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On pairwise K-connectedness, pairwise K-disconnectedness and pairwise K-compactness in vague fuzzy digital bi-structure spaces

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Abstract. In this paper, the concepts of pairwise vague fuzzy digital structure K-connected space, pairwise vague fuzzy digital structure super K-connected space, pairwise vague fuzzy digital structure strongly K-connected space, pairwise vague fuzzy digital structure K-extremally disconnected space, pairwise vague fuzzy digital structure K-totally disconnected spaces, pairwise vague fuzzy digital structure K-Hausdorff spaces, pairwise vague fuzzy digital structure K-compact spaces, vague fuzzy $\mathfrak{D}_{ij}$-$Kreg$ adherent convergent and vague fuzzy $\mathfrak{D}_{ij}$-K adherent convergent ($i, j = 1, 2$ and $i \neq j$) are introduced. Some interesting properties are established.

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Keywords: Pairwise vague fuzzy digital structure K-connected space, Pairwise vague fuzzy digital structure super K-connected space, Pairwise vague fuzzy digital structure strongly K-connected space, Pairwise vague fuzzy digital structure K-extremally disconnected space, Pairwise vague fuzzy digital structure K-totally disconnected spaces, Pairwise vague fuzzy digital structure K-Hausdorff spaces and pairwise vague fuzzy digital structure K-compact space.

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1. Introduction

The concept of fuzzy sets was introduced by Zadeh [11]. W.A.Gau. and D.J.Buehrer [2] were introduced the concept of vague fuzzy sets. T.Y. Kong and A. Rosenfeld [3] studied the idea of digital topology. Classical Digital Topology primarily concerns itself in the study of black-white images in the digital plane. It invariably needs to take care of the gray scale level images that can be represented by the concepts of fuzzy sets as noted by A.Rosenfeld [8, 9, 10]. Various topological relationships among parts of a digital picture notions such as connectedness play an important role in the image analysis. The concepts and well developed theories of connectedness and their related notions were discussed in [1, 4, 7]. The concept of vague fuzzy digital structure spaces and vague fuzzy digital bi-structure spaces were introduced by R.Narmada Devi [5, 6]. In this paper, various types of pairwise vague fuzzy digital structure K-connected space, pairwise vague fuzzy digital structure super K-connected space, pairwise vague fuzzy digital structure strongly K-connected space, pairwise vague fuzzy digital structure K-extremally disconnected space, pairwise vague fuzzy digital structure K-totally disconnected spaces, pairwise vague fuzzy digital structure K-Hausdorff spaces and pairwise vague fuzzy digital structure K-compact space are introduced. In this connection, some interesting properties are established.

2. Preliminaries

Definition 2.1 ([8])**.** Let $\sum$ be a rectangular array of integer-coordinate points. Thus the point $P = (x, y)$ of $\sum$ has four horizontal and vertical neighbors, namely $(x \pm 1, y)$ and $(x, y \pm 1)$; and it also has four diagonal neighbors, namely $(x \pm 1, y \pm 1)$ and $(x \pm 1, y \mp 1)$. We say that former points are 4-adjacent to, or 4-neighbors of P and we say that both types of neighbors are 8-adjacent to, or 8-neighbors of P. Note that if P is on the border of $\sum$, some of these neighbors may not exist.

Definition 2.2 ([2])**.** Let X be a nonempty fixed set and I be the closed interval $[0, 1]$. An vague fuzzy set(VFS) A is an object having form $A = \{\langle x, t_A(x), f_A(x)\rangle : x \in X\}$, where the mappings $t_A : X \longrightarrow I$ and $f_A : X \longrightarrow I$ denote degree of truth membership function(namely $t_A(x)$) and the degree of false membership (namely $f_A(x)$) for each element $x \in X$ to the set A, respectively. Further $0 \leq t_A(x) + f_A(x) \leq 1$ for each $x \in X$. For the sake of simplicity, we shall use the symbol $A = \langle x, t_A, f_A\rangle$ for the vague fuzzy set $A = \{\langle x, t_A(x), f_A(x)\rangle : x \in X\}$.

Definition 2.3 ([2])**.** Let X be a nonempty set and the VFSs A and B in the form $A = \{\langle x, t_A(x), f_A(x)\rangle : x \in X\}$, $B = \{\langle x, t_B(x), f_B(x)\rangle : x \in X\}$. Then

(i) $A \subseteq B$ iff $t_A(x) \leq t_B(x)$ and $f_A(x) \geq f_B(x)$ for all $x \in X$;
(ii) $\overline{A} = \{\langle x, f_A(x), t_A(x)\rangle : x \in X\}$;
(iii) $A = B$ iff $t_A(x) = t_B(x)$ and $f_A(x) = f_B(x)$, for every $x \in X$;
(iv) $A \cap B = \{\langle x, min(t_A(x), t_B(x)), max(f_A(x), f_B(x))\rangle : x \in X\}$;
(v) $A \cup B = \{\langle x, max(t_A(x), t_B(x)), min(f_A(x), f_B(x))\rangle : x \in X\}$;
(vi) $\tilde{0} = \langle x, 0, 1\rangle$ and $\tilde{1} = \langle x, 1, 0\rangle$.
(vii) $\overline{\overline{A}} = A$

Definition 2.4 ([5, 6])**.** Let $\sum$ be a rectangular array of integer-coordinate points. An vague fuzzy digital structure on $\sum$ is a family $\mathfrak{D}$ of an vague fuzzy sets in $\sum$ satisfying the following axioms:

(i) $\tilde{0}, \tilde{1} \in \mathfrak{D}$;
(ii) $G_1 \cap G_2 \in \mathfrak{D}$ for any $G_1, G_2 \in \mathfrak{D}$;
(iii) $\cup G_i \in \mathfrak{D}$ for arbitrary family $\{G_i \mid i \in J\} \subseteq \mathfrak{D}$.

Then the ordered pair $(\sum, \mathfrak{D})$ is called an vague fuzzy digital structure space. Every member in $\mathfrak{D}$ is said to be an vague fuzzy digital structure open set. The complement of a vague fuzzy digital structure open set A is a vague fuzzy digital structure closed set.

Definition 2.5 ([5]). Let $\sum$ be a rectangular array of integer-coordinate points. If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are two vague fuzzy digital structures on $\sum$, then a triple $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is said to be an vague fuzzy digital bi-structure space.

3. Characterizations of pairwise vague fuzzy digital structure K-connected spaces

Throughout this paper, $\sum$ represents a rectangular array of integer-coordinate points. That is, $\sum$ be a digital array of lattice points in either 2D (or) 3D.

Definition 3.1. Let $\sum$ be a rectangular array of integer-coordinate points and I be the closed interval [0,1]. A vague fuzzy digital set A is an object having the form $A = \{\langle P, t_A(P), f_A(P)\rangle : P \in \sum\}$ where the functions $t_A : \sum \to I$ and $f_A : \sum \to I$ denote the degree of truth membership (namely $t_A(P)$) and the degree of false membership (namely $f_A(P)$) and $0 \leq t_A(P) + f_A(P) \leq 1$, for each $P \in \sum$. For the sake of simplicity, we shall use the symbol $A = \langle P, t_A, f_A\rangle$ for the vague fuzzy digital set $A = \{\langle P, t_A(P), f_A(P)\rangle : P \in \sum\}$. The collection of all vague fuzzy digital sets on $\sum$ is denoted by $\zeta^{\sum}$.

Definition 3.2. Let $\sum$ be a rectangular array of integer-coordinate points. Let $A = \{\langle P, t_A(P), f_A(P)\rangle : P \in \sum\}$ and $B = \{\langle P, t_B(P), f_B(P)\rangle : P \in \sum\}$ be any two vague fuzzy digital sets in $\sum$. Then

(i) $A \subseteq B$ if and only if $t_A(P) \leq t_B(P)$ and $f_A(P) \geq f_B(P)$, for all $P \in \sum$.
(ii) $\overline{\mathrm{A}} = \{\langle P, f_A(P), t_A(P)\rangle : P \in \sum\}$.
(iii) $\overline{\overline{\mathrm{A}}} = A$.
(iv) $A \cap B = \{\langle P, t_A(P) \wedge t_B(P), f_A(P) \vee f_B(P)\rangle : P \in \sum\}$.
(v) $A \cup B = \{\langle P, t_A(P) \vee t_B(P), f_A(P) \wedge f_B(P)\rangle : P \in \sum\}$.
(vi) The empty vague fuzzy digital set in $\sum$ is defined and denoted as $\tilde{0} = \{\langle P, 0, 1\rangle : P \in \sum\}$.
(vii) The universal vague fuzzy digital set in $\sum$ is defined and denoted as $\tilde{1} = \{\langle P, 1, 0\rangle : P \in \sum\}$.

Definition 3.3. Let $\sum$ be a rectangular array of integer-coordinate points and let $A = \langle P, t_A, f_A\rangle$ be any vague fuzzy digital set in $\sum$. Then A is said to be a proper vague fuzzy digital set in $\sum$ if $A \neq \tilde{0}$ and $A \neq \tilde{1}$.

Definition 3.4. Let $\sum$ be a rectangular array of integer-coordinate points. A vague fuzzy digital structure on $\sum$ is a family $\mathfrak{D}$ of vague fuzzy digital sets in $\sum$ satisfying the following axioms:

(i) $\tilde{0}, \tilde{1} \in \mathfrak{D}$.
(ii) $G_1 \cap G_2 \in \mathfrak{D}$ for any $G_1, G_2 \in \mathfrak{D}$.
(iii) $\cup G_i \in \mathfrak{D}$ for arbitrary family $\{G_i \mid i \in J\} \subseteq \mathfrak{D}$.

Then the ordered pair $(\sum, \mathfrak{D})$ is called a vague fuzzy digital structure space. Every member in $\mathfrak{D}$ is called a vague fuzzy digital structure open set in $\sum$. The complement of a vague fuzzy digital structure open set A is a vague fuzzy digital structure closed set in $\sum$.

