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Application of intuitionistic fuzzy mathematical programming with exponential membership and quadratic non-membership functions in matrix games

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Abstract. The aim of this paper is to develop a new concept of optimization technique using Atanassov's intuitionistic fuzzy sets(IFS), to solve two-person zero sum matrix games in which each player has a intuitionistic fuzzy goal for each payoff. In this methodology, the solution concept for such games is defined by applying intuitionistic fuzzy mathematical programming with exponential membership and quadratic non-membership functions. It is noted that each matrix game with goals represented by Atanassov's IFS has a solution, which can be obtained through solving a pair of auxiliary non-linear programming models for two players. It is shown that the solution obtained from this methodology is better than the solution obtained from crisp equivalent problem. In addition, numerical example is also provided to illustrate the methodology.

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1. Introduction

Fuzziness in game theory has been investigated and applied by various researchers ([6], [18], [21], [28], [14]). However in many cases they do not represent exactly the real problems. In practical situation, due to insufficiency in the information available, it is not easy to describe the constraint conditions by ordinary fuzzy sets and the evaluation of membership values is not always possible up to decision makers(DM)'s satisfaction, consequently there remains an indeterministic part of which hesitation survives. The fuzzy set uses only a membership function to indicate the degree of belongingness to the fuzzy set under consideration. The degree of non-belongingness is just automatically the compliment of 1. However, a human being who expresses the degree of membership of a given element in a fuzzy set very often does not express corresponding degree of non-membership as the complement to 1. Sometimes it seems to be more natural to describe imprecise and uncertain opinions not only by membership functions. However, in some situations players also describe their negative feelings, i.e., degrees of dissatisfaction about the outcomes of the game. On the other hand, the players can only estimate their aspiration levels (goals) and/or their values with some imprecision. But it is possible that he/she is not so sure about it. In other words, there may be hesitation about the approximate payoff values. It is therefore most likely that the players have some indeterminacy or hesitation about these approximations. Fuzzy set theory is thus not enough to model the matrix game problems involving indeterminacy in aspiration levels of the players. Therefore it is reasonable to believe that there is some indeterminacy in estimating the aspiration levels. In such situation intuitionistic fuzzy(IF) set, introduced by Atanassov [4, 5], serve better our required purpose. IF set is characterized by two functions expressing the degree of membership and the degree of non-membership, respectively. The hesitation degree is equal to 1 minus both the degree of membership and the degree of non-membership. The IF set may express and describe information more abundant and flexible than the fuzzy sets when uncertain information is involved. The IF set has been applied to different areas such as decision making problem [29], medical diagnosis ([7],[9]) multi-attribute decision making problems [13] etc.

Intuitionistic fuzziness in matrix games can appear in so many ways, but two cases are seemed to be very natural. First one is the goal may be IF and other is the elements of the pay-off matrix are IF numbers ([12], [25], [22]). These two classes of fuzzy matrix games are referred as matrix games with IF goals and matrix games with IF pay-offs. Li and Nan [11] described a non-linear programming approach for the matrix games with pay-offs of IF sets. Nan and Li [16] discussed a method for solving matrix games with pay-offs of triangular IF numbers. Li et al. [15] implemented a bi-objective programming approach to solve matrix games with pay-offs of IF numbers. Seikh et al. [24] applied IF numbers to bi-matrix games. Aggarwal et al. [1, 2] extended some results in matrix games with fuzzy goals and fuzzy pay-offs [6] to IF scenario. Nan and Li [17] formulated a linear programming approach to matrix games with IF goals. Seikh et al.[26] investigated matrix games in intuitionistic fuzzy environment based on aspiration level approach.

In this paper, intuitionistic fuzzy optimization(IFO) is used to solve matrix games in the sense of degree of attainment of IF goals. The advantage of the IFO technique is that it gives the richest apparatus for formulation of optimization problems because this method can consider together the degree of acceptance and the degree of rejection. We assume that each player has a IF goal for the choice of the strategy in order to incorporate ambiguity of human judgement. We assume that, DMs want to optimize the degree of attainment of the IF goal. It is important to note that when implementing a matrix game to IF linear programming formulations, unique membership function and non-membership function are required to define. The degree of membership of a solution may be defined as the degree of acceptance and the degree of non-membership of a solution as the degree of rejection. The sum of degrees of acceptance and rejection is considered as less than or equal to 1.

