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Some observations on fuzzy cone metric spaces and fixed point theorems of contractive mappings

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Abstract. In an earlier paper [2], the concept of fuzzy cone metric space is introduced and establish some fixed point theorems in such space. In this paper, it is shown that every regular fuzzy cone is normal but not conversely . There are no fuzzy normal cones with normal constant $M < 1$. Some fixed point theorems are established in fuzzy cone metric spaces by omitting the assumption of normality and each fixed point theorem is justified by counter example.

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1. Introduction

Different types of generalized metric spaces and Banach spaces and fixed point results in such spaces are introduced by different authors, for references please see [1, 4, 3, 6, 9, 12, 13, 14, 15]. Cone metric space is one such generalized metric space introduced by H.Long-Guang et al. [8]. In this generalized metric space, authors replaced the real numbers by an ordering Banach space as the range set of the cone metric. They established some fixed point theorems in such space assuming normality condition. Sh. Rezapour et al. [11] established some important basic results in cone metric space. They proved that every regular cone is normal but not conversely. There are no normal cones with normal constant $M < 1$. They established some fixed point results by omitting the assumption of normality in some results in [8].

Following the concept of cone metric space introduced by H.Long-Guang et al [8], the idea of fuzzy cone metric space is introduced by the present author in [2] and proved some fixed point theorems in such space. In this paper, following the results of Sh. Rezapour et al.[11], it is shown that every regular fuzzy cone is normal fuzzy cone but not conversely and there are no fuzzy normal cone with normal constant $M < 1$. Finally some fixed point theorems are established by omitting the assumption of normality condition in some fixed point results in [2].

The organization of the paper is as follows:
Section 1, comprises some preliminary results which are used in this paper.
Some basic results of fuzzy cone metric spaces are studied in Section 2. In section 3, some fixed point theorems for contractive mappings are established.

2. Preliminaries

A fuzzy real number is a mapping $x : R \to [0\ ,\ 1]$ over the set R of all reals.
A fuzzy real number x is convex if $x(t) \geq min\ (x(s)\ ,\ x(r))$ where $s \leq t \leq r$.
α-level set of a fuzzy real number x is defined by $\{t \in R\ : x(t) \geq \alpha\}$ where $\alpha \in (0, 1]$.
If there exists a $t_0 \in R$ such that $x(t_0) = 1$, then x is called normal. For $0 < \alpha \leq 1$, α-level set of an upper semi continuous convex normal fuzzy real number η(denoted by $[\eta]_\alpha$) is a closed interval $[a_\alpha\ ,\ b_\alpha]$, where $a_\alpha = -\infty$ and $b_\alpha = +\infty$ are admissible. When $a_\alpha = -\infty$, for instance, then $[a_\alpha\ ,\ b_\alpha]$ means the interval $(-\infty\ ,\ b_\alpha]$. Similar is the case when $b_\alpha = +\infty$.
A fuzzy real number x is called non-negative if $x(t) = 0$, $\forall t < 0$.
Each real number r is considered as a fuzzy real number denoted by $\bar{r}$ and defined by
$\bar{r}(t) = 1$ if $t = r$ and $\bar{r}(t) = 0$ if $t \neq r$.
Kaleva [7] (Felbin [5]) denoted the set of all convex, normal, upper semicontinuous fuzzy real numbers by E (R(I)) and the set of all non-negative, convex, normal, upper semicontinuous fuzzy real numbers by $G(R^*(I))$.

A partial ordering " $\preceq$ " in E is defined by $\eta \preceq \delta$ if and only if $a^1_\alpha \leq a^2_\alpha$ and $b^1_\alpha \leq b^2_\alpha$ for all $\alpha \in (0\ ,\ 1]$ where $[\eta]_\alpha = [a^1_\alpha\ ,\ b^1_\alpha]$ and $[\delta]_\alpha = [a^2_\alpha\ ,\ b^2_\alpha]$. The strict inequality in E is defined by $\eta \prec \delta$ if and only if $a^1_\alpha < a^2_\alpha$ and $b^1_\alpha < b^2_\alpha$ for each $\alpha \in (0\ ,\ 1]$.

According to Mizumoto and Tanaka [10], the arithmetic operations $\oplus$, $\ominus$, $\odot$ on $E \times E$ are defined by

$$\begin{aligned}
(x \oplus y)(t) &= Sup_{s \in R} min\ \{x(s)\ ,\ y(t-s)\}, \quad t \in R \\
(x \ominus y)(t) &= Sup_{s \in R} min\ \{x(s)\ ,\ y(s-t)\}, \quad t \in R \\
(x \odot y)(t) &= Sup_{s \in R, s \neq \underline{0}} min\ \{x(s)\ ,\ y(\tfrac{t}{s})\}, \quad t \in R
\end{aligned}$$

Proposition 2.1 ([10])**.** *Let* $\eta, \delta \in E(R(I))$ *and* $[\eta]_\alpha = [a^1_\alpha, b^1_\alpha]$, $[\delta]_\alpha = [a^2_\alpha, b^2_\alpha]$, $\alpha \in (0, 1]$. *Then*

$$\begin{aligned}
[\eta \bigoplus \delta]_\alpha &= [a^1_\alpha + a^2_\alpha\ ,\ b^1_\alpha + b^2_\alpha] \\
[\eta \ominus \delta]_\alpha &= [a^1_\alpha - b^2_\alpha\ ,\ b^1_\alpha - a^2_\alpha] \\
[\eta \odot \delta]_\alpha &= [a^1_\alpha a^2_\alpha\ ,\ b^1_\alpha b^2_\alpha]
\end{aligned}$$

Definition 2.2 ([7])**.** A sequence $\{\eta_n\}$ in E is said to be convergent and converges to η denoted by $\lim\limits_{n\to\infty} \eta_n = \eta$ if $\lim\limits_{n\to\infty} a^n_\alpha = a_\alpha$ and $\lim\limits_{n\to\infty} b^n_\alpha = b_\alpha$ where $[\eta_n]_\alpha = [a^n_\alpha, b^n_\alpha]$ and $[\eta]_\alpha = [a_\alpha, b_\alpha]$ $\forall \alpha \in (0, 1]$.

Note 2.3 ([7])**.** If $\eta,\ \delta \in G(R^*(I))$ then $\eta \oplus \delta \in G(R^*(I))$.

Note 2.4 ([7]). For any scalar t, the fuzzy real number $t\eta$ is defined as $t\eta(s) = 0$ if t=0 otherwise $t\eta(s) = \eta(\frac{s}{t})$.