Notation 3.1. Let $(\sum, \mathfrak{D})$ be any vague fuzzy digital structure space. Then $\mathfrak{D}^c$ denotes the collection of all vague fuzzy digital structure closed sets in $\sum$.

Definition 3.5. Let $(\sum, \mathfrak{D})$ be any vague fuzzy digital structure space. Let $A = \langle P, t_A, f_A \rangle$ be a vague fuzzy digital set in $\sum$. Then the

(i) vague fuzzy digital structure kernal of A is defined and denoted as

$$VFKer_{\mathfrak{D}}(A) = \cap \{B = \langle P, t_B, f_B \rangle \mid B \in \mathfrak{D} \text{ and } A \subseteq B\}.$$

(ii) vague fuzzy digital structure co-kernal of A is defined and denoted as

$$VFCo\text{-}Ker_{\mathfrak{D}}(A) = \cup \{B = \langle P, t_B, f_B \rangle \mid B \in \mathfrak{D}^c \text{ and } B \subseteq A\}.$$

Definition 3.6. Let $(\sum, \mathfrak{D})$ be any vague fuzzy digital structure space. Let $A = \langle P, t_A, f_A \rangle$ be a vague fuzzy digital set in $\sum$. Then A is said to be a

(i) vague fuzzy digital structure kernal set if $VFKer_{\mathfrak{D}}(A) = A$.
(ii) vague fuzzy digital structure co-kernal set if $VFCo\text{-}Ker_{\mathfrak{D}}(A) = A$.

Remark 3.7. Let $(\sum, \mathfrak{D})$ be any vague fuzzy digital structure space. Let $A = \langle P, t_A, f_A \rangle$ be a vague fuzzy digital set in $\sum$. Then

(i) If A is a vague fuzzy digital structure kernal set, then $\overline{\mathrm{A}}$ is a vague fuzzy digital structure co-kernal set.
(ii) If A is a vague fuzzy digital structure co-kernal set, then $\overline{\mathrm{A}}$ is a vague fuzzy digital structure kernal set.
(iii) $\overline{VFKer_{\mathfrak{D}}(A)} = VFCo\text{-}Ker_{\mathfrak{D}}(\overline{\mathrm{A}})$.
(iv) $\overline{VFCo\text{-}Ker_{\mathfrak{D}}(A)} = VFKer_{\mathfrak{D}}(\overline{\mathrm{A}})$.
(v) $VFKer_{\mathfrak{D}}(\tilde{0}) = \tilde{0}$ and $VFCo\text{-}Ker_{\mathfrak{D}}(\tilde{0}) = \tilde{0}$.
(vi) $VFKer_{\mathfrak{D}}(\tilde{1}) = \tilde{1}$ and $VFCo\text{-}Ker_{\mathfrak{D}}(\tilde{1}) = \tilde{1}$.

Proof. The proof is simple. □

Definition 3.8. Let $\sum$ be a rectangular array of integer-coordinate points. A vague fuzzy digital bi-structure space is an ordered triple $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ where $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are two vague fuzzy digital structures on $\sum$.

Definition 3.9. Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space. Let $A = \langle P, t_A, f_A \rangle$ be a vague fuzzy digital set in $\sum$. Then A is said to be a

(i) vague fuzzy $\mathfrak{D}_{(1,2)}$ structure *Kreg*-set if $VFKer_{\mathfrak{D}_1}(VFCo\text{-}Ker_{\mathfrak{D}_2}(A)) = A$.
(ii) vague fuzzy $\mathfrak{D}_{(2,1)}$ structure *Kreg*-set if $VFKer_{\mathfrak{D}_2}(VFCo\text{-}Ker_{\mathfrak{D}_1}(A)) = A$.
(iii) pairwise vague fuzzy digital structure *Kreg*-set if it is both vague fuzzy $\mathfrak{D}_{(1,2)}$ structure *Kreg* set and vague fuzzy $\mathfrak{D}_{(2,1)}$ structure *Kreg* set.

Note 3.10. Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space. Then

(i) the complement of a vague fuzzy $\mathfrak{D}_{(1,2)}$ structure *Kreg*-set is a vague fuzzy $\mathfrak{D}_{(1,2)}$ structure *Co-Kreg*-set.
(ii) the complement of a vague fuzzy $\mathfrak{D}_{(2,1)}$ structure *Kreg*-set is a vague fuzzy $\mathfrak{D}_{(2,1)}$ structure *Co-Kreg*-set.
(iii) the complement of a pairwise vague fuzzy digital structure *Kreg*-set is a pairwise vague fuzzy digital structure *co-Kreg*-set.

Definition 3.11. Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space. Then $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is called a pairwise vague fuzzy digital structure kernal connected (in short, K-connected) space if and only if it has no proper vague fuzzy digital sets A and B which are vague fuzzy digital structure $\mathfrak{D}_1$ kernal set and vague fuzzy digital structure $\mathfrak{D}_2$ kernal set respectively such that $t_A + t_B \geq 1$ and $f_A + f_B \leq 1$.

Example 3.12. Let $\sum = \{P_1, P_2, P_3\}$ be a rectangular array of integer-coordinate points. Let

$A = \langle P, (\frac{P_1}{0}, \frac{P_2}{0.4}, \frac{P_3}{0}), (\frac{P_1}{1}, \frac{P_2}{0.3}, \frac{P_3}{1})\rangle$ and $B = \langle P, (\frac{P_1}{0.3}, \frac{P_2}{0}, \frac{P_3}{0.2}), (\frac{P_1}{0.1}, \frac{P_2}{1}, \frac{P_3}{0.8})\rangle$

be any two vague fuzzy digital sets in $\sum$. Then $\mathfrak{D}_1 = \{\tilde{0}, \tilde{1}, A\}$ and $\mathfrak{D}_2 = \{\tilde{0}, \tilde{1}, B\}$ are vague fuzzy digital structures on $\sum$. Hence, $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is a pairwise vague fuzzy digital structure K-connected space.

Definition 3.13. A vague fuzzy digital bi-structure space $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is called a pairwise vague fuzzy digital structure K-disconnected space if it is not a pairwise vague fuzzy digital structure K-connected space.

Proposition 3.14. *Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space. Then the following statements are equivalent:*

(i) $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ *is a pairwise vague fuzzy digital structure K-connected space.*

(ii) *There exist no vague fuzzy digital structure $\mathfrak{D}_1$ kernal set $A \neq \tilde{0}$ and vague fuzzy digital structure $\mathfrak{D}_2$ kernal set $B \neq \tilde{0}$ such that $t_A + t_B \geq 1$ and $f_A + f_B \leq 1$.*

(iii) *There exist no vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal set $C \neq \tilde{1}$ and vague fuzzy digital structure $\mathfrak{D}_2$ co-kernal set $D \neq \tilde{1}$ such that $t_C + t_D \leq 1$ and $f_C + f_D \geq 1$.*

(iv) $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ *contains no vague fuzzy digital set $A \neq \tilde{0}, \tilde{1}$ which is both vague fuzzy digital structure $\mathfrak{D}_1$ kernal set and vague fuzzy digital structure $\mathfrak{D}_2$ co-kernal set (or) it is both vague fuzzy digital structure $\mathfrak{D}_2$ kernal set and vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal set.*

Proof. **(i) ⇒ (ii)**

Assume that (i) is true. Then (ii) follows from Definition 3.11.

(ii) ⇒ (iii)

Assume that (ii) is true. Suppose that there exist vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal set $A \neq \tilde{1}$ and vague fuzzy digital structure $\mathfrak{D}_2$ co-kernal set $B \neq \tilde{1}$ such that $t_A + t_B \leq 1$ and $f_A + f_B \geq 1$. Then, $C = \overline{\mathrm{A}} \neq \tilde{0}$ is a non-zero vague fuzzy digital structure $\mathfrak{D}_1$ kernal set. Similarly, we get $D = \overline{B} \neq \tilde{0}$ is a non-zero vague fuzzy digital structure $\mathfrak{D}_2$ kernal set such that

$$t_C + t_D = t_{\overline{\mathrm{A}}} + t_{\overline{B}} = f_A + f_B \geq 1$$

and

$$f_C + f_D = f_{\overline{\mathrm{A}}} + f_{\overline{B}} = t_A + t_B \leq 1.$$

This is a contradiction. Hence, (ii)⇒(iii).

(iii) ⇒ (iv)

Assume that (iii) is true. Suppose that $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ contains a vague fuzzy digital set $A \neq \tilde{0}, \tilde{1}$ which is both vague fuzzy digital structure $\mathfrak{D}_1$ kernal set and vague fuzzy digital structure $\mathfrak{D}_2$ co-kernal set. Then, $\overline{\mathrm{A}}$ is a proper vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal set. By assumption, A is a vague fuzzy digital structure $\mathfrak{D}_2$ co-kernal set. Now, $t_A + t_{\overline{\mathrm{A}}} = t_A + f_A \leq 1$ and $f_A + f_{\overline{\mathrm{A}}} = t_A + f_A \leq 1$. This is a contradiction. Hence, (iii)⇒(iv).

(iv) ⇒ (i)

Assume that (iv) is true. Suppose that $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is not a pairwise vague fuzzy digital structure K-connected space. Then, $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ has proper vague fuzzy digital structure $\mathfrak{D}_1$ kernal set A and vague fuzzy digital structure $\mathfrak{D}_2$ kernal set B such that $t_A + t_B \geq 1$ and $f_A + f_B \leq 1$. That is, there exist vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal set $\overline{\mathrm{A}}$ and vague fuzzy digital structure $\mathfrak{D}_2$ co-kernal set $\overline{B}$ such that $t_{\overline{\mathrm{A}}} + t_{\overline{B}} \leq 1$ and $f_{\overline{\mathrm{A}}} + f_{\overline{B}} \geq 1$. This implies that $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ contains vague fuzzy digital set $A = \overline{B} \neq \tilde{0}, \tilde{1}$ which is both vague fuzzy digital structure $\mathfrak{D}_1$ kernal set and vague fuzzy digital structure $\mathfrak{D}_2$ co-kernal set. This is a contradiction. Hence, (iv)⇒(i). □

Definition 3.15. Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space. Then $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is called a pairwise vague fuzzy digital structure super kernal connected (in short, K-connected)space if it has no pairwise vague fuzzy digital structure $Kreg$-set A in $\sum$ such that $\tilde{0} \neq A \neq \tilde{1}$.