Several membership and non-membership functions have been employed in optimization techniques such as: linear, piece-wise linear, exponential, hyperbolic, logistic, parabolic, S-shaped, etc. Linear membership and non-membership functions are most commonly used ([6], [21]) because it is defined by fixing two points (upper and lower levels) of acceptability and rejectability. However, a linear membership function is not a suitable representation in many practical situations. The non-linear membership and non-membership functions are used more frequently ([8], [10], [27]) that provide a better representation. Furthermore, if the membership and non-membership functions are interpreted as the IF utility of decision maker used for describing levels of indifference, preference or aversion towards uncertainty, then non-linear membership and non-membership functions provide a better representation. For this reason, in this paper, we choose exponential membership function and quadratic non-membership function to establish IF environment.

The paper is organized as follows: In Section 2, some basic definitions and notations on IFS are recalled. In Section 3, the solution procedure of the matrix game on the basis of defining IFO model is described. In Section 4, numerical example is provided for illustration.

2. Preliminaries

2.1. **Intuitionistic fuzzy sets.** The IFS introduced by Atanassov [4, 5] is characterized by two functions expressing the degree of belongingness and the degree of non-belongingness respectively.

Definition 2.1. Let $U = \{x_1, x_2, \cdots, x_n\}$ be a finite universal set. An IFS $\widetilde{A}$ in a given universal set U is an object having the form

$$\widetilde{A} = \Big\{\langle x, \mu_{\widetilde{A}}(x), \nu_{\widetilde{A}}(x)\rangle \mid x \in U\Big\} \tag{2.1}$$

where the functions

$$\mu_{\widetilde{A}} : U \to [0,1] \quad \text{and} \quad \nu_{\widetilde{A}} : U \to [0,1]$$

define the degree of membership and the degree of non-membership of an element $x \in U$ to the set $A \subseteq U$, respectively, such that they satisfy the following condition:

$$0 \leq \mu_{\widetilde{A}}(x) + \nu_{\widetilde{A}}(x) \leq 1, \forall x \in U$$

which is known as *IF condition.* The degree of acceptance $\mu_{\widetilde{A}}(x)$ and of non-acceptance $\nu_{\widetilde{A}}(x)$ can be arbitrary. The set of all IFSs over U is defined by IFS(U).

Definition 2.2. Let $\widetilde{A}$ and $\widetilde{B}$ be two IFSs in the set U. The union and intersection of $\widetilde{A}$ and $\widetilde{B}$, Atanassov [4, 5], are defined as

$$\left.\begin{aligned} A \cup B &= \Big\{\langle x, \max(\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)), \min(\nu_{\widetilde{A}}(x), \nu_{\widetilde{B}}(x))\rangle | x \in U\Big\} \\ A \cap B &= \Big\{\langle x, \min(\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(x)), \max(\nu_{\widetilde{A}}(x), \nu_{\widetilde{B}}(x))\rangle | x \in U\Big\} \end{aligned}\right\}. \tag{2.2}$$

Note: From the above definitions we see that the numbers $\mu_{\widetilde{A}}(x)$ and $\nu_{\widetilde{A}}(x)$ reflect respectively the extent of acceptance and the degree of rejection of the element x to the set $\widetilde{A}$, and the value $\pi_{\widetilde{A}}(x) = 1 - \mu_{\widetilde{A}}(x) - \nu_{\widetilde{A}}(x)$ is the extent of indeterminacy between both.

2.2. **Intuitionistic fuzzy optimization model.** IFO, a method of uncertainty optimization, is put forward on the basis of fuzzy set. It is an extension of fuzzy optimization in which the degrees of rejection of objective(s) and constraints are considered together with the degrees of satisfaction ([3], [19]). According to IFO theory, we are to maximize the degree of acceptance of the IF objective(s) and constraints and to minimize the degree of rejection of IF objective(s) and constraints as follows:

$$\left.\begin{array}{c} \max\limits_{x\in\Re^n}\{\mu_k(x)\}; \\ \min\limits_{x\in\Re^n}\{\nu_k(x)\}; \\ \mu_k(x), \nu_k(x) \geq 0; \\ \mu_k(x) \geq \nu_k(x); \\ 0 \leq \mu_k(x) + \nu_k(x) \leq 1; \end{array}\right\} k = 1, 2, \ldots, n$$

where $\mu_k(x)$ denotes the degree of acceptance and $\nu_k(x)$ denotes the degree of rejection of x from the k^{th} IFS. The formula can be transformed to the following system

$$\begin{array}{cl} \max\xi, \quad \min\eta & \\ \xi \leq \mu_k(x); & k = 1, 2, \ldots, n \\ \eta \geq \nu_k(x); & k = 1, 2, \ldots, n \\ \xi \geq \eta; \quad \xi + \eta \leq 1; \quad \xi, \eta \geq 0 & \end{array}$$

where ξ denotes the minimal acceptable degree of objective(s) and constraints and η denotes the maximal degree of rejection of objective(s) and constraints. The IFO model can be changed into the following crisp optimization model as:

$$\left.\begin{array}{cl} \max\{\xi - \eta\} & \\ \xi \leq \mu_k(x); & k = 1, 2, \ldots, n \\ \eta \geq \nu_k(x); & k = 1, 2, \ldots, n \\ \xi \geq \eta; \quad \xi + \eta \leq 1; \quad \xi, \eta \geq 0 & \end{array}\right\} \tag{2.3}$$

which can be easily solved by various mathematical programming.