Definition of fuzzy norm on a linear space as introduced by C. Felbin is given below:

Definition 2.5 ([5]). Let X be a vector space over R.
Let $|| \ || \ : X \to R^*(I)$ and let the mappings
$L, U \ : [0 \ , \ 1] \times [0 \ , \ 1] \to [0 \ , \ 1]$ be symmetric, nondecreasing in both arguments and satisfy
$L(0 \ , \ 0) = \ 0$ and $U(1 \ , \ 1) = \ 1$.
Write
$[||x||]_\alpha \ = \ [||x||^1_\alpha \ , \ ||x||^2_\alpha]$ for $x \in X, \ 0 \ < \alpha \ \leq 1$ and suppose for all $x \in X, \ x \neq \underline{0}$, there exists $\alpha_0 \ \in (0 \ , \ 1]$ independent of x such that for all $\alpha \ \leq \ \alpha_0$,

(A) $||x||^2_\alpha \ < \ \infty$

(B) $\inf||x||^1_\alpha \ > \ 0$.

The quadruple $(X \ , \ || \ ||, \ L \ , \ U)$ is called a fuzzy normed linear space and $|| \ ||$ is a fuzzy norm if

(i) $||x|| \ = \ \bar{0}$ if and only if $x \ = \ \underline{0}$;

(ii)$||rx|| \ = \ |r|||x||, \ x \in X, \ r \in R$;

(iii) for all $x, y \ \in X$,

(a) whenever $s \ \leq \ ||x||^1_1, \ t \leq ||y||^1_1$ and $s + t \ \leq ||x + y||^1_1$,
$||x + y||(s + t) \ \geq \ L(||x||(s) \ , \ ||y||(t))$,

(b) whenever $s \ \geq \ ||x||^1_1, \ t \geq ||y||^1_1$ and $s + t \ \geq ||x + y||^1_1$,
$||x + y||(s + t) \ \leq \ U(||x||(s) \ , \ ||y||(t))$

Remark 2.6. Felbin proved that,
if $L \ = \bigwedge(\text{Min})$ and $U = \bigvee(\text{Max})$ then the triangle inequality (iii) in the Definition 1.1 is equivalent to
$||x + y|| \ \preceq ||x|| \ \bigoplus ||y||$.
Further $|| \ ||^i_\alpha; \ i = 1, 2$ are crisp norms on X for each $\alpha \in (0 \ , \ 1]$.

Definition 2.7 ([2]). Let $(E, || \ ||)$ be a fuzzy real Banach space where $|| \ || : E \to R^*(I)$.
Denote the range of $|| \ ||$ by $E^*(I)$. Thus $E^*(I) \subset R^*(I)$.

Definition 2.8 ([2]). A member $\eta \in A \subset R^*(I)$ is said to be an interior point if $\exists r > 0$ such that
$S(\eta, r) = \{\delta \in R^*(I) \ : \eta \ominus \delta \prec \bar{r}\} \subset A$.
Set of all interior points of A is called interior of A.

Definition 2.9 ([2]). A subset of F of $E^*(I)$ is said to be fuzzy closed if for any sequence $\{\eta_n\}$ such that $\lim\limits_{n\to\infty} \eta_n = \eta$ implies $\eta \in F$.

Definition 2.10 ([2]). A subset P of $E^*(I)$ is called a fuzzy cone if

(i) P is fuzzy closed, nonempty and $P \neq \{\bar{0}\}$;

(ii) $a, b \in R, \ a, b \geq 0, \ \eta, \delta \in P \Rightarrow a\eta \oplus b\delta \in P$.

Note 2.11. If $\eta \in P$ then $\ominus\eta \in P \quad \Rightarrow \eta = \bar{0}$.
For, suppose $[\eta]_\alpha = [\eta^1_\alpha \ , \ \eta^2_\alpha], \ \alpha \in (0, 1]$.

Since $\eta \in P \subset E^*(I)$, we have $\eta^1_\alpha,\ \eta^2_\alpha \geq 0\ \forall \alpha \in (0,1]$.
Now $[\ominus\eta]_\alpha = [-\eta^2_\alpha\ ,\ -\eta^1_\alpha],\ \alpha \in (0,1]$.
If $\eta \neq \bar{0}$, then $\eta^1_\alpha,\ \eta^2_\alpha > 0\ \forall \alpha \in (0,1]$.
i.e. $-\eta^2_\alpha\ \leq\ -\eta^1_\alpha < 0\ \forall \alpha \in (0,1]$.
This implies that $\ominus\eta$ does not belong to P. Hence $\eta = \bar{0}$.

Given a fuzzy cone $P \subset E^*(I)$, define a partial ordering $\leq$ with respect to P by $\eta \leq \delta$ iff $\delta \ominus \eta \in P$ and $\eta < \delta$ indicates that $\eta \leq \delta$ but $\eta \neq \delta$ while $\eta << \delta$ will stand for $\delta \ominus \eta \in$IntP where IntP denotes the interior of P.

The fuzzy cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,
with $\bar{0} \leq ||x|| \leq ||y||$ implies $||x|| \preceq K||y||$. The least positive number satisfying above is called the normal constant of P.
The fuzzy cone P is called regular if every increasing sequence which is bounded from above is convergent. That is if $\{x_n\}$ is a sequence in E such that $||x_1|| \leq ||x_2|| \leq .... \leq ||x_n|| \leq .... \leq ||y||$ for some $y \in E$, then there is $x \in E$ such that $||x_n - x|| \to \bar{0}$ as $n \to \infty$.
Equivalently, the fuzzy cone P is regular if every decreasing sequence which is bounded below is convergent.

In the following we always assume that E is a fuzzy real Banach space, P is a fuzzy cone in E with IntP $\neq \phi$ and $\leq$ is a partial ordering with respect to P.

Definition 2.12 ([2]). Let X be a nonempty set. Suppose the mapping
$d:\ X \times X \to E^*(I)$ satisfies
(Fd1) $\bar{0} \leq d(x,y)\ \forall x, y \in X$ and $d(x,y) = \bar{0}$ iff $x = y$;
(Fd2) $d(x,y) = d(y,x)\ \ \forall x, y \in X$;
(Fd3) $d(x,y) \leq d(x,z) \oplus d(z,y)\ \ \forall x, y, z \in X$.
Then d is called a fuzzy cone metric and (X, d) is called a fuzzy cone metric space.

