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On fuzzy supplement submodules

Yahya Talebi, Atefahe Darzi

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Abstract. In this paper, we introduce the concept of fuzzy supplement submodules of a module. We attempt to investigate various properties of such submodules. Also we define fuzzy coclosed submodules and study the relationship with fuzzy supplement submodules.

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Corresponding Author: Yahya Talebi (talebi@umz.ac.ir)

1. Introduction

After the introduction of fuzzy sets by Zadeh [12] a number of applications of this fundamental concept have come up. Rosenfeld [8] was the first one to define the concept of fuzzy subgroups of a group. Naegoita and Ralescu [5] applied this concept to modules and defined fuzzy submodules of a module. Consequently, fuzzy finitely generated submodules, fuzzy quotient modules [6], radical of fuzzy submodules, and primary fuzzy submodules [3, 10] were investigated. Saikia and Kalita [9] defined fuzzy essential submodules and investigated various characteristics of such submodule. Rahman and Saikia [7] defined fuzzy small submodules. In this paper we give the definition of fuzzy supplement submodules and fuzzy coclosed submodules of fuzzy module. We study some of its properties.

2. Basic definitions and results

In this section we briefly introduce some definitions and results of fuzzy sets and fuzzy submodules, which we need to develop our paper.

By R we mean a commutative ring with unity 1 and M denotes an R-module. The zero elements of R and M are 0 and θ, respectively.

Definition 2.1 ([2]). Let $K \leq M$. Then K is said to be a small submodule of M, if for all submodules L of M, $K + L = M$ implies that $L = M$. It is indicated by the notation $K \ll M$.

Definition 2.2 ([2]). Let $K \leq M$. If $K \ll M$, then we say M is a small cover of M/K.

Definition 2.3 ([2]). Let $N, L \leq M$. Then we say that N lies above K or K is a coessential submodule of N, if M/K is a small cover of M/N, that is; $N/K \ll M/K$.

Definition 2.4 ([2]). Let $N \leq M$. N is called coclosed in M if and only if N has no proper coessential submodule, that is; $\nexists K, N/K \ll M/K$.

Definition 2.5 ([11]). Let $N, L \leq M$. Then we call N a supplement of L, if N is minimal with respect to $N + L = M$. Equivalently, N is called a supplement of L if and only if $N + L = M$ and $N \cap L \ll N$.

Definition 2.6 ([11]). Let M be an R-module. If every submodule of M has a supplement, then M is called a supplemented module.

Definition 2.7 ([11]). Let $N, L \leq M$. Then we call N a weak supplement of L if and only if $N + L = M$ and $N \cap L \ll M$.

Definition 2.8 ([11]). Let M be an R-module. If every submodule of M has a weak supplement, then M is called a weakly supplemented module.

Definition 2.9 ([11]). If for every $K \leq M$ with $K + N = M$, there is a supplement L of K such that $L \subseteq N$, then it is said that K has ample supplements in M.

Definition 2.10 ([11]). If every submodule of M has ample supplements in M, then M is called amply supplemented. Every amply supplemented module is supplemented.

Definition 2.11 ([4]). By a fuzzy set of a module M we mean any mapping μ from M to $[0, 1]$. By $[0, 1]^M$ we will denote the set of all fuzzy subsets of M. The support of a fuzzy set μ, denoted by μ^*, is a subset of M defined by $\mu^* = \{x \in M \mid \mu(x) > 0\}$. The subset μ_* of M is defined as $\mu_* = \{x \in M \mid \mu(x) = \mu(\theta)\}$.

Definition 2.12 ([4]). If $N \subseteq M$ and $\alpha \in [0, 1]$, then α_N is defined as,

$$\alpha_N(x) = \begin{cases} \alpha, & if\ x \in N \\ 0, & otherwise. \end{cases}$$

When $\alpha = 1$, then 1_N is known as the characteristic function of N. We will denote the characteristic function of N as χ_N.

Definition 2.13 ([4]). If $\mu, \sigma \in [0, 1]^M$, then:
(1) $\mu \subseteq \sigma$ if and only if $\mu(x) \leq \sigma(x)$;
(2) $(\mu \cup \sigma)(x) = max\{\mu(x), \sigma(x)\} = \mu(x) \vee \sigma(x)$;
(3) $(\mu \cap \sigma)(x) = min\{\mu(x), \sigma(x)\} = \mu(x) \wedge \sigma(x)$;
(4) $(\mu+\sigma)(x) = sup\{min\{\mu(a), \sigma(b)\} \mid a, b \in M,\ x = a+b\} = \vee\{\mu(a)\wedge\sigma(b) \mid a, b \in M,\ x = a + b\}$.