3. Mathematical model of a matrix game

Let $i \in \{1, 2, \ldots, m\}$ be a pure strategy available for player I and $j \in \{1, 2, \ldots, n\}$ be a pure strategy available for player II. When player I chooses a pure strategy i and the player II chooses a pure strategy j, then a_{ij} is a payoff for player I and $-a_{ij}$ be a payoff for player II. The two person zero sum matrix game G can be represented by the pay-off matrix

$$A = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}.$$

3.1. **Mixed strategy.** Consider the game G with no saddle point, i.e. $\max_i\{\min_j a_{ij}\} \neq \min_j\{\max_i a_{ij}\}$. To solve such game, Neumann and Morgenstern [20] introduced the concept of mixed strategy in classical form. We denote the sets of all mixed strategies, called strategy spaces, available for players I, II by

$$S_I = \Big\{\mathbf{x} = (x_1, x_2, \cdots, x_m) \in \Re^m_+ | x_i \geq 0; i = 1, 2, \cdots, m \text{ and } \sum_{i=1}^m x_i = 1\Big\}$$

$$S_{II} = \Big\{\mathbf{y} = (y_1, y_2, \cdots, y_n) \in \Re^n_+ | y_j \geq 0; j = 1, 2, \cdots, n \text{ and } \sum_{j=1}^n y_j = 1\Big\},$$

where $\Re^m_+$ denotes the $m-$dimensional non-negative Euclidean space. Thus by a crisp two person zero sum game G we mean the triplet $G = (S_I, S_{II}, A)$. Since the player is uncertain about what strategy he/she will choose, he/she will choose a probability distribution over the set of alternatives available to him/her or a mixed strategy in terms of game theory.

Definition 3.1. (Expected payoff) If the mixed strategies $\mathbf{x}$ and $\mathbf{y}$ are proposed by the player I and player II respectively, then the expected pay-off of the player I when the player II uses the strategy $\mathbf{y}$ is defined by

$$E(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A\mathbf{y} = \sum_{i=1}^m \sum_{j=1}^n x_i a_{ij} y_j. \tag{3.1}$$

A pair of mixed strategies $(\mathbf{x}^*, \mathbf{y}^*)$ is said to be optimal if and only if

$$E(\mathbf{x}^*, \mathbf{y}) \geq E(\mathbf{x}^*, \mathbf{y}^*) \geq E(\mathbf{x}, \mathbf{y}^*) \text{ for all } \mathbf{x} \in S_I \text{ and } \mathbf{y} \in S_{II}.$$

Also $v^* = E(\mathbf{x}^*, \mathbf{y}^*) = {\mathbf{x}^*}^T A\mathbf{y}^*$ is known as the value of the game. The existing theory of crisp games has certain limitations because of uncertainties and ambiguous communication. The purpose of this paper is to obviate such difficulties. Now, we define the meaning of IF goal and try to explain how the players will play the game in IF environment.

3.2. **Matrix game with IF goal.** The IF goal models are described on the basis of maxmin and minmax principles of crisp matrix game theory. First, we define some terms which are useful in the solution procedure. Let the domain for the player I be defined by

$D = \Big\{\mathbf{x}^T A\mathbf{y} | (\mathbf{x}, \mathbf{y}) \in S_I \times S_{II} \subset \Re^m \times \Re^n\Big\} \subseteq \Re.$

Definition 3.2. (IF goal) An IF goal $\hat{G}_1$ for player I is defined as an IFS on D characterized by the membership and nonmembership functions

$$\mu_{\hat{G}_1} : D \to [0,1] \text{ and } \nu_{\hat{G}_1} : D \to [0,1]$$
$$\text{or simply, } \mu_1 : D \to [0,1] \text{ and } \nu_1 : D \to [0,1]$$

such that $0 \leq \mu_1(x) + \nu_1(x) \leq 1$. Similarly, an IF goal for player II is an IFS on D characterized by the membership function $\mu_{\hat{G}_2} : D \to [0,1]$ and nonmembership function $\nu_{\hat{G}_2} : D \to [0,1]$ such that $0 \leq \mu_{\hat{G}_2}(x) + \nu_{\hat{G}_2}(x) \leq 1$.

