differentiable : then (P_1') is equivalent to its following integral form

$$\begin{cases} \underline{y}(x,\alpha) = \underline{y}(0,\alpha) - \int_0^x f(t)\overline{y}(t,\alpha)dt \\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha) - \int_0^x f(t)\underline{y}(t,\alpha)dt \end{cases}$$

Hence, (P_1') is equivalent to

$$\left(\underline{y}(x,\alpha), \overline{y}(x,\alpha)\right) = \left(\underline{y}(0,\alpha), \overline{y}(0,\alpha)\right) - K\left(\underline{y}(x,\alpha), \overline{y}(x,\alpha)\right)$$

This identity can be written in the following form

$$(I+K)\left(\underline{y}(x,\alpha), \overline{y}(x,\alpha)\right) = \left(\underline{y}(0,\alpha), \overline{y}(0,\alpha)\right)$$

Therefore

$$\left(\underline{y}(x,\alpha), \overline{y}(x,\alpha)\right) = (I+K)^{-1}\left(\underline{y}(0,\alpha), \overline{y}(0,\alpha)\right)$$

Consequently (P_1') is equivalent to following equation

$$\left(\underline{y}(x,\alpha),\overline{y}(x,\alpha)\right) = (I - K + K^2 - K^3 + K^4 - \dots)\left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right)$$

which we can write

$$\begin{aligned}\left(\underline{y}(x,\alpha),\overline{y}(x,\alpha)\right) = (I + K^2 + K^4 + \dots)\left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right)\\ - (K + K^3 + K^5 + \dots)\left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right)\end{aligned}$$

Therefore

$$\left(\underline{y}(x,\alpha),\overline{y}(x,\alpha)\right) = \sum_{n=0}^{+\infty}\frac{(F(x))^{2n}}{(2n)!}\left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right) - \sum_{n=0}^{+\infty}\frac{(F(x))^{2n+1}}{(2n+1)!}\left(\overline{y}(0,\alpha),\underline{y}(0,\alpha)\right)$$

Thus

$$\left(\underline{y}(x,\alpha),\overline{y}(x,\alpha)\right) = \cosh(F(x))\left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right) - \sinh(F(x))\left(\overline{y}(0,\alpha),\underline{y}(0,\alpha)\right)$$

Then, we deduce that

$$\begin{cases}\underline{y}(x,\alpha) = \underline{y}(0,\alpha)\cosh(F(x)) - \overline{y}(0,\alpha)\sinh(F(x))\\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha)\cosh(F(x)) - \underline{y}(0,\alpha)\sinh(F(x))\end{cases}$$

(b) If $y(x)$ is (ii)-strongly generalized differentiable : then (P_1') is equivalent to its following integral form

$$\begin{cases}\underline{y}(x,\alpha) = \underline{y}(0,\alpha) - \int_0^x f(t)\underline{y}(t,\alpha)dt\\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha) - \int_0^x f(t)\overline{y}(t,\alpha)dt\end{cases}$$

which can be written as follows

$$\left(\underline{y}(x,\alpha),\overline{y}(x,\alpha)\right) = \left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right) - J\left(\underline{y}(x,\alpha),\overline{y}(x,\alpha)\right)$$

which can be expressed in the following form

$$(I + J)\left(\underline{y}(x,\alpha),\overline{y}(x,\alpha)\right) = \left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right)$$

Therefore

$$\left(\underline{y}(x,\alpha),\overline{y}(x,\alpha)\right) = (I + J)^{-1}\left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right)$$

Consequently (P_1') is equivalent to

$$\left(\underline{y}(x,\alpha),\overline{y}(x,\alpha)\right) = (I - J + J^2 - J^3 + J^4 - \dots)\left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right)$$

Therefore

$$\begin{aligned}\left(\underline{y}(x,\alpha),\overline{y}(x,\alpha)\right) &= \sum_{n=0}^{+\infty}\frac{(-F(x))^{n}}{n!}\left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right)\\ &= \exp(-F(x))\left(\underline{y}(0,\alpha),\overline{y}(0,\alpha)\right)\end{aligned}$$

Then finally, we have

$$\begin{cases}\underline{y}(x,\alpha) = \underline{y}(0,\alpha)\exp(-F(x))\\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha)\exp(-F(x))\end{cases}$$

So, we can resume the results above in the following proposition.

Proposition 4.3.