Definition 2.14 ([4])**.** Let M be an R-module. A fuzzy subset μ of M is said to be a fuzzy submodule, if for every $x, y \in M$ and $r \in R$ the following conditions are satisfied:
(*i*) $\mu(\theta) = 1$;
(*ii*) $\mu(x - y) \geq min\{\mu(x), \mu(y)\}$;
(*iii*) $\mu(rx) \geq \mu(x)$.

We denote the set of all fuzzy submodules of M by $F(M)$.

Definition 2.15 ([4])**.** Let $\mu, \nu \in F(M)$ such that $\mu \subseteq \nu$. Then the quotient of ν with respect to μ, is a fuzzy submodule of M/μ^*, denoted by ν/μ, and is defined as follows:

$$(\nu/\mu)([x]) = sup\{\nu(z) \mid z \in [x]\}, \ \forall x \in \nu^*$$

where $[x]$ denotes the coset $x + \mu^*$.

Proposition 2.16 ([4])**.** *If μ is a fuzzy submodule of M, then μ_* is a submodule of M.*

Proposition 2.17 ([4])**.** *Let $\mu \in F(M)$. Then $\mu_* = M$ if and only if $\mu = \chi_M$. Also if $\mu \subseteq \sigma$, then $\mu_* \subseteq \sigma_*$.*

Proposition 2.18 ([1])**.** *Let $\mu, \sigma \in F(M)$. Then $(\mu \cap \sigma)_* = \mu_* \cap \sigma_*$, $(\mu \cup \sigma)_* = \mu_* \cup \sigma_*$. Further, if μ and σ have finite images, then $(\mu + \sigma)_* = \mu_* + \sigma_*$.*

Definition 2.19 ([7])**.** Let M be an R-module and $\mu \in F(M)$. Then μ is said to be a fuzzy small submodule of M or (χ_M), if for any $\nu \in F(M)$, $\mu + \nu = \chi_M$ implies that $\nu = \chi_M$. It is indicated by the notation $\mu \ll_f M$ or $\mu \ll_f \chi_M$.

Definition 2.20 ([7])**.** Let $\mu \in F(M)$. If $\mu \ll_f M$, then we say χ_M (or M) is a fuzzy small cover of χ_M/μ or M/μ^*.

Definition 2.21 ([7])**.** Let $\mu, \sigma \in F(M)$ and $\mu \subseteq \sigma$. μ is said to be a fuzzy small submodule of σ or (σ^*), if for any fuzzy submodule γ of M, $\mu_{|_{\sigma^*}} + \gamma_{|_{\sigma^*}} = \chi_{\sigma^*}$ implies $\gamma_{|_{\sigma^*}} = \chi_{\sigma^*}$. It is indicated by the notation $\mu \ll_f \sigma$ or $\mu \ll_f \sigma^*$.

Definition 2.22 ([7])**.** Let $\mu, \sigma \in F(M)$ such that $\mu \subseteq \sigma$. Then we say σ lies above μ in M or μ is a fuzzy coessential submodule of σ, if M/μ is a fuzzy small cover of M/σ, that is; $\sigma/\mu \ll_f \chi_M/\mu$.

Proposition 2.23. *For fuzzy submodules $\mu \subseteq \nu$ of M the following properties hold:*

(1) *μ is fuzzy coessential submodule of ν in M if and only if $\mu + \sigma = \chi_M$ holds for all $\sigma \in F(M)$ with $\nu + \sigma = \chi_M$.*

(2) *$\nu \ll_f M$ if and only if $\mu \ll_f M$ and μ is fuzzy coessential submodule of ν.*

(3) *For fuzzy submodules $\gamma \subseteq \mu \subseteq \nu$ of M, γ is fuzzy coessential submodule of ν in M if and only if μ is a fuzzy coessential submodule of ν in M and γ is fuzzy coessential submodule of μ in M.*

(4) *Let $\delta \subseteq \sigma$ be fuzzy submodules of M. If μ is fuzzy coessential submodule of ν and δ is fuzzy coessential submodule of σ and $\nu + \sigma = \chi_M$, then $\mu + \delta = \chi_M$ and $\mu \cap \delta$ is fuzzy coessential submodule of $\nu \cap \sigma$.*

Proof. (1) By [7, Proposition 4.24].