Example 3.16. Let $\sum = \{P_1, P_2, P_3\}$ be a rectangular array of integer-coordinate points. Let

$A = \langle P, (\frac{P_1}{0.3}, \frac{P_2}{0.3}, \frac{P_3}{0.5}), (\frac{P_1}{0.4}, \frac{P_2}{0.5}, \frac{P_3}{0.5}) \rangle$ and $B = \langle P, (\frac{P_1}{0.2}, \frac{P_2}{0.2}, \frac{P_3}{0.5}), (\frac{P_1}{0.8}, \frac{P_2}{0.7}, \frac{P_3}{0.5}) \rangle$

be any two vague fuzzy digital sets in $\sum$. Then $\mathfrak{D}_1 = \{\tilde{0}, \tilde{1}, A\}$ and $\mathfrak{D}_2 = \{\tilde{0}, \tilde{1}, B\}$ are vague fuzzy digital structures on $\sum$. Hence, $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is a pairwise vague fuzzy digital structure super K-connected space.

Proposition 3.17. *Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space. Then the following statements are equivalent:*

- (i) $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ *is a pairwise vague fuzzy digital structure super K-connected space.*
- (ii) *The vague fuzzy digital structure $\mathfrak{D}_2$ co-kernal ((or) vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal) of a pairwise vague fuzzy digital structure $Kreg$-set which is different from $\tilde{0}$ is $\tilde{1}$.*
- (iii) *The vague fuzzy digital structure $\mathfrak{D}_2$ kernal ((or) vague fuzzy digital structure $\mathfrak{D}_1$ kernal) of a pairwise vague fuzzy digital structure Co-$Kreg$-set which is different from $\tilde{1}$ is $\tilde{0}$.*
- (iv) *There exist no vague fuzzy digital structure $\mathfrak{D}_1$ kernal set A and pairwise vague fuzzy digital structure $Kreg$-set B in $\sum$ such that $A \neq \tilde{0} \neq B$ and $A \subseteq \overline{B}$. ((or) There exist no vague fuzzy digital structure $\mathfrak{D}_2$ kernal set A and pairwise vague fuzzy digital structure $Kreg$-set B in $\sum$ such that $A \neq \tilde{0} \neq B$ and $A \subseteq \overline{B}$).*

Proof. **(i) ⇒ (ii)**

Assume that (i) is true. Suppose that there exists a pairwise vague fuzzy digital structure $Kreg$-set $A \neq \tilde{0}$ such that $VFCo\text{-}Ker_{\mathfrak{D}_2}(A) \neq \tilde{1}$. Now,

$$VFKer_{\mathfrak{D}_1}(VFCo\text{-}Ker_{\mathfrak{D}_2}(A)) \neq \tilde{1}. \tag{3.1}$$

Since A is a pairwise vague fuzzy digital structure $Kreg$-set,

$$VFKer_{\mathfrak{D}_1}(VFCo\text{-}Ker_{\mathfrak{D}_2}(A)) = A. \tag{3.2}$$

From (3.1) and (3.2), we have $A \neq \tilde{1}$. Thus, we find that $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ has a pairwise vague fuzzy digital structure $Kreg$-set A such that $\tilde{0} \neq A \neq \tilde{1}$. This is a contradiction. Hence, (i)⇒(ii).

(ii) ⇒ (iii)

Assume that (ii) is true. Suppose that there exists a pairwise vague fuzzy digital structure Co-$Kreg$-set $A \neq \tilde{1}$ in $\sum$ such that $VFKer_{\mathfrak{D}_2}(A) \neq \tilde{0}$. Now, $B = \overline{A} \neq \tilde{0}$ and B is non-zero pairwise vague fuzzy digital structure $Kreg$-set in $\sum$. Then $VFCo\text{-}Ker_{\mathfrak{D}_2}(B) = \overline{VFKer_{\mathfrak{D}_2}(\overline{B})} = \overline{VFKer_{\mathfrak{D}_2}(A)} \neq \tilde{1}$. This is a contradiction. Hence, (ii)⇒(iii).

(iii) ⇒ (iv)

Assume that (iii) is true. Let A be any vague fuzzy digital structure $\mathfrak{D}_1$ kernal set and B be pairwise vague fuzzy digital structure $Kreg$-set in $\sum$ such that $A \neq \tilde{0} \neq B$ and $A \subseteq \overline{B}$. Since $\overline{B}$ is a pairwise vague fuzzy digital structure Co-$Kreg$-set in $\sum$ and $\overline{B} \neq \tilde{1}$. By (iii), $VFKer_{\mathfrak{D}_1}(\overline{B}) = \tilde{0}$. Since A is a vague fuzzy digital structure $\mathfrak{D}_1$ kernal set and $A \subseteq \overline{B}$, $A = VFKer_{\mathfrak{D}_1}(A) \subseteq VFKer_{\mathfrak{D}_1}(\overline{B}) = \tilde{0}$. This is a contradiction. Hence, (iii)⇒(iv).

(iv) ⇒ (i)

Assume that (iv) is true. Let $\tilde{0} \neq A \neq \tilde{1}$ be any pairwise vague fuzzy digital structure $Kreg$-set in $\sum$. If we take $B = \overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(A)}$. Clearly, B is a vague fuzzy digital structure $\mathfrak{D}_2$ kernal set in $\sum$ and we get $B \neq \tilde{0}$. Because, otherwise we have, $B = \tilde{0}$. This implies that $\overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(A)} = \tilde{0}$ implies that $VFCo\text{-}Ker_{\mathfrak{D}_2}(A) = \tilde{1}$. Since A is a pairwise vague fuzzy digital structure $Kreg$-set in $\sum$, $VFKer_{\mathfrak{D}_1}(VFCo\text{-}Ker_{\mathfrak{D}_2}(A)) = A$. This implies that $VFKer_{\mathfrak{D}_1}(\tilde{1}) = \tilde{1} = A$. This is a contradiction to $A \neq \tilde{1}$. Also, we have $A \subseteq \overline{B}$. This is a contradiction. Hence, (iv)⇒(i). □

Remark 3.18. The Proposition 3.14. and Proposition 3.17. can be discussed for other case too.

Proposition 3.19. *Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space. Then the following statements are equivalent:*

(i) $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ *is a pairwise vague fuzzy digital structure super K-connected space.*

(ii) *There exist no vague fuzzy $\mathfrak{D}_{(2,1)}$ structure Kreg-set A and vague fuzzy $\mathfrak{D}_{(1,2)}$ structure Kreg-set B in $\sum$ such that*

$$A \neq \tilde{0} \neq B,\ A = \overline{VFCo\text{-}Ker_{\mathfrak{D}_1}(B)} \text{ and } B = \overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(A)}.$$

(iii) *There exist no vague fuzzy $\mathfrak{D}_{(2,1)}$ structure Co-Kreg-set A and vague fuzzy $\mathfrak{D}_{(1,2)}$ structure Co-Kreg-set B in $\sum$ such that*

$$A \neq \tilde{1} \neq B,\ A = \overline{VFKer_{\mathfrak{D}_1}(B)} \text{ and } B = \overline{VFKer_{\mathfrak{D}_2}(A)}.$$

Proof. **(i) ⇒ (ii)**

Assume that (i) is true. Suppose that there exist vague fuzzy $\mathfrak{D}_{(2,1)}$ structure $Kreg$-set A and vague fuzzy $\mathfrak{D}_{(1,2)}$ structure $Kreg$-set B in $\sum$ such that $A \neq \tilde{0} \neq B$,

$A = \overline{VFCo\text{-}Ker_{\mathfrak{D}_1}(B)}$ and $B = \overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(A)}$. Now,

$$\begin{aligned} VFKer_{\mathfrak{D}_1}(VFCo\text{-}Ker_{\mathfrak{D}_2}(A)) &= VFKer_{\mathfrak{D}_1}(\overline{B}) \\ &= \overline{VFCo\text{-}Ker_{\mathfrak{D}_1}(B)} = A. \end{aligned}$$

This implies that A is a vague fuzzy $\mathfrak{D}_{(1,2)}$ structure $Kreg$-set in $\sum$ and $A \neq \tilde{0}$. Also, $A \neq \tilde{1}$. If not, that is, $A = \tilde{1}$ implies that $\overline{VFCo\text{-}Ker_{\mathfrak{D}_1}(B)} = \tilde{1}$. This implies that $VFCo\text{-}Ker_{\mathfrak{D}_1}(B) = \tilde{0}$. Therefore, $B = \tilde{0}$ which is a contradiction. Hence, A is a pairwise vague fuzzy digital structure $Kreg$-set in $\sum$. This is a contradiction. Hence, (i)⇒(ii).

(ii) ⇒ (i)

Assume that (ii) is true. Suppose that $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is not a pairwise vague fuzzy digital structure super K-connected space. Let A be a pairwise vague fuzzy digital structure $Kreg$-set such that $\tilde{0} \neq A \neq \tilde{1}$. Now, if $B = \overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(A)}$, then $VFCo\text{-}Ker_{\mathfrak{D}_1}(B) = VFCo\text{-}Ker_{\mathfrak{D}_1}(\overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(A)})$.

