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Hausdorff fuzzy soft topological spaces

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Abstract. In this paper, we have studied Hausdorff separation axiom in a fuzzy soft topological space. Several basic desirable results have been proved. In particular, we have obtained a characterization of a Hausdorff fuzzy soft topological space, in terms of the diagonal set and it has been shown that Hausdorffness in a fuzzy soft topological space satisfies the productive, projective and hereditary properties.

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1. Introduction

There are many complicated problems arising in economics, engineering sciences, medical sciences, social sciences, environmental sciences etc. where the data is not crisp in nature. Classical mathematical tools cannot be successfully used to solve these problems due to the presence of various types of uncertainties involved in these problems.

There are theories e.g., theory of probability, theory of fuzzy sets [14], theory of intutionistic fuzzy sets [2, 3], theory of interval mathematics [3, 6] etc. But all of these theories have limitations/difficulties as mentioned by Molodtsov [10] and he felt that a possible reason for these difficulties was the inadequacy of the parametrization tool of these theories. As a consequence, Molodtsov [10] introduced the concept of soft sets which is a mathematical tool for dealing with uncertainties and is free from the difficulties in the previously mentioned theories.

Soft set theory has been applied in many directions e.g., stability and regularization [10], game theory and operations research [10], soft analysis [10], group theory [1] etc.

Later on Maji et al.[8] introduced and studied fuzzy soft sets. Topological structure of fuzzy soft sets was introduced and studied by Tanay and Kandemir [12]. It was further studied by Varol and Aygün [13] and Cetkin and Aygün [5] etc.

In this paper, we have studied Hausdorff separation axiom in a fuzzy soft topological space, in detail. Several basic desirable results have been established.

2. Preliminaries

Throughout this paper, X denotes a non empty set, called the universe, E the set of parameters for the universe X and $A \subseteq E$.

Definition 2.1 ([14]). A fuzzy set in X is a function $f : X \to [0,1]$. Now we define some basic fuzzy set operations as follows:

Let f and g be fuzzy sets in X. Then

(1) $f = g$ if $f(x) = g(x),\ \forall x \in X$.
(2) $f \subseteq g$ if $f(x) \leqslant g(x),\ \forall x \in X$.
(3) $(f \cup g)(x) = \max\{f(x), g(x)\},\ \forall x \in X$.
(4) $(f \cap g)(x) = \min\{f(x), g(x)\},\ \forall x \in X$.
(5) $f^c(x) = 1 - f(x),\ \forall x \in X$ (here f^c denotes the complement of f).

Support of a fuzzy set f in X, denoted by supp$f$, is defined as follows(cf [9]):

$$\text{supp}f = \{x \in X : f(x) > 0\}.$$

Definition 2.2 ([9]). Let Ω be an index set and $\{f_i : i \in \Omega\}$ be a family of fuzzy sets in X. Then their union $\bigcup\limits_{i\in\Omega} f_i$ and intersection $\bigcap\limits_{i\in\Omega} f_i$ are defined respectively as follows:

(1) $(\bigcup\limits_{i\in\Omega} f_i)(x)=\sup\{f_i(x) : i \in \Omega\},\ \forall x \in X$.
(2) $(\bigcap\limits_{i\in\Omega} f_i)(x)=\inf\{f_i(x) : i \in \Omega\},\ \forall x \in X$.

The constant fuzzy set in X, taking value $\alpha \in [0,1]$, will be denoted by α_X.

Definition 2.3 ([11]). A fuzzy point x_λ $(0 < \lambda < 1)$ in X is a fuzzy set in X given by

$$x_\lambda(x') = \begin{cases} \lambda, & \text{if } x' = x \\ 0, & \text{otherwise.} \end{cases}$$

Here x and λ are respectively called the support and value of x_λ.

Definition 2.4 ([10]). A pair (F, E) is called a soft set over X if F is a mapping from E to 2^X i.e., $F : E \to 2^X$, where 2^X is the powerset of X.

Definition 2.5 ([8]). A pair (f, E) is called a fuzzy soft set over X if f is a mapping from E to I^X i.e., $f : E \to I^X$, where I^X is the collection of all fuzzy sets in X.

Definition 2.6 ([13]). A fuzzy soft set f_A over X is a mapping from E to I^X i.e., $f_A : E \to I^X$ such that $f_A(e) \neq 0_X$, if $e \in A \subseteq E$ and $f_A(e) = 0_X$, otherwise, where 0_X denotes the empty fuzzy set in X, given by $0_X(x) = 0,\ \forall x \in X$.

Definition 2.7 ([13])**.** The universal fuzzy soft set 1_E over X is given by $1_E(e) = 1_X, \forall e \in E$ and the null fuzzy soft set 0_E over X is given by $0_E(e) = 0_X, \forall e \in E$, where 1_X denotes the absolute fuzzy set in X given by $1_X(x) = 1, \forall x \in X$.

From here onwards, we will denote by $\mathcal{F}(X, E)$, the set of all fuzzy soft sets over X.

Definition 2.8 ([13])**.** Let $f_A, g_B \in \mathcal{F}(X, E)$.Then

(1) f_A is said to be a fuzzy soft subset of g_B, denoted by $f_A \sqsubseteq g_B$, if $f_A(e) \subseteq g_B(e),\ \forall e \in E$.

(2) f_A and g_B are said to be equal, denoted by $f_A = g_B$, if $f_A \sqsubseteq g_B$ and $g_B \sqsubseteq f_A$.

(3) The union of f_A and g_B, denoted by $f_A \sqcup g_B$, is the fuzzy soft set over X defined by

$$(f_A \sqcup g_B)(e) = f_A(e) \cup g_B(e),\ \forall e \in E.$$

(4) The intersection of f_A and g_B, denoted by $f_A \sqcap g_B$, is the fuzzy soft set over X defined by

$$(f_A \sqcap g_B)(e) = f_A(e) \cap g_B(e),\ \forall e \in E.$$

Two fuzzy soft sets f_A and g_B over X are said to be disjoint if $f_A \sqcap g_B = 0_E$.

