Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 9, No. 2, (February 2015), pp. 235–246
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Uni-soft substructures of groups

Aslihan Sezgin Sezer, Naim Çağman, Akin Osman Atagün

Received 1 July 2014; Revised 25 July 2014; Accepted 15 August 2014

Abstract. In this paper, we introduce the notion of *uni*-soft subgroups and *uni*-soft normal subgroups of a group by using the definition of soft sets and investigate their related properties especially with respect to anti image, α-inclusions of soft sets and group homomorphism. We also obtain some significant relations between *int*-soft subgroups and *uni*-soft subgroups.

2010 AMS Classification: 03G25

Keywords: Soft sets, *int*-soft subgroups, *uni*-soft subgroups, *int*-soft normal subgroups, *uni*-soft normal subgroups, Anti image of soft sets, α-inclusions.

Corresponding Author: Aslıhan Sezgin Sezer (aslihan.sezgin@amasya.edu.tr)

1. Introduction

Since its inception by Molodtsov [23] in 1999, soft set theory has been regarded as a new mathematical tool for dealing with uncertainties and it has seen a wide-ranging applications in the mean of algebraic structures such as groups [2, 27], semirings [12], rings [1], BCK/BCI-algebras [16, 17, 18], BL-algebras [31], near-rings [25] and soft substructures and union soft substructures [6, 26].

Many related concepts with soft sets, especially soft set operations, have also undergone tremendous studies. Maji et al. [21] presented some definitions on soft sets and based on the analysis of several operations on soft sets Ali et al. [3] introduced several operations of soft sets, Sezgin and Atagün [28], Renukadevi and Shanthi [24] Ali et al. [4] studied on soft set operations as well.

The theory of soft set has also gone through remarkably rapid strides with a wide-ranging applications especially in soft decision making as in the following studies: [8, 9, 22] and some other fields as [5, 13, 14, 15, 30]. Soft set theory emphasizes a balanced coverage of both theory and practice. Nowadays, it has promoted a breadth of the discipline of Informations Sciences with intelligent systems, approximate reasoning, expert and decision support systems, self-adaptation and self-organizational systems, information and knowledge, modeling and computing with words.

In [6], Atagün and Sezgin introduce and study soft subrings and soft ideals of a ring by using Molodtsov's definition of the soft sets. Moreover, they introduce soft subfields of a field and soft submodule of a left R-module. We know from ring theory that any ring has some substructures, such as subrings and ideals. Just as any ring has some substructures, in the paper [6], the aim is to construct the soft substructures of rings and fields.

With the same idea, in [7], Atagün and Sezgin define the soft substructures of a group and semiring. They call them int-soft substructures of a group, since they use intersection and inclusion operations while constructing these structures. When defining these int-soft substructures, they first choose any subgroup or a normal subgroup of a group as the parameter set, that is, any substructure of a group. And these soft substructures are all on a group, namely, the universal set should be a group. In [11], Çağman et al. define a new kind of soft group, which is completely different from the soft group defined in [2]. In that paper, the aim is to construct a new kind of soft group, not soft substructures of a group. Therefore, the soft group defined in [11] is on any universal set, that is, the universal need not to be a group as in the case of [7]. In [19, 20], the study of soft int-group is extended and normal soft int-groups and their related properties are investigated, too.

In this paper, applying to soft set theory, we deal with the algebraic *uni*-soft substructures of a group. We define the notions of *uni*-soft subgroup and *uni*-soft normal subgroup and investigate their related properties by illustrating examples. Moreover, we characterize *uni*-soft substructures of a group with respect to *int*-soft substructures of a group, anti image, lower/upper α-inclusion of soft sets and group homomorphisms.

2. Preliminaries

Throughout this chapter, U refers to an initial universe, E is a set of parameters, $P(U)$ is the power set of U and $A, B \subseteq E$.

Definition 2.1 ([23])**.** A pair (F, A) is called a soft set over U, where F is a mapping given by

$$F : A \to P(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of ε-elements of the soft set (F, A), or as the set of ε-approximate elements of the soft set.

Note that from now on, a soft set (F, A) will be denoted by F_A.

Definition 2.2 ([3])**.** Let F_A and G_B be two soft sets over a common universe U such that $A \cap B \neq \varnothing$. The *restricted intersection* of F_A and G_B is denoted by $F_A \Cap G_B$, and is defined as $F_A \Cap G_B = (H, C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \cap G(c)$.

Definition 2.3 ([3])**.** Let F_A and G_B be two soft sets over a common universe U such that $A \cap B \neq \varnothing$. The *restricted union* of F_A and G_B is denoted by $F_A \cup_{\mathcal{R}} G_B$, and is defined as $F_A \cup_{\mathcal{R}} G_B = (H, C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \cup G(c)$.