The values of membership and non-membership functions for an IF goal can be interpreted as the degree of attainment [23] of the IF goal for a strategy of a payoff. According to property of IFS (seen in section 2.1 in this paper), the intersection of IF objective(s) and constraints is defined as

$$C = A \cap B = \Big\{ \langle x, \mu_C(x), \nu_C(x) \rangle | x \in X \Big\}$$
$$\text{where } \mu_C(x) = \min[\mu_A(x), \mu_B(x)]; \;\; \nu_C(x) = \max[\nu_A(x), \nu_B(x)]$$

where A denotes integrated IF objective and B denotes integrated IF constraint set. $\mu_C(x)$ denotes the degree of acceptance of IF decision set and $\nu_C(x)$ denotes the degree of rejection of IF decision set.

Definition 3.3. (Degree of attainment of IF goal) For any pair of mixed strategies $(\mathbf{x}, \mathbf{y}) \in S_I \times S_{II}$, the degree of attainment of the IF goal for player I is defined by the membership and non-membership functions as

$$\max_{\mathbf{x} \in S_I} \{\mu_1(\mathbf{x}, \mathbf{y})\} \text{ and } \min_{\mathbf{x} \in S_I} \{\nu_1(\mathbf{x}, \mathbf{y})\}. \tag{3.2}$$

The degree of attainment of the IF goal can be considered to be a concept of a degree of satisfaction and the degree of rejection, when the IF constraint can be replaced by expected pay-off. Let player I supposes that, player II will choose a strategy $\mathbf{y}$ so as to minimize player I's membership function μ_1 and non-membership function ν_1. Let us assume that, a player has no information about his opponent or the information is not useful for the decision making if he/she has. Here player I chooses a strategy so as to maximize the membership function μ_1 and minimize the non-membership function ν_1 of the IF goal. Similar for player II. Thus, when a player has two different strategies, he/she prefers the strategy possessing the higher membership function value and lower non-membership function value in comparison to the other.

Definition 3.4. (Maxmin Value) For any pair of mixed strategies $(\mathbf{x}, \mathbf{y}) \in S_I \times S_{II}$, the maxmin value with respect to a degree of attainment of the IF goal for player I is defined as

$$\max_{\mathbf{x} \in S_I} \min_{\mathbf{y} \in S_{II}} \mu_1(\mathbf{x}^T A \mathbf{y}) \text{ and } \min_{\mathbf{x} \in S_I} \max_{\mathbf{y} \in S_{II}} \nu_1(\mathbf{x}^T A \mathbf{y}). \tag{3.3}$$

Similarly, the minmax value with respect to the degree of attainment of the IF goal for player II is defined as

$$\max_{\mathbf{y} \in S_{II}} \min_{\mathbf{x} \in S_I} \mu_2(\mathbf{x}^T A \mathbf{y}) \text{ and } \max_{\mathbf{x} \in S_I} \min_{\mathbf{y} \in S_{II}} \nu_2(\mathbf{x}^T A \mathbf{y}). \tag{3.4}$$

Thus the player I wishes to determine $\mathbf{x}^* \in S_I$ (respectively $\mathbf{y}^* \in S_{II}$) such that the maxmin value with respect to the degree of attainment of the IF goal for player I is attained. Similarly, for player II. For this, we assume exponential membership function and quadratic non-membership functions $\{\mu_k(\mathbf{x}^T A\mathbf{y}), \nu_k(\mathbf{x}^T A\mathbf{y}); k = 1, 2\}$, for player I and player II respectively.

We now analyze the optimization problems for player I and player II so as to obtain a solution of the given matrix game with respect to the degree of attainment of the IF goal. By using the above definitions for the IF game, we construct the following IF programming problem for player I and II respectively.

3.3. **Optimization problem for player I.** We construct the membership function $\mu_1(\mathbf{x}^T A\mathbf{y})$ and non-membership function $\nu_1(\mathbf{x}^T A\mathbf{y})$ of the IF goal for the player I as (depicted in Fig.1):

$$\mu_1(\mathbf{x}^T A\mathbf{y}) = \begin{cases} 1; & \mathbf{x}^T A\mathbf{y} \geq \overline{a} \\ \frac{e^{-\alpha\left(\frac{\mathbf{x}^T A\mathbf{y} - \underline{a}}{\overline{a} - \underline{a}}\right)} - 1}{e^{-\alpha} - 1}; & \underline{a} < \mathbf{x}^T A\mathbf{y} < \overline{a} \\ 0; & \mathbf{x}^T A\mathbf{y} \leq \underline{a} \end{cases}$$

$$\nu_1(\mathbf{x}^T A\mathbf{y}) = \begin{cases} 1; & \mathbf{x}^T A\mathbf{y} \leq \underline{a} \\ \left(\frac{\overline{a} - \mathbf{x}^T A\mathbf{y}}{\overline{a} - \underline{a}}\right)^2; & \underline{a} < \mathbf{x}^T A\mathbf{y} < \overline{a} \\ 0; & \mathbf{x}^T A\mathbf{y} \geq \overline{a} \end{cases}$$

where α is a parameter that measure the degree of vagueness and is called the shape parameter.

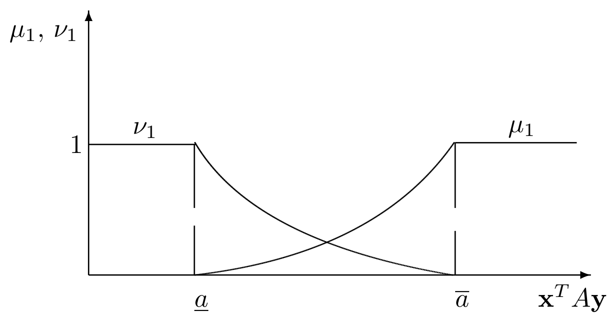

FIGURE 1. Optimization problem for player I

Also $\underline{a}$ and $\overline{a}$ are the tolerances of the expected pay-off $\mathbf{x}^T A\mathbf{y}$ and $\mu_1(\mathbf{x}^T A\mathbf{y})$ should be determined in objective allowable region $[\underline{a}, \overline{a}]$. For player I, $\underline{a}$ and $\overline{a}$ are the pay-offs giving the worst and the best degree of satisfaction respectively. Although $\underline{a}$ and $\overline{a}$ would be any scalars with $\overline{a} > \underline{a}$, Nishizaki and Sakawa [21] suggested that, parameters $\underline{a}$ and $\overline{a}$ can be taken as

$$\overline{a} = \max_x \max_y \mathbf{x}^T A\mathbf{y} = \max_i \max_j a_{ij}$$

$$\underline{a} = \min_x \min_y \mathbf{x}^T A\mathbf{y} = \min_i \min_j a_{ij}.$$

Thus player I is not satisfied by the pay-off less than $\underline{a}$ but is fully satisfied by the pay-off greater than $\underline{a}$. Thus, the conditions $\underline{a} \leq \min_{i,j} a_{ij}$ and $\overline{a} \geq \max_{i,j} a_{ij}$ hold.

For these membership and non-membership functions of the IF goal for player I from (2.3) a maxmin solution with respect to the degree of attainment of the aggremented IF goal can be obtained by solving the following crisp mathematical problem:

$$(3.5)\qquad \left.\begin{array}{c} \max\{\xi_1-\eta_1\} \\ \text{S.T.}\ \ \xi_1 \le \dfrac{e^{-\alpha\left(\frac{\mathbf{x}^TA\mathbf{y}-\underline{a}}{\overline{a}-\underline{a}}\right)}-1}{e^{-\alpha}-1}; \\ \eta_1 \ge \left(\dfrac{\overline{a}-\mathbf{x}^TA\mathbf{y}}{\overline{a}-\underline{a}}\right)^2; \\ e^T\mathbf{x}=1;\ \ \xi_1\ge\eta_1;\ \ \xi_1+\eta_1\le 1;\ \ \mathbf{x},\xi_1,\eta_1\ge 0, \end{array}\right\}$$

where ξ_1 and η_1 denote respectively the minimal acceptance degree and the maximal rejection degree of constraints fixed by the player I. Also $e=(1,1,\cdots,1)^T$, is an unit vector in $\Re^m_+$.

From (3.5) we see that the constraints are separable in the decision variable $\mathbf{x}$. Thus the model can be changed into the following optimization model.

$$(3.6)\qquad \left.\begin{array}{c} \max\{\xi_1-\eta_1\} \\ \text{S.T.}\ \ \dfrac{1}{e^{-\alpha}-1}e^{-\alpha\left[\sum\limits_{i=1}^{m}\frac{a_{ij}}{\overline{a}-\underline{a}}x_i-\frac{\underline{a}}{\overline{a}-\underline{a}}\right]}-1\ge\xi_1;\ \ j=1,2,\ldots,n \\ \left[\sum\limits_{i=1}^{m}\dfrac{a_{ij}}{\overline{a}-\underline{a}}x_i-\dfrac{\overline{a}}{\overline{a}-\underline{a}}\right]^2\le\eta_1;\ \ j=1,2,\ldots,n \\ e^T\mathbf{x}=1;\ \ \xi_1\ge\eta_1;\ \ \xi_1+\eta_1\le 1;\ \ \mathbf{x},\xi_1,\eta_1\ge 0 \end{array}\right\}$$

For the choice of exponential membership and quadratic non-membership functions of the IF goals, solution of (3.6) is equal to degree of attainment of the IF goal for the matrix game. Since S_I is convex and the membership and non-membership functions both are continuous within $[\underline{a},\overline{a}]$, quite naturally these two functions meet at a point somewhere in $[\underline{a},\overline{a}]$. Therefore, the existence of solution of the game is guaranteed from the equation (3.6). Also if $(\mathbf{x}^*,\xi_1^*,\eta_1^*)$ is an optimal solution of (3.6) then $\mathbf{x}^*$ is an optimal strategy for player I and $(\xi_1^*-\eta_1^*)$ is the degree of attainment to which the aspiration level of player I can be met by choosing to play the strategy $\mathbf{x}^*$.

3.4. **Optimization problem for player II.** Similarly, for player II, we consider the membership function $\mu_2(\mathbf{x}^TA\mathbf{y})$ and non-membership function $\nu_2(\mathbf{x}^TA\mathbf{y})$ of the IF goal as:

$$\mu_2(\mathbf{x}^TA\mathbf{y})=\begin{cases} 0; & \mathbf{x}^TA\mathbf{y}\le\underline{a} \\ \dfrac{e^{-\beta\left(\frac{\overline{a}-\mathbf{x}^TA\mathbf{y}}{\overline{a}-\underline{a}}\right)}-e^{-\beta}}{1-e^{-\beta}}; & \underline{a}<\mathbf{x}^TA\mathbf{y}<\overline{a} \\ 1; & \mathbf{x}^TA\mathbf{y}\ge\overline{a} \end{cases}$$

$$\nu_2(\mathbf{x}^TA\mathbf{y})=\begin{cases} 0; & \mathbf{x}^TA\mathbf{y}\ge\overline{a} \\ \left(\frac{\mathbf{x}^TA\mathbf{y}-\underline{a}}{\overline{a}-\underline{a}}\right)^2; & \underline{a}<\mathbf{x}^TA\mathbf{y}<\overline{a} \\ 1; & \mathbf{x}^TA\mathbf{y}\le\overline{a}, \end{cases}$$

where β is the shape parameter that measures the degree of vagueness. Also $\underline{a}$ and $\overline{a}$ are the tolerances of the expected pay-off $\mathbf{x}^T A\mathbf{y}$ and $\mu_2(\mathbf{x}^T A\mathbf{y})$ should be determined in objective allowable region $[\underline{a}, \overline{a}]$.

As in case of player I, we have the following certain mathematical problem as

$$(3.7)\quad \left.\begin{array}{c} \max\{\xi_2 - \eta_2\} \\ \text{S.T.}\;\; \xi_2 \geq \dfrac{e^{-\beta\left(\dfrac{\overline{a} - \mathbf{x}^T A\mathbf{y}}{\overline{a} - \underline{a}}\right)} - e^{-\beta}}{1 - e^{-\beta}}; \\ \eta_2 \leq \left(\dfrac{\mathbf{x}^T A\mathbf{y} - \underline{a}}{\overline{a} - \underline{a}}\right)^2; \\ e^T\mathbf{y} = 1;\;\; \xi_2 \geq \eta_2;\;\; \xi_2 + \eta_2 \leq 1;\;\; \mathbf{y}, \xi_2, \eta_2 \geq 0, \end{array}\right\}$$

where ξ_2 denotes the maximum acceptance degree of constraints and η_2 denotes the minimal rejection degree of constraints fixed by the player II and $e = (1, 1, \cdots, 1)^T$, is an unit vector in $\Re^n_+$.

From (3.7) we see that the constraints are separable in the decision variable y. Thus the model can be changed into the following optimization model.

$$(3.8)\quad \left.\begin{array}{c} \max\{\eta_2 - \xi_2\} \\ \text{S.T}\;\; \dfrac{e^{-\beta\left[-\sum_{j=1}^{n}\dfrac{a_{ij}}{\overline{a} - \underline{a}}y_j + \dfrac{\overline{a}}{\overline{a} - \underline{a}}\right]} - e^{-\beta}}{1 - e^{-\beta}} \leq \xi_2;\;\; i = 1, 2, \ldots, m \\ \left[\sum_{j=1}^{n}\dfrac{a_{ij}}{\overline{a} - \underline{a}}y_i - \dfrac{\underline{a}}{\overline{a} - \underline{a}}\right]^2 \geq \eta_2;\;\; i = 1, 2, \ldots, m \\ e^T\mathbf{y} = 1;\;\; \eta_2 \geq \xi_2;\;\; \xi_2 + \eta_2 \leq 1;\;\; \mathbf{y}, \xi_2, \eta_2 \geq 0 \end{array}\right\}$$

The optimal solution $(\mathbf{y}^*, \xi_2^*, \eta_2^*)$ obtained from the mathematical programming problem (3.8) for which $\mathbf{y}^*$ gives an optimal strategy for player II and $(\eta_2^* - \xi_2^*)$ gives the degree of attainment to which the aspiration level of player II can be met by choosing to play the strategy $\mathbf{y}^*$. Thus, once an optimal solutions $(\mathbf{x}^*, \xi_1^*, \eta_1^*)$ and $(\mathbf{y}^*, \xi_2^*, \eta_2^*)$ of the mathematical programming problems (3.6) and (3.8) have been obtained for given α, β (shape parameters) , $\mathbf{x}^*$ and $\mathbf{y}^*$ give an equilibrium solution of the matrix game. The value of the game can be determined by evaluating $\mathbf{x}^{*T} A\mathbf{y}^*$.

4. Numerical example

Consider a two-person zero-sum crisp matrix game G, whose pay-off matrix is given by

$$A = \begin{pmatrix} 9 & 1 & 4 \\ 1 & 6 & 3 \\ 5 & 2 & 8 \end{pmatrix}.$$

The optimal solution(crisp) of the game G is given by $(\mathbf{x}^*, \mathbf{y}^*, v^*)$ where $\mathbf{x}^* = (0.375, 0.541667, 0.083333)^T$, $\mathbf{y}^* = (0.291667, 0.55556, 0.152778)^T$ and the value of the game is $v^* = 3.79167$.

Next we consider the IF versions of the game G. The tolerances are arbitrarily fixed for each membership and non-membership functions of the players. Let the IF goals of the player I be represented by the following membership and non-membership functions,

$$\mu_1(\mathbf{x}^T A\mathbf{y}) = \begin{cases} 1; & \mathbf{x}^T A\mathbf{y} \geq 9 \\ \frac{e^{-\alpha\left(\frac{\mathbf{x}^T A\mathbf{y}-1}{8}\right)}-1}{e^{-\alpha}-1}; & 1 < \mathbf{x}^T A\mathbf{y} < 9 \\ 0; & \mathbf{x}^T A\mathbf{y} \leq 1, \end{cases}$$

$$\nu_1(\mathbf{x}^T A\mathbf{y}) = \begin{cases} 1; & \mathbf{x}^T A\mathbf{y} \leq 1 \\ \left(\frac{9-\mathbf{x}^T A\mathbf{y}}{8}\right)^2; & 1 < \mathbf{x}^T A\mathbf{y} < 9 \\ 0; & \mathbf{x}^T A\mathbf{y} \geq 9. \end{cases}$$

Using (3.6) we get the following mathematical programming problem for player I.

$$\max\{\xi_1 - \eta_1\}$$

$$\left.\begin{array}{c} e^{-\frac{\alpha}{8}[9x_1+x_2+5x_3-1]} - (e^{-\alpha}-1)\xi_1 - 1 \geq 0 \\ e^{-\frac{\alpha}{8}[x_1+6x_2+2x_3-1]} - (e^{-\alpha}-1)\xi_1 - 1 \geq 0 \\ e^{-\frac{\alpha}{8}[4x_1+3x_2+8x_3-1]} - (e^{-\alpha}-1)\xi_1 - 1 \geq 0 \\ 81x_1^2 + x_2^2 + 25x_3^2 + 18x_1x_2 + 10x_2x_3 \\ +90x_1x_3 - 162x_1 - 18x_2 - 90x_3 - 64\eta_1 + 81 \leq 0 \\ x_1^2 + 36x_2^2 + 4x_3^2 + 12x_1x_2 + 24x_2x_3 \\ +4x_1x_3 - 18x_1 - 108x_2 - 36x_3 - 64\eta_1 + 81 \leq 0 \\ 16x_1^2 + 9x_2^2 + 64x_3^2 + 24x_1x_2 + 48x_2x_3 \\ +64x_1x_3 - 72x_1 - 54x_2 - 144x_3 - 64\eta_1 + 81 \leq 0 \\ x_1 + x_2 + x_3 = 1 \\ \xi_1 + \eta_1 \leq 1; \quad \xi_1 \geq \eta_1; \quad \mathbf{x}, \xi_1, \eta_1 \geq 0. \end{array}\right\} \tag{4.1}$$