(5) Let Ω be an index set and $\{(f_A)_i : i \in \Omega\}$ be a family of fuzzy soft sets over X. Then their union $\bigsqcup_{i \in \Omega} (f_A)_i$ and intersection $\sqcap_{i \in \Omega} (f_A)_i$ are defined, respectively as follows:

(a) $(\bigsqcup_{i \in \Omega} (f_A)_i)(e) = \bigcup_{i \in \Omega} (f_A)_i(e),\ \forall e \in E$.

(b) $(\sqcap_{i \in \Omega} (f_A)_i)(e) = \bigcap_{i \in \Omega} (f_A)_i(e),\ \forall e \in E$.

(6) The complement of f_A, denoted by f_A^c, is the fuzzy soft set over X, defined by

$$f_A^c(e) = 1_X - f_A(e),\ \forall e \in E.$$

Definition 2.9 ([4])**.** Let $\mathcal{F}(X, E)$ and $\mathcal{F}(Y, K)$ be the collection of all the fuzzy soft sets over X and Y respectively and E, K be the parameters sets for the universe X and Y respectively. Let $\varphi : X \to Y$ and $\psi : E \to K$ be two maps. Then the fuzzy soft mapping from X to Y is a pair (φ, ψ) and is denoted by

$$(\varphi, \psi) : \mathcal{F}(X, E) \to \mathcal{F}(Y, K).$$

(1) Let $f_A \in \mathcal{F}(X, E)$. Then the image of f_A under the fuzzy soft mapping (φ, ψ) is a fuzzy soft set over Y, denoted by $(\varphi, \psi) f_A$ and is defined as

$$(\varphi, \psi) f_A(k)(y) = \begin{cases} \sup\limits_{\varphi(x)=y} \ \sup\limits_{\psi(e)=k} f_A(e)(x), & \text{if } \varphi^{-1}(y) \neq \phi \text{ and } \psi^{-1}(k) \neq \phi \\ 0, & \text{otherwise.} \end{cases}$$

$$\forall y \in Y, \forall k \in K.$$

(2) Let $g_B \in \mathcal{F}(Y, K)$. Then the inverse image of g_B under the fuzzy soft mapping (φ, ψ) is a fuzzy soft set over X, denoted by $(\varphi, \psi)^{-1} g_B$ and is defined as

$$(\varphi, \psi)^{-1} g_B(e)(x) = g_B(\psi(e))(\varphi(x)),\ \forall e \in E, \forall x \in X.$$

Definition 2.10 ([13])**.** Let $f_A \in \mathcal{F}(X,E)$ and $g_B \in \mathcal{F}(Y,K)$. Then the fuzzy soft product of f_A and g_B, denoted by $f_A \times g_B$, is a fuzzy soft set over $X \times Y$ and is defined by

$$(f_A \times g_B)(e,k) = f_A(e) \times g_B(k),\ \forall (e,k) \in E \times K$$

and for $(x,y) \in X \times Y$,

$$(f_A(e) \times g_B(k))(x,y) = \min\{f_A(e)(x), g_B(k)(y)\}.$$

Definition 2.11 ([12, 13])**.** A fuzzy soft topological space is a pair (X,τ) consisting of a non empty set X and a family τ of fuzzy soft sets over X satisfying the following conditions :

(1) $0_E, 1_E \in \tau$.
(2) If $f_A, g_B \in \tau$, then $f_A \sqcap g_B \in \tau$.
(3) If $(f_A)_j \in \tau, \forall j \in \Omega$, where Ω is some index set, then $\bigsqcup_{j\in\Omega}(f_A)_j \in \tau$.

Then τ is called a fuzzy soft topology over X. Members of τ are called fuzzy soft open sets. A fuzzy soft set g_B over X is called fuzzy soft closed if $(g_B)^c \in \tau$.

In particular, $\tau^o = \{0_E, 1_E\}$ and $\tau^1 = \mathcal{F}(X,E)$ are fuzzy soft topologies over X.

Theorem 2.12. *Let (X,τ) be a fuzzy soft topological space relative to the parameters set E and $G \subseteq E$. Then (X,τ_G) is a fuzzy soft topology over X where*

$$\tau_G = \{f_A \mid_G : f_A \in \tau\},$$

relative to the parameters set G.

Proof. (1) $0_G = 0_E \mid_G, 1_G = 1_E \mid_G$, therefore 0_G and $1_G \in \tau_G$.
(2) Let f_{G_1}, $f_{G_2} \in \tau_G$. Then $f_{G_1} \sqcap f_{G_2} \in \tau_G$ since $f_{G_1} = (f_A)_1 \mid_G$ and $f_{G_2} = (f_A)_2 \mid_G$, so $(f_{G_1} \sqcap f_{G_2}) = ((f_A)_1 \mid_G) \sqcap ((f_A)_2 \mid_G) = ((f_A)_1 \sqcap (f_A)_2) \mid_G$.
(3) Let $\{f_{G_i} : i \in \Omega\}$ be a family of members of τ_G. Then $\bigsqcup_{i\in\Omega} f_{G_i} \in \tau_G$ since $f_{G_i} = (f_A)_i \mid_G$, so $\bigsqcup_{i\in\Omega} f_{G_i} = (\bigsqcup_{i\in\Omega}(f_A)_i) \mid_G$.

□

Definition 2.13. Let (X,τ) be a fuzzy soft topological space and $G \subseteq E$. Then (X,τ_G) defined in the above theorem is called a fuzzy soft subspace of (X,τ).