(a) *If $y(x)$ is (i)-strongly generalized differentiable, then the solution of* (P'_1) *is given by*

$$\begin{cases} \underline{y}(x,\alpha) = \underline{y}(0,\alpha)\cosh(F(x)) - \overline{y}(0,\alpha)\sinh(F(x)) \\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha)\cosh(F(x)) - \underline{y}(0,\alpha)\sinh(F(x)) \end{cases}$$

(b) *If $y(x)$ is (ii)-strongly generalized differentiable, then the solution of* (P'_1) *is given by*

$$\begin{cases} \underline{y}(x,\alpha) = \underline{y}(0,\alpha)\exp(-F(x)) \\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha)\exp(-F(x)) \end{cases}$$

Remark 4.4. Assume that $\lim_{x\to\infty} F(x) = \infty$ and $len\,(y(0,\alpha)) > 0$ i.e $y_0 \in E \setminus \mathbb{R}$.

(1) Under (i)-strong generalized differentiability, then

$$len\,(y(x,\alpha)) = e^{F(x)} len\,(y(0,\alpha))$$

Therefore

$$len\,(y(x,\alpha)) \to \infty \text{ as } x \to \infty.$$

So, this solution is asymptotically unstable.

(2) Under (ii)-strong generalized differentiability, then

$$len\,(y(x,\alpha)) = e^{-F(x)} len\,(y(0,\alpha))$$

Therefore

$$len\,(y(x,\alpha)) \to 0 \text{ as } x \to \infty.$$

So, this solution is asymptotically stable.

If f is negative and $\lim_{x\to\infty} F(x) = -\infty$, the solution became asymptotically stable in the first case, and asymptotically unstable in the second case.
To solve (P'_1) under the condition $\lim_{x\to\infty} F(x) = \infty$ (respectively $\lim_{x\to\infty} F(x) = -\infty$) and $len\,(y(0,\alpha)) > 0$, we can choose the (ii)-differentiability (respectively (i)-differentiability)as the appropriate kind of differentiability, because in this case the behavior of the fuzzy solution is the same as the deterministic solution (for more details see [2] and [**?**]).

4.3. **Inhomogeneous first order fuzzy linear differential equations.**
We consider the following first order fuzzy inhomogeneous linear differential equations:

$$(P_2) \qquad \begin{cases} y'(x) = f(x)y(x) + P(x) \\ y(0) = y_0 \in E \end{cases}$$

and

$$(P'_2) \qquad \begin{cases} y'(x) = -f(x)y(x) + P(x) \\ y(0) = y_0 \in E \end{cases}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nonnegative crisp function defined on $\mathbb{R}_+$ and P is a (crisp) continuous function.
Using the variation of constants formula and the calculs above, we deduce easily the following results.

Proposition 4.5.

(a) *If $y(x)$ is (i)-strongly generalized differentiable, then the solution of (P_2) is given by*

$$\begin{cases} \underline{y}(x,\alpha) = \underline{y}(0,\alpha)e^{F(x)} + e^{F(x)}\int_0^x P(t)e^{-F(t)}dt \\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha)e^{F(x)} + e^{F(x)}\int_0^x P(t)e^{-F(t)}dt \end{cases}$$

(b) *If $y(x)$ is (ii)-strongly generalized differentiable, then the solution of (P_2) is*

$$\begin{cases} \underline{y}(x,\alpha) = \underline{y}(0,\alpha)\cosh(F(x)) + \overline{y}(0,\alpha)\sinh(F(x)) + e^{F(x)}\int_0^x P(t)e^{-F(t)}dt \\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha)\cosh(F(x)) + \underline{y}(0,\alpha)\sinh(F(x)) + e^{F(x)}\int_0^x P(t)e^{-F(t)}dt \end{cases}$$

Proposition 4.6.

(a) *If $y(x)$ is (i)-strongly generalized differentiable, then the solution of (P_2') is given by*

$$\begin{cases} \underline{y}(x,\alpha) = \underline{y}(0,\alpha)\cosh(F(x)) - \overline{y}(0,\alpha)\sinh(F(x)) + e^{-F(x)}\int_0^x P(t)e^{F(t)}dt \\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha)\cosh(F(x)) - \underline{y}(0,\alpha)\sinh(F(x)) + e^{-F(x)}\int_0^x P(t)e^{F(t)}dt \end{cases}$$

(b) *If $y(x)$ is (ii)-strongly generalized differentiable, then the solution of (P_2') is*

$$\begin{cases} \underline{y}(x,\alpha) = \underline{y}(0,\alpha)e^{-F(x)} + e^{-F(x)}\int_0^x P(t)e^{F(t)}dt \\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha)e^{-F(x)} + e^{-F(x)}\int_0^x P(t)e^{F(t)}dt \end{cases}$$

5. Numeric examples

The following examples 5.1 and 5.2 were studied in [2], under one type of differentiability.