(2) By [7, Proposition 4.22].

(3) Suppose μ is fuzzy coessential submodule of ν and γ is fuzzy coessential submodule of μ. Let $\sigma \in F(M)$ and $\nu + \sigma = \chi_M$. Since μ is fuzzy coessential submodule of ν so $\mu + \sigma = \chi_M$ and since γ is fuzzy coessential submodule of μ so $\gamma + \sigma = \chi_M$. Thus γ is fuzzy coessential submodule of ν in M.

Conversely, suppose γ is fuzzy coessential submodule of ν. Let $\sigma \in F(M)$ and $\mu + \sigma = \chi_M$. Then $\nu + \sigma = \chi_M$ (since $\mu \subseteq \nu$) and since γ is fuzzy coessential submodule of ν so $\gamma + \sigma = \chi_M$. Thus γ is fuzzy coessential submodule of μ.
Now, let $\sigma \in F(M)$ and $\nu + \sigma = \chi_M$. Since γ is fuzzy coessential submodule of ν, so $\gamma + \sigma = \chi_M$. Therefore, $\mu + \sigma = \chi_M$ (since $\gamma \subseteq \mu$). Thus μ is fuzzy coessential submodule of ν.

(4) Suppose $\chi_M = \nu + \sigma$. Since μ is fuzzy coessential submodule of ν, so $\chi_M = \mu + \sigma$ and since δ is fuzzy coessential submodule of σ, so $\chi_M = \mu + \delta$.

Now, $\mu \cap \sigma$ is fuzzy coessential submodule of $\nu \cap \sigma$ and $\mu \cap \delta$ is fuzzy coessential submodule of $\mu \cap \sigma$, so by (3) we have $\mu \cap \delta$ is fuzzy coessential submodule of $\nu \cap \sigma$. □

Proposition 2.24 ([7])**.** *Let $N \leq M$. Then $N \ll M$ if and only if $\chi_N \ll_f M$.*

Proposition 2.25 ([7])**.** *Let $\mu \in F(M)$. Then $\mu \ll_f M$ if and only if $\mu_* \ll M$.*

Proposition 2.26 ([7])**.** *Let $\mu, \sigma \in F(M)$. Then $\mu \ll_f \sigma$ if and only if $\mu_* \ll \sigma_*$.*

3. Fuzzy supplement submodules and fuzzy coclosed submodules

Definition 3.1. Let $\nu \in F(M)$. ν is called fuzzy coclosed in M if and only if ν has no proper fuzzy coessential submodule, that is; $\nexists \mu$, $\nu/\mu \ll_f \chi_M/\mu$.

Proposition 3.2. *If σ is fuzzy coclosed in M, then σ_* is coclosed in M.*

Proof. Let σ be a fuzzy coclosed in M. Assume K be a submodule of σ_* such that $\sigma_*/K \ll M/K$. So $\sigma/\chi_K \ll_f \chi_M/\chi_K$ and since σ is fuzzy coclosed in M , thus $\chi_K = \sigma$ and therefore $K = \sigma_*$. Hence σ_* is coclosed in M. □

Remark 3.3. Inverse of previous proposition is not essentially true.

Lemma 3.4. *Let $\mu, \sigma \in F(M)$ such that $\mu \subseteq \sigma$. If σ is fuzzy coclosed in M and $\mu \ll_f M$, then $\mu \ll_f \sigma$.*

Proof. Assume $\nu \subseteq \sigma$ and $\mu + \nu = \sigma$. Let $\delta \in F(M)$ and $\nu \subseteq \delta$ such that $\chi_M/\nu = \sigma/\nu + \delta/\nu$. Thus $\chi_M = \sigma + \delta = \mu + \nu + \delta$ and since $\mu \ll_f M$, therefore $\chi_M = \nu + \delta$. Since $\nu \subseteq \delta$, thus $\chi_M = \delta$. Hence $\sigma/\nu \ll_f \chi_M/\nu$ and since σ is fuzzy coclosed in M, thus σ has no proper fuzzy coessential submodule. So $\nu = \sigma$ and hence $\mu \ll_f \sigma$. □

Definition 3.5. Let $\mu, \sigma \in F(M)$ such that $\mu \subseteq \sigma$. μ is called fuzzy coclosure of σ in M, if μ is fuzzy coessential submodule of σ and μ has no proper fuzzy coessential submodule.