Definition 2.14 ([13])**.** A fuzzy soft topology τ_1 is called finer than a fuzzy soft topology τ_2 if $\tau_2 \subseteq \tau_1$ and then τ_2 is called coarser than τ_1.

Clearly, τ^o and τ^1 are the coarsest and finest fuzzy soft topologies over X, respectively.

Definition 2.15 ([13])**.** Let (X,τ) be a fuzzy soft topological space. Then a subfamily $\mathcal{B}$ of τ is called a base for τ if every member of τ can be written as a union of members of $\mathcal{B}$.

Definition 2.16 ([13])**.** Let (X,τ) be a fuzzy soft topological space. Then a subfamily $\mathcal{S}$ of τ is called a subbase for τ if the family of finite intersection of its members forms a base for τ.

Definition 2.17 ([13]). A fuzzy soft topology τ over X is said to be generated by a subfamily $\mathcal{S}$ of fuzzy soft set over X if every member of τ is a union of finite intersection of members of $\mathcal{S}$.

Definition 2.18 ([13]). Let $\{(X_i, \tau_i)\}_{i\in\Omega}$ be a family of fuzzy soft topological spaces and for each $i \in \Omega$, we have a fuzzy soft mapping

$$(\varphi, \psi)_i : X \to (X_i, \tau_i).$$

Then the fuzzy soft topology τ over X is said to be initial with respect to the family $\{(\varphi, \psi)_i\}_{i\in\Omega}$ if τ has as subbase the set

$$\mathcal{S} = \{(\varphi, \psi)_i^{-1}(f_{A_i}) : i \in \Omega,\ f_{A_i} \in \tau_i\}$$

i.e., the fuzzy soft topology τ over X is generated by $\mathcal{S}$.

Definition 2.19 ([13]). Let $\{(X_i, \tau_i)\}_{i\in\Omega}$ be a family of fuzzy soft topological spaces. Then their product is defined as the fuzzy soft topological space (X, τ) where $X = \prod\limits_i X_i$ and τ is the fuzzy soft topology over X which is initial with respect to the family $\{(p_{X_i}, q_{E_i})\}_{i\in\Omega}$, $p_{X_i} : \prod\limits_i X_i \to X_i$ and $q_{E_i} : \prod\limits_i E_i \to E_i$, $i \in \Omega$ are the projection maps i.e., τ is generated by

$$\{(p_{X_i}, q_{E_i})^{-1}(f_{A_i}) : i \in \Omega, f_{A_i} \in \tau_i\}.$$

In particular, let (X_1, τ_1) and (X_2, τ_2) be two fuzzy soft topological spaces, then their product is $(X_1 \times X_2, \tau)$ where τ is generated by the set,

$$\mathcal{S} = \{(p_{X_1}, q_{E_1})^{-1}(f_{A_1}), (p_{X_2}, q_{E_2})^{-1}(g_{A_2}) : f_{A_1} \in \tau_1,\ g_{A_2} \in \tau_2\}.$$

Note that

$$\begin{aligned}(p_{X_1}, q_{E_1})^{-1}(f_{A_1})(e_1, e_2)(x, y) &= f_{A_1}(q_{E_1}(e_1, e_2))(p_{X_1}(x, y))\\ &= f_{A_1}(e_1)(x)\\ &= (f_{A_1} \times 1_{E_2})(e_1, e_2)(x, y),\ \forall (x, y) \in X_1 \times X_2.\end{aligned}$$

Therefore, $(p_{X_1}, q_{E_1})^{-1}(f_{A_1}) = f_{A_1} \times 1_{E_2}$.
Similarly, $(p_{X_2}, q_{E_2})^{-1}(g_{A_2}) = 1_{E_1} \times g_{A_2}$.

So, $\mathcal{S}$ has the following form

$$\mathcal{S} = \{f_{A_1} \times 1_{E_2},\ 1_{E_1} \times g_{A_2}\ :\ f_{A_1} \in \tau_1,\ g_{A_2} \in \tau_2\}$$

and τ has a base $\mathcal{B}$ of the form

$$\mathcal{B} = \{f_{A_1} \times g_{A_2} : f_{A_1} \in \tau_1,\ g_{A_2} \in \tau_2\}.$$

Mahanta and Das ([7]) had given the following definitions:

Definition 2.20 ([7]). A fuzzy soft set g_A is said to be a fuzzy soft point, denoted by e_{g_A}, if for the element $e \in A$, $g_A(e) \neq 0_X$ and $g_A(e') = 0_X$, $\forall e' \in A - \{e\}$.

Definition 2.21 ([7]). A fuzzy soft point e_{g_A} is said to be in a fuzzy soft set h_A, denoted by $e_{g_A} \widetilde{\in} h_A$ if for the element $e \in A$, $g_A(e) \leq h_A(e)$.

We observe from the above definitions that the result given by the authors [7], in Theorem 2.5(v) i.e.,

$$e_{g_A} \tilde{\in} \bigsqcup \{h_{\lambda B} : \lambda \in \Lambda\} \Leftrightarrow \exists\, \lambda \in \Lambda \text{ such that } e_{g_A} \tilde{\in} h_{\lambda B}$$

does not hold good. A counter example is given as follows:

Example 2.22. Consider the fuzzy soft point e_{g_A} such that

$$e_{g_A}(e) = \alpha_X, \quad \alpha \in (0,1)$$

and the family $\{h_{\lambda B} : 0 < \lambda < \alpha\}$ of fuzzy soft sets over X such that

$$h_{\lambda B}(e') = \begin{cases} (\alpha - \lambda)_X, & \text{if } e' = e \\ 0_X, & \text{otherwise.} \end{cases}$$

Then $e_{g_A} \tilde{\in} \bigsqcup\limits_{0<\lambda<\alpha} h_{\lambda B}$ but $e_{g_A} \tilde{\notin} h_{\lambda B}$ for any λ such that $0 < \lambda < \alpha$.