Example 2.13. Let $(E, ||\ ||')$ be a real Banach space.
Define $||\ ||:\ E \to R^*(I)$ by

$$||x||(t) = \begin{cases} \frac{||x||'}{t} & \text{if } t \geq |x||', x \neq \underline{0} \\ 1 & \text{if } t = ||x||' = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then $[||x||]_\alpha = [||x||'\ ,\ \frac{||x||'}{\alpha}]\ \ \forall \alpha \in (0,1]$.
It is easy to verify that,
(i) $||x|| = \bar{0}$ iff $x = \underline{0}$ (ii) $||rx|| = |r|||x||$ (iii) $||x+y|| \preceq ||x|| \oplus ||y||$.
Thus $(E, ||\ ||)$ is a fuzzy normed linear space in Felbin's sense (where L = min and U = max). Let $\{x_n\}$ be a Cauchy sequence in $(E, ||\ ||)$
So, $\lim_{m,n\to\infty} ||x_n - x_m|| = \bar{0}$.
$\Rightarrow \lim_{m,n\to\infty} ||x_n - x_m|| = 0$
$\Rightarrow \{x_n\}$ is a Cauchy sequence in $(E, ||\ ||')$.
Since $(E, ||\ ||')$ is complete, $\exists x \in E$ such that $\lim_{m,n\to\infty} ||x_n - x||' = 0$.
i.e. $\lim_{n\to\infty} ||x_n - x|| = \bar{0}$.
Thus $(E, ||\ ||)$ is a real fuzzy Banach space.
Define $P = \{\eta \in E^*(I)\ :\eta \succeq \bar{0}\}$.
(i) P is fuzzy closd.

For, consider a sequence $\{\delta_n\}$ in P such that $\lim_{n\to\infty} \delta_n \to \delta$.
i.e. $\lim_{n\to} \delta^1_{n,\alpha} = \delta^1_\alpha$ and $\lim_{n\to} \delta^2_{n,\alpha} = \delta^2_\alpha$ where $[\delta_n]_\alpha = [\delta^1_{n,\alpha}\ ,\ \delta^2_{n,\alpha}]$ and $[\delta]_\alpha = [\delta^1_\alpha\ ,\ \delta^2_\alpha]\ \ \forall \alpha \in (0,1]$.
Now $\delta_n \succeq \bar{0}\ \ \forall n$.
So, $\delta^1_{n,\alpha} \geq 0$ and $\delta^2_{n,\alpha} \geq 0\ \ \forall \alpha \in (0,1]$.
$\Rightarrow \lim_{n\to} \delta^1_{n,\alpha} \geq 0$ and $\lim_{n\to} \delta^2_{n,\alpha} \geq 0\ \ \forall \alpha \in (0,1]$
$\Rightarrow \delta^1_\alpha \geq 0$ and $\delta^2_\alpha \geq 0\ \ \forall \alpha \in (0,1]$
$\Rightarrow \delta \succeq \bar{0}$.
So $\delta \in P$. Hence P is fuzzy closed.
(ii) It is obvious that, $a, b \in R,\ a, b \geq 0\ \eta, \delta \in P \Rightarrow a\eta \oplus b\delta \in P$.
Thus P is a fuzzy cone in E.
Now choose the ordering $\leq$ as $\preceq$ and define $d : E \times E \to E^*(I)$ by $d(x,y) = ||x-y||$. Then it is easy to verify that d satisfies the conditions (Fd1) to (Fd3). Hence $(E\ ,\ d)$ is a fuzzy cone metric space.

Definition 2.14 ([2])**.** Let (X,d) be a fuzzy cone metric space. Let$\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $\bar{0} << ||c||$ there is a positive integer N such that for all $n > N,\ d(x_n, x) << ||c||$, then $\{x_n\}$ is said to be convergent and converges to x and x is called the limit of $\{x_n\}$. We denote it by $\lim\limits_{n\to\infty} x_n = x$.

Definition 2.15 ([2])**.** Let (X,d) be a fuzzy cone metric space and $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $\bar{0} << ||c||$, there exists a natural number N such that $\forall m, n > N,\ d(x_n, x_m) << ||c||$, then $\{x_n\}$ is called a Cauchy sequence in X.

Definition 2.16 ([2])**.** Let (X,d) be a fuzzy cone metric space. If every Cauchy sequence is convergent in X, then X is called a complete fuzzy cone metric space.

Note 2.17. In a fuzzy cone metric space, every convergent sequence is Cauchy (please see Lemma 3.13[2]).
For, let $\{x_n\}$ converges to x. So for any $c \in E$ with $\bar{0} << ||c||$ there exists a natural number N such that $\forall m, n > N,\ d(x_n, x) << ||\frac{c}{2}||$ and $d(x_m, x) << ||\frac{c}{2}||$.
Hence $d(x_n, x_m) \leq d(x_n, x) \oplus d(x, x_m) << ||c||\ \ \forall m, n > N$.
Thus $\{x_n\}$ is a Cauchy sequence.
Converse of the result may not hold.
To verify it, first we prove the following result:
Let $X = C[0,1]$. Define a metric d by $d(x,y) = \int_0^1 |x(t) - y(t)|dt$ (integral is taken in the sense of Riemann).
Consider a sequence $\{x_n\}$ given by

$$x_n(t) = \begin{cases} 0 & \text{if } t \in [0, \frac{1}{2}] \\ 1 & \text{if } t \in [a_n, 1] \end{cases}$$

where $a_n = \frac{1}{2} + \frac{1}{n}$.
It is easy to verify that $\{x_n\}$ is Cauchy.
For $x \in X$, we have
$d(x_n, x) = \int_0^1 |x_n(t) - x(t)|dt = \int_0^{\frac{1}{2}} |x(t)|dt + \int_{\frac{1}{2}}^{a_n} |x_n(t) - x(t)|dt + \int_{a_n}^1 |1 - x(t)|dt$.
Since the integrands are non-negative, so is each integral on the right. Hence

$d(x_n, x) \to 0$ would imply that each integral approaches to zero. Since x is continuous, we should have

$$x(t) = \begin{cases} 0 & \text{if } t \in [0, \frac{1}{2}] \\ 1 & \text{if } t \in (\frac{1}{2}, 1] \end{cases}$$

which is impossible for a continuous function. Hence $\{x_n\}$ does not converge.
Choose $E = X = C[0,1]$, the Banach space w.r.t. norm $|| \; ||'$ defined by
$||x||' = \bigvee_{0 \le t \le 1} |x(t)|$.
Define $|| \; || : X \to R^*(I)$ by

$$||x||(t) = \begin{cases} 1 & \text{if } t = ||x||' \\ 0 & \text{otherwise} \end{cases}$$