Lemma 3.6. *Let $\mu, \sigma \in F(M)$ such that $\mu \subseteq \sigma$. If $\sigma = \mu + \nu$ and $\nu \ll_f M$, then μ is fuzzy coessential submodule of σ in M.*

Proof. Suppose $\delta \in F(M)$ such that $\chi_M = \sigma + \delta$. Thus $\chi_M = \mu + \nu + \delta$ and since $\nu \ll_f M$, hence $\chi_M = \mu + \delta$. By proposition 2.23, μ is fuzzy coessential submodule of σ. □

Definition 3.7. Let $\mu, \sigma \in F(M)$. Then we call μ a fuzzy supplement of σ, if μ is minimal with respect to $\mu + \sigma = \chi_M$. Equivalently, μ is called a fuzzy supplement of σ if and only if $\mu + \sigma = \chi_M$ and $\mu \cap \sigma \ll_f \mu$.

A fuzzy submodule μ of M is called a fuzzy supplement, if there is a fuzzy submodule σ of M such that μ is a fuzzy supplement of σ.

Definition 3.8. Let $\mu, \sigma \in F(M)$. μ is called a weak fuzzy supplement of σ if and only if $\mu + \sigma = \chi_M$ and $\mu \cap \sigma \ll_f M$.

A fuzzy submodule μ of M is called a weak fuzzy supplement, if there is a fuzzy submodule σ of M such that μ is a weak fuzzy supplement of σ .

Clearly any fuzzy supplement submodule is a weak fuzzy supplement submodule.

Proposition 3.9. *Let $\mu, \nu \in F(M)$. If μ and σ have finite images, then μ is a fuzzy supplement of ν if and only if μ_* is a supplement of ν_*.*

Proof. Let μ be a fuzzy supplement of ν. So $\mu + \nu = \chi_M$ and $\mu \cap \nu \ll_f \mu$. Thus $(\mu+\nu)_* = M$ and $(\mu\cap\nu)_* \ll \mu_*$. By proposition 2.18, $\mu_*+\nu_* = M$ and $\mu_*\cap\nu_* \ll \mu_*$. Hence μ_* is a supplement of ν_*.

Conversely, assume μ_* is a supplement of ν_*. So $\mu_* + \nu_* = M$ and $\mu_* \cap \nu_* \ll \mu_*$. By proposition 2.18, $(\mu+\nu)_* = M$ and $(\mu\cap\nu)_* \ll \mu_*$. Therefore we get $(\mu+\nu)_* = M$ and $(\mu\cap\nu)_* \ll \mu_*$. Thus $\mu+\nu = \chi_M$ and $\mu\cap\nu \ll_f \mu$. Hence μ is a fuzzy supplement of ν. □

Corollary 3.10. *Let $\mu, \nu \in F(M)$. Then μ is a weak fuzzy supplement of ν if and only if μ_* is a weak supplement of ν_*.*

Definition 3.11. R-module M (or χ_M) is said to be a fuzzy hollow module, if every proper fuzzy submodule of M is fuzzy small submodule in M.

Remark 3.12. (a) Let μ be a fuzzy hollow submodule of an R-module M. If μ is not fuzzy small in M, then there exists $\nu \in F(M)$ such that $\nu \neq \chi_M$ and $\mu+\nu = \chi_M$. Since μ is fuzzy hollow, $\mu \cap \nu \ll_f \mu$ .Thus μ is fuzzy supplement in M.

(b) Let $\mu, \nu \in F(M)$, $\mu \subseteq \nu$. By 2.23, ν lies above μ if and only if $\nu + \sigma = \chi_M$ implies $\mu + \sigma = \chi_M$ for all $\sigma \in F(M)$. If ν is minimal with respect $\nu + \sigma = \chi_M$ for some σ, then there cannot be a fuzzy submodule μ of ν such that ν lies above μ. Thus ν is fuzzy coclosed.

Proposition 3.13. *Let $\mu \in F(M)$. Consider the following statement:*

(i) μ is a fuzzy supplement in M;

(ii) μ is a fuzzy coclosed in M;

(iii) For all $\sigma \subseteq \mu$, $\sigma \ll_f M$ implies $\sigma \ll_f \mu$.

Then $i \Rightarrow ii \Rightarrow iii$ holds and if μ is weak fuzzy supplement in M, then $iii \Rightarrow i$ holds.