To retain the above result, in the definition 2.21, '$g_A(e) \leq h_A(e)$' must be replaced by '$g_A(e) < h_A(e)$ (i.e., $g_A(e)(x) < h_A(e)(x)$, $\forall x \in X$)'. In view of this modification, any fuzzy soft point $e_{g_A} \tilde{\in} h_A$ only if $h_A(e)(x) > 0, \forall x \in X$. In this situation, no pair of distinct fuzzy soft points e_{g_A} and e_{k_A} can be separated by disjoint fuzzy soft open sets, which is a requirement in the definition of Hausdorffness [7] in a fuzzy soft topological space.

Therefore, we give an alternative definition of a 'fuzzy soft point' and 'belonging of a fuzzy soft point to a fuzzy soft set', as follows.

Definition 2.23. A fuzzy soft point e_{x_λ} over X is a fuzzy soft set over X defined as follows:

$$e_{x_\lambda}(e') = \begin{cases} x_\lambda, & \text{if } e' = e \\ 0_X, & \text{if } e' \in E - \{e\}, \end{cases}$$

where x_λ is the fuzzy point([11]) in X with support x and value λ, $\lambda \in (0,1)$.

A fuzzy soft point e_{x_λ} is said to belong to a fuzzy soft set f_A, denoted by $e_{x_\lambda} \in f_A$ if $\lambda < f_A(e)(x)$.Two fuzzy soft points e_{x_λ} and e'_{y_s} are said to be distinct if $x \neq y$ or $e \neq e'$.

Example 2.24. Let $X = \{x^1, x^2\}$ and $E = \{e^1, e^2\}$ be the universe set and the parameters set for the universe X, respectively. Then the fuzzy soft point $(e^1)_{(x^1)_{0.5}}$ is a fuzzy soft set over X given by

$$(e^1)_{(x^1)_{0.5}}(e) = \begin{cases} (x^1)_{0.5}, & \text{if } e = e^1 \\ 0_X, & \text{if } e = e^2. \end{cases}$$

Proposition 2.25. *Let $\{(f_A)_i : i \in \Omega\}$ be a family of fuzzy soft sets over X , then $e_{x_\lambda} \in \bigsqcup\limits_{i\in\Omega} (f_A)_i$ iff $e_{x_\lambda} \in (f_A)_i$ for some $i \in \Omega$.*

Proof. First, suppose that $e_{x_\lambda} \in (f_A)_i$ for some $i \in \Omega$. Then,

$$\begin{array}{ll} & \lambda < (f_A)_i(e)(x) \\ \Rightarrow & \lambda < (f_A)_i(e)(x) \leqslant \sup\limits_{j\in\Omega}(f_A)_j(e)(x) \\ \Rightarrow & e_{x_\lambda} \in \bigsqcup\limits_{j\in\Omega}(f_A)_j \end{array}$$

Conversely, let $e_{x_\lambda} \in \bigsqcup\limits_{j\in\Omega}(f_A)_j$, then

$$\begin{array}{ll} & \lambda < (\bigsqcup\limits_{j\in\Omega}(f_A)_j)(e)(x) \\ \Rightarrow & \lambda < \sup\limits_{j\in\Omega}(f_A)_j(e)(x) \\ \Rightarrow & \lambda < (f_A)_i(e)(x), \text{ for some } i \in \Omega \\ \Rightarrow & e_{x_\lambda} \in (f_A)_i. \end{array}$$

□

Proposition 2.26. *A fuzzy soft set f_A over X is the union of all the fuzzy soft points belonging to it i.e.,*

$$f_A = \bigsqcup\{e_{x_\lambda} : e_{x_\lambda} \in f_A\}.$$

Proof. It is easy to see that $\bigsqcup\{e_{x_\lambda} : e_{x_\lambda} \in f_A\} \sqsubseteq f_A$.

Conversely, to show that $f_A \sqsubseteq \bigsqcup\{e_{x_\lambda} : e_{x_\lambda} \in f_A\}$. First we note that $f_A(e')(x') = 0$, if $e' \notin A$ or $x' \notin \text{supp} f_A(e')$. Next consider the case when $e' \in A$, $x' \in \text{supp} f_A(e')$. Then,

$$\begin{array}{rcl} \bigsqcup\{e_{x_\lambda} : e_{x_\lambda} \in f_A\}(e')(x') & = & \sup\{e_{x_\lambda}(e')(x') : e_{x_\lambda} \in f_A\} \\ & = & \sup\{e'_{x'_\lambda}(e')(x') : e'_{x'_\lambda} \in f_A\} \\ & = & \sup\{\lambda : e'_{x'_\lambda} \in f_A\}, \\ & = & f_A(e')(x') \end{array}$$

Thus, $f_A \sqsubseteq \bigsqcup\{e_{x_\lambda} : e_{x_\lambda} \in f_A\}$. Hence $f_A = \bigsqcup\{e_{x_\lambda} : e_{x_\lambda} \in f_A\}$. □

Proposition 2.27. *Let (X, τ) be a fuzzy soft topological space. Then a fuzzy soft set f_A is fuzzy soft open iff $\forall e_{x_r} \in f_A$, there exists a basic fuzzy soft open set g_B such that $e_{x_r} \in g_B \sqsubseteq f_A$.*

Proof. First, suppose that the fuzzy soft set f_A over X is open and $\mathcal{B}$ denotes a base for τ. Then $f_A = \bigsqcup\limits_{i\in\Omega}(g_B)_i$, where Ω is an index set and $(g_B)_i \in \mathcal{B}, \forall i \in \Omega$.

Let $e_{x_r} \in f_A$. Then, $e_{x_r} \in \bigsqcup\limits_{i\in\Omega}(g_B)_i \Rightarrow e_{x_r} \in (g_B)_i \sqsubseteq f_A$ for some $i \in \Omega$.

Conversely, assume that $\forall e_{x_r} \in f_A$, there exists a basic fuzzy soft open set $(g_B)_{e_{x_r}}$ such that

$$e_{x_r} \in (g_B)_{e_{x_r}} \sqsubseteq f_A.$$

Now, taking union, we get

$$\bigsqcup\{e_{x_r} : e_{x_r} \in f_A\} \sqsubseteq \bigsqcup (g_B)_{e_{x_r}} \sqsubseteq f_A$$

implying that

$$f_A = \bigsqcup_{e_{x_r} \in f_A} (g_B)_{e_{x_r}}.$$

Hence f_A is fuzzy soft open. □

3. Hausdorff fuzzy soft topological space

Mahanta and Das ([7]) had introduced Hausdorffness in a fuzzy soft topological space using the definitions of a 'fuzzy soft point' and 'belonging of a fuzzy soft point to a fuzzy soft set', in his sense. Here we define Hausdorffness in a fuzzy soft topological space in terms of the modified definitions of a 'fuzzy soft point' and 'belonging', given in definition 2.23.

Definition 3.1. Let (X, τ) be a fuzzy soft topological space. Then (X, τ) is said to be Hausdorff if for each pair of distinct fuzzy soft points e_{x_λ}, e'_{y_s} over X, there exist fuzzy soft open sets f_A and g_B such that $e_{x_\lambda} \in f_A$, $e'_{y_s} \in g_B$ and $f_A \sqcap g_B = 0_E$.

We give an example of Hausdorff fuzzy soft topological spaces as follows:

Example 3.2. Let $X = \{x^1, x^2\}$ and $E = \{e^1, e^2\}$ be the universe set and the parameters set for the universe X, respectively. Consider the collection τ of fuzzy soft sets over X,

$$\tau = \{0_E, 1_E, F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, F_{10}, F_{11}, F_{12}, F_{13}, F_{14}\},$$

where $F_i's$ are as follows:

$$F_1(e^1) = \{\frac{x^1}{1}, \frac{x^2}{0}\}, F_1(e^2) = \{\frac{x^1}{0}, \frac{x^2}{0}\}; \quad F_2(e^1) = \{\frac{x^1}{0}, \frac{x^2}{1}\}, F_2(e^2) = \{\frac{x^1}{0}, \frac{x^2}{0}\};$$

$$F_3(e^1) = \{\frac{x^1}{0}, \frac{x^2}{0}\}, F_3(e^2) = \{\frac{x^1}{1}, \frac{x^2}{0}\}; \quad F_4(e^1) = \{\frac{x^1}{0}, \frac{x^2}{0}\}, F_4(e^2) = \{\frac{x^1}{0}, \frac{x^2}{1}\};$$

$$F_5(e^1) = \{\frac{x^1}{1}, \frac{x^2}{1}\}, F_5(e^2) = \{\frac{x^1}{0}, \frac{x^2}{0}\}; \quad F_6(e^1) = \{\frac{x^1}{0}, \frac{x^2}{1}\}, F_6(e^2) = \{\frac{x^1}{1}, \frac{x^2}{0}\};$$

$$F_7(e^1) = \{\frac{x^1}{0}, \frac{x^2}{0}\}, F_7(e^2) = \{\frac{x^1}{1}, \frac{x^2}{1}\}; \quad F_8(e^1) = \{\frac{x^1}{1}, \frac{x^2}{0}\}, F_8(e^2) = \{\frac{x^1}{0}, \frac{x^2}{1}\};$$

$$F_9(e^1) = \{\frac{x^1}{1}, \frac{x^2}{0}\}, F_9(e^2) = \{\frac{x^1}{1}, \frac{x^2}{0}\}; \quad F_{10}(e^1) = \{\frac{x^1}{0}, \frac{x^2}{1}\}, F_{10}(e^2) = \{\frac{x^1}{0}, \frac{x^2}{1}\};$$

$$F_{11}(e^1) = \{\frac{x^1}{1}, \frac{x^2}{1}\}, F_{11}(e^2) = \{\frac{x^1}{1}, \frac{x^2}{0}\}; \quad F_{12}(e^1) = \{\frac{x^1}{0}, \frac{x^2}{1}\}, F_{12}(e^2) = \{\frac{x^1}{1}, \frac{x^2}{1}\};$$

$$F_{13}(e^1) = \{\frac{x^1}{1}, \frac{x^2}{0}\}, F_{13}(e^2) = \{\frac{x^1}{1}, \frac{x^2}{1}\}; \quad F_{14}(e^1) = \{\frac{x^1}{1}, \frac{x^2}{1}\}, F_{14}(e^2) = \{\frac{x^1}{0}, \frac{x^2}{1}\}.$$

Then, clearly τ is fuzzy soft topology over X. Also, for every pair of distinct fuzzy soft points, there exist disjoint fuzzy soft open sets over X containing them. Hence (X, τ) is a Hausdorff fuzzy soft topological space .

Theorem 3.3. *A fuzzy soft topological space (X, τ) is Hausdorff iff the fuzzy soft set f_{Δ_E} over $X \times X$ is closed, where f_{Δ_E} is given by :*

$$f_{\Delta_E}(e_1, e_2) = \begin{cases} \chi_{\Delta_X}, & if\, e_1 = e_2 \\ 0_{X\times X}, & if\, e_1 \neq e_2. \end{cases}$$

Proof. First, let us assume that (X, τ) is Hausdorff. To show that f_{Δ_E} is fuzzy soft closed, equivalently, $(f_{\Delta_E})^c$ is fuzzy soft open, choose a fuzzy soft point $(e, e^{'})_{(x,y)_\lambda} \in (f_{\Delta_E})^c$. Now e_{x_λ} and e'_{y_λ} are distinct fuzzy soft points over X. Since (X, τ) is Hausdorff, there exist fuzzy soft open sets f_A and g_B such that

$$e_{x_\lambda} \in f_A,\ e'_{y_\lambda} \in g_B \text{ and } f_A \sqcap g_B = 0_E.$$

Now, consider $f_A \times g_B$. Then

$$(e, e')_{(x,y)_\lambda} \in f_A \times g_B \sqsubseteq (f_{\Delta_E})^c$$

as shown below:
Since $e_{x_\lambda} \in f_A$ and $e'_{y_\lambda} \in g_B$, so we have

$$\begin{aligned} & \lambda < f_A(e)(x) \text{ and } \lambda < g_B(e')(y) \\ \Rightarrow\quad & \lambda < \min\{f_A(e)(x), g_B(e')(y)\} = (f_A \times g_B)(e, e')(x, y) \\ \Rightarrow\quad & (e, e')_{(x,y)_\lambda} \in f_A \times g_B \end{aligned}$$

Next, for $f_A \times g_B \sqsubseteq (f_{\Delta_E})^c$, we proceed as follows:
case I : $e_1 \neq e_2$, this inclusion is trivially satisfied.
case II : $e_1 = e_2$, we need only to show that

$$(f_A \times g_B)(e_1, e_1)(x, x) = 0,\ \forall e_1 \in E, \forall x \in X$$

which is true, since we have

$$\begin{aligned} & f_A \sqcap g_B = 0_E \\ \Rightarrow\quad & (f_A(e_1) \cap g_B(e_1))(x) = 0,\ \forall e_1 \in E, \forall x \in X \\ \Rightarrow\quad & \min\{f_A(e_1)(x), g_B(e_1)(x)\} = 0,\ \forall e_1 \in E, \forall x \in X \\ \Rightarrow\quad & (f_A \times g_B)(e_1, e_1)(x, x) = 0,\ \forall e_1 \in E, \forall x \in X \end{aligned}$$

Conversely, let f_{Δ_E} be fuzzy soft closed. To show that (X, τ) is fuzzy soft Hausdorff, let e_{x_r} and e'_{y_s} be two distinct fuzzy soft points over X. Then $(e, e')_{(x,y)_\lambda} \in (f_{\Delta_E})^c$, where $\lambda = \max(\mathrm{r}, \mathrm{s})$. Now since $(f_{\Delta_E})^c$ is fuzzy soft open, there exists a basic fuzzy soft open set, say $f_A \times g_B$, such that

$$\begin{aligned} & (e, e')_{(x,y)_\lambda} \in f_A \times g_B \sqsubseteq (f_{\Delta_E})^c \\ \Rightarrow\quad & \lambda < (f_A \times g_B)(e, e')(x, y) \\ \Rightarrow\quad & \lambda < \min\{f_A(e)(x), g_B(e')(y)\} \\ \Rightarrow\quad & \lambda < f_A(e)(x) \text{ and } \lambda < g_B(e')(y) \\ \Rightarrow\quad & r \leq \lambda < f_A(e)(x) \text{ and } s \leq \lambda < g_B(e')(y) \\ \Rightarrow\quad & e_{x_r} \in f_A \text{ and } e'_{y_s} \in g_B \end{aligned}$$

Further, since $f_A \times g_B \sqsubseteq (f_{\Delta_E})^c$, we have

$$\begin{aligned}&(f_A \times g_B)(e_1, e_1)(x, x) = 0,\ \forall e_1 \in E, \forall x \in X\\ \Rightarrow\ &\min\{f_A(e_1)(x), g_B(e_1)(x)\} = 0,\ \forall e_1 \in E, \forall x \in X\\ \Rightarrow\ &f_A(e_1) \cap g_B(e_1) = 0_X,\ \forall e_1 \in E\\ \Rightarrow\ &f_A \sqcap g_B = 0_E.\end{aligned}$$

□

Theorem 3.4. *If $\{(X_i, \tau_i); i \in \Omega\}$ is a family of fuzzy soft topological spaces, then the fuzzy soft product topological space,*

$$(X, \tau) = \prod_{i\in\Omega}(X_i, \tau_i)$$

is Hausdorff iff each coordinate fuzzy soft topological space (X_i, τ_i) is Hausdorff.

Proof. First, let us assume that each (X_i, τ_i), $i \in \Omega$ is Hausdorff.
Let $(\prod_{i\in\Omega} e_i)_{(\prod_{i\in\Omega} x_i)_r}$ and $(\prod_{i\in\Omega} e'_i)_{(\prod_{i\in\Omega} y_i)_s}$ be any pair of distinct fuzzy soft points over X. Then $\prod_{j\in\Omega} x_j \neq \prod_{j\in\Omega} y_j$ or $\prod_{j\in\Omega} e_j \neq \prod_{j\in\Omega} e'_j$. Let $\prod_{j\in\Omega} x_j \neq \prod_{j\in\Omega} y_j$, then $x_i \neq y_i$ for some $i \in \Omega$. Consider two fuzzy soft points $(e_i)_{(x_i)_r}$ and $(e'_i)_{(y_i)_s}$ over X_i which are distinct as $x_i \neq y_i$. Since (X_i, τ_i) is Hausdorff, so there exist two fuzzy soft open sets f_{A_i} and g_{B_i} such that

$$(e_i)_{(x_i)_r} \in f_{A_i},\ \ (e'_i)_{(y_i)_s} \in g_{B_i}\ \text{ and }\ f_{A_i} \sqcap g_{B_i} = 0_{E_i}.$$

Now, consider two fuzzy soft open sets over X as follows:

$$f_A = \prod_{j\in\Omega} f^1_{A_j} \text{ and } g_B = \prod_{j\in\Omega} g^1_{B_j},$$

where $f^1_{A_j} = 1_{E_j} = g^1_{B_j}$, $j \neq i$ and $f^1_{A_i} = f_{A_i}$, $g^1_{B_i} = g_{B_i}$. It is easy to see that f_A and g_B are disjoint fuzzy soft open sets such that $(\prod_{i\in\Omega} e_i)_{(\prod_{i\in\Omega} x_i)_r} \in f_A$ and $(\prod_{i\in\Omega} e'_i)_{(\prod_{i\in\Omega} y_i)_s} \in g_B$. The other case can be handled similarly.