Similarly, let the IF goals of the player II be represented by the following membership and non-membership functions,

$$\mu_2(\mathbf{x}^T A\mathbf{y}) = \begin{cases} 0; & \mathbf{x}^T A\mathbf{y} \leq 1 \\ \frac{e^{-\beta\left(\frac{9-\mathbf{x}^T A\mathbf{y}}{8}\right)}-e^{-\beta}}{1-e^{-\beta}}; & 1 < \mathbf{x}^T A\mathbf{y} < 9 \\ 1; & \mathbf{x}^T A\mathbf{y} \geq 9 \end{cases}$$

$$\nu_2(\mathbf{x}^T A\mathbf{y}) = \begin{cases} 1; & \mathbf{x}^T A\mathbf{y} \leq 1 \\ \left(\frac{\mathbf{x}^T A\mathbf{y}-1}{8}\right)^2; & 1 < \mathbf{x}^T A\mathbf{y} < 9 \\ 0; & \mathbf{x}^T A\mathbf{y} \geq 9. \end{cases}$$

With the help of (3.8) we get the following mathematical programming problem for player II.

$$\max\{\eta_2 - \xi_2\}$$

Subject to

$$
(4.2)\quad \left.\begin{array}{c}
e^{-\frac{\beta}{8}[9-9y_1-y_2-4y_3]} - e^{-\beta} - (1-e^{-\beta})\xi_2 \leq 0 \\
e^{-\frac{\beta}{8}[9-y_1-6y_2-3y_3]} - e^{-\beta} - (1-e^{-\beta})\xi_2 \leq 0 \\
e^{-\frac{\beta}{8}[9-5y_1-2y_2-8y_3]} - e^{-\beta} - (1-e^{-\beta})\xi_2 \leq 0 \\
81y_1^2 + y_2^2 + 16y_3^2 + 18y_1y_2 + 8y_2y_3 \\
+72y_1y_3 - 18y_1 - 2y_2 - 8y_3 - 64\eta_2 + 1 \geq 0 \\
y_1^2 + 36y_2^2 + 9y_3^2 + 12y_1y_2 + 36y_2y_3 \\
+6y_1y_3 - 2y_1 - 12y_2 - 6y_3 - 64\eta_2 + 1 \geq 0 \\
25y_1^2 + 4y_2^2 + 64y_3^2 + 20y_1y_2 + 32y_2y_3 \\
+80y_1y_3 - 10y_1 - 4y_2 - 16y_3 - 64\eta_2 + 1 \geq 0 \\
y_1 + y_2 + y_3 = 1 \\
\xi_2 + \eta_2 \leq 1; \quad \eta_2 \geq \xi_2; \quad \mathbf{y}, \xi_2, \eta_2 \geq 0.
\end{array}\right\}
$$

The best values of the optimal solutions are obtained by solving (4.1), and (4.2) using LINGO software are given by, for $\alpha = 1$ and for $\beta = 3$,

$$
\begin{aligned}
x^* &= (0.3043478, 0.5652174, 0.1304348)^T, \quad \xi_1^* = 0.60255, \quad \eta_1^* = 0.387448 \\
y^* &= (0.3043478, 0.5217391, 0.1739130)^T, \quad \xi_2^* = 0.1712481, \quad \eta_2^* = 0.397448.
\end{aligned}
$$

This choice corresponds to the situation where player I aspires to win more than 0.60255, but is satisfied if he/she wins more than 0.387448. Similarly for player II aspires not to lose more than 0.397448 but he/she will be satisfied if he/she loses at most 0.1712481. The degree of attainment of the IF goals for the players I and II are 0.2051 and 0.2262 respectively. The value of the game with IF goal is given by $x^{*T}Ay^* = 3.95652$, which is better value than our analogous crisp one.

Remark: In general, the solution of the IFO problem is different from the solution of the analogous crisp problems, and the degrees of satisfaction of the given objective or constraint in an IFO problem can be higher or lower. This depends on the formulation of the respective functions of acceptance and rejection.

5. Conclusion

In this paper, two-person zero sum matrix games in which goals are represented by Atanassov's IFS has been considered. The solutions are obtained by solving two non-linear programming problems which have been formulated by choosing exponential membership and quadratic non-membership functions. One of the advantages of using exponential membership function(which is a continuous function) is the flexibility in changing the shape parameters. Also, the players can easily minimize the worse cases and maximize the better cases. In the process of solution it is desirable that the shape parameters are heuristically and experimentally decided by the players, because, after obtaining the optimum value in any situation, if the player is not satisfied with the outputs, he/she may perform the analysis again by re-choosing the shape parameters until a better optimal solution is obtained. Also, The degree of membership and the degree of non-membership functions have been considered together so that the sum of both values is less than or equal to 1. Further, the solution concept is defined in the sense of degree of attainment of the intuitionistic fuzzy goals. With the help of numerical illustration, we have shown that the IF version of two person zero sum matrix game gives better results than that of our crisp situation.

Although, in the proposed methodology we have considered matrix games with IF goals, a general methodology for the solution of matrix games in which the goals as well as the pay-offs are described by IFS will be investigated in near future. This theory can be applied in decision making theory such as economics, operations research, management, war science, etc.

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