Taking complement on both sides,

$$\begin{aligned} \overline{VFCo\text{-}Ker_{\mathfrak{D}_1}(B)} &= \overline{VFCo\text{-}Ker_{\mathfrak{D}_1}(\overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(A)})} \\ &= \overline{\overline{VFKer_{\mathfrak{D}_1}(VFCo\text{-}Ker_{\mathfrak{D}_2}(A))}} \\ &= VFKer_{\mathfrak{D}_1}(VFCo\text{-}Ker_{\mathfrak{D}_2}(A)) = A. \end{aligned}$$

Clearly, $B \neq \tilde{0}$ and B is a vague fuzzy $\mathfrak{D}_{(2,1)}$ structure $Kreg$-set in $\sum$. Hence there exist vague fuzzy $\mathfrak{D}_{(1,2)}$ structure $Kreg$-set A and vague fuzzy $\mathfrak{D}_{(2,1)}$ structure $Kreg$-set B respectively in $\sum$ such that $A \neq \tilde{0} \neq B$, $A = \overline{VFCo\text{-}Ker_{\mathfrak{D}_1}(B)}$ and $B = \overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(A)}$. This is a contradiction. Hence, (ii)⇒(i).

(ii) ⇒ (iii)

Assume that (ii) is true. Suppose that there exist vague fuzzy $\mathfrak{D}_{(2,1)}$ structure $Co\text{-}Kreg$-set A and vague fuzzy $\mathfrak{D}_{(1,2)}$ structure $Co\text{-}Kreg$-set B in $\sum$ such that $A \neq \tilde{1} \neq B$, $A = \overline{VFKer_{\mathfrak{D}_1}(B)}$ and $B = \overline{VFKer_{\mathfrak{D}_2}(A)}$. Then, there exist vague fuzzy $\mathfrak{D}_{(2,1)}$ structure $Kreg$-set $C = \overline{\mathrm{A}}$ and vague fuzzy $\mathfrak{D}_{(1,2)}$ structure $Kreg$-set $D = \overline{B}$ in $\sum$ such that $C \neq \tilde{0} \neq D$. Now,

$$\begin{aligned} C &= \overline{\mathrm{A}} \\ &= \overline{\overline{VFKer_{\mathfrak{D}_1}(B)}} \\ &= \overline{VFCo\text{-}Ker_{\mathfrak{D}_1}(\overline{B})} \\ &= \overline{VFCo\text{-}Ker_{\mathfrak{D}_1}(D)}. \end{aligned}$$

Similarly, $D = \overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(C)}$ can also be proved. This is a contradiction. Hence, (ii)⇒(iii).

(iii) ⇒ (ii) It is obvious. □

Definition 3.20. Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space. Let ${\mathfrak{D}_i}^c = \{A \mid \overline{\mathrm{A}} \in \mathfrak{D}_i, i = 1 \text{ (or) } 2\}$ be any family of vague fuzzy digital sets of $\sum$. Then

(i) $\mathfrak{D}_1 \cup \mathfrak{D}_2 = \{A \mid A \in \mathfrak{D}_1 \text{ (or) } A \in \mathfrak{D}_2\}$.
(ii) ${\mathfrak{D}_1}^c \cup {\mathfrak{D}_2}^c = \{A \mid A \in {\mathfrak{D}_1}^c \text{ (or) } A \in {\mathfrak{D}_2}^c\}$.

Definition 3.21. Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space. Then $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is called a pairwise vague fuzzy digital structure strongly kernal connected (in short, K-connected) space if and only if it has no proper vague fuzzy digital structure co-kernal sets $A, B \in {\mathfrak{D}_1}^c \cup {\mathfrak{D}_2}^c$ such that $t_A + t_B \leq 1$ and $f_A + f_B \geq 1$.

If $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is not a pairwise vague fuzzy digital structure strongly kernal connected space, then it is called a pairwise vague fuzzy digital structure weakly kernal connected (in short, K-connected) space.

Example 3.22. Let $\sum = \{P_1, P_2, P_3\}$ be a rectangular array of integer-coordinate points. Let

$A = \langle P, (\frac{P_1}{0.3}, \frac{P_2}{0.3}, \frac{P_3}{0.4}), (\frac{P_1}{0.4}, \frac{P_2}{0.5}, \frac{P_3}{0.5})\rangle$ and $B = \langle P, (\frac{P_1}{0.2}, \frac{P_2}{0.2}, \frac{P_3}{0.4}), (\frac{P_1}{0.3}, \frac{P_2}{0.4}, \frac{P_3}{0.4})\rangle$

be any two vague fuzzy digital sets in $\sum$.

Then $\mathfrak{D}_1 = \{\tilde{0}, \tilde{1}, A\}$ and $\mathfrak{D}_2 = \{\tilde{0}, \tilde{1}, B\}$ are vague fuzzy digital structures on $\sum$. Hence, $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is a pairwise vague fuzzy digital structure strongly K-connected space.

Proposition 3.23. *A vague fuzzy digital bi-structure space $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is a pairwise vague fuzzy digital structure strongly K-connected space if and only if there exist no proper vague fuzzy digital structure kernal sets $A, B \in \mathfrak{D}_1 \cup \mathfrak{D}_2$ such that $t_A + t_B \geq 1$ and $f_A + f_B \leq 1$.*

Proof. Suppose that $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is a pairwise vague fuzzy digital structure weakly K-connected space if and only if there exist proper vague fuzzy digital structure co-kernal sets $C, D \in {\mathfrak{D}_1}^c \cup {\mathfrak{D}_2}^c$ such that $t_C + t_D \leq 1$ and $f_C + f_D \geq 1$ if and only if there exist proper vague fuzzy digital structure kernal sets $A, B \in \mathfrak{D}_1 \cup \mathfrak{D}_2$ where $A = \overline{C}$ and $B = \overline{D}$ such that $t_A + t_B = t_{\overline{C}} + t_{\overline{D}} = f_C + f_D \geq 1$ and $f_A + f_B = f_{\overline{C}} + f_{\overline{D}} = t_C + t_D \leq 1$. This implies that $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is a pairwise vague fuzzy digital structure strongly K-connected space if and only if there exist no proper vague fuzzy digital structure kernal sets $A, B \in \mathfrak{D}_1 \cup \mathfrak{D}_2$ such that $t_A + t_B \geq 1$ and $f_A + f_B \leq 1$. □

Remark 3.24. Every pairwise vague fuzzy digital structure strongly K-connected space is a pairwise vague fuzzy digital structure K-connected space.

Proof. The proof is obvious. □

Note 3.25. The converse of the above remark need not be true and it is seen in the following example.

Example 3.26. Let $\sum = \{P_1, P_2, P_3\}$ be a rectangular array of integer-coordinate points. Let

$A = \langle P, (\frac{P_1}{0}, \frac{P_2}{0.4}, \frac{P_3}{0}), (\frac{P_1}{1}, \frac{P_2}{0.3}, \frac{P_3}{1})\rangle$ and $B = \langle P, (\frac{P_1}{0.3}, \frac{P_2}{0}, \frac{P_3}{0.2}), (\frac{P_1}{0.1}, \frac{P_2}{1}, \frac{P_3}{0.8})\rangle$

be any two vague fuzzy digital sets in $\sum$. Then $\mathfrak{D}_1 = \{\tilde{0}, \tilde{1}, A\}$ and $\mathfrak{D}_2 = \{\tilde{0}, \tilde{1}, B\}$ are vague fuzzy digital structures on $\sum$. Hence, $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is a pairwise vague fuzzy digital structure K-connected space. But $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is not a pairwise vague fuzzy digital structure strongly K-connected space.

Definition 3.27. Let $\sum$ be a rectangular array of integer-coordinate points and let Ω be any subset of $\sum$. The digital characteristic function of Ω is denoted and defined as

$$\chi_\Omega(P) = \begin{cases} 1, & if \quad P \in \Omega; \\ 0, & if \quad P \notin \Omega. \end{cases}$$

Notation 3.2. Let $\sum$ be a rectangular array of integer-coordinate points and let Ω be any subset of $\sum$. Then a vague fuzzy digital set $\widehat{\chi_\Omega}$ in $\sum$ is defined as $\widehat{\chi_\Omega} = \{\langle P, \chi_\Omega(P), 1 - \chi_\Omega(P)\rangle : P \in \sum\}$.

Definition 3.28. Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space and let Ω be any subset of $\sum$. Then $\mathfrak{D}_1/\Omega = \{A/\Omega = \widehat{\chi_\Omega} \cap A \mid A \in \mathfrak{D}_1\}$ and $\mathfrak{D}_2/\Omega = \{B/\Omega = \widehat{\chi_\Omega} \cap B \mid B \in \mathfrak{D}_2\}$ are vague fuzzy digital structures on Ω and are called the induced vague fuzzy digital structures on Ω. The pair $(\Omega, \mathfrak{D}_1/\Omega, \mathfrak{D}_2/\Omega)$ is called a vague fuzzy digital bi-structure subspace of vague fuzzy digital bi-structure space $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$.

Notation 3.3. Let $(\Omega, \mathfrak{D}_1/\Omega, \mathfrak{D}_2/\Omega)$ be any vague fuzzy digital bi-structure subspace of vague fuzzy digital bi-structure space $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$. Then
$\tilde{1}_\Omega = \{\langle P, 1, 0\rangle : P \in \Omega\}$ denotes the universal vague fuzzy digital set in Ω.

Proposition 3.29. *Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space and let Ω be any subset of $\sum$. Then the following statements are equivalent:*

- (i) *$(\Omega, \mathfrak{D}_1/\Omega, \mathfrak{D}_2/\Omega)$ is a pairwise vague fuzzy digital structure strongly K-connected subspace of vague fuzzy digital bi-structure space $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$.*
- (ii) *For any proper vague fuzzy digital structure kernal sets $A, B \in \mathfrak{D}_1 \cup \mathfrak{D}_2$, $t_{A/\Omega} + t_{B/\Omega} \geq 1$ and $f_{A/\Omega} + f_{B/\Omega} \leq 1$ implies that either $A/\Omega = \tilde{1}_\Omega$ (or) $B/\Omega = \tilde{1}_\Omega$.*

Proof. **(ii) ⇒ (i)**
Suppose that $(\Omega, \mathfrak{D}_1/\Omega, \mathfrak{D}_2/\Omega)$ is not a pairwise vague fuzzy digital structure strongly K-connected subspace of $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$. Then, there exist proper vague fuzzy digital structure co-kernal sets $A, B \in (\mathfrak{D}_1/\Omega)^c \cup (\mathfrak{D}_2/\Omega)^c$ such that $t_A + t_B \leq 1$ and $f_A + f_B \geq 1$. Therefore, there exist proper vague fuzzy digital structure kernal sets $C, D \in \mathfrak{D}_1 \cup \mathfrak{D}_2$ such that $C/\Omega = \overline{\mathrm{A}}, D/\Omega = \overline{B}$. Then,

$$t_{C/\Omega} + t_{D/\Omega} = t_{\overline{\mathrm{A}}} + t_{\overline{B}} = f_A + f_B \geq 1$$

and

$$f_{C/\Omega} + f_{D/\Omega} = f_{\overline{\mathrm{A}}} + f_{\overline{B}} = t_A + t_B \leq 1.$$

Since C and D are proper vague fuzzy digital structure kernal sets in $\sum$, C/Ω and D/Ω are proper vague fuzzy digital structure kernal sets in Ω. This implies that $C/\Omega \neq \tilde{1}_\Omega$ and $D/\Omega \neq \tilde{1}_\Omega$. This is a contradiction. Hence, (ii)⇒(i).
(i) ⇒ (ii)
Suppose that there exist proper vague fuzzy digital structure kernal sets $C, D \in \mathfrak{D}_1 \cup \mathfrak{D}_2$ such that $t_{C/\Omega} + t_{D/\Omega} \geq 1$ and $f_{C/\Omega} + f_{D/\Omega} \leq 1$ but both $C/\Omega \neq \tilde{1}_\Omega$ and $D/\Omega \neq \tilde{1}_\Omega$. By Proposition 3.23., $(\Omega, \mathfrak{D}_1/\Omega, \mathfrak{D}_2/\Omega)$ is not a pairwise vague fuzzy digital structure strongly K-connected subspace of $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$. This is a contradiction. Hence, (i)⇒(ii). □

Proposition 3.30. *Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space. Let Ω be any subset of $\sum$ such that $\widehat{\chi_\Omega} \in {\mathfrak{D}_1}^c \cup {\mathfrak{D}_2}^c$. Then $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is a pairwise vague fuzzy digital structure strongly K-connected space implies that $(\Omega, \mathfrak{D}_1/\Omega, \mathfrak{D}_2/\Omega)$ is a pairwise vague fuzzy digital structure strongly K-connected subspace.*

Proof. Assume that $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is a pairwise vague fuzzy digital structure strongly K-connected space. Let Ω be any subset of $\sum$ such that $\widehat{\chi_\Omega} \in {\mathfrak{D}_1}^c \cup {\mathfrak{D}_2}^c$. We want to show that $(\Omega, \mathfrak{D}_1/\Omega, \mathfrak{D}_2/\Omega)$ is a pairwise vague fuzzy digital structure strongly K-connected subspace. Suppose that $(\Omega, \mathfrak{D}_1/\Omega, \mathfrak{D}_2/\Omega)$ is not a pairwise vague fuzzy digital structure strongly K-connected subspace. Then, there exist proper vague fuzzy digital structure co-kernal sets $A, B \in (\mathfrak{D}_1/\Omega)^c \cup (\mathfrak{D}_2/\Omega)^c$ such that

$$t_A + t_B \leq 1 \text{ and } f_A + f_B \geq 1. \tag{3.3}$$

Hence, there exist proper vague fuzzy digital structure co-kernal sets $C, D \in {\mathfrak{D}_1}^c \cup {\mathfrak{D}_2}^c$ such that $A = C/\Omega, B = D/\Omega$. Now, consider the proper vague fuzzy digital structure co-kernal sets $C \cap \widehat{\chi_\Omega}$ and $D \cap \widehat{\chi_\Omega}$ in Ω. Since $\widehat{\chi_\Omega} \in {\mathfrak{D}_1}^c \cup {\mathfrak{D}_2}^c$, $C \cap \widehat{\chi_\Omega} \in {\mathfrak{D}_1}^c \cup {\mathfrak{D}_2}^c$ and $D \cap \widehat{\chi_\Omega} \in {\mathfrak{D}_1}^c \cup {\mathfrak{D}_2}^c$. Further from (3.3),

$$t_{C \cap \widehat{\chi_\Omega}} + t_{D \cap \widehat{\chi_\Omega}} \leq 1 \text{ and } f_{C \cap \widehat{\chi_\Omega}} + f_{D \cap \widehat{\chi_\Omega}} \geq 1. \tag{3.4}$$

Thus, $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is not a pairwise vague fuzzy digital structure strongly K-connected space, which is a contradiction. Therefore, $(\Omega, \mathfrak{D}_1/\Omega, \mathfrak{D}_2/\Omega)$ is a pairwise vague fuzzy digital structure strongly K-connected subspace. □

4. A view on disconnectedness in vague fuzzy digital bi-structure spaces

Definition 4.1. Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space. Then $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is called a pairwise vague fuzzy digital structure kernal extremally (in short, K-extremally) disconnected space if vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal of each vague fuzzy digital structure $\mathfrak{D}_2$ kernal set is a vague fuzzy digital structure $\mathfrak{D}_2$ kernal set and vague fuzzy digital structure $\mathfrak{D}_2$ co-kernal of each vague fuzzy digital structure $\mathfrak{D}_1$ kernal set is a vague fuzzy digital structure $\mathfrak{D}_1$ kernal set.

Example 4.2. Let $\sum = \{P_1, P_2, P_3\}$ be a rectangular array of integer-coordinate points. Let

$A = \langle P, (\frac{P_1}{0.3}, \frac{P_2}{0.3}, \frac{P_3}{0.5}), (\frac{P_1}{0.4}, \frac{P_2}{0.4}, \frac{P_3}{0.5}) \rangle$ and $B = \langle P, (\frac{P_1}{0.4}, \frac{P_2}{0.4}, \frac{P_3}{0.5}), (\frac{P_1}{0.3}, \frac{P_2}{0.3}, \frac{P_3}{0.5}) \rangle$

be any two vague fuzzy digital sets in $\sum$.

Then $\mathfrak{D}_1 = \{\tilde{0}, \tilde{1}, A\}$ and $\mathfrak{D}_2 = \{\tilde{0}, \tilde{1}, B\}$ are vague fuzzy digital structures on $\sum$. Hence, $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is a pairwise vague fuzzy digital structure K-extremally disconnected space.

Proposition 4.3. *Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space. Then the following statements are equivalent:*

(i) $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ *is a pairwise vague fuzzy digital structure K-extremally disconnected space.*

(ii) *Whenever A is a vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal set, $VFKer_{\mathfrak{D}_2}(A)$ is a vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal set. Similarly, whenever B is a vague fuzzy digital structure $\mathfrak{D}_2$ co-kernal set, $VFKer_{\mathfrak{D}_1}(B)$ is a vague fuzzy digital structure $\mathfrak{D}_2$ co-kernal set.*

(iii) *Whenever A is a vague fuzzy digital structure $\mathfrak{D}_1$ kernal (resp., $\mathfrak{D}_2$ kernal) set, we have* $VFCo\text{-}Ker_{\mathfrak{D}_1}(\overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(A)}) = VFKer_{\mathfrak{D}_2}(\overline{A})$ *(resp.,* $VFCo\text{-}Ker_{\mathfrak{D}_2}(\overline{VFCo\text{-}Ker_{\mathfrak{D}_1}(A)}) = VFKer_{\mathfrak{D}_1}(\overline{A})$*).*

Proof. **(i) ⇒ (ii)**

Assume that (i) is true. Let A be a vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal set in $\sum$. Then, $\overline{\text{A}}$ is a vague fuzzy digital structure $\mathfrak{D}_1$ kernal set. Then by (i), $VFCo\text{-}Ker_{\mathfrak{D}_2}(\overline{\text{A}})$ is a vague fuzzy digital structure $\mathfrak{D}_1$ kernal set. Clearly, $\overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(\overline{\text{A}})}$ is a vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal set. But,

$$\overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(\overline{\text{A}})} = VFKer_{\mathfrak{D}_2}(\overline{\overline{\text{A}}}) = VFKer_{\mathfrak{D}_2}(A).$$

Thus, $VFKer_{\mathfrak{D}_2}(A)$ is a vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal set. Similarly, we can show that $VFKer_{\mathfrak{D}_1}(B)$ is a vague fuzzy digital structure $\mathfrak{D}_2$ co-kernal set, whenever B is a vague fuzzy digital structure $\mathfrak{D}_2$ co-kernal set. Hence, (i)⇒(ii).

(ii) ⇒ (iii)

Assume that (ii) is true. Suppose that A is a vague fuzzy digital structure $\mathfrak{D}_1$ kernal set in $\sum$. Then, $\overline{\text{A}}$ is a vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal set. By assumption, $VFKer_{\mathfrak{D}_2}(\overline{\text{A}}) = \overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(A)}$ is a vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal set. Now,

$$\begin{aligned} VFCo\text{-}Ker_{\mathfrak{D}_1}(\overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(A)}) &= VFCo\text{-}Ker_{\mathfrak{D}_1}(VFKer_{\mathfrak{D}_2}(\overline{\text{A}})) \\ &= VFKer_{\mathfrak{D}_2}(\overline{\text{A}}). \end{aligned}$$

Similarly, we can show that $VFCo\text{-}Ker_{\mathfrak{D}_2}(\overline{VFCo\text{-}Ker_{\mathfrak{D}_1}(A)}) = VFKer_{\mathfrak{D}_1}(\overline{\text{A}})$, whenever A is a vague fuzzy digital structure $\mathfrak{D}_2$ kernal set in $\sum$. Hence, (ii)⇒(iii).

(iii) ⇒ (i)

Assume that (iii) is true. Let A be a vague fuzzy digital structure $\mathfrak{D}_1$ kernal set. Then,

$$VFCo\text{-}Ker_{\mathfrak{D}_1}(\overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(A)}) = VFKer_{\mathfrak{D}_2}(\overline{\text{A}}) = \overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(A)} \tag{4.1}$$

From (4.1), $\overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(A)}$ is a vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal set. Therefore, $VFCo\text{-}Ker_{\mathfrak{D}_2}(A)$ is a vague fuzzy digital structure $\mathfrak{D}_1$ kernal set. Similarly, we can show that $VFCo\text{-}Ker_{\mathfrak{D}_1}(B)$ is a vague fuzzy digital structure $\mathfrak{D}_2$ kernal set, whenever B is a vague fuzzy digital structure $\mathfrak{D}_2$ kernal set. Hence, (iii)⇒(i). □

Proposition 4.4. *Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space. Then the following statements are equivalent:*

(i) *$(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is a pairwise vague fuzzy digital structure K-extremally disconnected space.*

(ii) *For each pair of vague fuzzy digital structure $\mathfrak{D}_1$ kernal set A and vague fuzzy digital structure $\mathfrak{D}_2$ kernal set B with $\overline{A} = VFCo\text{-}Ker_{\mathfrak{D}_1}(B)$ and $\overline{B} = VFCo\text{-}Ker_{\mathfrak{D}_2}(A)$, we have*

$$VFCo\text{-}Ker_{\mathfrak{D}_2}(A) = VFKer_{\mathfrak{D}_1}(\overline{B}) \text{ and } VFCo\text{-}Ker_{\mathfrak{D}_1}(B) = VFKer_{\mathfrak{D}_2}(\overline{A}).$$

Proof. **(i) ⇒ (ii)**

Assume that (i) is true. Let A and B be any two vague fuzzy digital structure $\mathfrak{D}_1$ kernal set and vague fuzzy digital structure $\mathfrak{D}_2$ kernal set in $\sum$ respectively with $\overline{\mathrm{A}} = VFCo\text{-}Ker_{\mathfrak{D}_1}(B)$ and $\overline{B} = VFCo\text{-}Ker_{\mathfrak{D}_2}(A)$. Now,

$$\begin{aligned} VFCo\text{-}Ker_{\mathfrak{D}_2}(A) &= VFCo\text{-}Ker_{\mathfrak{D}_2}(\overline{VFCo\text{-}Ker_{\mathfrak{D}_1}(B)}) \\ &= VFCo\text{-}Ker_{\mathfrak{D}_2}(VFKer_{\mathfrak{D}_1}(\overline{B})) \\ &= VFKer_{\mathfrak{D}_1}(\overline{B}). \end{aligned}$$

Similarly,

$$\begin{aligned} VFCo\text{-}Ker_{\mathfrak{D}_1}(B) &= VFCo\text{-}Ker_{\mathfrak{D}_1}(\overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(A)}) \\ &= VFCo\text{-}Ker_{\mathfrak{D}_1}(VFKer_{\mathfrak{D}_2}(\overline{\mathrm{A}})) \\ &= VFKer_{\mathfrak{D}_2}(\overline{\mathrm{A}}). \end{aligned}$$

Hence, (i)⇒(ii).

(ii) ⇒ (i)

Assume that (ii) is true. Let $A = \overline{VFCo\text{-}Ker_{\mathfrak{D}_1}(B)}$ and $B = \overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(A)}$. Clearly, A and B are vague fuzzy digital structure $\mathfrak{D}_1$ kernal set and vague fuzzy digital structure $\mathfrak{D}_2$ kernal set respectively. By (ii),

$$VFCo\text{-}Ker_{\mathfrak{D}_2}(A) = VFKer_{\mathfrak{D}_1}(\overline{B}) \text{ and } VFCo\text{-}Ker_{\mathfrak{D}_1}(B) = VFKer_{\mathfrak{D}_2}(\overline{\mathrm{A}}).$$

Thus, $VFCo\text{-}Ker_{\mathfrak{D}_2}(A)$ and $VFCo\text{-}Ker_{\mathfrak{D}_1}(B)$ are vague fuzzy digital structure $\mathfrak{D}_1$ kernal set and vague fuzzy digital structure $\mathfrak{D}_2$ kernal set respectively. Hence, (ii)⇒(i). □

Definition 4.5. Let $\sum$ be a rectangular array of integer-coordinate points and $P \in \sum$. Let $r \in I_0 = (0,1]$ and $s \in I_1 = [0,1)$ such that $r + s \leq 1$. Then a vague fuzzy digital set $P_{r,s} = \langle P, t_{P_{r,s}}, f_{P_{r,s}} \rangle$ is called a vague fuzzy digital point in $\sum$ where

$$t_{P_{r,s}}(R) = \begin{cases} r, & \text{if P = R;} \\ 0, & \text{otherwise.} \end{cases} \quad \text{and } f_{P_{r,s}}(R) = \begin{cases} s, & \text{if P = R;} \\ 1, & \text{otherwise.} \end{cases}$$

for $R \in \sum$. The vague fuzzy digital point $P_{r,s}$ is contained in the vague fuzzy digital set A (that is, $P_{r,s} \in A$) in $\sum$ if and only if $r < t_A(P)$ and $s > f_A(P)$.

Definition 4.6. Let $(\sum, \mathfrak{D})$ be any vague fuzzy digital structure space. Let $A = \langle P, t_A, f_A \rangle$ be any vague fuzzy digital set in $\sum$. Then A is called a vague fuzzy digital structure kernal neighbourhood of a vague fuzzy digital point $P_{r,s}$ if there exists a vague fuzzy digital structure kernal set $B = \langle P, t_B, f_B \rangle$ in $\sum$ such that $P_{r,s} \in B \subseteq A$.

Definition 4.7. Let $(\sum, \mathfrak{D})$ be any vague fuzzy digital structure space. A family $\mathfrak{B}$ of $\mathfrak{D}$ is called a vague fuzzy digital structure base for $\mathfrak{D}$ if each member of $\mathfrak{D}$ is a union of some members of $\mathfrak{B}$.

Definition 4.8. Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space. Then $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is called a pairwise vague fuzzy digital structure K-Hausdorff ((or) K-T_2) space if for each two distinct vague fuzzy digital points $P_{r,s}$ and $Q_{m,n}$ in $\sum$, there exists a vague fuzzy digital structure $\mathfrak{D}_1$ kernal neighbourhood A of

$P_{r,s}$ which does not contain $Q_{m,n}$ and a vague fuzzy digital structure $\mathfrak{D}_2$ kernal neighbourhood B of $Q_{m,n}$ which does not contain $P_{r,s}$.

Definition 4.9. Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space. Then $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is called a pairwise vague fuzzy digital structure kernal totally (in short, K-totally) disconnected space if and only if for any two distinct vague fuzzy digital points $P_{r,s}$ and $Q_{m,n}$ in $\sum$ there exist vague fuzzy digital structure disconnection $t_A + t_B \leq 1$ and $f_A + f_B \leq 1$ of $\sum$ such that A and B are vague fuzzy digital structure $\mathfrak{D}_1$ kernal set and vague fuzzy digital structure $\mathfrak{D}_2$ kernal set with $P_{r,s} \in A$ and $Q_{m,n} \in B$.

Remark 4.10. Every pairwise vague fuzzy digital structure K-totally disconnected space is a pairwise vague fuzzy digital structure K-Hausdorff space.

Proof. The proof is obvious. □

Proposition 4.11. *Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space. Suppose*

(i) $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ *is a pairwise vague fuzzy digital structure K-Hausdorff space.*

(ii) $\mathfrak{D}_1$ *has a vague fuzzy digital structure base whose members are vague fuzzy digital structure $\mathfrak{D}_2$ co-kernal sets (or) $\mathfrak{D}_2$ has a vague fuzzy digital structure base whose members are vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal sets.*

Then $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is a pairwise vague fuzzy digital structure K-totally disconnected space.

Proof. Let $P_{r,s}$ and $Q_{m,n}$ be any two distinct vague fuzzy digital points in $\sum$. Then, there exists a vague fuzzy digital structure $\mathfrak{D}_1$ kernal neighbourhood $A = \langle P, t_A, f_A \rangle$ of $P_{r,s}$ which does not contain $Q_{m,n}$ and a vague fuzzy digital structure $\mathfrak{D}_2$ kernal neighbourhood $B = \langle P, t_B, f_B \rangle$ of $Q_{m,n}$ which does not contain $P_{r,s}$. Suppose that $\mathfrak{D}_1$ has a vague fuzzy digital structure base whose members are vague fuzzy digital structure $\mathfrak{D}_2$ co-kernal sets. Then, there exists a vague fuzzy digital structure basic $\mathfrak{D}_1$ kernal set $C = \langle P, t_C, f_C \rangle$ such that $\overline{C} = \langle P, f_C, t_C \rangle$ is a vague fuzzy digital structure $\mathfrak{D}_2$ co-kernal set and $P_{r,s} \in C \subseteq A$. Then, $t_C + t_{\overline{C}} = t_C + f_C \leq 1$ and $f_C + f_{\overline{C}} = f_C + t_C \leq 1$.

Here $C \in \mathfrak{D}_1, \overline{C} \in \mathfrak{D}_2, P_{r,s} \in C$ and $Q_{m,n} \in \overline{C}$. Therefore, $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is a pairwise vague fuzzy digital structure K-totally disconnected space. Similarly, we can prove that $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is a pairwise vague fuzzy digital structure K-totally disconnected space when $\mathfrak{D}_2$ has a vague fuzzy digital structure base whose members are vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal sets. □

5. Pairwise vague fuzzy digital structure K-compact spaces

Definition 5.1. Let $(\sum, \mathfrak{D})$ be any vague fuzzy digital structure space and let A be a vague fuzzy digital set in $\sum$. Then A is called a vague fuzzy digital structure $Kreg$-set in $\sum$ if $A = VFKer_{\mathfrak{D}}(VFCo\text{-}Ker_{\mathfrak{D}}(A))$. The complement of a vague fuzzy digital structure $Kreg$-set in $\sum$ is a vague fuzzy digital structure *co-$Kreg$*-set in $\sum$.

Remark 5.2. Let $(\sum, \mathfrak{D})$ be any vague fuzzy digital structure space. Then

(i) Every vague fuzzy digital structure $Kreg$-set is a vague fuzzy digital structure open set.
(ii) Every vague fuzzy digital structure *co*-$Kreg$-set is a vague fuzzy digital structure closed set.

Proof. The proof is simple. □

Definition 5.3. Let $(\sum, \mathfrak{D})$ be any vague fuzzy digital structure space and let A be any vague fuzzy digital set in $\sum$. A family $\mathcal{A}$ of vague fuzzy digital structure kernal sets in $\sum$ is called a vague fuzzy digital structure kernal cover of A if and only if $A \subseteq \cup_{i\in J}\{B_i \mid B_i \in \mathcal{A}\}$.

Definition 5.4. Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space. Then $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is said to be a vague fuzzy digital structure $\mathfrak{D}_{(1,2)}$ K-compact space if for every proper vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal set A and every vague fuzzy digital structure $\mathfrak{D}_2$ kernal cover $\mathcal{A}$ of A, there exists a finite subfamily of $\mathcal{A}$ the vague fuzzy digital structure $\mathfrak{D}_2$ co-kernal of whose members covers A.

Similarly, a vague fuzzy digital structure $\mathfrak{D}_{(2,1)}$ K-compact space can also be defined in the same manner.

Definition 5.5. Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space. Then $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is said to be a pairwise vague fuzzy digital structure kernal compact (in short, K-compact) space if it is both vague fuzzy digital structure $\mathfrak{D}_{(1,2)}$ K-compact and vague fuzzy digital structure $\mathfrak{D}_{(2,1)}$ K-compact.

Note 5.6. Let $\sum$ be a rectangular array of integer-coordinate points and let Ω be a subset of $\sum$. Then $\overline{\widehat{\chi_\Omega}} = \widehat{\chi_\Upsilon}$ where $\Upsilon = \sum - \Omega$.

Proposition 5.7. *Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any pairwise vague fuzzy digital structure K-compact space. If Ω is a subset of $\sum$ such that $\widehat{\chi_\Omega}$ is a proper vague fuzzy digital structure $\mathfrak{D}_1$ Co-Kreg-set and $\mathcal{F}$ is a family of vague fuzzy digital structure $\mathfrak{D}_2$ co-kernal sets of $\sum$ such that $(\cap_{A\in\mathcal{F}}A) \cap \widehat{\chi_\Omega} = \tilde{0}$, then there exists a finite number of elements say $A_1, A_2, ..., A_n$ such that $(\cap_{k=1}^{n} VFKer_{\mathfrak{D}_2}(A_k)) \cap \widehat{\chi_\Omega} = \tilde{0}$.*

Proof. Suppose that $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be a pairwise vague fuzzy digital structure K-compact space. Let Ω be any subset of $\sum$ such that $\widehat{\chi_\Omega}$ is a proper vague fuzzy digital structure $\mathfrak{D}_1$ *Co-Kreg*-set and $\mathcal{F}$ is a family of vague fuzzy digital structure $\mathfrak{D}_2$ co-kernal sets of $\sum$ such that $\cap\{A \cap \widehat{\chi_\Omega} \mid A \in \mathcal{F}\} = \tilde{0}$. Now, $(\cap_{A\in\mathcal{F}}A) \cap \widehat{\chi_\Omega} = \tilde{0}$, which implies that $\cap_{A\in\mathcal{F}}A \subseteq \overline{\widehat{\chi_\Omega}}$. Therefore, $\widehat{\chi_\Omega} \subseteq \overline{\cap_{A\in\mathcal{F}}A} = \cup\{\overline{\mathrm{A}} \mid A \in \mathcal{F}\}$.

Hence, $\{\overline{\mathrm{A}} \mid A \in \mathcal{F}\}$ is a vague fuzzy digital structure $\mathfrak{D}_2$ K cover of $\widehat{\chi_\Omega}$ which is a vague fuzzy digital structure $\mathfrak{D}_1$ *Co-Kreg*-set in $\sum$. By assumption, we have a finite collection, say $\overline{A_1}, \overline{A_2}, ..., \overline{A_n}$ such that

$$\begin{aligned} \widehat{\chi_\Omega} &\subseteq \cup_{k=1}^{n} VFCo\text{-}Ker_{\mathfrak{D}_2}(\overline{A_k}) \\ &= \cup_{k=1}^{n} \overline{VFKer_{\mathfrak{D}_2}(A_k)} \\ &= \overline{\cap_{k=1}^{n} VFKer_{\mathfrak{D}_2}(A_k)}. \end{aligned}$$

This implies that $\cap_{k=1}^{n} VFKer_{\mathfrak{D}_2}(A_k) \subseteq \overline{\widehat{\chi_\Omega}} = \widehat{\chi_\Upsilon}$. Therefore,

$$\widehat{\chi_\Omega} \cap (\cap_{k=1}^{n} VFKer_{\mathfrak{D}_2}(A_k)) \subseteq \widehat{\chi_\Omega} \cap \widehat{\chi_\Upsilon} = \tilde{0}.$$

This implies that $\widehat{\chi_\Omega} \cap (\cap_{k=1}^{n} VFKer_{\mathfrak{D}_2}(A_k)) = \tilde{0}$. □

Proposition 5.8. *Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any pairwise vague fuzzy digital structure K-compact space. If Ω is a subset of $\sum$ such that $\widehat{\chi_\Omega}$ is a proper vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal set and $\mathcal{F}$ is a family of vague fuzzy digital structure $\mathfrak{D}_2$ co-kernal sets of $\sum$ such that $(\cap_{A\in\mathcal{F}} A) \cap \widehat{\chi_\Omega} = \tilde{0}$, then there exists a finite number of elements say $A_1, A_2, ..., A_n$ such that $(\cap_{k=1}^{n} VFKer_{\mathfrak{D}_2}(A_k)) \cap \widehat{\chi_\Omega} = \tilde{0}$.*

Proof. Suppose that $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be a pairwise vague fuzzy digital structure compact space. Let Ω be any subset of $\sum$ such that $\widehat{\chi_\Omega}$ is a proper vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal set and $\mathcal{F}$ is a family of vague fuzzy digital structure $\mathfrak{D}_2$ co-kernal sets of $\sum$ such that $\cap\{A \cap \widehat{\chi_\Omega} \mid A \in \mathcal{F}\} = \tilde{0}$. Now, $(\cap_{A\in\mathcal{F}} A) \cap \widehat{\chi_\Omega} = \tilde{0}$, which implies that $\cap_{A\in\mathcal{F}} A \subseteq \overline{\widehat{\chi_\Omega}}$. Therefore,

$$\widehat{\chi_\Omega} \subseteq \overline{\cap_{A\in\mathcal{F}} A} = \cup\{\overline{\mathrm{A}} \mid A \in \mathcal{F}\}.$$

Hence, $\{\overline{\mathrm{A}} \mid A \in \mathcal{F}\}$ is a vague fuzzy digital structure $\mathfrak{D}_2$ kernal cover of $\widehat{\chi_\Omega}$ which is a vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal set in $\sum$. By assumption, we have a finite collection, say $\overline{A_1}, \overline{A_2}, ..., \overline{A_n}$ such that

$$\begin{aligned} \widehat{\chi_\Omega} &\subseteq \cup_{k=1}^{n} VFCo\text{-}Ker_{\mathfrak{D}_2}(\overline{A_k}) \\ &= \cup_{k=1}^{n} \overline{VFKer_{\mathfrak{D}_2}(A_k)} \\ &= \overline{\cap_{k=1}^{n} VFKer_{\mathfrak{D}_2}(A_k)}. \end{aligned}$$

This implies that $\cap_{k=1}^{n} VFKer_{\mathfrak{D}_2}(A_k) \subseteq \overline{\widehat{\chi_\Omega}} = \widehat{\chi_\Upsilon}$. Therefore,

$$\widehat{\chi_\Omega} \cap (\cap_{k=1}^{n} VFKer_{\mathfrak{D}_2}(A_k)) \subseteq \widehat{\chi_\Omega} \cap \widehat{\chi_\Upsilon} = \tilde{0}.$$

This implies that $\widehat{\chi_\Omega} \cap (\cap_{k=1}^{n} VFKer_{\mathfrak{D}_2}(A_k)) = \tilde{0}$. □

Definition 5.9. Let $(\sum, \mathfrak{D})$ be any vague fuzzy digital structure space and let $\mathcal{B}$ be a family of vague fuzzy digital sets in $\sum$. Then $\mathcal{B}$ is said to be a vague fuzzy digital structure filter base in $\sum$ if for every finite subfamily $\mathcal{B}_o$ of $\mathcal{B}$, $\cap\mathcal{B}_o \neq \tilde{0}$.

Note 5.10. A family $\mathcal{B}$ is a vague fuzzy digital structure K-filter base in $\sum$ if every elements in $\mathcal{B}$ is a vague fuzzy digital structure kernal sets in $\sum$.

Definition 5.11. Let $(\sum, \mathfrak{D})$ be any vague fuzzy digital structure space and let $A = \langle P, \mu_A, \gamma_A \rangle$ and $B = \langle P, \mu_B, \gamma_B \rangle$be any two vague fuzzy digital sets in $\sum$. Then A is said to be a

(i) vague fuzzy digital structure kernal neighbourhood of B if there exists a vague fuzzy digital structure kernal set C in $(\sum, \mathfrak{D})$ such that $B \subseteq C \subseteq A$.
(ii) vague fuzzy digital structure $Kreg$ neighbourhood of B if there exists a vague fuzzy digital structure $Kreg$-set C in $(\sum, \mathfrak{D})$ such that $B \subseteq C \subseteq A$.

Definition 5.12. Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space. A vague fuzzy digital structure $\mathfrak{D}_j$-filter base $\mathcal{F}$ is said to be a vague fuzzy digital structure $\mathfrak{D}_{ij}$-$Kreg$ adherent convergent if every vague fuzzy digital structure $\mathfrak{D}_i$ $Kreg$ neighbourhood of the vague fuzzy digital structure $\mathfrak{D}_j$-K adherent set of $\mathcal{F}$

contains an elements of $\mathcal{F}$ ($i \neq j$ and $i, j = 1, 2$) where the vague fuzzy digital structure $\mathfrak{D}_j$-K adherent set of $\mathcal{F}$ is defined as

$$\cap\{VFCo\text{-}Ker_{\mathfrak{D}_j}(A) \mid A \in \mathcal{F}\}.$$

Proposition 5.13. *If* $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ *is a pairwise vague fuzzy digital structure* K*-compact space, then every vague fuzzy digital structure* $\mathfrak{D}_2$ K*-filter base is vague fuzzy digital structure* $\mathfrak{D}_{12}$ $Kreg$*-adherent convergent.*

Proof. Suppose that $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be a pairwise vague fuzzy digital structure K-compact space and let $\mathcal{F}$ be any vague fuzzy digital structure $\mathfrak{D}_2$ K-filter base with a vague fuzzy digital structure $\mathfrak{D}_2$-K adherent set A of $\mathcal{F}$. Let B be a vague fuzzy digital structure $Kreg$ neighbourhood of A. Thus, $A = \cap\{VFCo\text{-}Ker_{\mathfrak{D}_2}(C) \mid C \in \mathcal{F}\}$ and and $A \subseteq B$ and $\overline{B}$ is a vague fuzzy digital structure $\mathfrak{D}_1$ Co-$Kreg$-set in $\sum$. Now,

$$\begin{aligned}\overline{B} &\subseteq \overline{\mathrm{A}}\\ &= \overline{\cap\{VFCo\text{-}Ker_{\mathfrak{D}_2}(C) \mid C \in \mathcal{F}\}}\\ &= \cup\{\overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(C)} \mid C \in \mathcal{F}\}.\end{aligned}$$

Therefore, $\{\overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(C)} \mid C \in \mathcal{F}\}$ is a vague fuzzy digital structure $\mathfrak{D}_2$ kernal cover of vague fuzzy digital structure $\mathfrak{D}_1$ Co-$Kreg$-set $\overline{B}$. Since $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is a pairwise vague fuzzy digital structure K-compact space, we can find a subfamily, say $\{\overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(C_k)} \mid k = 1, 2, ..., n\}$ such that

$$\begin{aligned}\overline{B} &\subseteq \cup_{k=1}^{n} VFCo\text{-}Ker_{\mathfrak{D}_2}(\overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(C_k)})\\ &= \cup_{k=1}^{n} \overline{VFKer_{\mathfrak{D}_2}(VFCo\text{-}Ker_{\mathfrak{D}_2}(C_k))}\\ &= \overline{\cup_{k=1}^{n} VFKer_{\mathfrak{D}_2}(VFCo\text{-}Ker_{\mathfrak{D}_2}(C_k))}.\end{aligned}$$

This implies that $\cup_{k=1}^{n} VFKer_{\mathfrak{D}_2}(VFCo\text{-}Ker_{\mathfrak{D}_2}(C_k)) \subseteq B$. Further,

$$C_k \subseteq VFKer_{\mathfrak{D}_2}(VFCo\text{-}Ker_{\mathfrak{D}_2}(C_k)), \text{ for } k = 1, 2, ..., n.$$

Therefore, $\cap_{k=1}^{n} C_k \subseteq VFKer_{\mathfrak{D}_2}(VFCo\text{-}Ker_{\mathfrak{D}_2}(C_k))$. Hence, $\cap_{k=1}^{n} C_k \subseteq B$. This implies that $B \in \mathcal{F}$. Hence the proof is completed. □

Definition 5.14. Let $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be any vague fuzzy digital bi-structure space. A vague fuzzy digital structure $\mathfrak{D}_j$-K filter base $\mathcal{F}$ is said to be a vague fuzzy digital structure $\mathfrak{D}_{ij}$-K adherent convergent if every vague fuzzy digital structure $\mathfrak{D}_i$ kernal neighbourhood of the vague fuzzy digital structure $\mathfrak{D}_j$-K adherent set of $\mathcal{F}$ contains an elements of $\mathcal{F}$ ($i \neq j$ and $i, j = 1, 2$) where the vague fuzzy digital structure $\mathfrak{D}_j$-K adherent set of $\mathcal{F}$ is defined as $\cap\{VFCo\text{-}Ker_{\mathfrak{D}_j}(A) \mid A \in \mathcal{F}\}$.

Proposition 5.15. *If* $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ *is a pairwise vague fuzzy digital structure* K*-compact space, then every vague fuzzy digital structure* $\mathfrak{D}_2$*-*K *filter base is vague fuzzy digital structure* $\mathfrak{D}_{12}$*-*K *adherent convergent.*

Proof. Suppose that $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ be a pairwise vague fuzzy digital structure K-compact space and let $\mathcal{F}$ be any vague fuzzy digital structure $\mathfrak{D}_2$ K-filter base with a vague fuzzy digital structure $\mathfrak{D}_2$-K adherent set A of $\mathcal{F}$. Let B be a vague fuzzy digital structure $\mathfrak{D}_1$ kernal neighbourhood of A. Thus, $A = \cap\{VFCo\text{-}Ker_{\mathfrak{D}_2}(C) \mid C \in \mathcal{F}\}$ and and $A \subseteq B$ and $\overline{B}$ is a vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal set in $\sum$. Now,

$$\overline{B} \subseteq \overline{\mathrm{A}}$$

$$= \overline{\cap\{VFCo\text{-}Ker_{\mathfrak{D}_2}(C) \mid C \in \mathcal{F}\}}$$
$$= \cup\{\overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(C)} \mid C \in \mathcal{F}\}.$$

Therefore, $\{\overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(C)} \mid C \in \mathcal{F}\}$ is a vague fuzzy digital structure $\mathfrak{D}_2$ kernal cover of vague fuzzy digital structure $\mathfrak{D}_1$ co-kernal set $\overline{B}$. Since $(\sum, \mathfrak{D}_1, \mathfrak{D}_2)$ is a pairwise vague fuzzy digital structure K-compact space, we can find a subfamily, say $\{\overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(C_k)} \mid k = 1, 2, ..., n\}$ such that

$$\overline{B} \subseteq \cup_{k=1}^{n} VFCo\text{-}Ker_{\mathfrak{D}_2}(\overline{VFCo\text{-}Ker_{\mathfrak{D}_2}(C_k)})$$
$$= \cup_{k=1}^{n} \overline{VFKer_{\mathfrak{D}_2}(VFCo\text{-}Ker_{\mathfrak{D}_2}(C_k))}$$
$$= \overline{\cup_{k=1}^{n} VFKer_{\mathfrak{D}_2}(VFCo\text{-}Ker_{\mathfrak{D}_2}(C_k))}.$$

This implies that $\cup_{k=1}^{n} VFKer_{\mathfrak{D}_2}(VFCo\text{-}Ker_{\mathfrak{D}_2}(C_k)) \subseteq B$. Further,

$$C_k \subseteq VFKer_{\mathfrak{D}_2}(VFCo\text{-}Ker_{\mathfrak{D}_2}(C_k)), \text{ for } k = 1, 2, ..., n.$$

Therefore, $\cap_{k=1}^{n} C_k \subseteq VFKer_{\mathfrak{D}_2}(VFCo\text{-}Ker_{\mathfrak{D}_2}(C_k))$. Hence, $\cap_{k=1}^{n} C_k \subseteq B$. This implies that $B \in \mathcal{F}$. Hence the proof is completed. □

Remark 5.16. The Proposition 5.7., Proposition 5.8., Proposition 5.13. and Proposition 5.15. are also can be discussed for other case too.

References

[1] U. V. Fatteh and D. S. Bassan, Fuzzy Connectedness and its stronger forms, J. Math. Anal. Appl. 111(2) (1985) 449–464.

[2] W. A. Gau and D. J. Buehrer, Vague sets, IEEE Transactions on Systems and Cybernetics 23 (1993) 610–614.

[3] T. Y. Kong and A. Rosenfeld, Digital Topology, Introduction and Survey, Computer Vision, Graphics and Image Processsing 48 (1989) 357–393.

[4] R. Lowen, Connectedness in fuzzy topological spaces, Rocky Mountain J. Math. 11(3) (1981) 427–433.

[5] R. Narmada Devi, E. Roja and M. Sudha, Compactness and connectedness in vague fuzzy digital bi-structure space, International Journal of Mathematics and Computer Applications Research 3(4) (2013) 19–30.

[6] R. Narmada Devi, E. Roja and M. K. Uma, Deformations in vague fuzzy digital structure space, J. Fuzzy Math. (in press).

[7] S. E. Rodabaugh, Connectivity and the L-fuzzy unit interval, Rocky Mountain J. Math. 12(1) (1982) 113–121.

[8] A. Rosenfeld, Fuzzy digital topology, Inform. and Control 40(1) (1979) 76–87.

[9] A. Rosenfeld, On connectivity properties of gray scale pictures, Pattern Recognition Letters 16 (1983) 47–50.

[10] A. Rosenfeld, The fuzzy geometry of image subsets, Pattern Recognition Letters 2 (1984) 311–317.

[11] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

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