Proof. $(i) \Rightarrow (ii)$ Assume that μ is a fuzzy supplement of $\nu \in F(M)$. For all fuzzy submodules $\sigma \subseteq \mu$ such that μ lies above σ, we have that $\mu + \nu = \chi_M$ implies

$\sigma + \nu = \chi_M$. By the minimality of μ with respect to this property, we get $\mu = \sigma$. Hence μ is fuzzy coclosed.

$(ii) \Rightarrow (iii)$ Let $\sigma \ll_f M$ and $\sigma \subseteq \mu$. Assume $\mu = \sigma + \nu$ for $\nu \subseteq \mu$. Then for every $\delta \in F(M)$ with $\mu + \delta = \chi_M$, we get $\sigma + \nu + \delta = \chi_M$. Since $\sigma \ll_f M$, so $\nu + \delta = \chi_M$. Thus ν is fuzzy coessential of μ. By the coclosure of μ, we get $\nu = \mu$. Thus $\sigma \ll_f \mu$.

$(iii) \Rightarrow (i)$ Assume μ is a weak fuzzy supplement of $\nu \in F(M)$, so $\mu \cap \nu \ll_f M$. By assumption, $\mu \cap \nu \ll_f \mu$. Thus μ is a supplement of ν in M. □

Corollary 3.14. *Any fuzzy direct summand of M is fuzzy coclosed in M.*

Proposition 3.15. *Let $\mu, \sigma \in F(M)$ such that $\mu \subseteq \sigma$.*

(i) *If σ is fuzzy coclosed in M, then σ/μ is fuzzy coclosed in χ_M/μ.*

(ii) *Assume σ is fuzzy supplement in M. Then μ is fuzzy coclosed in σ if and only if μ is fuzzy coclosed in M.*

Proof. (i) Since σ is fuzzy coclosed in M, for every proper fuzzy submodule δ/μ of σ/μ, $(\sigma/\mu)/(\delta/\mu) \simeq \sigma/\delta$ is not fuzzy small in $(\chi_M/\mu)/(\delta/\mu) \simeq \chi_M/\delta$ (since σ/δ is not small in χ_M/δ). So σ/μ is fuzzy coclosed in χ_M/μ.

(ii) Let σ be fuzzy supplement of $\delta \subseteq \chi_M$. Assume μ is a fuzzy coclosed in M. Since whenever $\mu/\nu \ll_f \sigma/\nu$, we get $\mu/\nu \ll_f \chi_M/\nu$ as $\sigma/\nu \subseteq \chi_M/\nu$, so μ is fuzzy coclosed in σ.

Conversely, assume that μ is fuzzy coclosed in σ and that μ lies above a proper fuzzy submodule $\gamma \subset \mu$ in M. Since μ is fuzzy coclosed in σ and μ does not lie above γ in σ, hence there exists a proper fuzzy submodule λ of σ containing γ such that $\mu/\gamma + \lambda/\gamma = \sigma/\gamma$ holds. Hence $\chi_M = \sigma + \delta = \mu + \lambda + \delta$ implies $\chi_M = \gamma + \lambda + \delta = \lambda + \delta$ (since μ lies above γ in M). But since σ is a fuzzy supplement of δ in M, we get $\lambda = \sigma$, a contradiction to λ being proper fuzzy submodule of σ. Hence μ is fuzzy coclosed in M. □

Definition 3.16. R-module M (or χ_M) is called fuzzy supplemented, if every fuzzy submodule has a fuzzy supplement in M.

Definition 3.17. R-module M (or χ_M) is called weakly fuzzy supplemented, if every fuzzy submodule has a weak fuzzy supplement in M.

Lemma 3.18. *Any fuzzy supplemented module is weakly fuzzy supplemented.*

Proof. Suppose M be fuzzy supplemented and $\mu \in F(M)$. So there is $\nu \in F(M)$ such that $\mu + \nu = \chi_M$ and $\mu \cap \nu \ll_f \nu$. Thus $\mu \cap \nu \ll_f M$, so ν is a weak fuzzy supplement of μ. Hence M is weakly fuzzy supplemented. □

Proposition 3.19. *Let fuzzy submodules of R-module M have finite images. Then R-module M is fuzzy supplemented if and only if M is supplemented module.*

Proof. First let M be fuzzy supplemented. Let $N \leq M$ and μ be the characteristic function on N. So $\mu \in F(M)$ and $\mu_* = N$. Since M is fuzzy supplemented, so there is $\nu \in F(M)$ such that $\mu + \nu = \chi_M$ and $\mu \cap \nu \ll_f M$. Hence $(\mu + \nu)_* = M$ and $(\mu \cap \nu)_* \ll M$. By proposition 2.18, $\mu_* + \nu_* = M$ and $\mu_* \cap \nu_* \ll M$ which ν_* is submodule of M. Hence ν_* is a supplement of N. This implies M is supplemented module.