Then $[||x||]_\alpha = [||x||' \; , \; ||x||'] \;\; \forall \alpha \in (0,1]$.
Then it can be shown that $(X \; , \; || \; ||)$ is a complete fuzzy normed linear space (Felbin's sense , L= min and U= max).
Now we define $\rho : \; E \times E \to E^*(I)$ by

$$\rho(x,y)(t) = \begin{cases} \frac{d(x,y)}{t} & \text{if } t \ge d(x,y), x \ne y \\ 1 & t = d(x,y) = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then $[\rho(x,y)]_\alpha = [d(x,y) \; , \; \frac{d(x,y)}{t}] \;\; \forall \alpha \in (0,1]$.
It is easy to verify that $(X \; , \; \rho)$ is a fuzzy cone metric space if we choose the ordering $\le$ as $\preceq$.
Since $\rho^1_\alpha(x,y) = d(x,y)$ and $\rho^2_\alpha(x,y) = \frac{d(x,y)}{\alpha} \;\; \alpha \in (0,1]$, it follows that if $\{x_n\}$ is a Cauchy sequence in (X, d) iff it is a Cauchy sequence in $(X \; , \; \rho)$. So if we consider the sequence $\{x_n\}$ defined as above, then it is a Cauchy in $(X \; , \; \rho)$ but not convergent.

Following is an example of a regular fuzzy cone metric space:

Example 2.18. Let $X = R$. Define $|| \; || : \; X \to R^*(I)$ by

$$||x||(t) = \begin{cases} 1 & \text{if } t = |x| \\ 0 & \text{otherwise} \end{cases}$$

Then $[||x||]_\alpha = [|x| \; , \; |x|] \;\; \forall \alpha \in (0,1]$.
It can be shown that $(X \; , \; || \; ||)$ is a complete fuzzy normed linear space (Felbin's sense where L = min and U= max).
Now we define $d : \; E \times E \to E^*(I)$ where $E = R$ by

$$d(x,y)(t) = \begin{cases} \frac{|x|}{t} & \text{if } t \ge |x|, x \ne \underline{0} \\ 1 & \text{if } t = |x| = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then $[d(x,y)]_\alpha = [|x| \; , \; \frac{|x|}{\alpha}] \;\; \forall \alpha \in (0,1]$.
It is easy to verify that $(X \; , \; d)$ is a fuzzy cone metric space if we choose the ordering $\le$ as $\preceq$, usual ordering of fuzzy real numbers and the cone P is given by $P = \{\eta : \; \eta \succeq \bar{0}\}$.
This is also a regular fuzzy cone metric space.
For, consider a sequence $\{x_n\}$ in E such that

$||x_1|| \preceq ||x_2|| \preceq ..... \preceq ||x_n|| \preceq .... \preceq ||x||$ for some $x \in E$.
This implies that,
$||x_1||_\alpha^i \leq ||x_2||_\alpha^i \leq ..... \leq ||x_n||_\alpha^n \leq .... \leq ||x||_\alpha^i$ for $i = 1, 2$ and $\alpha \in (0, 1]$.
i.e. $|x_1| \leq |x_2| \leq ..... \leq |x_n| \leq .... \leq |x|$. Thus the sequence $\{|x_n|\}$ is convergent and hence the sequence $\{x_n\}$ is convergent and converges to some $y \in E$.
So $\lim\limits_{n\to\infty} |x_n - y| = 0$.
i.e. $\lim\limits_{n\to\infty} d_\alpha^i(x_n\ ,\ y) = 0$ for $i = 1, 2$ and $\alpha \in (0, 1]$.
i.e. $\lim\limits_{n\to\infty} d(x_n\ ,\ y) = \bar{0}$.
Hence P is regular.

3. Some observations on fuzzy cone metric spaces

In this Section it is shown that every regular fuzzy cone metric space is normal but not conversely and there does not exist any normal fuzzy cone with normal constant $M < 1$.

Lemma 3.1. *Every regular fuzzy cone metric space is normal.*

Proof. Let $(X\ ,\ d)$ be a fuzzy cone metric space and P be a regular fuzzy cone and if possible suppose that P is not normal.
Thus for each $n \geq 1$, $\exists ||x_n||,\ ||y_n|| \in P$ where $x_n,\ y_n \in E$ such that
$||x_n|| \ominus ||y_n|| \in P$ and $n^2||x_n|| \prec ||y_n||$ (3.1.1).
Clearly $x_n \neq \theta\ \ \forall n$.
For, if $x_n = \theta$ for some n then $||x_n|| = \bar{0}$ and so $\ominus||y_n|| \in P$.
This implies that $||y_n|| = \bar{0}$ which contradicts (3.1.1).
For each $n \geq 1$,
put $||z_n|| = \frac{||x_n||}{\bigvee\limits_{\alpha\in(0,1]} ||x_n||_\alpha^2}$ ("Sup" exists because of (A) in Felbin's definition).
$||z_n'|| = \frac{||y_n||}{\bigvee\limits_{\alpha\in(0,1]} ||x_n||_\alpha^2}$.
Then $||z_n|| \ominus ||z_n'|| \in P\ \ \forall n$.
We have $||z_n||_\alpha^1 \leq 1$ and $||z_n||_\alpha^2 \leq 1 \forall n,\ \ \forall \alpha \in (0, 1]$ (3.1.2).

Now $[\sum\limits_{n=1}^{\infty} \frac{||z_n||}{n^2}]_\alpha = [\sum\limits_{n=1}^{\infty} \frac{||z_n||_\alpha^1}{n^2}\ ,\ \sum\limits_{n=1}^{\infty} \frac{||z_n||_\alpha^2}{n^2}]\ \ \forall \alpha \in (0, 1]$.
From (3.1.2), it follows that $\sum\limits_{n=1}^{\infty} \frac{||z_n||_\alpha^1}{n^2}$ and $\sum\limits_{n=1}^{\infty} \frac{||z_n||_\alpha^2}{n^2}$ both are convergent for each $\alpha \in (0, 1]$.
Since P is closed, thus $\exists z \in E$ such that $||z|| \in P$ and $||z|| = \sum\limits_{n=1}^{\infty} \frac{||z_n||}{n^2}$.
Now we have,
$\bar{0} \leq ||z_1'|| \leq ||z_1'|| \oplus \frac{1}{2^2}||z_2'|| \leq ||z_1'|| \oplus \frac{1}{2^2}||z_2'|| \oplus \frac{1}{3^2}||z_3'|| \leq ..... \leq ||z||$.