Conversely, let us assume that the fuzzy soft product space (X, τ) is Hausdorff. Now, let $(e_i)_{(x_i)_r}$ and $(e'_i)_{(y_i)_s}$ be two distinct fuzzy soft points over X_i. Then $x_i \neq y_i$ or $e_i \neq e'_i$. Let $x_i \neq y_i$. Consider two fuzzy soft points $(\prod_{j\in\Omega} e_j)_{(\prod_{j\in\Omega} x_j)_r}$ and $(\prod_{j\in\Omega} e'_j)_{(\prod_{j\in\Omega} y_j)_s}$ over X, where $\prod_{j\in\Omega} x_j$ and $\prod_{j\in\Omega} y_j$ have identical j^{th} coordinates for $j \neq i$ and have i^{th} coordinates as x_i and y_i respectively and $\prod_{j\in\Omega} e_j$ and $\prod_{j\in\Omega} e'_j$ have identical j^{th} coordinates for $j \neq i$ and have i^{th} coordinates as e_i and e'_i respectively. Then $(\prod_{j\in\Omega} e_j)_{(\prod_{j\in\Omega} x_j)_r}$ and $(\prod_{j\in\Omega} e'_j)_{(\prod_{j\in\Omega} y_j)_s}$ are distinct fuzzy soft points over X. Since (X, τ) is Hausdorff, there exist two fuzzy soft open sets g_A and h_B such that

$$(\prod_{j\in\Omega} e_j)_{(\prod_{j\in\Omega} x_j)_r} \in g_A,\ (\prod_{j\in\Omega} e'_j)_{(\prod_{j\in\Omega} y_j)_s} \in h_B\ \text{ and }\ g_A \sqcap h_B = 0_E.$$

Now, since g_A and h_B are fuzzy soft open, so we can find basic fuzzy soft open sets $\prod_{j\in\Omega} g_{A_j}$, $\prod_{j\in\Omega} h_{B_j}$ such that

$$(\prod_{j\in\Omega} e_j)_{(\prod_{j\in\Omega} x_j)_r} \in \prod_{j\in\Omega} g_{A_j} \sqsubseteq g_A$$

and

$$(\prod_{j\in\Omega} e'_j)_{(\prod_{j\in\Omega} y_j)_s} \in \prod_{j\in\Omega} h_{B_j} \sqsubseteq h_B$$

Now,

$$\begin{aligned} & (\prod_{j\in\Omega} e_j)_{(\prod_{j\in\Omega} x_j)_r} \in \prod_{j\in\Omega} g_{A_j} \\ \Rightarrow \quad & r < \inf_j g_{A_j}(e_j)(x_j) \\ \Rightarrow \quad & r < g_{A_j}(e_j)(x_j), \ \forall j \in \Omega \qquad (3.1) \\ \Rightarrow \quad & r < g_{A_i}(e_i)(x_i) \\ \Rightarrow \quad & (e_i)_{(x_i)_r} \in g_{A_i} \end{aligned}$$

Similarly, $(e'_i)_{(y_i)_s} \in h_{B_i}$.
Since, $r \in (0,1)$, so from (3.1), we get

$$g_{A_j}(e_j)(x_j) > 0, \quad \forall j \in \Omega. \tag{3.2}$$

Similarly,

$$h_{B_j}(e'_j)(y_j) > 0, \quad \forall j \in \Omega. \tag{3.3}$$

Next, we have to show that

$$g_{A_i} \sqcap h_{B_i} = 0_{E_i}$$

Suppose on the contrary that,

$$g_{A_i} \sqcap h_{B_i} \neq 0_{E_i}$$

Then there exists $p_i \in E_i$, $z_i \in X_i$ such that

$$g_{A_i}(p_i)(z_i) > 0 \quad \text{and} \quad h_{B_i}(p_i)(z_i) > 0 \tag{3.4}$$

Construct a fuzzy soft point $(\prod_{j\in\Omega} e''_j)_{(\prod_{j\in\Omega} z^1_j)_\lambda}$ over X such that

$$e''_j = \begin{cases} e_j, & \text{if } j \neq i \\ p_i, & \text{if } j = i, \end{cases}$$

and

$$z^1_j = \begin{cases} x_j, & \text{if } j \neq i \\ z_i, & \text{if } j = i, \end{cases}$$

Now, for $z = \prod_{j\in\Omega} z^1_j$, from (3.2) and (3.4), we get

$$\prod_{j\in\Omega} g_{A_j}(\prod_{j\in\Omega} e''_j)(z) \quad = \quad \inf_j g_{A_j}(e''_j)(z^1_j) > 0$$

Similarly, from (3.3) and (3.4), we get

$$\prod_{j\in\Omega} h_{B_j}(\prod_{j\in\Omega} e_j^{''})(z) > 0$$

This gives us, $g_A(\prod_{j\in\Omega} e_j^{''})(z) > 0$, since $\prod_{j\in\Omega} g_{A_j} \sqsubseteq g_A$.
Similarly, $h_B(\prod_{j\in\Omega} e_j^{''})(z) > 0$, since $\prod_{j\in\Omega} h_{B_j} \sqsubseteq h_B$
implying that

$$g_A \sqcap h_B \neq 0_E,$$

which is a contradiction.
The other case can be handled similarly. □

Proposition 3.5. *Subspace of a Hausdorff fuzzy soft topological space is also Hausdorff.*

Proof. The proof is easy, hence is omitted. □

Definition 3.6 ([13]). Let (X_1, τ_1) and (X_2, τ_2) be two fuzzy soft topological spaces. Then a fuzzy soft mapping

$$(\varphi, \psi) : (X_1, \tau_1) \to (X_2, \tau_2)$$

is said to be fuzzy soft continuous if $(\varphi, \psi)^{-1} f_B \in \tau_1, \forall f_B \in \tau_2$.