Example 5.1.

$$(P_3) \qquad \begin{cases} y'(x) = y(x) \\ y(0) = y_0 \in E \end{cases}$$

Here $f(x) = 1$, then $F(x) = x$.

(a) If $y(x)$ is (i)-strongly generalized differentiable, then the solution of (P_3) is

$$\begin{cases} \underline{y}(x,\alpha) = \underline{y}(0,\alpha)e^{x} \\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha)e^{x} \end{cases}$$

this is the solution given in [2] for the same example, but it is asymptotically unstable.

(b) If $y(x)$ is (ii)-strongly generalized differentiable, then the solution of (P_3) is

$$\begin{cases} \underline{y}(x,\alpha) = \underline{y}(0,\alpha)\cosh(x) + \overline{y}(0,\alpha)\sinh(x) \\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha)\cosh(x) + \underline{y}(0,\alpha)\sinh(x) \end{cases}$$

this solution (not given in [2]) is asymptotically stable.

Example 5.2.

$$(P_3') \qquad \begin{cases} y'(x) = -y(x) + x + 1 \\ y(0) = y_0 \in E \end{cases}$$

Taking $P(x) = x + 1$ in (P_3'), we get

(a) If $y(x)$ is (i)-strongly generalized differentiable, then the solution of (P_3') is

$$\begin{cases} \underline{y}(x,\alpha) = \underline{y}(0,\alpha)e^{-x} + x \\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha)e^{-x} + x \end{cases}$$

this is the solution given in [2], which is asymptotically stable.

(b) If $y(x)$ is (ii)-strongly generalized differentiable, then the solution of (P_3') is

$$\begin{cases} \underline{y}(x,\alpha) = \underline{y}(0,\alpha)\cosh(x) - \overline{y}(0,\alpha)\sinh(x) + x \\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha)\cosh(x) - \underline{y}(0,\alpha)\sinh(x) + x \end{cases},$$

this solution (not given in [2]) is asymptotically unstable.

Example 5.3. We consider the equation with variable coefficient $f(x) = x^n$:

$$(P_4) \qquad \begin{cases} y'(x) = x^n y(x) \\ y(0) = y_0 \in E \end{cases}$$

(a) If $y(x)$ is (i)-strongly generalized differentiable, then the solution of (P_4) is

$$\begin{cases} \underline{y}(x,\alpha) = \underline{y}(0,\alpha)\exp\left(\frac{x^{n+1}}{n+1}\right) \\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha)\exp\left(\frac{x^{n+1}}{n+1}\right) \end{cases}$$

this solution is asymptotically unstable.

(b) If $y(x)$ is (ii)-strongly generalized differentiable, then the solution of (P_4) is

$$\begin{cases} \underline{y}(x,\alpha) = \underline{y}(0,\alpha)\cosh\left(\frac{x^{n+1}}{n+1}\right) + \overline{y}(0,\alpha)\sinh\left(\frac{x^{n+1}}{n+1}\right) \\ \overline{y}(x,\alpha) = \overline{y}(0,\alpha)\cosh\left(\frac{x^{n+1}}{n+1}\right) + \underline{y}(0,\alpha)\sinh\left(\frac{x^{n+1}}{n+1}\right) \end{cases}$$

this solution is asymptotically stable.

6. Conclusion

Using operator method, a general form of the solution for linear first order differential equation $y' = f(x)y + P(x)$ is given, where f is continuous nonnegative (respectively nonpositive) function. For future research, one can apply this method whenever f changes sign on $\mathbb{R}$.

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Said Melliani (melliani@fstbm.ac.ma)
Laboratoire de Mathématiques Appliquées & Calcul Scientifique, Sultan Moulay Slimane University, PO Box 523, 23000 Beni Mellal, Morocco

El Hassan El Jaoui (eljaouihass@gmail.com)
Laboratoire de Mathématiques Appliquées & Calcul Scientifique, Sultan Moulay Slimane University, PO Box 523, 23000 Beni Mellal, Morocco

Lalla Saadia Chadli (chadli@fstbm.ac.ma)
Laboratoire de Mathématiques Appliquées & Calcul Scientifique, Sultan Moulay Slimane University, PO Box 523, 23000 Beni Mellal, Morocco