Since P is a regular fuzzy cone, so $\sum_{n=1}^{\infty} \frac{||z'_n||}{n^2}$ is convergent.
i.e. $\sum_{n=1}^{\infty} \frac{||z'_n||_\alpha^1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{||z'_n||_\alpha^2}{n^2}$ and are convergent for each $\alpha \in (0,1]$.
Thus $\lim_{n\to\infty} \frac{||z'_n||_\alpha^1}{n^2} = \lim_{n\to\infty} \frac{||z'_n||_\alpha^2}{n^2} = 0 \ \forall \alpha \in (0,1]$ (3.1.3).
From (3.1.1), we get $n^2||x_n||_\alpha^1 < ||y_n||_\alpha^1$ and $n^2||x_n||_\alpha^2 < ||y_n||_\alpha^2 \ \forall n, \ \forall \alpha \in (0,1]$.
Now $n^2||x_n||_\alpha^2 < ||y_n||_\alpha^2 \ \forall n, \ \forall \alpha \in (0,1]$
$\Rightarrow n^2||x_n||_\alpha^2 < ||z'_n||_\alpha^2 \bigvee_{\alpha\in(0,1]} ||x_n||_\alpha^2 \ \forall n, \ \forall \alpha \in (0,1]$
$\Rightarrow \exists \alpha_0 \in (0,1]$ such that
$n^2||x_n||_\alpha^2 < ||z'_n||_\alpha^2||x_n||_{\alpha_0}^2 \ \forall n, \ \forall \alpha \in (0,1]$
$\Rightarrow n^2||x_n||_{\alpha_0}^2 < ||z'_n||_{\alpha_0}^2||x_n||_{\alpha_0}^2 \ \forall n$
$\Rightarrow \frac{||z'_n||_{\alpha_0}^2}{n^2} > 1 \ \forall n$
$\Rightarrow \lim_{n\to\infty} \frac{||z'_n||_{\alpha_0}^2}{n^2} \geq 1$-which contradicts (3.1.3).
Hence P is normal.
Converse result may not be true and is justified by the Example 3.3. □

Lemma 3.2. *There is no fuzzy cone with normal constant $M < 1$.*

Proof. Let $(X\ ,\ d)$ be a fuzzy cone metric space and if possible suppose that P be the fuzzy normal cone with normal constant $M < 1$.
Choose a non-zero element $||x|| \in P$ where $x(\neq \theta) \in E$ and $0 < \epsilon < 1$ such that $M < 1 - \epsilon$.
Now $||x|| \ominus (1-\epsilon)||x|| = \epsilon||x|| \in P$. So $(1-\epsilon)||x|| \leq \epsilon||x||$.
We have, $M||x||_\alpha^1 < (1-\epsilon)||x||_\alpha^1$ and $M||x||_\alpha^2 < (1-\epsilon)||x||_\alpha^2 \ \forall \alpha \in (0,1]$.
Thus $M||x|| \prec (1-\epsilon)||x||$. This contradicts the fact that P is a fuzzy normal cone. □

Following is an example to justify that there exists fuzzy normal cone with normal constant 1.

Example 3.3. Let $E = C[0\ ,\ 1]$, the space of all real valued continuous functions with norm $||\ ||'$ (supnorm) given by $||f||' = \bigvee_{t\in[0,1]} |f(t)|$.
Define $||\ || :\ E \to E^*(I)$ by

$$||f||(t) = \begin{cases} 1 & \text{if } t \geq ||f||' \\ 0 & \text{if } t < ||f||'. \end{cases}$$

Then $(E\ ,\ ||\ ||)$ is a Felbin's type fuzzy normed linear space where $[||f||]_\alpha = [||f||_\alpha^1\ ,\ ||f||_\alpha^2] = [||f||'\ ,\ ||f||']\ \forall \alpha \in (0,1]$. Define $P = \{||f|| \in E^*(I) :\ ||f|| \succeq \bar{0}\}$.
Now take $||f||,\ ||g|| \in P$ such that $\bar{0} \leq ||f|| \leq ||g||$.
Thus $||g|| \ominus ||f|| \in P$. i.e. $||g|| \ominus ||f|| \succeq \bar{0}$.
i.e. $||g||_\alpha^1 - ||f||_\alpha^2 \geq 0$ and $||g||_\alpha^2 - ||f||_\alpha^1 \geq 0\ \forall \alpha \in (0,1]$.
i.e. $||g||' - ||f||' \geq 0$. i.e. $||g||' \geq ||f||'$.
This implies that $||g||_\alpha^1 \geq ||f||_\alpha^1$ and $||g||_\alpha^2 \geq ||f||_\alpha^2\ \forall \alpha \in (0,1]$.

i.e. $||f|| \preceq ||g||$.
Since f and g are arbitrary, it follows that normal constant of P is 1.

Now consider the sequence $\{f \ , f^2, \ f^3, \ ......, f^n, ...\}$ in E defined by $f^n(t) = \{f(t)\}^n \ = t^n$.
Then $||f|| \geq ||f^2|| \geq ||f^3|| \geq ..... \geq \bar{0}$ is a decreasing sequence in P and bounded below.
If possible suppose that $\exists g \in E$ such that $\lim\limits_{n\to\infty} ||f^n - g|| = \bar{0}$.
i.e. $\lim\limits_{n\to\infty} sup_{t\in[0,1]}|t^n - g(t)| = 0$
$\Rightarrow$ for each $\epsilon > 0$ there exists a positive integer N such that $\bigvee\limits_{t\in[0,1]} |t^n - g(t)| < \epsilon \ \ \forall n \geq N$
$\Rightarrow \bigvee\limits_{t\in[0,1]} |t^n - g(t)| < \epsilon \ \ \forall n \geq N \ \forall t \in [0, 1]$
$\Rightarrow \lim_{n\infty} |t^n - g(t)| = 0 \ \forall t \in [0, 1]$.
From above it follows that

$$g(t) = \begin{cases} 0 & \text{for } t \in [0 \ , \ 1) \\ 1 & \text{for } t = 1. \end{cases}$$

So g is not a member of E. Hence P is not regular.