Theorem 3.7 ([13]). *Let $(\varphi, \psi) : (X_1, \tau_1) \to (X_2, \tau_2)$ be a fuzzy soft mapping and $\mathcal{B}$ be a base for τ_2. Then (φ, ψ) is fuzzy soft continuous iff $(\varphi, \psi)^{-1} f_B \in \tau_1, \forall f_B \in \mathcal{B}$.*

Now we prove the following theorem,

Theorem 3.8. *Let (φ, ψ) and (φ', ψ') be two fuzzy soft continuous maps between fuzzy soft topological spaces (X_1, τ_1) and (X_2, τ_2) relative to the parameters sets E, E' respectively, where (X_2, τ_2) is Hausdorff. Then the fuzzy soft set h_A over X_1 defined as follows:*

$$h_A(e)(x) = \begin{cases} 1, & \textit{if } e \in A_1, x \in B_1 \\ 0, & \textit{otherwise} \end{cases}$$

where

$$A_1 = \{e \in E : \psi(e) = \psi'(e)\} \textit{ and } B_1 = \{x \in X_1 : \varphi(x) = \varphi'(x)\},$$

is fuzzy soft closed.

Proof. Here $(\varphi, \psi) : (X_1, \tau_1) \to (X_2, \tau_2)$ and $(\varphi', \psi') : (X_1, \tau_1) \to (X_2, \tau_2)$ are two fuzzy soft continuous maps. Now we define

$$((\varphi, \psi), (\varphi', \psi')) : (X_1, \tau_1) \to (X_2 \times X_2, \tau_2 \times \tau_2)$$

as the fuzzy soft map given by

$$((\varphi, \psi), (\varphi', \psi'))h_A = ((\varphi, \varphi'), (\psi, \psi'))h_A, \ \forall h_A \in \mathcal{F}(X_1, E)$$

where $(\psi, \psi')\colon E \to E' \times E'$, $(\varphi, \varphi')\colon X_1 \to X_2 \times X_2$ are given by

$$(\varphi, \varphi')(x) = (\varphi(x), \varphi'(x)), \ \forall x \in X_1, \ (\psi, \psi')(e) = (\psi(e), \psi'(e)), \ \forall e \in E.$$

Now we show that $((\varphi,\psi),(\varphi',\psi'))$ is fuzzy soft continuous. For this, consider a basic fuzzy soft open set $f_{A'}\times g_{B'}$ over $X_2\times X_2$.
Then,

$$\begin{aligned}
&((\varphi,\psi),(\phi',\psi'))^{-1}(f_{A'}\times g_{B'})(e)(x)\\
&= ((\varphi,\varphi'),(\psi,\psi'))^{-1}(f_{A'}\times g_{B'})(e)(x),\ \forall e\in E,\forall x\in X_1\\
&= (f_{A'}\times g_{B'})(\psi(e),\psi'(e))(\varphi(x),\varphi'(x)),\ \forall e\in E,\forall x\in X_1\\
&= \min\{f_{A'}(\psi(e))(\varphi(x)),g_{B'}(\psi'(e))(\varphi'(x))\},\ \forall e\in E,\forall x\in X_1\\
&= ((\varphi,\psi)^{-1}f_{A'}\sqcap(\varphi',\psi')^{-1}g_{B'})(e)(x),\ \forall e\in E,\forall x\in X_1
\end{aligned}$$

Since, (φ,ψ) and (φ',ψ') both are fuzzy soft continuous maps from (X_1,τ_1) to (X_2,τ_2), so we have $(\varphi,\psi)^{-1}f_{A'}\sqcap(\varphi',\psi')^{-1}g_{B'}\in\tau_1$. Hence $((\varphi,\psi),(\varphi',\psi'))$ is fuzzy soft continuous. Therefore $((\varphi,\psi),(\varphi',\psi'))^{-1}f_{\Delta_{E'}}$ is fuzzy soft closed over X_1, since $f_{\Delta_{E'}}$ is fuzzy soft closed over $X_2\times X_2$. Now we show that $((\varphi,\psi),(\varphi',\psi'))^{-1}f_{\Delta_{E'}}=h_A$, as follows:

$$\begin{aligned}
&((\varphi,\psi),(\varphi',\psi'))^{-1}(f_{\Delta_{E'}})(e)(x)\\
&= ((\varphi,\varphi'),(\psi,\psi'))^{-1}(f_{\Delta_{E'}})(e)(x),\ \forall e\in E,\forall x\in X_1\\
&= (f_{\Delta_{E'}})(\psi(e),\psi'(e))(\varphi(x),\varphi'(x)),\ \forall e\in E,\forall x\in X_1\\
&= \begin{cases}1, & \text{if } \psi(e)=\psi'(e) \text{ and } \varphi(x)=\varphi'(x)\\ 0, & \text{otherwise}\end{cases}\\
&= h_A(e)(x),\ \forall e\in E,\forall x\in X_1.
\end{aligned}$$

□

4. Conclusion

Molodtsov ([10]) initiated the theory of soft sets. This theory has been applied in many directions (cf.[10], [1] etc.). Fuzzy soft sets were introduced by Maji et al.([8]). Fuzzy soft topology was defined by Tanay and Kandemir ([12]). In this paper, we have studied Hausdorffness in a fuzzy soft topological space in detail. Several basic results have been proved.

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