Conversely, assume M is supplemented module. Let $\mu \in F(M)$, so $\mu_* \leq M$. Since M is supplemented, thus there is $K \leq M$ such that $\mu_* + K = M$ and $\mu_* \cap K \ll M$. Suppose ν is the characteristic function on K, so $\nu \in F(M)$ and $\nu_* = K$. Therefore $\mu_* + \nu_* = M$ and $\mu_* \cap \nu_* \ll M$. So by proposition 2.18 , $(\mu + \nu)_* = M$ and $(\mu \cap \nu)_* \ll M$. Thus $\mu + \nu = \chi_M$ and $\mu \cap \nu \ll_f M$. Hence ν is fuzzy supplement of μ and so M is fuzzy supplemented module. □

Corollary 3.20. *R-module M is weakly fuzzy supplemented if and only if M is weakly supplemented module.*

Example 3.21. Semi-simple module is fuzzy supplemented.

Example 3.22. The Z-module Q is weakly fuzzy supplemented.

Proposition 3.23. *Assume that M is weakly fuzzy supplemented and $\mu, \sigma \in F(M)$ such that $\chi_M = \mu + \sigma$. Then μ has a weak fuzzy supplement as ν in M such that $\nu \subseteq \sigma$.*

Proof. Let $\chi_M = \mu + \sigma$. Since M is weakly fuzzy supplemented, so there is $\delta \in F(M)$ such that $\chi_M = (\mu \cap \sigma) + \delta$ and $(\mu \cap \sigma) \cap \delta \ll_f M$. Thus $\sigma = \chi_M \cap \sigma = ((\mu \cap \sigma) + \delta) \cap \sigma = (\mu \cap \sigma) + (\delta \cap \sigma)$. So $\chi_M = \mu + (\mu \cap \sigma) + (\delta \cap \sigma)$ and $\mu \cap (\delta \cap \sigma) \ll_f M$. Since $\mu \cap \sigma \subseteq \mu$, so $\chi_M = \mu + (\delta \cap \sigma)$. Hence $\delta \cap \sigma$ is a weak fuzzy supplement of μ such that $\delta \cap \sigma \subseteq \sigma$. □

Proposition 3.24. *Let M be weakly fuzzy supplemented and $\mu \in F(M)$. Then the following conditions are equivalent:*
(i) μ is fuzzy supplement;
(ii) μ is fuzzy coclosed.

Proof. (i⇒ii) By proposition 3.13.

(ii⇒i) Let μ be fuzzy coclosed in M. Since M is weakly fuzzy supplement, so μ has a weak supplement as σ such that $\chi_M = \mu + \sigma$ and $\mu \cap \sigma \ll_f M$. By lemma 3.4, $\mu \cap \sigma \ll_f \mu$. Thus μ is fuzzy supplement of σ in M. Therefore μ is fuzzy supplement submodule in M. □

4. Amply fuzzy supplemented modules

Definition 4.1. If for every $\nu \in F(M)$ with $\mu + \nu = \chi_M$ there is a supplement σ of μ such that $\sigma \subseteq \nu$, then it is said that μ has ample fuzzy supplement in M.

Definition 4.2. If every fuzzy submodule of M has a ample fuzzy supplement in M, then M (or χ_M) is called amply fuzzy supplemented.

Every amply fuzzy supplemented module is fuzzy supplemented.