4. Fixed point theorems in fuzzy cone metric spaces

In this Section some fixed point theorems of contractive mappings are established in fuzzy cone metric spaces without considering normal cone.

Theorem 4.1. *Let (X, d) be a complete fuzzy cone metric space and the mapping $T : X \to X$ satisfies the contractive condition $d(Tx, Ty) \leq kd(x, y) \ \ \forall x, y \in X$ where $k \in [0, 1)$ is a constant. Then T has a unique fixed point in X. For any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.*

Proof. Choose $x_0 \in X$.
Set $x_1 = Tx_0, \ x_2 = Tx_1 = T^2x_0, ..., x_{n+1} = Tx_n = T^{n+1}x_0, ...$
we have,
$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq kd(x_n, x_{n-1}) \leq k^2d(x_{n-1}, x_{n-2}) \leq .... \leq k^n d(x_1, x_0)$.
So for $n > m$, we have
$d(x_n, x_m) \leq d(x_n, x_{n-1}) \oplus d(x_{n-1}, x_{n-2}) \oplus .... \oplus (x_{m+1}, x_m)$.
i.e. $d(x_n, x_m) \leq (k^{n-1} + k^{n-2} + ... + k^m)d(x_1, x_0)$.
i.e. $d(x_n, x_m) \leq \frac{k^m}{1-k} d(x_1, x_0)$ (4.1.1).
Let $||z|| \in$ IntP be given. i.e. $||z|| >> \bar{0}$ (where $z \in E$).
Choose $\epsilon > 0$ such that $||z|| \oplus N_\epsilon(\bar{0}) \subset$P where
$N_\epsilon(\bar{0}) = \{||y|| \in E^*(I)$ where $y \in E : \ ||y|| \prec \bar{\epsilon}\}$.
Again choose a natural number N_1 such that $\frac{k^m}{1-k} d(x_1, x_0) \ \in N_\epsilon(\bar{0}) \ \ \forall m \geq N_1$.
Thus $\frac{k^m}{1-k} d(x_1, x_0) \ << ||z|| \ \ \forall m \geq N_1$.
So from (4.1.1), we have
$d(x_n, x_m) \leq \frac{k^m}{1-k} d(x_1, x_0) \ << ||z|| \ \ \forall m \geq N_1$.
Therefore $\{x_n\}$ is a Cauchy sequence in $(X \ , \ d)$.

By completeness of X, there is $x^* \in X$ such that $x_n \to x^*$.
Now,
Choose a natural number N_2 such that $d(x_n\ ,\ x^*) << \frac{||z||}{2}\ \ \forall n \geq N_2$.
Now we have,
$d(Tx^*, x^*) \leq d(Tx_n, Tx^*) \oplus d(Tx_n, x^*) \leq kd(x_n, x^*) \oplus d(x_{n+1}, x^*)$
$\Rightarrow d(Tx^*, x^*)\ \leq kd(x_n, x^*) \oplus d(x_{n+1}, x^*)$
$\Rightarrow d(Tx^*, x^*)\ \leq d(x_n, x^*) \oplus d(x_{n+1}, x^*)$ (since $1 - k > 0$)
This implies that $\forall n \geq N_2$,
$d(Tx^*, x^*) <<\ \frac{||z||}{2} \oplus \frac{||z||}{2}\ = ||z||$.
i.e. $d(Tx^*, x^*) <<\ \frac{||z||}{m}\ \ \forall m \geq 1$,
$\Rightarrow \frac{||z||}{m} \ominus d(Tx^*, x^*) \in P\ \ \forall m \geq 1$.
Since $\frac{||z||}{m} \to \bar{0}$ as $m \to \infty$ and P is closed we have
$\ominus d(Tx^*, x^*) \in P$. Thus $d(Tx^*, x^*) = \bar{0}$. i.e. $Tx^* = x^*$
So x^* is a fixed point of T.
For uniqueness, suppose that y^* is another fixed point of T. We have
$d(x^*, y^*) = d(Tx^*, Ty^*) \leq kd(x^*, y^*)$
$\Rightarrow (k - 1)d(x^*, y^*) \in P$
$\Rightarrow d(x^*, y^*) = \bar{0}$ (since $k - 1 < 0$)
$\Rightarrow x^* = y^*$. □

Theorem 4.1 is justified by the following example.

Example 4.2. Let E be the real Banach space where E=R.
Define $||\ ||:\ E \to R^*(I)$ by

$$||x||(t) = \begin{cases} 1 & \text{if } t = |x| \\ 0 & \text{otherwise} \end{cases}$$

Then $[||x||]_\alpha = [|x|\ ,\ |x|]\ \ \forall \alpha \in (0, 1]$.
It is easy to verify that,
(i) $||x|| = \bar{0}$ iff $x = \underline{0}$ (ii) $||rx|| = |r|||x||$ (iii) $||x + y|| \preceq ||x|| \oplus ||y||$.
Thus $(E, ||\ ||)$ is a complete fuzzy normed linear space (since E is complete).
If we define $d:\ X \times X \to E^*(I)$ where X=R by

$$d(x, y)(t) = \begin{cases} \frac{|x-y|}{t} & \text{if } t \geq |x - y|, x \neq y \\ 1 & \text{if } t = |x - y| = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then $[d(x, y)]_\alpha = [|x - y|\ ,\ \frac{|x-y|}{\alpha}]\ \ \forall \alpha \in (0, 1]$.
It can be verified that d is a complete fuzzy (since R is complete) cone metric if we choose the ordering " $\leq$ " as " $\preceq$ " and the cone P is given by $P = \{\eta \in E^*(I):\ \eta \succeq \bar{0}\}$.
Define a function $f:\ X \to X$ by

$$f(x) = \begin{cases} \frac{\beta}{1+\beta}x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

where $0 \leq \beta < 2$.
We have $d(fx\ ,\ fy) = d(\frac{\beta}{1+\beta}x\ ,\ \frac{\beta}{1+\beta}y$.

Then $[d(fx\ ,\ fy)]_\alpha = [\frac{\beta}{1+\beta}|x-y|\ ,\ \frac{\beta}{\alpha(1+\beta)}|x-y|]\ \ \forall\alpha \in (0,1]$.
Now $d^1_\alpha(fx, fy) = \frac{\beta}{1+\beta}|x-y| = k|x-y| = kd^1_\alpha(x,y),\ \ \ k \in [0,1)$.
Similarly $d^2_\alpha(fx, fy) = \frac{k}{\alpha}|x-y| = kd^2_\alpha(x,y)$.
So $d(fx, fy) \preceq kd(x,y)$. Thus f satisfies the condition of the Theorem 4.1 and $x = \underline{0}$ is the unique fixed of point of f.
Again for any fixed x_0, iterative sequence $\{x_n\}$ is given by $x_n = (\frac{\beta}{1+\beta})^n x_0$.
So $\lim_{n\to\infty} x_n = \underline{0}$.

Theorem 4.3. *Let (X,d) be a complete fuzzy cone metric space and the mapping $T : X \to X$ satisfies the the contractive condition $d(Tx,Ty) \le k(d(Tx,x) \oplus d(Ty,y))\ \forall x,y \in X$, where $k \in [0,\frac{1}{2})$ is a constant. Then T has a unique fixed point in X. Also for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.*

Proof. Choose $x_0 \in X,\ n \ge 1$.
Set $x_1 = Tx_0,\ x_2 = Tx_1 = T^2x_0, ..., x_{n+1} = Tx_n = T^{n+1}x_0, ...$
We have,
$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \le k(d(Tx_n, x_n) \oplus d(Tx_{n-1}, x_{n-1})) = k(d(x_{n+1}, x_n) \oplus d(x_n, x_{n-1}))$.
So, $d(x_{n+1}, x_n) \le \frac{k}{1-k} d(x_n, x_{n-1}) = hd(x_n, x_{n-1})$ where $h = \frac{k}{1-k}$.
For $n > m$, we have
$d(x_n, x_m) \le d(x_n, x_{n-1}) \oplus d(x_{n-1}, x_{n-2}) \oplus ...... \oplus d(x_{m+1}, x_m)$.
i.e. $d(x_n, x_m) \le (h^{n-1} + h^{n-2} + ... + h^m)d(x_1, x_0)\ \ =\ \ \frac{h^m}{1-h}d(x_1, x_0)$.
Let $\bar{0} << ||c||\ (c \in E)$ be given. Choose a natural number N_1 such that
$\frac{h^m}{1-h}d(x_1, x_0) << ||c||\ \ \forall m \ge N_1$.
Thus $\forall m, n \ge N_1$ we get $d(x_n, x_m) << ||c||$. So $\{x_n\}$ is a Cauchy sequence in $(X\ ,\ d)$.
By completeness of X, there is $x^* \in X$ such that $x_n \to x^*$.
Choose a natural number N_2 such that $d(x_{n+1}\ ,\ x_n) << \frac{(1-k)}{2k}||c||$ and
$d(x_{n+1}\ ,\ x^*) << \frac{(1-k)}{2}||c||\ \ \forall n \ge N_2$.
Thus $\forall n \ge N_2$ we have
$d(Tx^*, x^*)\ \le\ d(Tx_n\ ,\ Tx^*) \oplus d(Tx_n\ ,\ x^*)\ \le\ k(d(Tx_n\ ,\ x_n) \oplus d(Tx^*, x^*)) \oplus d(x_{n+1}\ ,\ x^*)$
$\Rightarrow d(Tx^*, x^*) \le \frac{1}{1-k}(kd(Tx_n, x_n) \oplus d(x_{n+1}, x^*)) << \frac{||c||}{2} \oplus \frac{||c||}{2} = ||c||$
$\Rightarrow d(Tx^*, x^*) << \frac{||c||}{m}\ \ \ \forall m \ge 1$ (since $||c|| \in IntP\ \ \Rightarrow s||c|| \in IntP\ \forall s > 0$)
$\Rightarrow \frac{||c||}{m} \ominus d(Tx^*, x^*) \in P\ \ \forall m \ge 1$.
Since $\frac{||c||}{m} \to \bar{0}$ as $m \to \infty$ and P is closed, it follows that
$\ominus d(Tx^*, x^*) \in P$. So $d(Tx^*, x^*) = \bar{0}$. i.e. $Tx^* = x^*$.
For uniqueness, if suppose that y^* is another fixed point of T. Then
$d(x^*\ ,\ y^*) = d(Tx^*\ ,\ Ty^*) \le k(d(Tx^*\ ,\ x^*) \oplus d(Ty^*\ ,\ y^*)) = \bar{0}$.
i.e. $d(x^*\ ,\ y^*) = \bar{0}$. i.e. $x^* = y^*$. □

Theorem 4.3. is justified by the following example:

Example 4.4. In Example 4.2, define $T(x) = \frac{x}{4},\ \ \ x \in X$.
We have $d^1_\alpha(Tx, Ty) = d^1_\alpha(\frac{x}{4}\ ,\ \frac{y}{4}) = |\frac{x}{4} - \frac{y}{4}| = \frac{1}{4}|x-y|$.
Now $d^1_\alpha(Tx, x) + d^1_\alpha(Ty, y) = d^1_\alpha(\frac{x}{4}\ ,\ x) + d^1_\alpha(\frac{y}{4}\ ,\ y) = |\frac{x}{4} - x| + |\frac{y}{4} - y|$.

i.e. $d^1_\alpha(Tx,x)+d^1_\alpha(Ty,y)=\frac{3}{4}\{|x|+|y|\}\geq\frac{3}{4}|x-y|$.
$\Rightarrow \frac{1}{3}\{d^1_\alpha(Tx,x)+d^1_\alpha(Ty,y)\}\geq\frac{1}{4}|x-y|=d^1_\alpha(Tx,Ty)$ (4.4.1).
Similarly $\frac{1}{3}\{d^2_\alpha(Tx,x)+d^2_\alpha(Ty,y)\}\geq d^2_\alpha(Tx,Ty)$ (4.4.2).
From (4.4.1) and (4.4.2), we have $\frac{1}{3}\{d(Tx,x)\oplus d(Ty,y)\}\succeq d(Tx,Ty)$.
i.e. $d(Tx,Ty)\preceq k\{d(Tx,x)\oplus d(Ty,y)\}$ where $k=\frac{1}{3}\in[0,\frac{1}{2})$.
Thus T satisfies the condition of Theorem 4.3. Here $\underline{0}$ is the unique fixed point of T.
Also any iterative sequence in X converges to $\underline{0}$.

Theorem 4.5. *Let (X,d) be a complete fuzzy cone metric space and the mapping $T:X\to X$ satisfies the the contractive condition $d(Tx,Ty)\leq k(d(Tx,y)\oplus d(x,Ty))$ $\forall x,y\in X$, where $k\in[0,\frac{1}{2})$ is a constant. Then T has a unique fixed point in X. For any $x\in X$, iterative sequence $\{T^nx\}$ converges to the fixed point.*

Proof. Choose $x_0\in X,\ n\geq 1$.
Set $x_1=Tx_0,\ x_2=Tx_1=T^2x_0,........,x_{n+1}=Tx_n=T^{n+1}x_0,.....$
We have,
$d(x_{n+1},x_n)=d(Tx_n,Tx_{n-1})\leq k(d(Tx_n,x_{n-1})\oplus d(Tx_{n-1},x_n))=k(d(x_{n+1},x_{n-1})\oplus d(x_n,x_n))=kd(x_{n+1},x_{n-1})$.
i.e. $d(x_{n+1},x_n)\leq k(d(x_{n+1}\ ,\ x_n)\oplus d(x_n\ ,\ x_{n-1}))$
$\Rightarrow d(x_{n+1},x_n)\leq\frac{k}{1-k}d(x_n,x_{n-1})=hd(x_n,x_{n-1})$ where $h=\frac{k}{1-k}$.
For $n>m$, we have
$d(x_n,x_m)\leq d(x_n,x_{n-1})\oplus d(x_{n-1},x_{n-2})\oplus......\oplus d(x_{m+1},x_m)$.
i.e. $d(x_n,x_m)\leq(h^{n-1}+h^{n-2}+......+h^m)d(x_1,x_0)\ =\ \frac{h^m}{1-h}d(x_1,x_0)$.
Let $\bar{0}<<||z||$ where $z\in E$ be given. Choose a natural number N_1 such that $\frac{h^m}{1-h}d(x_1,x_0)<<||z||\ \ \forall m\geq N_1$.
Thus $\forall n>m\geq N_1$ we get $d(x_n,x_m)<<||z||$.
So $\{x_n\}$ is a Cauchy sequence in $(X\ ,\ d)$. By completeness of X, there is $x^*\in X$ such that $x_n\to x^*$.
Choose a natural number N_2 such that $d(x_n\ ,\ x^*)<<\frac{(1-k)}{2k}||z||$ and
$d(x_{n+1}\ ,\ x^*)<<\frac{(1-k)}{2(1+k)}||z||\ \ \forall n\geq N_2$.
Thus $\forall n\geq N_2$ we have
$d(Tx^*,x^*)\ \leq\ d(Tx_n\ ,\ Tx^*)\oplus d(Tx_n\ ,\ x^*)\ \leq\ k(d(Tx^*\ ,\ x_n)\oplus d(Tx_n,x^*))\oplus d(x_{n+1}\ ,\ x^*)$
$\Rightarrow d(Tx^*,x^*)\leq k(d(Tx^*\ ,\ x^*)\oplus d(x_n,x^*)\oplus d(x_{n+1}\ ,\ x^*))\oplus d(x_{n+1}\ ,\ x^*)$
$\Rightarrow d(Tx^*,x^*)\leq\frac{k}{1-k}d(x_n,x^*)\oplus\frac{1+k}{1-k}d(x_{n+1},x^*)$
i.e $d(Tx^*,x^*)<<\frac{||z||}{2}\oplus\frac{||z||}{2}=||z||$
$\Rightarrow d(Tx^*,x^*)<<\frac{||c||}{m}\ \ \forall m\geq 1$ (since $||c||\in IntP\ \Rightarrow s||c||\in IntP\ \forall s>0$)
$\Rightarrow\frac{||c||}{m}\ominus d(Tx^*,x^*)\in P\ \ \forall m\geq 1$.
Since $\frac{||c||}{m}\to\bar{0}$ as $m\to\infty$ and P is closed, it follows that
$\ominus d(Tx^*,x^*)\in P$. So $d(Tx^*,x^*)=\bar{0}$. i.e. $Tx^*=x^*$.
For uniqueness, if suppose that y^* is another fixed point of T. Then
$d(x^*\ ,\ y^*)=d(Tx^*\ ,\ Ty^*)\leq k(d(Tx^*\ ,\ y^*)\oplus d(Ty^*\ ,\ x^*))$.
i.e. $d(x^*\ ,\ y^*)\leq 2kd(x^*\ ,\ y^*)$.
This implies that $d(x^*\ ,\ y^*)=\bar{0}$ (since $2k-1<0$). i.e. $x^*=y^*$. □

Theorem 4.5 is justified by the following example:

Example 4.6. In Example 4.2, define $T(x) = \frac{x}{4}, \quad x \in X$.
We have $d^1_\alpha(Tx, Ty) = \frac{1}{4}|x - y|$.
Now, $d^1_\alpha(Tx, y) + d^1_\alpha(x, Ty) = d^1_\alpha(\frac{x}{4}\ ,\ y) + d^1_\alpha(x\ ,\ \frac{y}{4}) = |\frac{x}{4} - y| + |x - \frac{y}{4}|$.
i.e. $d^1_\alpha(Tx, y) + d^1_\alpha(x, Ty) \geq |\frac{x}{4} - y + x - \frac{y}{4}| = \frac{5}{4}|x - y|$.
$\Rightarrow \frac{1}{5}\{d^1_\alpha(Tx, y)+d^1_\alpha(x, Ty)\} \geq \frac{1}{4}|x-y| = d^1_\alpha(Tx, Ty)$ (4.6.1).
Similarly $\frac{1}{5}\{d^2_\alpha(Tx, y)+d^2_\alpha(x, Ty)\} \geq \frac{1}{4}|x-y| = d^2_\alpha(Tx, Ty)$ (4.6.2).
From (4.6.1) and (4.6.2), it follows that,
$d(Tx, Ty) \preceq k(d(Tx, y) \oplus d(x, Ty))$ where $k = \frac{1}{5} \in [0, \frac{1}{2})$.
Thus T satisfies the condition of Theorem 4.5. Here $\underline{0}$ is the unique fixed point of T.
Also any iterative sequence in X converges to $\underline{0}$.

5. Conclusion

In this paper, it is shown that every regular fuzzy cone is normal but not conversely . There are no fuzzy normal cones with normal constant $M < 1$. Some fixed point theorems are established in fuzzy cone metric spaces by omitting the assumption of normality. I think that there is a large scope of developing more results of fuzzy functional analysis particularly in the field of generalized fuzzy metric spaces.

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