Definition 2.4 ([3])**.** The *relative complement* of the soft set F_A over U is denoted by F_A^r, where $F^r : A \to P(U)$ is a mapping given as $F^r(\alpha) = U \setminus F(\alpha)$ for all $\alpha \in A$.

In [11], a different type of soft group, called soft int-group is defined in the following manner:

Definition 2.5 ([11])**.** Let G be a group and F_G be a soft set over U. Then, F_G is called a soft int-group over U if it satisfies the following conditions:

(1) $F(xy) \supseteq F(x) \cap F(y)$
(2) $F(x^{-1}) = F(x)$

for all $x, y \in G$.

In the same paper [11], it is showed that these two conditions are equivalent to $F(xy^{-1}) \supseteq F(x) \cap F(y)$ for all $x, y \in G$. In [7], in order to construct the soft substructures of a group, Atagün and Sezgin defined the int-soft subgroup of a group in the following manner:

Definition 2.6 ([7])**.** Let H be a subgroup of a group G and F_H be a soft set over G. If for all $x, y \in H$, $F(xy^{-1}) \supseteq F(x) \cap F(y)$, then the soft set F_H is called an *int-soft subgroup* of G and denoted by $F_H \widetilde{<} G$.

For the sake of the brevity, *int*-soft subgroup of a group is abbreviated by ISS in what follows. Here note that, in [11], the parameter set of a soft int-group should be any group, however in [7], the parameter set of an int-soft subgroup should be selected as a subgroup of the universal set. Since every group is a subgroup of itself, soft int-group defined in [11] is a special case of int-soft subgroup defined in [7] as regards the parameter sets.

In [11], normal soft int-group is defined basing on Abelian soft set and in [19] normal soft int-groups and their properties are studied in detail. And in [7], while defining int-soft subgroup of a group, Atagün and Sezgin choose the parameter set of the soft set as a subgroup of the group, they choose the parameter set of the soft set as a normal subgroup of the group in order to define the int-soft normal subgroup of a group as following:

Definition 2.7 ([7])**.** Let N be a normal subgroup of a group G and F_N be a soft set over G. Then, F_N is called an *int-soft normal subgroup* of G if

(1) $F(xy^{-1}) \supseteq F(x) \cap F(y)$
(2) $F(gxg^{-1}) \supseteq F(x)$

for all $x, y \in N$ and $g \in G$, and it is denoted by $F_N \widetilde{\lhd} G$.

For the sake of ease, we denote an *int*-soft normal subgroup of a group by $ISNS$ in what follows. The same arguments are valid for the small difference between normal soft int-groups and int-soft normal subgroups as in the case of soft int-group and int-soft subgroup as explained above.

Definition 2.8 ([11, 10])**.** Let F_A be a soft set over U and α be a subset of U. Then *upper α-inclusion* and *lower α-inclusion* of F_A, denoted by $F_A^{\supseteq\alpha}$ and $F_A^{\subseteq\alpha}$, are defined as

$$F_A^{\supseteq\alpha} = \{x \in A \mid F(x) \supseteq \alpha\} \quad \text{and} \quad F_A^{\subseteq\alpha} = \{x \in A \mid F(x) \subseteq \alpha\}$$

respectively.

Definition 2.9 ([11]). Let F_A and G_B be soft sets over U and Ψ be a function from A to B. Then we can define the soft set $\Psi(F_A)$ over U, where $\Psi(F_A) : B \to P(U)$ is a set valued function defined by

$$\Psi(F_A)(b) = \begin{cases} \bigcup\{F(a) \mid a \in A \ \ and \ \ \Psi(a) = b\}, & \text{if } \ \Psi^{-1}(b) \neq \varnothing, \\ \varnothing, & otherwise \end{cases}$$

for all $b \in B$. Here, $\Psi(F_A)$ is called the *image* of F_A under Ψ. Moreover we can define a soft set $\Psi^{-1}(G_B)$ over U, where $\Psi^{-1}(G_B) : A \to P(U)$ is a set-valued function defined by $\Psi^{-1}(G_B)(a) = G(\Psi(a))$ for all $a \in A$. Then, $\Psi^{-1}(G_B)$ is called the *preimage (or inverse image)* of G_B under Ψ.

Definition 2.10 ([29]). Let F_A and G_B be soft sets over U and Ψ be a function from A to B. Then we can define the soft set $\Psi^\star(F_A)$ over U, where $\Psi^\star(F_A) : B \to P(U)$ is a set-valued function defined by

$$\Psi^\star(F_A)(b) = \begin{cases} \bigcap\{F(a) \mid a \in A \ \ and \ \ \Psi(a) = b\}, & \text{if } \ \Psi^{-1}(b) \neq \varnothing, \\ \varnothing, & otherwise \end{cases}$$

for all $b \in B$. Here, $\Psi^\star(F_A)$ is called the *anti image* of F_A under Ψ.

Theorem 2.11 ([29]). *Let F_H and T_K be soft sets over U, F_H^r, T_K^r be their relative soft sets, respectively and Ψ be a function from H to K. Then,*

(1) $\Psi^{-1}(T_K^r) = (\Psi^{-1}(T_K))^r$.
(2) $\Psi(F_H^r) = (\Psi^\star(F_H))^r$ *and* $\Psi^\star(F_H^r) = (\Psi(F_H))^r$.

3. *uni*-soft subgroups

Throughout this paper, G denotes an arbitrary group with identity e. If H is a subgroup of G and N is a normal subgroup of G, then they are denoted by $H < G$ and $N \lhd G$, respectively.

Definition 3.1. Let H be a subgroup of G and F_H be a soft set over G. If for all $x, y \in H$, $F(xy^{-1}) \subseteq F(x) \cup F(y)$, then the soft set F_H is called a *uni-soft subgroup* of G and denoted by $F_H \widetilde{<}_u G$.

For sake of the brevity, *uni*-soft subgroup of a group is abbreviated by USS throughout this paper.

Example 3.2. Consider the multiplicative group $G = \{1, -1, i, -i\}$, $H_1 = G < G$ and the soft set F_{H_1} over G, where $F : H_1 \to P(G)$ is a set-valued function defined by $F(x) =< x >$ for all $x \in H_1$. Then $F(1) = \{1\}$, $F(-1) = \{-1, 1\}$ and $F(i) = F(-i) = \{-1, 1, i, -i\} = G$. One can easily show that $F_{H_1} \widetilde{<}_u G$.
However, if we consider the additive group $\mathbb{Z}_6$, $H_2 = \mathbb{Z}_6$ as the subgroup of $\mathbb{Z}_6$ and the soft set T_{H_2} over $\mathbb{Z}_6$, where $T : H_2 \to P(\mathbb{Z}_6)$ is a set-valued function defined by $T(x) =< x >$ for all $x \in H_2$, then $T(0) = \{0\}$, $T(1) = T(5) = \mathbb{Z}_6$, $T(2) = T(4) = \{0, 2, 4\}$ and $T(3) = \{0, 3\}$. Since $T(2 + 3^{-1}) = T(2 + 3) = T(5) = \mathbb{Z}_6 \nsubseteq T(2) \cup T(3) = \{0, 2, 3, 4\}$, T_{H_2} is not a USS of G.

As seen from the example, although we define the two soft sets exactly the same, one of them is a USS and the other is not. Also it is easy to see that if we take the subgroup of G as $H = \{e\}$, then it is obvious that F_H is a USS as well as ISS of G no matter how F is defined. Thus, every group has at least one USS and one ISS.

Proposition 3.3. *If $F_H \widetilde{<}_u G$, then $F(e) \subseteq F(x)$ for all $x \in H$.*

Proof. Since F_H is a USS of G, then

$$F(e) = F(xx^{-1}) \subseteq F(x) \cup F(x) = F(x)$$

for all $x \in H$. □

Theorem 3.4. *Let F_H be a soft set over G such that $H < G$. Then, F_H is a USS of G if and only if $F(xy) \subseteq F(x) \cup F(y)$ and $F(x^{-1}) = F(x)$ for all $x, y \in H$.*

Proof. Suppose that $F(xy) \subseteq F(x) \cup F(y)$ and $F(x^{-1}) = F(x)$ for all $x, y \in H$. Then,

$$F(xy^{-1}) \subseteq F(x) \cup F(y^{-1}) = F(x) \cup F(y)$$

for all $x, y \in H$. Thus, F_H is a USS of G. Conversely, assume that F_H is a USS of G. Then

$$F(xy^{-1}) \subseteq F(x) \cup F(y)$$

for all $x, y \in H$. If we choose $x = e$, then for all $y \in H$,

$$F(ey^{-1}) = F(y^{-1}) \subseteq F(e) \cup F(y) = F(y),$$

by Proposition 3.3. Similarly,

$$F(y) = F((y^{-1})^{-1}) \subseteq F(y^{-1}),$$

thus $F(y^{-1}) = F(y)$ for all $y \in H$. Since F_H is a USS of G by assumption, it follows that

$$F(xy) \subseteq F(x) \cup F(y^{-1}) = F(x) \cup F(y).$$

Thus, the proof is completed. □

Sometimes it is difficult to show only one condition when illustrating that a soft set is a USS of a group. In these conditions, we prefer to use Theorem 3.4 as in the case of Theorem 3.12 and Theorem 3.13.

Theorem 3.5. *Let F_H be a USS of G and $x \in H$. Then,*

$$F(x) = F(e) \Leftrightarrow F(xy) = F(yx) = F(y)$$

for all $y \in H$.

Proof. Suppose that $F(xy) = F(yx) = F(y)$ for all $y \in H$. Then by choosing $y = e$, we obtain that $F(x) = F(e)$. Conversely, assume that $F(x) = F(e)$. Then by Proposition 3.3, we have

$$F(e) = F(x) \subseteq F(y), \quad \forall y \in H \tag{3.1}$$

Since F_H is a USS of G,

$$F(xy) \subseteq F(x) \cup F(y) = F(y), \quad \forall y \in H.$$

Moreover, for all $y \in H$,

$$\begin{aligned} F(y) &= F((x^{-1}x)y) \\ &= F(x^{-1}(xy)) \\ &\subseteq F(x^{-1}) \cup F(xy) \\ &= F(x) \cup F(xy) \\ &= F(xy) \end{aligned}$$

by (3.1). It follows that $F(xy) = F(y)$ is satisfied for all $y \in H$.
Now, let $x \in H$. Then, for all $y \in H$,

$$\begin{aligned} F(yx) &= F(yx(yy^{-1})) \\ &= F(y(xy)y^{-1}) \\ &\subseteq F(y) \cup F(xy) \cup F(y) \\ &= F(y) \cup F(xy) \\ &= F(y) \cup F(y) \\ &= F(y) \end{aligned}$$

Furthermore, for all $y \in H$,

$$\begin{aligned} F(y) &= F(y(xx^{-1})) \\ &= F((yx)x^{-1}) \\ &\subseteq F(yx) \cup F(x) \\ &= F(yx) \end{aligned}$$

by (3.1). It follows that $F(yx) = F(y)$ and so $F(xy) = F(yx) = F(y)$ for all $y \in H$, which completes the proof. □

In [7], Atagün and Sezgin showed that the intersection, the sum and the product of two ISS of G is an ISS of G. We show that the restricted union of two USS of G is a USS of G with the following theorem:

Theorem 3.6. *If $F_{H_1} \widetilde{<}_u G$ and $T_{H_2} \widetilde{<}_u G$, then $F_{H_1} \cup_{\mathcal{R}} T_{H_2} \widetilde{<}_u G$.*

Proof. By Definition 2.3, let $F_{H_1} \cup_{\mathcal{R}} T_{H_2} = (F, H_1) \cup_{\mathcal{R}} (T, H_2) = (Q, H_1 \cap H_2)$, where $Q(x) = F(x) \cup T(x)$ for all $x \in H_1 \cap H_2 \neq \varnothing$. Since H_1 and H_2 are subgroups of G, then so is $H_1 \cap H_2$. Let $x, y \in H_1 \cap H_2$, then

$$\begin{aligned} Q(xy^{-1}) &= F(xy^{-1}) \cup T(xy^{-1}) \\ &\subseteq (F(x) \cup F(y)) \cup (T(x) \cup T(y)) \\ &= (F(x) \cup T(x)) \cup (F(y) \cup T(y)) \\ &= Q(x) \cup Q(y) \end{aligned}$$

Therefore $F_{H_1} \cup_{\mathcal{R}} T_{H_2} = Q_{H_1 \cap H_2} \widetilde{<}_u G$. □

Theorem 3.7. *If $F_H \widetilde{<}_u G$, then $H_F = \{x \in H \mid F(x) = F(e)\}$ is a subgroup of G.*

Proof. It is obvious that $e \in H_F$ and $\varnothing \neq H_F \subseteq G$. We need to show that $xy^{-1} \in H_F$ for all $x, y \in H_F$, which means that $F(xy^{-1}) = F(e)$ has to be satisfied. Since $x, y \in H_F$, then $F(x) = F(y) = F(e)$. By Proposition 3.3, $F(e) \subseteq F(xy^{-1})$ for all $x, y \in H_F$. Since F_H is a USS of G, then $F(xy^{-1}) \subseteq F(x) \cup F(y) = F(e)$ for all $x, y \in H_F$. Therefore H_F is a subgroup of G. □

Theorem 3.8 ([11])**.** *Let F_H be a soft set over G and α be a subset of G such that $F(e) \supseteq \alpha$. If F_H is an ISS of G, then $F_{\overline{H}}^{\supseteq \alpha}$ is a subgroup of G.*

Theorem 3.9. *Let F_H be a soft set over G and α be a subset of G such that $\alpha \supseteq F(e)$. If F_H is a USS of G, then $F_{\overline{H}}^{\subseteq \alpha}$ is a subgroup of G.*

Proof. Since $\alpha \supseteq F(e)$, thus $e \in F_H^{\subseteq\alpha}$ and $\varnothing \neq F_H^{\subseteq\alpha} \subseteq G$. Let $x, y \in F_H^{\subseteq\alpha}$, then $F(x) \subseteq \alpha$ and $F(y) \subseteq \alpha$. We need to show that $xy^{-1} \in F_H^{\subseteq\alpha}$. Since F_H is a USS of G, $F(xy^{-1}) \subseteq F(x) \cup F(y) \subseteq \alpha \cup \alpha = \alpha$. Thus, the proof is completed. □

Example 3.10. Let us define a soft set F_H over a group G, where $F : H \to P(G)$ is a set valued function by $F(x) = \alpha$ ($\alpha \subseteq G$) for all $x \in H$. It is obvious that F_H is a USS of G. Moreover $F_H^{\subseteq\alpha} = \{x \in H \mid F(x) \subseteq \alpha\} = H$. Since H is a subgroup of G, then so is $F_H^{\subseteq\alpha}$.

The following theorem gives the relation between int-soft subgroups and uni-soft subgroups of a group.

Theorem 3.11. *Let F_H be a soft set over G. Then, F_H is a USS of G if and only if F_H^r is an ISS of G.*

Proof. Let F_H be a USS of G. Then,

$$\begin{aligned} F^r(xy^{-1}) &= G \setminus F(xy^{-1}) \\ &\supseteq G \setminus ((F(x) \cup F(y)) \\ &= (G \setminus F(x)) \cap (G \setminus F(y)) \\ &= F^r(x) \cap F^r(y). \end{aligned}$$

for all $x, y \in H$. Thus, F_H^r is an ISS of G. Conversely, let F_H^r be an ISS of G. Then,

$$\begin{aligned} F(xy^{-1}) &= G \setminus F^r(xy^{-1}) \\ &\subseteq G \setminus ((F^r(x) \cap F^r(y)) \\ &= (G \setminus F^r(x)) \cup (G \setminus F^r(y)) \\ &= F(x) \cup F(y). \end{aligned}$$

for all $x, y \in H$. Thus, F_H is a USS of G. □

Theorem 3.11 shows that if a soft set is a USS of G, then its relative complement is an ISS of G and vice versa.

Theorem 3.12 ([11])**.** *Let F_H and be a soft set over G and Ψ be a group epimorphism from H to K. If F_H is an ISS of G, then $\Psi(F_H)$ is an ISS of G.*

Theorem 3.13 ([11])**.** *Let T_K be a soft set over G and Ψ be a group homomorphism from H to K. If T_K is an ISS of G, then $\Psi^{-1}(T_K)$ is an ISS of G.*

Theorem 3.14. *Let T_K be a soft set over G and Ψ be a group homomorphism from H to K. If T_K is a USS of G, then $\Psi^{-1}(T_K)$ is a USS of G.*

Proof. Let T_K be a USS of G. Then, T_K^r is an ISS of G by Theorem 3.11 and $\Psi^{-1}(T_K^r)$ is an ISS of G by Theorem 3.13. Thus, $\Psi^{-1}(T_K^r) = (\Psi^{-1}(T_K))^r$ is an ISS of G by Theorem 2.11 (i). Therefore, $\Psi^{-1}(T_K)$ is a USS of G by Theorem 3.11. □

Theorem 3.15. *Let F_H be a soft set over G and Ψ be a group epimorphism from H to K. If F_H is a USS of G, then $\Psi^\star(F_H)$ is a USS of G.*

Proof. Let F_H be a USS of G. Then, F_H^r is an ISS of G by Theorem 3.11 and $\Psi(F_H^r)$ is an ISS of G by Theorem 3.12. Thus, $\Psi(F_H^r) = (\Psi^\star(F_H))^r$ is an ISS of G by Theorem 2.11 (ii). Therefore, $\Psi^\star(F_H)$ is a USS of G by Theorem 3.11. $\square$

4. *uni*-soft normal subgroups

In this section, we define *uni*-soft normal subgroups of a group and investigate its related properties.

Definition 4.1. Let N be a normal subgroup of G and F_N be a soft set over G. Then, F_N is called a *uni-soft normal subgroup* of G if

(1) $F(xy^{-1}) \subseteq F(x) \cup F(y)$
(2) $F(gxg^{-1}) \subseteq F(x)$

for all $x, y \in N$ and $g \in G$, and it is denoted by $F_N \widetilde{\lhd_u} G$.

For the sake of ease, a *uni*-soft normal subgroup of a group by $USNS$ in what follows.

Example 4.2. Let $G = S_3$ be the symmetric group and $N = A_3 = \{e, (123), (132)\}$ be the normal subgroup of G and the soft set F_N over G, where $F : N \to P(G)$ is a set-valued function defined by $F(e) = \{e\}$ and $F(123) = F(132) = \{e, (123), (132)\}$. One can easily show that $F_N \widetilde{\lhd_u} G$.
Again consider $N = A_3 = \{e, (123), (132)\} \lhd G$ and the soft set T_N over G, where $T : N \to P(G)$ is a set-valued function defined by $T(e) = \{e\}$, $T(123) = \{e, (13), (23)\}$ and $T(132) = \{e, (132), (123)\}$. Then,

$$\begin{aligned} T((12)(123)(12)^{-1}) &= T((12)(123)(12)) \\ &= T(132) \\ &= \{e, (123), (132)\} \\ &\nsubseteq T(123) = \{e, (13), (23)\}. \end{aligned}$$

Thus, T_N is not a $USNS$ of G.

In [7], Atagün and Sezgin showed that the intersection, the sum and the product of two $ISNS$ of G is an $ISNS$ of G. We show that the restricted union of two $USNS$ of G is a $USNS$ of G with the following theorem:

Theorem 4.3. *If $F_{N_1} \widetilde{\lhd_u} G$ and $T_{N_2} \widetilde{\lhd_u} G$, then $F_{N_1} \cup_{\mathcal{R}} T_{N_2} \widetilde{\lhd_u} G$.*

Proof. By Definition 2.3, let $F_{N_1} \cup_{\mathcal{R}} T_{N_2} = (F, N_1) \cup_{\mathcal{R}} (T, N_2) = (W, N_1 \cap N_2)$, where $W(x) = F(x) \cup T(x)$ for all $x \in N_1 \cap N_2 \neq \varnothing$. Since $N_1, N_2 \lhd G$, then $N_1 \cap N_2 \lhd G$. Then, for all $x \in N_1 \cap N_2$ and $g \in G$,

$$\begin{aligned} W(xy^{-1}) &= F(xy^{-1}) \cup T(xy^{-1}) \\ &\subseteq (F(x) \cup F(y)) \cup (T(x) \cup T(y)) \\ &= (F(x) \cup T(x)) \cup (F(y) \cup T(y)) \\ &= W(x) \cup W(y) \end{aligned}$$

and

$$\begin{aligned} W(gxg^{-1}) &= F(gxg^{-1}) \cup T(gxg^{-1}) \\ &\subseteq F(x) \cup T(x) \\ &= W(x) \end{aligned}$$

Therefore $F_{N_1} \cup_{\mathcal{R}} T_{N_2} = W_{N_1 \cap N_2} \widetilde{\lhd_u} G$. □

It is well-known that every subgroup of an Abelian group G is also a normal subgroup of G. A similar relation exists for USS and $USNS$ of a group G. We have the following proposition:

Proposition 4.4. *Let H be a subgroup of G, where G is an Abelian group and F_H be a soft set over G. If F_H is a USS of G, then it is also a USS of G.*

Proof. Since H is a subgroup of an Abelian group of G, then $H \lhd G$. Moreover,

$$F(gxg^{-1}) = F(gg^{-1}x) = F(x) \subseteq F(x)$$

for all $x \in H$ and $g \in G$. Thus, $F_H \widetilde{\lhd_u} G$. □

Corollary 4.5. *Every $USNS$ of a group G is a USS of G; however the converse is true when G is an abelian group.*

Theorem 4.6. *If $F_N \widetilde{\lhd_u} G$, then $N_F = \{x \in N \mid F(x) = F(e)\}$ is a normal subgroup of G.*

Proof. If $F_N \widetilde{\lhd_u} G$, then so is $F_N \widetilde{<_u} G$. Since in Theorem 3.7, it is shown that if $F_N \widetilde{<_u} G$, then N_F is a subgroup of G, we only show that for all $x \in N_F$ and $g \in G$, $gxg^{-1} \in N_F$ which means that $F(gxg^{-1}) = F(e)$ is satisfied. Since $x \in N_F$, then $F(x) = F(e)$. By Proposition 3.3, $F(e) \subseteq F(gxg^{-1})$. Since F_N is a USS of G, then $F(gxg^{-1}) \subseteq F(x) = F(e)$ for all $x \in N_F$ and $g \in G$. Thus, the proof is completed. □

Theorem 4.7 ([19]). *Let F_H be a soft set over G and α be a subset of G such that $F(e) \supseteq \alpha$. If F_H is an $ISNS$ of G, then $F_{\overline{H}}^{\supseteq \alpha}$ is a normal subgroup of G.*

Theorem 4.8. *Let F_H be a soft set over G and α be a subset of G such that $\alpha \supseteq F(e)$. If F_H is a $USNS$ of G, then $F_{\overline{H}}^{\subseteq \alpha}$ is a normal subgroup of G.*

Proof. If F_H is a $USNS$ of G, then so is a USS of G. Since in Theorem 3.9, it is illustrated that if F_H is a USS of G, then $F_{\overline{H}}^{\subseteq \alpha}$ is a subgroup of G, we only show that for all $x \in F_{\overline{H}}^{\subseteq \alpha}$ and $g \in G$, $gxg^{-1} \in F_{\overline{H}}^{\subseteq \alpha}$. Let $x \in F_{\overline{H}}^{\subseteq \alpha}$, then $F(x) \subseteq \alpha$ and it follows that $F(gxg^{-1}) \subseteq F(x) \subseteq \alpha$. Thus, the proof is completed. □

The following theorem gives the main relation between *int*-soft normal subgroups and *uni*-soft normal subgroups of a group.

Theorem 4.9. *Let F_N be a soft set over G. Then, F_N is a $USNS$ of G if and only if F_N^r is an $ISNS$ of G.*

Proof. Let F_N be a $USNS$ of G, then F_N is a USS of G. Since the first condition of Definition 4.1 is shown in Theorem 3.11, we only show that the second condition of Definition 4.1 is satisfied. Let $x \in N$ and $g \in G$, then

$$\begin{aligned} F^r(gxg^{-1}) &= G \setminus F(gxg^{-1}) \\ &\supseteq G \setminus F(x) \\ &= F^r(x). \end{aligned}$$

Thus, F_N^r is an $ISNS$ of G. The converse can be shown similarly, therefore omitted. □

Definition 4.10. Let F_N be a USS(or $USNS$) of G. Then,

(1) F_N is said to be *trivial* if $F(x) = \{e\}$ for all $x \in N$.
(2) F_N is said to be *whole* if $F(x) = G$ for all $x \in N$.

Proposition 4.11. *Let F_{N_1} and T_{N_2} be USS (or $USNS$) of G, then*

(1) *If F_{N_1} and T_{N_2} are trivial USS (or $USNS$) of G, then $F_{N_1} \cup_{\mathcal{R}} T_{N_2}$ is a trivial USS (or $USNS$) of G.*
(2) *If F_{N_1} and T_{N_2} are whole USS (or $USNS$) of G, then $F_{N_1} \cup_{\mathcal{R}} T_{N_2}$ is a whole USS (or $USNS$) of G.*
(3) *If F_{N_1} is a trivial USS (or $USNS$) of G and T_{N_2} is a whole USS (or $USNS$) of G, then $F_{N_1} \cup_{\mathcal{R}} T_{N_2}$ is a whole USS (or $USNS$) of G.*

Proof. The proof is obvious, hence omitted. □

Theorem 4.12. *Let G_1 and G_2 be two groups and $(F_1, H_1)\widetilde{\lhd_u}G_1$, $(F_2, H_2)\widetilde{\lhd_u}G_2$. If $f : H_1 \to H_2$ is a group homomorphism, then*

(1) *If f is an epimorphism, then $(F_1, f^{-1}(H_2))\widetilde{\lhd_u}G_1$,*
(2) *$(F_2, f(H_1))\widetilde{\lhd_u}G_2$,*
(3) *$(F_1, kerf)\widetilde{\lhd_u}G_1$.*

Proof. (1) Since $H_1 < G_1$, $H_2 < G_2$ and $f : H_1 \to H_2$ is a group epimorphism, then it is clear that $f^{-1}(H_2) < G_1$. Since $(F_1, H_1)\widetilde{\lhd_u}G_1$ and $f^{-1}(H_2) \subseteq H_1$, $F_1(xy^{-1}) \subseteq F_1(x) \cup F_1(y)$ and $F_1(gxg^{-1}) \subseteq F_1(x)$ for all $x, y \in f^{-1}(H_2)$ and $g \in G$. Hence $(F_1, f^{-1}(H_2))\widetilde{\lhd_u}G_1$.

(2) Since $H_1 < G_1$, $H_2 < G_2$ and $f : H_1 \to H_2$ is a group homomorphism, then $f(H_1) < G_2$. Since $f(H_1) \subseteq H_2$, the result is obvious by Definition 4.1.

(3) Since $kerf < G_1$ and $kerf \subseteq H_1$, the rest of the proof is clear by Definition 4.1. □

Corollary 4.13. *Let $(F_1, H_1)\widetilde{\lhd_u}G_1$, $(F_2, H_2)\widetilde{\lhd_u}G_2$ and $f : H_1 \to H_2$ is a group homomorphism, then $(F_2, \{e_{G_2}\})\widetilde{\lhd_u}G_2$.*

Proof. By Theorem 4.12(3), $(F_1, kerf)\widetilde{\lhd_u}G_1$. Then $(F_2, f(Kerf)) =$
$(F_2, \{e_{G_2}\})\widetilde{\lhd_u}G_2$ by Theorem 4.12(2). □

5. Conclusion

Throughout this paper, we deal with the algebraic *uni*-soft substructures of a group. We first have introduced *uni*-soft subgroups and *uni*-soft normal subgroups. Then, we have investigated the relations between *uni*-soft subgroups and *uni*-soft normal subgroups under certain conditions of the group. Furthermore, we have characterized *uni*-soft substructures of a group with respect anti image and lower α-inclusion of a soft set and group homomorphism. To extend this work, one could study the *uni*-soft substructures of other algebraic structures such as near-rings.

References

[1] U. Acar, F. Koyuncu and B. Tanay, Soft sets and soft rings, Comput. Math. Appl. 59 (2010) 3458–3463.
[2] H. Aktaș and N. Çağman, Soft sets and soft groups, Inform. Sci. 177 (2007) 2726–2735.
[3] M. I. Ali, F. Feng, X. Liu, W. K. Min and M. Shabir, On some new operations in soft set theory, Comput. Math. Appl. 57 (2009) 1547–1553.
[4] M.I. Ali, M. Shabir and M. Naz, Algebraic structures of soft sets associated with new operations, Comput. Math. Appl. 61(9) (2011) 2647–2654.
[5] M. I. Ali, A note on soft sets, rough soft sets and fuzzy soft sets, Appl. Soft Comput. 11(4) (2011) 3329–3332.
[6] A. O. Atagün and A. Sezgin, Soft substructures of rings, fields and modules, Comput. Math. Appl. 61(3) (2011) 592–601.
[7] A. O. Atagün and A. Sezgin, Int-soft substructures of groups and semirings, submitted.
[8] N. Çağman and S. Enginoğlu, Soft matrix theory and its decision making, Comput. Math. Appl. 59 (2010) 3308–3314.
[9] N. Çağman and S. Enginoğlu, Soft set theory and uni-int decision making, European J. Oper. Res. 207(2) (2010) 848–855.
[10] N. Çağman, A. Sezgin and A. O. Atagün, α-inclusions and their applications to group theory, submitted.
[11] N. Çağman, F. Çıtak and H. Aktaş, Soft int-groups and its applications to group theory, Neural Comput. Appl. 21(1) (2012) 151–158.
[12] F. Feng, Y. B. Jun and X. Zhao, Soft semirings, Comput. Math. Appl. 56 (2008) 2621–2628.
[13] F. Feng, Y. B. Jun, X. Y. Liu and L. F. Li, An adjustable approach to fuzzy soft set based decision making, J. Comput. Appl. Math. 234 (2010) 10–20.
[14] F. Feng, C. Li, B. Davvas and M. I. Ali, Soft sets combined with fuzzy sets and rough sets: a tentative approach, Soft Comput. 14(6) (2010) 899–911.
[15] F. Feng, X. Y. Liu, V. Leoreanu-Fotea and Y. B. Jun, Soft sets and soft rough sets, Inform. Sci. 181 (2011) 1125–1137.
[16] Y. B. Jun, Soft BCK/BCI-algebras, Comput. Math. Appl. 56 (2008) 1408–1413.
[17] Y. B. Jun and C. H. Park, Applications of soft sets in ideal theory of BCK/BCI-algebras, Inform. Sci. 178 (2008) 2466–2475.
[18] Y. B. Jun, K. J. Lee and J. Zhan, Soft p-ideals of soft BCI-algebras, Comput. Math. Appl. 58 (2009) 2060–2068.
[19] K. Kaygısız, Normal soft int-groups, arxiv:1209.3157v1 [mat.GR] 14 Sep 2012.
[20] K. Kaygısız, On soft int-groups, Ann. Fuzzy Math. Inform. 4(2) (2012) 365–375.
[21] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.
[22] P. K. Maji, A. R. Roy and R. Biswas, An application of soft sets in a decision making problem, Comput. Math. Appl. 44 (2002) 1077–1083.
[23] D. Molodtsov, Soft set theory-first results, Comput. Math. Appl. 37 (1999) 19–31.
[24] V. Renukadevi and S. D. Shanthi, Properties of soft sets associatied with new operations, Ann. Fuzzy Math. Inform. 8(1) (2014) 7–17.
[25] A. Sezgin, A. O. Atagün and E. Aygün, A note on soft near-rings and idealistic soft near-rings, Filomat 25(1) (2011) 53–68.

[26] A. Sezgin, A. O. Atagün and N. Çağman, Union soft substructures of near-rings and N-groups, Neural Comput. Appl. 21 (Issue 1-Supplement) (2012) 133–143.

[27] A. Sezgin and A. O. Atagün, Soft groups and normalistic soft groups, Comput. Math. Appl. 62(2) (2011) 685–698.

[28] A. Sezgin and A. O. Atagün, On operations of soft sets, Comput. Math. Appl. 61(5) (2011) 1457–1467.

[29] A. Sezgin Sezer, N. Çağman and A. O. Atagün, Soft uni-groups and its applications to group theory, submitted.

[30] M. Shabir and A. Ahmad, On soft ternary semigroups, Ann. Fuzzy Math. Inform. 3(1) (2012) 9–59.

[31] J. Zhan and Y. B. Jun, Soft BL-algebras based on fuzzy sets, Comput. Math. Appl. 59(6) (2010) 2037–2046.

Aslihan Sezgin Sezer (aslihan.sezgin@amasya.edu.tr)
Department of Statistics, Faculty of Arts and Science, Amasya University, 05100, Amasya, Turkey

Naim Çağman (naim.cagman@gop.edu.tr)
Department of Mathematics, Faculty of Arts and Science, Gaziosmanpaşa University, 60250, Tokat, Turkey

Akin Osman Atagün (aosman.atagun@bozok.edu.tr)
Department of Mathematics, Faculty of Arts and Science, Bozok University, 66100, Yozgat Turkey