Proposition 4.3. *Let fuzzy submodules of R-module M have finite images. If R-module M is amply fuzzy supplemented, then M is amply supplemented.*

Proof. Let M be amply fuzzy supplemented. let $N \leq M$ with $N + K = M$. Suppose μ and ν are the characteristic function on N and K. So $\mu, \nu \in F(M)$ and $\mu_* = N$, $\nu_* = K$. Thus $\mu_* + \nu_* = M$. By proposition 2.18, $\mu_* + \nu_* = (\mu + \nu)_* = M$, so $\mu + \nu = \chi_M$. Since M is amply fuzzy supplemented, so there is a fuzzy supplement σ of μ such that $\sigma \subseteq \nu$. So by proposition 3.9, σ_* is a supplement of μ_* such that

$\sigma_* \subseteq \nu_*$. Since $\sigma \in F(M)$, so $\sigma_* = L \leq M$ and hence L is a supplement of N such that $L \subseteq K$. Therefore M is amply supplemented. □

Proposition 4.4. *An R-module M is amply fuzzy supplemented if and only if M is weakly fuzzy supplemented and any fuzzy submodule of M has a fuzzy coclosure in M.*

Proof. Let M be amply fuzzy supplemented, so M is weakly fuzzy supplemented. Let $\mu \in F(M)$, thus μ has a fuzzy supplement as σ. So $\mu + \sigma = \chi_M$ and $\mu \cap \sigma \ll_f \sigma$, this implies $\mu \cap \sigma \ll_f M$. Since M is amply fuzzy supplemented, so there is a fuzzy supplement ν of σ such that $\nu \subseteq \mu$. Thus $\nu + \sigma = \chi_M$ and $\nu \cap \sigma \ll_f \nu$. We will prove ν is a fuzzy coclosure of μ in M. ν is fuzzy coclosed in M because ν is a fuzzy supplement submodule in M. Since $\mu = \mu \cap \chi_M = \mu \cap (\nu + \sigma) = (\mu \cap \nu) + (\mu \cap \sigma) = \nu + (\mu \cap \sigma)$ and $\mu \cap \sigma \ll_f M$, by lemma 3.18, ν is a fuzzy coessential submodule of μ in M. Therefore ν is fuzzy coclosure of μ in M.

Conversely, let M is weakly fuzzy supplemented and any fuzzy submodule of M has a fuzzy coclosure in M. Let $\mu, \sigma \in F(M)$ and $\chi_M = \mu + \sigma$. Since M is weakly fuzzy supplemented and by proposition 2.18, there is a weak fuzzy supplement ν of μ such that $\nu \subseteq \sigma$. Thus $\chi_M = \mu + \nu$ and $\mu \cap \nu \ll_f M$. Assume δ is a fuzzy coclosure of ν in M. So by proposition 2.23, $\chi_M = \mu + \delta$. Since δ is a fuzzy coclosed in M and $\delta \cap \mu \ll_f M$, so by lemma 3.4, $\delta \cap \mu \ll_f \delta$. This implies δ is a fuzzy supplement of μ in M such that $\delta \subseteq \sigma$. Hence M is amply fuzzy supplemented. □

Proposition 4.5. *Let M is amply fuzzy supplemented module. Then,*

(i) Any fuzzy supplement of a fuzzy submodule of M is a amply fuzzy supplemented module.

(ii) Any fuzzy direct summand of M is a amply fuzzy supplemented module.

Proof. (i) Let $\mu \in F(M)$ and σ is a fuzzy supplement of μ such that for fuzzy submodules ν and δ of σ, $\sigma = \delta + \nu$. Since σ is a fuzzy supplement of μ, so $\chi_M = \mu + \sigma = \mu + \delta + \nu$ and since M is amply fuzzy supplemented, thus there is a fuzzy supplement γ of $\mu + \delta$ such that $\gamma \subseteq \nu$. Also, $\delta \subseteq \mu + \delta$, so $\delta \cap \gamma \subseteq (\mu + \delta) \cap \gamma$. Since γ is a fuzzy supplement of $\delta + \mu$, thus $\delta \cap \gamma \subseteq (\mu + \delta) \cap \gamma \ll_f \gamma$. Hence $\chi_M = \mu + \delta + \gamma$ and $\delta \cap \gamma \ll_f \gamma$. Now, by the minimality of σ with respect to this property, we get $\sigma \subseteq \delta + \gamma$. Since $\delta \subseteq \sigma$ and $\gamma \subseteq \nu \subseteq \sigma$, so $\delta + \gamma \subseteq \sigma$. Hence $\sigma = \delta + \gamma$, this implies γ is a fuzzy supplement of δ in σ and so σ is amply fuzzy supplemented.

(ii) Since any fuzzy direct summand is a fuzzy supplement, so by (i), any fuzzy direct summand is a amply fuzzy supplemented module. □

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Yahya Talebi (talebi@umz.ac.ir)
Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran

Atefahe Darzi (a.darzi@stu.umz.ac.ir)
Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran