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On soft linear spaces and soft normed linear spaces

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Abstract. In the present paper a notion of soft linear space and soft norm on a soft linear space have been presented and some of their properties have been studied. Soft vectors in soft linear spaces are introduced and their properties are investigated. Completeness of soft normed linear spaces, equivalent soft norms and convex soft sets are defined and some of their properties are examined in soft normed linear space settings.

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1. Introduction

In 1999, Molodtsov [16] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties while modeling the problems in engineering, physics, computer science, economics, social sciences, and medical sciences. Following his work Maji et al. ([13], [14]) introduced several operations on soft sets and applied soft sets to decision making problems. Some new operations on soft sets were described by Ali et al. in ([2]). Chen et al. [3] presented a new definition of soft set parametrization reduction and some works in this line have been found in ([12], [17], [21]). Soft group was introduced by Aktas and Cagman [1] and soft BCK/BCI – algebras and its application in ideal theory was investigated by Jun ([10], [11]). Feng et al. [8] worked on soft semirings, soft ideals and idealistic soft semirings. Some works on semigroups and soft ideals over a semi-group has been found in ([18]). The idea of a soft topological space was first given by M. Shabir, M. Naz [19]. Mappings between soft sets were described by P. Majumdar, S. K. Samanta [15]. Feng et al. [9] worked on soft sets combined with fuzzy sets and rough sets. Recently in ([4], [5]) we have introduced notions of soft real sets, soft real numbers, soft complex sets, soft complex numbers and some of their basic properties have been investigated. Some applications of soft real sets and soft real numbers have been presented in real life problems. Two different notions of 'soft metric' are presented in ([6], [7]) and some properties of soft metric spaces are studied in both cases.

In this paper we have introduced notions of soft linear space and soft normed linear space. In Section 2, some preliminary results are given. In Section 3, a notion of 'soft linear space' is given and various properties of soft linear spaces are studied. In Section 4, a definition of 'soft vector' in a soft linear space is given and various properties of soft vectors are studied in details with examples and counter examples. A notion of 'soft norm' in a soft linear space is introduced in Section 5. It has been shown that every 'soft normed linear space' is also a 'soft metric space' [6]. In that section, completeness of soft normed linear spaces, equivalent soft norms and convex soft sets are studied in soft normed linear space settings. Section 6 concludes the paper.

2. Preliminaries

Definition 2.1 ([16])**.** Let U be an universe and E be a set of parameters. Let $\mathcal{P}(U)$ denote the power set of U and A be a non-empty subset of E. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to \mathcal{P}(U)$. In other words, a soft set over U is a parametrized family of subsets of the universe U. For $\varepsilon \in A,\ F(\varepsilon)$ may be considered as the set of ε − approximate elements of the soft set (F, A).

Definition 2.2 ([9])**.** For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) if

(1) $A \subseteq B$ and
(2) for all $e \in A,\ F(e) \subseteq G(e)$. We write $(F, A)\widetilde{\subset}(G, B)$.

(F, A) is said to be a soft superset of (G, B), if (G, B) is a soft subset of (F, A). We denote it by $(F, A)\widetilde{\supset}(G, B)$.

Definition 2.3 ([9])**.** Two soft sets (F, A) and (G, B) over a common universe U are said to be equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 2.4 ([9])**.** The complement of a soft set (F, A) is denoted by $(F, A)^c = (F^c, A)$, where $F^c : A \to \mathcal{P}(U)$ is a mapping given by $F^c(\alpha) = U - F(\alpha)$, for all $\alpha \in A$.

Definition 2.5 ([14])**.** A soft set (F, E) over U is said to be an *absolute* soft set denoted by $\check{U}$ if for all $\varepsilon \in E,\ F(\varepsilon) = U$.

Definition 2.6 ([14])**.** A soft set (F, E) over U is said to be a *null* soft set denoted by Φ if for all $\varepsilon \in E,\ F(\varepsilon) = \varnothing$.

Definition 2.7 ([14])**.** The union of two soft sets (F, A) and (G, B) over the common universe U is the soft set (H, C), where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & if\ e \in A - B \\ G(e) & if\ e \in B - A \\ F(e) \cup G(e) & if\ e \in A \cap B. \end{cases}$$

We express it as $(F, A)\,\widetilde{\cup}\,(G, B) = (H, C)$.

The following definition of intersection of two soft sets is given as that of the bi-intersection in [8].

Definition 2.8 ([8])**.** The intersection of two soft sets (F, A) and (G, B) over the common universe U is the soft set (H, C), where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e) \cap G(e)$. We write $(F, A) \widetilde{\cap} (G, B) = (H, C)$.

Let X be an initial universal set and E be the non-empty set of parameters.

Definition 2.9 ([19])**.** The difference (H, E) of two soft sets (F, E) and (G, E) over X, denoted by $(F, E) \backslash (G, E)$, is defined by $H(e) = F(e) \backslash G(e)$ for all $e \in E$.

Proposition 2.10 ([19])**.** *Let (F, E) and (G, E) be two soft sets over X. Then*

(i). $((F, E)\widetilde{\cup}(G, E))^c = (F, E)^c \widetilde{\cap} (G, E)^c$

(ii). $((F, E)\widetilde{\cap}(G, E))^c = (F, E)^c \widetilde{\cup} (G, E)^c$.

Definition 2.11 ([4])**.** Let X *be* a non-empty set and E be a non-empty parameter set. Then a function $\varepsilon : E \to X$ is said to be a soft element of X. A soft element ε of X is said to belongs to a soft set A of X, which is denoted by $\varepsilon \widetilde{\in} A$, if $\varepsilon(e) \in A(e)$, $\forall e \in E$. Thus for a soft set A of X with respect to the index set E, we have $A(e) = \left\{ \varepsilon(e), \varepsilon \widetilde{\in} A \right\}$, $e \in E$.

It is to be noted that every singleton soft set (a soft set (F, E) for which $F(e)$ is a singleton set, $\forall e \in E$) can be identified with a soft element by simply identifying the singleton set with the element that it contains $\forall e \in E$.

Definition 2.12 ([4])**.** Let R be the set of real numbers and $\mathfrak{B}(R)$ the collection of all non-empty bounded subsets of R and A taken as a set of parameters. Then a mapping $F : A \to \mathfrak{B}(R)$ is called a *soft real set*. It is denoted by (F, A). If specifically (F, A) is a singleton soft set, then after identifying (F, A) with the corresponding soft element, it will be called a *soft real number*.

We use notations $\tilde{r}$, $\tilde{s}$, $\tilde{t}$ to denote soft real numbers whereas $\overline{r}$, $\overline{s}$, $\bar{t}$ will denote a particular type of soft real numbers such that $\overline{r}(\lambda) = r$, for all $\lambda \in A$ etc. For example $\overline{0}$ is the soft real number where $\overline{0}(\lambda) = 0$, for all $\lambda \in A$.

Definition 2.13 ([6])**.** For two soft real numbers $\tilde{r}, \tilde{s}$ we define

(i). $\tilde{r} \widetilde{\leq} \tilde{s}$ if $\tilde{r}(\lambda) \leq \tilde{s}(\lambda)$, for all $\lambda \in A$.

(ii). $\tilde{r} \widetilde{\geq} \tilde{s}$ if $\tilde{r}(\lambda) \geq \tilde{s}(\lambda)$, for all $\lambda \in A$.

(iii). $\tilde{r} \tilde{<} \tilde{s}$ if $\tilde{r}(\lambda) < \tilde{s}(\lambda)$, for all $\lambda \in A$.

(iv). $\tilde{r} \tilde{>} \tilde{s}$ if $\tilde{r}(\lambda) > \tilde{s}(\lambda)$, for all $\lambda \in A$.

Definition 2.14 ([6])**.** A soft real number $\tilde{r}$ is said to be non-negative if $\tilde{r}(\lambda) \geq 0$, for all $\lambda \in A$. We denote the set of all non-negative soft real numbers by $\mathcal{R}(A)^*$.

Remark 2.15. Let X be a non-empty set. Let $\check{X}$ be the absolute soft set i.e., $F(\lambda) = X$, $\forall \lambda \in A$, where $(F, A) = \check{X}$. Let $\mathcal{S}(\check{X})$ be the collection all soft sets (F, A) over X for which $F(\lambda) \neq \varnothing$, for all $\lambda \in A$ together with the null soft set Φ. Let $(F, A)(\neq \Phi) \in \mathcal{S}(\check{X})$, then the collection of all soft elements of (F, A) will be denoted by $SE(F, A)$. For a collection $\mathfrak{B}$ of soft elements of $\check{X}$, the soft set generated by $\mathfrak{B}$ is denoted by $SS(\mathfrak{B})$.

Definition 2.16 ([6]). A mapping $d : SE(\check{X}) \times SE(\check{X}) \to \mathcal{R}(A)^*$, is said to be a *soft metric* on the soft set $\check{X}$ if d satisfies the following conditions:

(M1). $d(\tilde{x}, \tilde{y}) \tilde{\geq} \overline{0}$, for all $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$.
(M2). $d(\tilde{x}, \tilde{y}) = \overline{0}$ if and only if $\tilde{x} = \tilde{y}$.
(M3). $d(\tilde{x}, \tilde{y}) = d(\tilde{y}, \tilde{x})$ for all $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$.
(M4). For all $\tilde{x}, \tilde{y}, \tilde{z} \tilde{\in} \check{X}, d(\tilde{x}, \tilde{z}) \tilde{\leq} d(\tilde{x}, \tilde{y}) + d(\tilde{y}, \tilde{z})$

The soft set $\check{X}$ with a soft metric d on $\check{X}$ is said to be a *soft metric space* and is denoted by $(\check{X}, d, A)$ or $(\check{X}, d)$. (M1), (M2), (M3) and (M4) are said to be soft metric axioms.

Theorem 2.17 ([6]). *(Decomposition Theorem) If a soft metric d satisfies the condition:*
(M5).For $(\xi, \eta) \in X \times X$, *and* $\lambda \in A$, $\{d(\tilde{x}, \tilde{y})(\lambda) : \tilde{x}(\lambda) = \xi, \tilde{y}(\lambda) = \eta\}$ *is a singleton set, and if for* $\lambda \in A$, $d_\lambda : X \times X \to \mathcal{R}^+$ *is defined by* $d_\lambda(\tilde{x}(\lambda), \tilde{y}(\lambda)) = d(\tilde{x}, \tilde{y})(\lambda)$, $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$; *then* d_λ *is a metric on* X.

Definition 2.18 ([6]). $(\check{X}, d)$ be a soft metric space, $\tilde{r}$ be a non-negative soft real number and $\tilde{a} \tilde{\in} \check{X}$. By an *open ball* with centre $\tilde{a}$ and radius $\tilde{r}$, we mean the collection of soft elements of $\check{X}$ satisfying $d(\tilde{x}, \tilde{a}) \tilde{<} \tilde{r}$.
The *open ball* with centre $\tilde{a}$ and radius $\tilde{r}$ is denoted by $B(\tilde{a}, \tilde{r})$.
Thus $B(\tilde{a}, \tilde{r}) = \{\tilde{x} \tilde{\in} \check{X}; d(\tilde{x}, \tilde{a}) \tilde{<} \tilde{r}\} \subset SE(\check{X})$.
$SS(B(\tilde{a}, \tilde{r}))$ will be called a *soft open ball* with centre $\tilde{a}$ and radius $\tilde{r}$.

Definition 2.19 ([6]). Let $\mathfrak{B}$ be a collection of soft elements of $\check{X}$ in a soft metric space $(\check{X}, d)$. Then a soft element $\tilde{a}$ is said to be an *interior element* of $\mathfrak{B}$ if there exists a positive soft real number $\tilde{r}$ such that $\tilde{a} \in B(\tilde{a}, \tilde{r}) \subset \mathfrak{B}$.

Definition 2.20 ([6]). Let $(\check{X}, d)$ be a soft metric space and $\mathfrak{B}$ be a non-null collection of soft elements of $\check{X}$. Then $\mathfrak{B}$ is said to be '*open in* $\check{X}$ *with respect to d*' or '*open in* $(\check{X}, d)$' if all elements of $\mathfrak{B}$ are interior elements of $\mathfrak{B}$.

Definition 2.21 ([6]). Let $(\check{X}, d)$ be a soft metric space and (Y, A) be a non-null soft subset of $\mathcal{S}(\check{X})$ in $(\check{X}, d)$. Then (Y, A) is said to be '*soft open in* $\check{X}$ *with respect to d*' if there is a collection $\mathfrak{B}$ of soft elements of (Y, A) such that $\mathfrak{B}$ is open with respect to d and $(Y, A) = SS(\mathfrak{B})$.

Definition 2.22 ([6]). Let $(\check{X}, d)$ be a soft metric space. A soft set (Y, A) of $\mathcal{S}(\check{X})$, is said to be '*soft closed in* $\check{X}$ *with respect to d*' if its complement $(Y, A)^c$ is soft open in $(\check{X}, d)$.

Definition 2.23 ([6]). Let $(\check{X}, d)$ be a soft metric space and $\mathfrak{B}$ be a collection of soft elements of $\check{X}$. A soft element $\tilde{a} \tilde{\in} \mathfrak{B}$ is said to be a *limit element* of $\mathfrak{B}$, if every open ball $B(\tilde{a}, \tilde{r})$ containing $\tilde{a}$ in $(\check{X}, d)$ contains at least one element of $\mathfrak{B}$ different from $\tilde{a}$.
The set of all limit elements of $\mathfrak{B}$ is said to be the derived set of $\mathfrak{B}$ and is denoted by $\mathfrak{B}^d$.

Definition 2.24 ([6]). Let $(\check{X}, d)$ be a soft metric space and $(Y, A) \in \mathcal{S}(\check{X})$. A soft element $\tilde{a} \tilde{\in} \check{X}$ is said to be a *soft limit element* of (Y, A), if every open ball $B(\tilde{a}, \tilde{r})$ containing $\tilde{a}$ in $(\check{X}, d)$ contains at least one soft element of (Y, A) different from $\tilde{a}$.

A soft limit element of a soft set (Y, A) may or may not belong to the soft set (Y, A). The set of all soft limit elements of (Y, A) is said to be the derived set of (Y, A) and is denoted by $(Y, A)^d$.

Definition 2.25 ([6])**.** Let $(\check{X}, d)$ be a soft metric space and $\mathfrak{B}$ be a collection of soft elements of $\check{X}$. Then the collection of all soft elements of $\mathfrak{B}$ and limit elements of $\mathfrak{B}$ in $(\check{X}, d)$ is said to be the *closure* of $\mathfrak{B}$ in $(\check{X}, d)$. It is denoted by $\widetilde{\mathfrak{B}}$.

Definition 2.26 ([6])**.** Let $(\check{X}, d)$ be a soft metric space and (Y, A) be a soft subset of $\mathcal{S}(\check{X})$. Then the collection of all soft elements of (Y, A) and soft limit elements of (Y, A) in $(\check{X}, d)$ is said to be the *soft closure* of (Y, A) in $(\check{X}, d)$. It is denoted by $\overline{(Y, A)}$.

Definition 2.27 ([6])**.** Let $\{\tilde{x}_n\}$ be a sequence of soft elements in a soft metric space $(\check{X}, d)$. The sequence $\{\tilde{x}_n\}$ is said to be convergent in $(\check{X}, d)$ if there is a soft element $\tilde{x}\tilde{\in}\check{X}$ such that $d(\tilde{x}_n, \tilde{x}) \to \overline{0}$ as $n \to \infty$.

This means for every $\widetilde{\varepsilon}\tilde{>}\overline{0}$, chosen arbitrarily, there exists a natural number $N = N(\widetilde{\varepsilon})$, such that $\overline{0}\tilde{\leq}d(\tilde{x}_n, \tilde{x})\tilde{<}\widetilde{\varepsilon}$, whenever $n > N$. i.e., $n > N \Longrightarrow \tilde{x}_n \in B(\tilde{x},\ \widetilde{\varepsilon})$.

We denote this by $\tilde{x}_n \to \tilde{x}$ as $n \to \infty$ or by $\lim_{n\to\infty} \tilde{x}_n\ = \tilde{x}$. $\tilde{x}$ is said to be the limit of the sequence $\tilde{x}_n$ as $n \to \infty$.

3. Soft vector/linear spaces

Let V be a vector space over a field K and let A be a parameter set. A soft set (F, A) where $F : A \to \wp(V)$ will be denoted by F only.

Definition 3.1. (Sums and Scalar products of soft sets) Let $F_1, F_2, \ldots, F_n$ be n soft sets over V. Then $F = F_1 + F_2 + \cdots + F_n$ is a soft set over V and is defined as $F(\lambda) = \{x_1 + x_2 + \cdots + x_n; x_i \in F_i(\lambda), i = 1, 2, \ldots, n\},\ \forall\lambda \in A$.
Let $\alpha \in K$ be a scalar and F be a soft sets over V, then αF is a soft set over V and is defined as follows: $\alpha F = G,\ G(\lambda) = \{\alpha x; x \in F(\lambda)\},\ \lambda \in A$.

Definition 3.2. Let V be a vector space over a field K and let A be a parameter set. Let G be a soft set over V. Now G is said to be a soft vector space or soft linear space of V over K if $G(\lambda)$ is a vector subspace of V, $\forall\lambda \in A$.

Example 3.3. Consider the Euclidian n-dimensional space $\mathcal{R}^n$ over $\mathcal{R}$. Let $A = \{1, 2, 3, \ldots, n\}$ be the set of parameters. Let $G : A \to \wp(\mathcal{R}^n)$ be defined as follows:

$$G(i) = \{t \in \mathcal{R}^n; i - th\ co - ordinate\ of\ t\ is\ 0\},\ i = 1, 2, \ldots, n.$$

Then G is a soft vector space or soft linear space of $\mathcal{R}^n$ over $\mathcal{R}$.

Proposition 3.4. *$\alpha(F + G) = \alpha F + \alpha G$ for all soft sets F, G over V and $\alpha \in K$.*

Proof. $[\alpha(F + G)](\lambda) = \{\alpha z; z \in (F + G)(\lambda)\}$
$= \{\alpha(x + y); x \in F(\lambda), y \in G(\lambda)\}$
$= \{\alpha x + \alpha y; x \in F(\lambda), y \in G(\lambda)\}$, ($V$ is a vector space)
Again $(\alpha F + \alpha G)(\lambda) = \{x' + y'; x' \in \alpha F(\lambda), y' \in \alpha G(\lambda)\}$
$= \{\alpha x'' + \alpha y''; x'' \in F(\lambda), y'' \in G(\lambda)\}$.
Hence the result follows. □

Lemma 3.5. *Let $F_1, F_2, \ldots, F_n$, $G_1, G_2, \ldots, G_m$ be soft sets over V and let $F = F_1 + F_2 + \cdots + F_n$ and $G = G_1 + G_2 + \cdots + G_m$. Then $H = F + G$ is soft set, where $H = F_1 + F_2 + \cdots + F_n + G_1 + G_2 + \cdots + G_m$.*

Proof. The proof is straightforward and hence omitted. □

Definition 3.6. Let $x \in V$ and F be a soft set over V. Then $x + F$ is a soft set over V defined as follows:

$$(x + F)(\lambda) = \{x + y; y \in F(\lambda)\}, \lambda \in A.$$

Lemma 3.7. *Let U be an ordinary subset of V and let F be a soft set over V. Then $U + F$ is a soft set over V defined as follows:*

$$(U + F)(\lambda) = \bigcup_{x \in U} \{x + y; y \in F(\lambda)\}, \lambda \in A.$$

$$U + F = \bigcup_{x \in U} (x + F)$$

Proof. Follows from Definition 3.6 and the Definition 2.7. □

Definition 3.8. (Soft Vector Subspaces) Let F be a soft vector space of V over K. Let $G : A \to \wp(V)$ be a soft set over V. Then G is said to be a soft vector subspace of F if

(i). for each $\lambda \in A, G(\lambda)$ is a vector subspace of V over K
(ii). $F(\lambda) \supseteq G(\lambda)$, $\forall \lambda \in A$.

Theorem 3.9. *A soft subset G of a soft vector space F is a soft vector subspace of F if and only if for all scalars $\alpha, \beta \in K, \alpha G + \beta G \tilde{\subseteq} G$.*

Proof. Let G be a soft vector subspace of F of V over K.
Let $\lambda \in A$, $(\alpha G + \beta G)(\lambda) = \{x' + y'; x' \in \alpha G(\lambda), y' \in \beta G(\lambda), \alpha, \beta \in K\}$
$= \{\alpha x + \beta y; x, y \in G(\lambda), \alpha, \beta \in K\} \subset G(\lambda)$,
[Now, $G(\lambda)$ is a vector subspace over K and $x, y \in G(\lambda)$, $\alpha, \beta \in K \Rightarrow \alpha x + \beta y \in G(\lambda)$]
So, $\alpha G + \beta G \tilde{\subseteq} G$, and the given condition is satisfied.
Conversely, let the given condition hold.
We have, $(\alpha G + \beta G)(\lambda) = \{\alpha x + \beta y; x, y \in G(\lambda)\}$, $\forall \lambda \in A$.
By the given condition, $\alpha G + \beta G \tilde{\subseteq} G$ i.e., $\{\alpha x + \beta y; x, y \in G(\lambda)\} \subset G(\lambda), \forall \lambda \in A$.
i.e., for all $x, y \in G(\lambda)$ and for every scalar $\alpha, \beta \in K$, $\alpha x + \beta y \in G(\lambda)$
This implies $G(\lambda)$ is a vector subspace of F over K. This is true for all $\lambda \in A$.
Also since G is a soft subset of F, $F(\lambda) \supseteq G(\lambda)$, $\forall \lambda \in A$.
Hence G is a soft vector subspace of F. □

Proposition 3.10. *If F and G be two soft vector subspaces of H over K and α be a scalar, then $F + G$ and αF are soft vector subspaces of H over K.*

Proof. The proof is straight forward and hence imitted. □

Proposition 3.11. *If $\{F_i\}$ be a family of soft vector subspaces of H over K, then $G = \bigcap_i F_i$ is a soft vector subspaces of H over K.*

Proof. The proof is straight forward and hence omitted. □

4. Soft vectors in soft vector spaces

In this section we introduce the concept of soft vectors in soft vector spaces and study some of their basic properties.

Definition 4.1. Let G be a soft vector space of V over K. Then a soft element of G is said to be a soft vector of G. In a similar manner a soft element of the soft set (K, A) is said to be a soft scalar, K being the scalar field.

Example 4.2. Consider the soft vector space G as Example 3.3. Let $\tilde{x}$ be a soft element of G as the following;
$\tilde{x}\,(i) = (1, 1, .., 0_{i-th}, .., 1) \in \mathcal{R}^n, i = 1, 2, .., n$. Then $\tilde{x}$ is a soft vector of G.

Definition 4.3. A soft vector $\tilde{x}$ in a soft vector space G is said to be the null soft vector if $\tilde{x}\,(\lambda) = \theta,\ \forall \lambda \in A$, θ being the zero element of V. It will be denoted by Θ. A soft vector is said to be non-null if it is not a null soft vector.

Definition 4.4. Let $\tilde{x}, \tilde{y}$ be soft vectors of G and $\tilde{k}$ be a soft scalar. Then the addition $\tilde{x} + \tilde{y}$ of $\tilde{x}, \tilde{y}$ and scalar multiplication $\tilde{k}.\tilde{x}$ of $\tilde{k}$ and $\tilde{x}$ are defined by
$(\tilde{x} + \tilde{y})\,(\lambda) = \tilde{x}\,(\lambda) + \tilde{y}\,(\lambda)\,,\ \left(\tilde{k}.\tilde{x}\right)(\lambda) = \tilde{k}\,(\lambda)\,.\tilde{x}\,(\lambda)\,,\ \forall \lambda \in A$. Obviously, $\tilde{x} + \tilde{y}, \tilde{k}.\tilde{x}$ are soft vectors of G.

Theorem 4.5. *In a soft vector space G of V over K,*
(i). $\overline{0}.\widetilde{\alpha} = \Theta$, *for all* $\widetilde{\alpha} \widetilde{\in} G$;
(ii). $\tilde{k}.\Theta = \Theta$, *for all soft scalar* $\tilde{k}$.
(iii). $\left(-\overline{1}\right).\widetilde{\alpha} = -\widetilde{\alpha}$, *for all* $\widetilde{\alpha} \widetilde{\in} G$.

Proof. (i) We have, $\left(\overline{0}.\widetilde{\alpha}\right)(\lambda) = \overline{0}\,(\lambda)\,.\widetilde{\alpha}\,(\lambda) = 0.\widetilde{\alpha}\,(\lambda) = \theta,\ \forall \lambda \in A$.
This implies $\overline{0}.\widetilde{\alpha} = \Theta$, for all $\widetilde{\alpha} \widetilde{\in} G$.
(ii) $\left(\tilde{k}.\Theta\right)(\lambda) = \tilde{k}\,(\lambda)\,.\Theta\,(\lambda) = \tilde{k}\,(\lambda)\,.\theta = \theta,\ \forall \lambda \in A$.
This implies $\tilde{k}.\Theta = \Theta$, for all soft scalar $\tilde{k}$.
(iii) $\left(\left(-\overline{1}\right).\widetilde{\alpha}\right)(\lambda) = \left(-\overline{1}\right)(\lambda)\,.\widetilde{\alpha}\,(\lambda) = (-1)\,.\widetilde{\alpha}\,(\lambda) = -\widetilde{\alpha}\,(\lambda) = (-\widetilde{\alpha})\,(\lambda)\,,$
$\forall \lambda \in A$.
This implies $\left(-\overline{1}\right).\widetilde{\alpha} = -\widetilde{\alpha}$, for all $\widetilde{\alpha} \widetilde{\in} G$. □

Remark 4.6. However, $\tilde{k}.\widetilde{\alpha} = \Theta$ does not necessarily imply that either $\tilde{k} = \overline{0}$ or $\widetilde{\alpha} = \Theta$. For example let us consider the soft vector space as Example 3.3. Let $\tilde{k}\,(1) = 1$, and $\tilde{k}\,(i) = 0$, for $i = 2, 3, .., n$ and $\widetilde{\alpha}\,(1) = \theta$, and $\widetilde{\alpha}\,(i) = (1, 1, .., 0_{i-th}, .., 1) \in \mathcal{R}^n, i = 2, 3, .., n$. Then $\left(\tilde{k}.\widetilde{\alpha}\right)(1) = \tilde{k}\,(1)\,.\widetilde{\alpha}\,(1) = 1.\theta = \theta = \Theta\,(1)$ and
$\left(\tilde{k}.\widetilde{\alpha}\right)(i) = \tilde{k}\,(i)\,.\widetilde{\alpha}\,(i) = 0.\,(1, 1, .., 0_{i-th}, .., 1) = \theta = \Theta\,(i)$, for $i = 2, 3, .., n$.
So, $\tilde{k}.\widetilde{\alpha} = \Theta$, but neither $\tilde{k} = \overline{0}$ nor $\widetilde{\alpha} = \Theta$.

Theorem 4.7. *A non-null soft subset (W, A) of a soft vector space G of V over K, is a soft subspace of G if and only if $\widetilde{\alpha}, \widetilde{\beta} \widetilde{\in} (W, A)$ and $\tilde{k}, \tilde{s}$ be soft scalars then $\tilde{k}.\widetilde{\alpha} + \tilde{s}.\widetilde{\beta} \widetilde{\in} (W, A)$.*

Proof. Let (W, A) be a soft vector subspace of G of V over K. Let $\widetilde{\alpha}, \widetilde{\beta} \widetilde{\in} (W, A)$ and $\tilde{k}, \tilde{s}$ be soft scalars, then

$\left(\tilde{k}.\widetilde{\alpha}+\tilde{s}.\widetilde{\beta}\right)(\lambda) = \tilde{k}(\lambda).\widetilde{\alpha}(\lambda)+\tilde{s}(\lambda).\widetilde{\beta}(\lambda) \in W(\lambda),\ \forall\lambda\in A$. ($W(\lambda)$ is a vector subspace of V for each $\lambda\in A$, $\tilde{k}(\lambda),\ \tilde{s}(\lambda)\in K$, $\widetilde{\alpha}(\lambda)+\widetilde{\beta}(\lambda)\in W(\lambda),\ \forall\lambda\in A$.)

$$\therefore \tilde{k}.\widetilde{\alpha}+\tilde{s}.\widetilde{\beta}\widetilde{\in}(W,A).$$

Conversely, let the given condition be satisfied.
Then for all soft scalars $\tilde{k},\tilde{s}$ and soft vectors $\widetilde{\alpha},\widetilde{\beta}\widetilde{\in}(W,A)$, $\tilde{k}.\widetilde{\alpha}+\tilde{s}.\widetilde{\beta}\widetilde{\in}(W,A)$.
i.e., $\left(\tilde{k}.\widetilde{\alpha}+\tilde{s}.\widetilde{\beta}\right)(\lambda)\in W(\lambda),\forall\lambda\in A$, i.e., $\tilde{k}(\lambda).\widetilde{\alpha}(\lambda)+\tilde{s}(\lambda).\widetilde{\beta}(\lambda)\in W(\lambda),\forall\lambda\in A$;
This implies $W(\lambda)$ is a vector subspace of V for each $\lambda\in A$. Also it is obvious that, $W(\lambda)\subset V(\lambda),\ \forall\lambda\in A$.
Hence, (W,A) is a soft vector subspace of G of V over K. □

Definition 4.8. Let G be a soft vector space of V over K. Let $\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\widetilde{\in}G$. A soft vector $\widetilde{\beta}$ in G is said to be a linear combination of the soft vectors $\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n$ if $\widetilde{\beta}$ can be expressed as $\widetilde{\beta}=\tilde{c}_1.\widetilde{\alpha}_1+\tilde{c}_2.\widetilde{\alpha}_2+..+\tilde{c}_n.\widetilde{\alpha}_n$, for some soft scalars $\tilde{c}_1,\tilde{c}_2,\ldots,\tilde{c}_n$.

Example 4.9. Consider the soft vector space G as Example 3.3.
Let $\widetilde{\alpha}_i=(1,1,..,0_{i-th},..,1)\in\mathcal{R}^n, i=1,2,3$. Then $\widetilde{\alpha}_1+\widetilde{\alpha}_2,\ \widetilde{\alpha}_1+\widetilde{\alpha}_2+\widetilde{\alpha}_3,\ \overline{2}.\widetilde{\alpha}_1+\overline{5},\ \widetilde{\alpha}_2+\widetilde{\alpha}_3$ are linear combinations of $\widetilde{\alpha}_1,\widetilde{\alpha}_2,\widetilde{\alpha}_3$.

Definition 4.10. A finite set of soft vectors $\{\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\}$ of a soft vector space G is said to be linearly dependent in G if there exists soft scalars $\tilde{c}_1,\tilde{c}_2,\ldots,\tilde{c}_n$ not all $\overline{0}$ such that

$$\tilde{c}_1.\widetilde{\alpha}_1+\tilde{c}_2.\widetilde{\alpha}_2+\cdots+\tilde{c}_n.\widetilde{\alpha}_n=\Theta \tag{4.1}$$

An arbitrary set S of soft vectors of G is said to be linearly dependent in G if there exists a finite subset of S which is linearly dependent in G.

Definition 4.11. Let $\{\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\}$ be a set of soft vectors of a soft vector space G such that $\widetilde{\alpha}_i(\lambda)\neq\theta$ for any $\lambda\in A$ and $i=1,2,..,n$. Then $\{\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\}$ is said to be linearly independent in G if for any set of soft scalars $\tilde{c}_1,\tilde{c}_2,\ldots,\tilde{c}_n$, $\tilde{c}_1.\widetilde{\alpha}_1+\tilde{c}_2.\widetilde{\alpha}_2+\cdots+\tilde{c}_n.\widetilde{\alpha}_n=\Theta$ implies $\tilde{c}_1=\tilde{c}_2=\cdots=\tilde{c}_n=\overline{0}$.

Proposition 4.12. *A set $S=\{\widetilde{\alpha}_1,\widetilde{\alpha}_2,\ldots,\widetilde{\alpha}_n\}$ of soft vectors in a soft vector space G over V is linearly independent if and only if the sets $S(\lambda)=\{\widetilde{\alpha}_1(\lambda),\widetilde{\alpha}_2(\lambda),\ldots,\widetilde{\alpha}_n(\lambda)\}$ are linearly independent in V, $\forall\lambda\in A$.*

Proof. Let S be linearly independent.
Let $\lambda_0\in A$, and $S(\lambda_0)=\{\widetilde{\alpha}_1(\lambda_0),\widetilde{\alpha}_2(\lambda_0),\ldots,\widetilde{\alpha}_n(\lambda_0)\}$. Let $c_1.\ \widetilde{\alpha}_1(\lambda_0)+c_2.\ \widetilde{\alpha}_2(\lambda_0)+\cdots+c_n.\ \widetilde{\alpha}_n(\lambda_0)=\theta$. Let us consider any set of soft scalars $\tilde{c}_1,\ \tilde{c}_2,\ \ldots,\tilde{c}_n$, such that $\tilde{c}_1(\lambda_0)=c_1,\tilde{c}_2(\lambda_0)=c_2\ldots\tilde{c}_n(\lambda_0)=c_n$ and $\tilde{c}_1(\mu)=\tilde{c}_2(\mu)=\cdots=\tilde{c}_n(\mu)=0,\ \forall\mu(\neq\lambda_0)\in A$. Now $\tilde{c}_1.\ \widetilde{\alpha}_1+\tilde{c}_2.\ \widetilde{\alpha}_2+\cdots+\tilde{c}_n.\ \widetilde{\alpha}_n=\Theta$.
Since S is linearly independent, $\tilde{c}_1=\tilde{c}_2=\cdots=\tilde{c}_n=\overline{0}$.
$\Longrightarrow \tilde{c}_1(\lambda_0)=\tilde{c}_2(\lambda_0)=\cdots=\tilde{c}_n(\lambda_0)=0$
$\Longrightarrow c_1=c_2=\cdots=c_n=0$.
$\Longrightarrow S(\lambda_0)=\{\widetilde{\alpha}_1(\lambda_0),\widetilde{\alpha}_2(\lambda_0),\ldots,\widetilde{\alpha}_n(\lambda_0)\}$ is linearly independent in V.
Since $\lambda_0\in A$ is arbitrary, it follows that $S(\lambda)=\{\widetilde{\alpha}_1(\lambda),\widetilde{\alpha}_2(\lambda),\ldots,\widetilde{\alpha}_n(\lambda)\}$, are linearly independentin V, $\forall\lambda\in A$.
Conversely let $S(\lambda)=\{\widetilde{\alpha}_1(\lambda),\widetilde{\alpha}_2(\lambda),\ldots,\widetilde{\alpha}_n(\lambda)\}$ be linearly independent in V, $\forall\lambda\in A$.

Let $\tilde{c}_1,\ \tilde{c}_2,\ \ldots, \tilde{c}_n$ be any set of soft scalars such that $\tilde{c}_1.\ \widetilde{\alpha}_1 + \tilde{c}_2.\ \widetilde{\alpha}_2 + .. + \tilde{c}_n.\ \widetilde{\alpha}_n = \Theta$. Then $\tilde{c}_1(\lambda).\ \widetilde{\alpha}_1(\lambda) + \tilde{c}_2(\lambda).\ \widetilde{\alpha}_2(\lambda) + \cdots + \tilde{c}_n(\lambda).\ \widetilde{\alpha}_n(\lambda) = \theta \Rightarrow \tilde{c}_1(\lambda) =\ \tilde{c}_2(\lambda) = \cdots = \tilde{c}_n(\lambda) = 0,\ \forall \lambda \in A \Rightarrow \tilde{c}_1 =\ \tilde{c}_2 = \cdots = \tilde{c}_n = \overline{0}$.
Hence $S = \{\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\}$ is linearly independent. □

Proposition 4.13. *A set* $S = \{\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\}$ *of soft vectors in a soft vector space* G *over* V *is linearly dependent if and only if the sets*
$S(\lambda) = \{\widetilde{\alpha}_1(\lambda), \widetilde{\alpha}_2(\lambda), \ldots, \widetilde{\alpha}_n(\lambda)\}$, *are linearly dependent in* V *for some* $\lambda \in A$.

Proof. Let S be linearly dependent. Then there is a set of soft scalars $\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n$, not all equal to $\overline{0}$ such that $\tilde{c}_1.\widetilde{\alpha}_1 + \tilde{c}_2.\widetilde{\alpha}_2 + \cdots + \tilde{c}_n.\widetilde{\alpha}_n = \Theta$. Then $\tilde{c}_1(\lambda).\widetilde{\alpha}_1(\lambda) + \tilde{c}_2(\lambda).\widetilde{\alpha}_2(\lambda) + \cdots + \tilde{c}_n(\lambda).\widetilde{\alpha}_n(\lambda) = \theta,\ \forall \lambda \in A$ and there is at least one $\lambda_0 \in A$, such that $\tilde{c}_1(\lambda_0), \tilde{c}_2(\lambda_0), \ldots, \tilde{c}_n(\lambda_0)$ are not all zeros. Then $\tilde{c}_1(\lambda_0).\widetilde{\alpha}_1(\lambda_0) + \tilde{c}_2(\lambda_0).\widetilde{\alpha}_2(\lambda_0) + \cdots + \tilde{c}_n(\lambda_0).\widetilde{\alpha}_n(\lambda_0) = \theta$ and $\tilde{c}_1(\lambda_0), \tilde{c}_2(\lambda_0), \ldots, \tilde{c}_n(\lambda_0)$ are not all zeros. Proving that $S(\lambda_0) = \{\widetilde{\alpha}_1(\lambda_0), \widetilde{\alpha}_2(\lambda_0), \ldots, \widetilde{\alpha}_n(\lambda_0)\}$ is linearly dependent.
Conversely let $S(\lambda_0) = \{\widetilde{\alpha}_1(\lambda_0), \widetilde{\alpha}_2(\lambda_0), \ldots, \widetilde{\alpha}_n(\lambda_0)\}$, be linearly dependent for some $\lambda_0 \in A$. Then there is a set of scalars $c_1, c_2, \ldots, c_n$ not all zeros such that $c_1.\widetilde{\alpha}_1(\lambda_0) + c_2.\widetilde{\alpha}_2(\lambda_0) + \cdots + c_n.\widetilde{\alpha}_n(\lambda_0) = \theta$.
Let $\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n$ be a set of soft scalars such that $\tilde{c}_i(\lambda_0) = c_i$, and $\tilde{c}_i(\lambda) = 0$ for $\lambda \in A \backslash \{\lambda_0\}$, for $i = 1, 2, \ldots, n$. Then $\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n$, are not all equal to $\overline{0}$ and $\tilde{c}_1.\widetilde{\alpha}_1 + \tilde{c}_2.\widetilde{\alpha}_2 + \cdots + \tilde{c}_n.\widetilde{\alpha}_n = \Theta$. Hence S is linearly dependent. □

5. Soft norm and soft normed linear spaces

Let X be a vector space over a field K, X is also our initial universe set and A be a non-empty set of parameters. Let $\check{X}$ be the absolute soft vector space i.e., $\check{X}(\lambda) = X,\ \forall \lambda \in A$. We use the notation $\tilde{x},\ \tilde{y},\ \tilde{z}$ to denote soft vectors of a soft vector space and $\tilde{r},\ \tilde{s},\ \tilde{t}$ to denote soft real numbers.

5.1. Definitions and examples of soft norm and soft normed linear spaces.

Definition 5.1. Let $\check{X}$ be the absolute soft vector space i.e., $\check{X}(\lambda) = X,\ \forall \lambda \in A$. Then a mapping $\|.\| : SE(\check{X}) \to R(A)^*$ is said to be a soft norm on the soft vector space $\check{X}$ if $\|.\|$ satisfies the following conditions:

(N1). $\|\tilde{x}\| \widetilde{\geq} \overline{0}$, for all $\tilde{x} \widetilde{\in} \check{X}$;
(N2). $\|\tilde{x}\| = \overline{0}$ if and only if $\tilde{x} = \Theta$;
(N3). $\|\widetilde{\alpha}.\tilde{x}\| = |\widetilde{\alpha}|\,\|\tilde{x}\|$ for all $\tilde{x} \widetilde{\in} \check{X}$ and for every soft scalar $\widetilde{\alpha}$;
(N4). For all $\tilde{x}, \tilde{y} \widetilde{\in} \check{X}$, $\|\tilde{x} + \tilde{y}\| \widetilde{\leq} \|\tilde{x}\| + \|\tilde{y}\|$.

The soft vector space $\check{X}$ with a soft norm $\|.\|$ on $\check{X}$ is said to be a soft normed linear space and is denoted by $(\check{X}, \|.\|, A)$ or $(\check{X}, \|.\|)$. (N1), (N2), (N3) and (N4) are said to be soft norm axioms.

Example 5.2. Let $\mathcal{R}(A)$ be the set of all soft real numbers. We define $\|.\| : \mathcal{R}(A) \to \mathcal{R}(A)^*$, by, $\|\tilde{x}\| = |\tilde{x}|$, for all $\tilde{x} \widetilde{\in}\ \mathcal{R}(A)$, where $|\tilde{x}|$ denotes the modulus of soft real numbers. Then $\|.\|$ satisfied all the soft norm axioms so, $\|.\|$ is a soft norm on $\mathcal{R}(A)$ and $(\mathcal{R}(A), \|.\|, A)$ or $(\mathcal{R}(A), \|.\|)$ is a soft normed linear space.

Proposition 5.3. *Every parametrized family of crisp norms* $\{\|.\|_\lambda : \lambda \in A\}$ *on a crisp vector space* X *can be considered as a soft norm on the soft vector space* $\check{X}$.

Proof. Let $\check{X}$ be the absolute soft vector space over a field K, A be a non-empty set of parameters. Let $\{\|.\|_\lambda : \lambda \in A\}$ be a family of crisp norms on the vector space X. Let $\tilde{x}\tilde{\in}\check{X}$, then $\tilde{x}(\lambda) \in X$, for every $\lambda \in A$. Let us define a mapping $\|.\| : \check{X} \to R(A)^*$ by $\|\tilde{x}\|(\lambda) = \|\tilde{x}(\lambda)\|_\lambda,\ \forall\lambda \in A,\ \forall\tilde{x}\tilde{\in}\check{X}$.
Then $\|.\|$ is a soft norm on $\check{X}$.
To verify it we now verify the conditions (N1), (N2), (N3) and (N4) for soft norm.
(N1). We have $\|\tilde{x}\|(\lambda) = \|\tilde{x}(\lambda)\|_\lambda \geq 0, \forall\lambda \in A,\ \forall\tilde{x}\tilde{\in}\check{X}$,
$\therefore \|\tilde{x}\| \tilde{\geq} \bar{0}$, for all $\tilde{x}\tilde{\in}\check{X}$.
(N2). $\|\tilde{x}\| = \Theta$
$\Longleftrightarrow \|\tilde{x}\|(\lambda) = \theta,\ \forall\lambda \in A$
$\Longleftrightarrow \|\tilde{x}(\lambda)\|_\lambda = \theta,\ \forall\lambda \in A$
$\Longleftrightarrow \tilde{x}(\lambda) = \theta,\ \forall\lambda \in A$
$\Longleftrightarrow \tilde{x} = \Theta$
(N3). We have, $\|\widetilde{\alpha}.\tilde{x}\|(\lambda) = \|\widetilde{\alpha}(\lambda).\tilde{x}(\lambda)\|_\lambda$
$= |\widetilde{\alpha}(\lambda)|\,\|\tilde{x}(\lambda)\|_\lambda,\ [\because \|\widetilde{\alpha}(\lambda).\tilde{x}(\lambda)\|_\lambda = |\widetilde{\alpha}(\lambda)|\|\tilde{x}(\lambda)\|_\lambda,\ \forall\lambda \in A]$
$= (|\alpha|\,\|\tilde{x}\|)(\lambda),\ \forall\lambda \in A.$
So, $\|\widetilde{\alpha}.\tilde{x}\| = |\widetilde{\alpha}|\,\|\tilde{x}\|$, for all $\tilde{x}\tilde{\in}\check{X}$ and for every soft scalar $\widetilde{\alpha}\tilde{\in}\tilde{K}$.
(N4). For all $\tilde{x}, \tilde{y}\tilde{\in}\check{X}$,
$[\|\tilde{x}\| + \|\tilde{y}\|](\lambda) = \|\tilde{x}\|(\lambda) + \|\tilde{y}\|(\lambda)$
$= \|\tilde{x}(\lambda)\|_\lambda + \|\tilde{y}(\lambda)\|_\lambda$
$\geq \|\tilde{x}(\lambda) + \tilde{y}(\lambda)\|_\lambda$, [by the property of triangle inequality of $\|.\|_\lambda$]
$= \|\tilde{x} + \tilde{y}\|(\lambda), \forall\lambda \in A.$
$\therefore \|\tilde{x}\| + \|\tilde{y}\| \tilde{\geq} \|\tilde{x} + \tilde{y}\|$.
Thus (N4) is satisfied.
$\therefore \|.\|$ is a soft norm on $\check{X}$ and consequently $(\check{X}, \|.\|)$ is a soft normed linear space. □

Proposition 5.4. *Every crisp norm $\|.\|_X$ on a crisp vector space X can be extended to a soft norm on the soft vector space $\check{X}$.*

Proof. First we construct the absolute soft vector space $\check{X}$ using a non-empty set of parameters A.
Let us define a mapping $\|.\| : SE(\check{X}) \to R(A)^*$ by $\|\tilde{x}\|(\lambda) = \|\tilde{x}(\lambda)\|_X, \forall\lambda \in A,\ \forall\tilde{x}\tilde{\in}\check{X}$.
Then using the same procedure as Proposition 5.3, it can be easily proved that $\|.\|$ is a soft norm on $\check{X}$.
This soft norm is generated using the crisp norm $\|.\|_X$ and it is said to be the soft norm generated by $\|.\|_X$. □

Theorem 5.5. *(Decomposition Theorem) Every soft norm $\|.\|$ satisfies the condition (A) .... For $\xi \in X$, and $\lambda \in A$, $\{\|\tilde{x}\|(\lambda) : \tilde{x}(\lambda) = \xi\}$ is a singleton set.*
And hence each soft norm $\|.\|$ can be decomposed into a family of crisp norms $\{\|.\|_\lambda,\ \lambda \in A\}$, where $\|.\|_\lambda : X \to \mathcal{R}^+$ is defined by the following:
for each $\xi \in X$, $\|\xi\|_\lambda = \|\tilde{x}\|(\lambda)$, with $\tilde{x} \tilde{\in} \check{X}$ such that $\tilde{x}(\lambda) = \xi$.

Proof. First we prove the condition (A) holds for each soft norm. The proof of the first part of this theorem is as presented in [20].
Let $\tilde{x}$ be any soft element of $\check{X}$ such that $\tilde{x}(\mu) = \theta$, where $\mu \in A$. Let us consider a soft scalar $\tilde{\alpha}$ such that $\tilde{\alpha}(\mu) = 1$ and $\tilde{\alpha}(\lambda) = 0,\ \forall\lambda \in A - \{\mu\}$. Then $\tilde{\alpha}.\tilde{x} = \Theta$.

We have by (N3), $\|\tilde{\alpha}.\tilde{x}\| = |\tilde{\alpha}| \, \|\tilde{x}\|$
$\Longrightarrow \|\Theta\| = |\tilde{\alpha}| \, \|\tilde{x}\|$
$\Longrightarrow \overline{0} = |\tilde{\alpha}| \, \|\tilde{x}\|$
$\Longrightarrow 0 = |\tilde{\alpha}(\mu)| \, \|\tilde{x}\|(\mu)$, in particular.
$\Longrightarrow 0 = 1. \, \|\tilde{x}\|(\mu)$
$\Longrightarrow 0 = \|\tilde{x}\|(\mu)$
Thus we see that if for any soft element $\tilde{x}$ of $\check{X}$, $\tilde{x}(\mu) = \theta \implies \|\tilde{x}\|(\mu) = 0$.
Next let $\xi \in X$ and $\mu \in A$ be arbitrary. Let $\tilde{x},\ \tilde{y} \tilde{\in} \check{X}$ be any two soft elements such that $\tilde{x}(\mu) = \tilde{y}(\mu) = \xi$.
Then $(\tilde{x} - \tilde{y})(\mu) = \theta$ and hence by the above argument we get $\|\tilde{x} - \tilde{y}\|(\mu) = 0$.
We also have by (N4), $\|\tilde{x}\| = \|\tilde{x} - \tilde{y} + \tilde{y}\| \tilde{\leq} \|\tilde{x} - \tilde{y}\| + \|\tilde{y}\|$
i.e., $\|\tilde{x}\| - \|\tilde{y}\| \tilde{\leq} \|\tilde{x} - \tilde{y}\|$
$\Longrightarrow (\|\tilde{x}\| - \|\tilde{y}\|)(\mu) \leq \|\tilde{x} - \tilde{y}\|(\mu)$, in particular.
$\Longrightarrow \|\tilde{x}\|(\mu) - \|\tilde{y}\|(\mu) \leq \|\tilde{x} - \tilde{y}\|(\mu)$.
In a similar way it can be shown that,
$\|\tilde{y}\|(\mu) - \|\tilde{x}\|(\mu) \leq \|\tilde{y} - \tilde{x}\|(\mu) = \|\tilde{x} - \tilde{y}\|(\mu)$.
Thus we get, $|\, \|\tilde{x}\|(\mu) - \|\tilde{y}\|(\mu) \,| \leq \|\tilde{x} - \tilde{y}\|(\mu) \leq 0$. [since $\|\tilde{x} - \tilde{y}\|(\mu) = 0$.]
$\Longrightarrow |\, \|\tilde{x}\|(\mu) - \|\tilde{y}\|(\mu) \,| = 0$
$\Longrightarrow \|\tilde{x}\|(\mu) = \|\tilde{y}\|(\mu)$
$\Longrightarrow \{\|\tilde{x}\|(\mu) : \tilde{x}(\mu) = \xi\}$ is a singleton set.
$\Longrightarrow$ The condition (A) is satisfied.
Now we prove the second part of the theorem.
Clearly $\|.\|_{\lambda} : X \to \mathcal{R}^{+}$ is a rule that assign a vector of X to a non-negative crisp real number $\forall \lambda \in A$. Now the well defined property of $\|.\|_{\lambda}$, $\forall \lambda \in A$ follows from the condition (A) and the soft norm axioms gives the norm conditions of $\|.\|_{\lambda}$, $\forall \lambda \in A$. Thus every soft norm gives a parameterized family of crisp norms. With this point of view, it also follows that, a soft norm, is a particular 'soft mapping' as defined by P. Majumdar, et al. in [15] where $\|.\| : A \to (\mathcal{R}^{+})^{X}$. □

Proposition 5.6. *Let* $(\check{X}, \|.\|, A)$ *be a soft normed linear space. Let us define* $d : SE(\check{X}) \times SE(\check{X}) \to R(A)^{*}$ *by* $d(\tilde{x}, \tilde{y}) = \|\tilde{x} - \tilde{y}\|$*, for all* $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$*. Then* d *is a soft metric on* $\check{X}$*.*

Proof. We have, (M1). $d(\tilde{x}, \tilde{y}) = \|\tilde{x} - \tilde{y}\| \tilde{\geq} \overline{0}$, for all $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$. [using (N1)]
(M2). $d(\tilde{x}, \tilde{y}) = \overline{0} \Longleftrightarrow \|\tilde{x} - \tilde{y}\| = \overline{0} \Longleftrightarrow \tilde{x} = \tilde{y}$. [using (N2)]
(M3). $d(\tilde{x}, \tilde{y}) = \|\tilde{x} - \tilde{y}\| = \|\tilde{y} - \tilde{x}\| = d(\tilde{y}, \tilde{x})$ for all $\tilde{x}, \tilde{y} \tilde{\in} \check{X}$. [using (N3)]
(M4). $d(\tilde{x}, \tilde{y}) + d(\tilde{y}, \tilde{z}) = \|\tilde{x} - \tilde{y}\| + \|\tilde{y} - \tilde{z}\| \tilde{\geq} \|\tilde{x} - \tilde{z}\| = d(\tilde{x}, \tilde{z})$. [using (N4)]
So, d is a soft metric on $\check{X}$. d is said to be the soft metric induced by the soft norm $\|.\|$. From the above proposition it also follows that every soft normed linear space is also a soft metric space. □

Proposition 5.7. *(Translation invariance) A soft metric* d *induced by a soft norm* $\|.\|$ *on a soft normed linear space* $(\check{X}, \|.\|)$ *satisfies*

(1). $d(\tilde{x} + \tilde{a},\ \tilde{y} + \tilde{a}) = d(\tilde{x}, \tilde{y})$;
(2). $d(\tilde{\alpha}.\tilde{x},\ \tilde{\alpha}.\tilde{y}) = |\tilde{\alpha}| \, d(\tilde{x}, \tilde{y})$*, for all* $\tilde{x},\ \tilde{y} \tilde{\in} \check{X}$ *and for every soft scalar* $\tilde{\alpha}$*.*

Proof. We have, (1). $d(\tilde{x} + \tilde{a},\ \tilde{y} + \tilde{a}) = \|(\tilde{x} + \tilde{a}) - (\tilde{y} + \tilde{a})\| = \|\tilde{x} - \tilde{y}\| = d(\tilde{x}, \tilde{y})$;

$$(2).\ d(\widetilde{\alpha}.\tilde{x},\ \widetilde{\alpha}.\tilde{y}) = \|\widetilde{\alpha}.\tilde{x} - \widetilde{\alpha}.\tilde{y}\| = |\widetilde{\alpha}|\,\|\tilde{x} - \tilde{y}\| = |\widetilde{\alpha}|\,d(\tilde{x}, \tilde{y}). \qquad \square$$

Definition 5.8. Let $(\check{X}, \|.\|)$ be a soft normed linear space and (Y, A) be a non-null member of $\mathcal{S}(\check{X})$. Then the mapping $\|.\|_Y : SE(Y, A) \to \mathcal{R}(A)^*$ given by $\|\tilde{x}\|_Y = \|\tilde{x}\|$ for all $\tilde{x}\widetilde{\in}(Y, A)$ is a soft norm on (Y, A). This norm $\|.\|_Y$ is known as the relative norm induced on (Y, A) by $\|.\|$. The soft normed linear space $(Y, \|.\|_Y, A)$ is called a normed subspace or simply a subspace of the soft normed linear space $(\check{X}, \|.\|, A)$.

5.2. **Sequences and their convergence in soft normed linear spaces.**

Definition 5.9. Let $(\check{X}, \|.\|, A)$ be a soft normed linear space and $\tilde{r}\tilde{>}\overline{0}$ be a soft real number. We define the followings;

$$B(\tilde{x}, \tilde{r}) = \{\tilde{y}\widetilde{\in}\check{X} : \|\tilde{x} - \tilde{y}\| \tilde{<} \tilde{r}\} \subset SE(\check{X}) \tag{5.1}$$

$$\overline{B}(\tilde{x}, \tilde{r}) = \{\tilde{y}\widetilde{\in}\check{X} : \|\tilde{x} - \tilde{y}\| \widetilde{\leq} \tilde{r}\} \subset SE(\check{X}) \tag{5.2}$$

$$S(\tilde{x}, \tilde{r}) = \{\tilde{y}\widetilde{\in}\check{X} : \|\tilde{x} - \tilde{y}\| = \tilde{r}\} \subset SE(\check{X}) \tag{5.3}$$

$B(\tilde{x}, \tilde{r}), \overline{B}(\tilde{x}, \tilde{r})$ and $S(\tilde{x}, \tilde{r})$ are respectively called an open ball, a closed ball and a sphere with centre at $\tilde{x}$ and radius $\tilde{r}$. $SS(B(\tilde{x}, \tilde{r})), SS(\overline{B}(\tilde{x}, \tilde{r}))$ and $SS(S(\tilde{x}, \tilde{r}))$ are respectively called a soft open ball, a soft closed ball and a soft sphere with centre at $\tilde{x}$ and radius $\tilde{r}$.

Definition 5.10. A sequence of soft elements $\{\tilde{x}_n\}$ in a soft normed linear space $(\check{X}, \|.\|, A)$ is said to be convergent and converges to a soft element $\tilde{x}$ if $\|\tilde{x}_n - \tilde{x}\| \to \overline{0}$ as $n \to \infty$.This means for every $\widetilde{\varepsilon}\tilde{>}\overline{0}$, chosen arbitrarily, there exists a natural number $N = N(\widetilde{\varepsilon})$, such that $\overline{0}\widetilde{\leq} \|\tilde{x}_n - \tilde{x}\| \tilde{<} \widetilde{\varepsilon}$, whenever $n > N$.
i.e., $n > N \implies \tilde{x}_n \in B(\tilde{x}, \widetilde{\varepsilon})$. We denote this by $\tilde{x}_n \to \tilde{x}$ as $n \to \infty$ or by $\lim_{n\to\infty} \tilde{x}_n = \tilde{x}$. $\tilde{x}$ is said to be the limit of the sequence $\tilde{x}_n$ as $n \to \infty$.

Example 5.11. Let us consider the set $\mathcal{R}$ of all real numbers endowed with the usual norm $\|.\|$. Let $(\check{\mathcal{R}}, \|.\|)$ or $(\check{\mathcal{R}}, \|.\|, A)$ be the soft norm generated by the crisp norm $\|.\|$, where A is the non-empty set of parameters. Let $(Y, A)\widetilde{\subset}\check{\mathcal{R}}$ such that $Y(\lambda) = (0, 1]$ in the real line, $\forall\lambda \in A$. Let us choose a sequence $\{\tilde{x}_n\}$ of soft elements of (Y, A) where $\tilde{x}_n(\lambda) = \frac{1}{n}$, $\forall n \in \mathbb{N}, \forall\lambda \in A$. Then there is no $\tilde{x}\widetilde{\in}(Y, A)$ such that $\tilde{x}_n \to \tilde{x}$ in $(Y, \|.\|_Y, A)$. However the sequence $\{\tilde{y}_n\}$ of soft elements of (Y, A) where $\tilde{y}_n(\lambda) = \frac{1}{2}$, $\forall n \in \mathbb{N}, \forall\lambda \in A$ is convergent in $(Y, \|.\|, A)$ and converges to $\overline{\frac{1}{2}}$.

Theorem 5.12. *Limit of a sequence in a soft normed linear space, if exists is unique.*

Proof. If possible let there exists a sequence $\{\tilde{x}_n\}$ of soft elements in a soft normed linear space $(\check{X}, d)$ such that $\lim_{n\to\infty} \tilde{x}_n = \tilde{x}$, $\lim_{n\to\infty} \tilde{x}_n = \tilde{x}'$, where $\tilde{x} \neq \tilde{x}'$. Then there is at least one $\lambda \in A$ such that
$\left\|\tilde{x} - \tilde{x}'\right\|(\lambda) \neq 0$. We consider a positive real number ε_λ satisfying $0 < \varepsilon_\lambda < \frac{1}{2}\left\|\tilde{x} - \tilde{x}'\right\|(\lambda)$. Let $\widetilde{\varepsilon}\tilde{>}\overline{0}$ with $\widetilde{\varepsilon}(\lambda) = \varepsilon_\lambda$.

Since $\tilde{x}_n \to \tilde{x}$, $\tilde{x}_n \to \tilde{x}'$. Corresponding to $\widetilde{\varepsilon}\tilde{>}\overline{0}$, there exists natural numbers $N_1 = N_1(\widetilde{\varepsilon})$, $N_2 = N_2(\widetilde{\varepsilon})$ such that $n > N_1$ Implies $\tilde{x}_n \in B(\tilde{x}, \widetilde{\varepsilon})$ Implies $\left\|\tilde{x} - \tilde{x}'\right\| \tilde{<} \widetilde{\varepsilon}$
This implies $\left\|\tilde{x} - \tilde{x}'\right\|(\lambda) < \varepsilon_\lambda$, in particular.
Also, $n > N_2$ Implies $\tilde{x}_n \in B(\tilde{x}', \widetilde{\varepsilon})$ Implies $\left\|\tilde{x} - \tilde{x}'\right\| \tilde{<} \widetilde{\varepsilon}$ Implies $\left\|\tilde{x} - \tilde{x}'\right\|(\lambda) < \varepsilon_\lambda$, in particular.
Hence for all $n > N = \max\{N_1, N_2\}$,
$\left\|\tilde{x} - \tilde{x}'\right\|(\lambda) \leq \|\tilde{x}_n - \tilde{x}\|(\lambda) + \left\|\tilde{x}_n - \tilde{x}'\right\|(\lambda) < 2\varepsilon_\lambda$
So, $\varepsilon_\lambda > \frac{1}{2}\left\|\tilde{x} - \tilde{x}'\right\|(\lambda)$. This contradicts our hypothesis.
Hence the result follows. □

Definition 5.13. A sequence $\{\tilde{x}_n\}$ of soft elements in $(\check{X}, d)$ is said to be bounded if the set $\{\|\tilde{x}_n - \tilde{x}_m\|; m, n \in N\}$ of soft real numbers is bounded, i.e., there exists $\widetilde{M}\tilde{>}\overline{0}$ such that $\|\tilde{x}_n - \tilde{x}_m\| \tilde{\leq} \widetilde{M}$, $\forall m, n \in N$.

Definition 5.14. A sequence $\{\tilde{x}_n\}$ of soft elements in a soft normed linear space $(\check{X}, \|.\|, A)$ is said to be a Cauchy sequence in $\check{X}$ if corresponding to every $\widetilde{\varepsilon}\tilde{>}\overline{0}$, there exists $m \in N$ such that $\|\tilde{x}_i - \tilde{x}_j\| \tilde{\leq} \widetilde{\varepsilon}$, $\forall\ i, j \geq m$ i.e., $\|\tilde{x}_i - \tilde{x}_j\| \to \overline{0}$ as $i, j \to \infty$.

Theorem 5.15. *Every convergent sequence in a soft normed linear space is Cauchy and every Cauchy sequence is bounded.*

Proof. Let $\{\tilde{x}_n\}$ be a convergent sequence of soft elements with limit $\tilde{x}$(say) in $(\check{X}, \|.\|)$. Then corresponding to each $\widetilde{\varepsilon}\tilde{>}\overline{0}$, there exists $m \in N$ such that $\tilde{x}_n \in B\left(\tilde{x}, \frac{\widetilde{\varepsilon}}{2}\right)$ i.e., $\|\tilde{x} - \tilde{x}_n\| \tilde{\leq} \frac{\widetilde{\varepsilon}}{2}$, $\forall n \geq m$.
Then for $i, j \geq m$, $\|\tilde{x}_i - \tilde{x}_j\| \tilde{\leq} \|\tilde{x}_i - \tilde{x}\| + \|\tilde{x} - \tilde{x}_j\| \tilde{<} \frac{\widetilde{\varepsilon}}{2} + \frac{\widetilde{\varepsilon}}{2} = \widetilde{\varepsilon}$. Hence $\{\tilde{x}_n\}$ is a Cauchy sequence.
Next let $\{\tilde{x}_n\}$ be a Cauchy sequence of soft elements in $(\check{X}, \|.\|)$. Then there exists $m \in N$ such that $\|\tilde{x}_i - \tilde{x}_j\| \tilde{<} \overline{1}$, $\forall i, j \geq m$. Take $\tilde{M}$ with
$\tilde{M}(\lambda) = \max_{1 \leq i,j \leq m}\{\|\tilde{x}_i - \tilde{x}_j\|(\lambda)\}$, $\forall \lambda \in A$. Then for $1 \leq i \leq m$ and $j \geq m$, $\|\tilde{x}_i - \tilde{x}_j\| \tilde{\leq} \|\tilde{x}_i - \tilde{x}_m\| + \|\tilde{x}_m - \tilde{x}_j\| \tilde{<} \tilde{M} + \overline{1}$.
Thus, $\|\tilde{x}_i - \tilde{x}_j\| \tilde{<} \tilde{M} + \overline{1}$, $\forall i, j \in N$ and consequently the sequence is bounded. □

Definition 5.16. A soft subset (Y, A) with $Y(\lambda) \neq \varnothing$, $\forall \lambda \in A$, in a soft normed linear space $(\check{X}, \|.\|, A)$ is said to be bounded if there exists a soft real number $\tilde{k}$ such that $\|\tilde{x}\| \tilde{\leq} \tilde{k}$, $\forall \tilde{x} \tilde{\in} (Y, A)$.

Definition 5.17. Let $(\check{X}, \|.\|, A)$ be a soft normed linear space. Then $\check{X}$ is said to be complete if every Cauchy sequence in $\check{X}$ converges to a soft element of $\check{X}$ i.e., every complete soft normed linear space is called a soft Banach space.

Theorem 5.18. *Let $(\check{X}, \|.\|, A)$ be a soft normed linear space. Then*

(i). *if $\tilde{x}_n \to \tilde{x}$ and $\tilde{y}_n \to \tilde{y}$ then $\tilde{x}_n + \tilde{y}_n \to \tilde{x} + \tilde{y}$.*
(ii). *if $\tilde{x}_n \to \tilde{x}$ and $\widetilde{\lambda}_n \to \widetilde{\lambda}$ then $\widetilde{\lambda}_n.\tilde{x}_n \to \widetilde{\lambda}.\tilde{x}$, where $\{\widetilde{\lambda}_n\}$ is a sequence of soft scalars.*
(iii). *if $\{\tilde{x}_n\}$ and $\{\tilde{y}_n\}$ are Cauchy sequences in $\check{X}$ and $\{\widetilde{\lambda_n}\}$ is a Cauchy sequence of soft scalars, then $\{\tilde{x}_n + \tilde{y}_n\}$ and $\{\widetilde{\lambda_n}.\tilde{x}_n\}$ are also Cauchy sequences in $\check{X}$.*

Proof. (i) Since $\tilde{x}_n \to \tilde{x}$ and $\tilde{y}_n \to \tilde{y}$, for $\widetilde{\varepsilon}\tilde{>}\overline{0}$, there exists positive integers N_1, N_2 such that $\|\tilde{x}_n - \tilde{x}\| \tilde{<} \frac{\tilde{\varepsilon}}{2}$, $\forall n \geq N_1$ and $\|\tilde{y}_n - \tilde{y}\| \tilde{<} \frac{\tilde{\varepsilon}}{2}$, $\forall n \geq N_2$. Let $N = \max\{N_1, N_2\}$, then both the above relations hold for $n \geq N$.
Then $\|(\tilde{x}_n + \tilde{y}_n) - (\tilde{x} + \tilde{y})\| = \|\tilde{x}_n + \tilde{y}_n - \tilde{x} - \tilde{y}\| \widetilde{\leq} \|\tilde{x}_n - \tilde{x}\| + \|\tilde{y}_n - \tilde{y}\| \tilde{<} \frac{\tilde{\varepsilon}}{2} + \frac{\tilde{\varepsilon}}{2} = \widetilde{\varepsilon}$, $\forall n \geq N$.

$$\implies \tilde{x}_n + \tilde{y}_n \to \tilde{x} + \tilde{y}.$$

(ii) Since $\tilde{x}_n \to \tilde{x}$, for $\widetilde{\varepsilon}\tilde{>}\overline{0}$, there is a positive integer N such that $\|\tilde{x}_n - \tilde{x}\| \tilde{<} \widetilde{\varepsilon}$, $\forall n \geq N$.
Now, $\|\tilde{x}_n\| = \|\tilde{x}_n - \tilde{x} + \tilde{x}\| \widetilde{\leq} \|\tilde{x}_n - \tilde{x}\| + \|\tilde{x}\| \tilde{<} \widetilde{\varepsilon} + \|\tilde{x}\|$, $\forall n \geq N$.

$$\implies \|\tilde{x}_n\| \tilde{<} \widetilde{\varepsilon} + \|\tilde{x}\|, \ \forall n \geq N. \tag{5.4}$$

Thus the sequence $\{\|\tilde{x}_n\|\}$ is bounded.
Now, $\left\|\widetilde{\lambda}_n.\tilde{x}_n - \widetilde{\lambda}.\tilde{x}\right\| = \left\|\widetilde{\lambda}_n.\tilde{x}_n - \widetilde{\lambda}.\tilde{x}_n + \widetilde{\lambda}.\tilde{x}_n - \widetilde{\lambda}.\tilde{x}\right\| = \left\|\tilde{x}_n(\widetilde{\lambda}_n - \widetilde{\lambda}) + \widetilde{\lambda}(\tilde{x}_n - \tilde{x})\right\|$
$\widetilde{\leq} \left\|\tilde{x}_n(\widetilde{\lambda}_n - \widetilde{\lambda})\right\| + \left\|\widetilde{\lambda}(\tilde{x}_n - \tilde{x})\right\| = \left|\widetilde{\lambda}_n - \widetilde{\lambda}\right| \|\tilde{x}_n\| + \left|\widetilde{\lambda}\right| \|\tilde{x}_n - \tilde{x}\|$

$$\implies \left\|\widetilde{\lambda}_n.\tilde{x}_n - \widetilde{\lambda}.\tilde{x}\right\| \widetilde{\leq} \left|\widetilde{\lambda}_n - \widetilde{\lambda}\right| \|\tilde{x}_n\| + \left|\widetilde{\lambda}\right| \|\tilde{x}_n - \tilde{x}\| \tag{5.5}$$

Since $\tilde{x}_n \to \tilde{x}$ and $\widetilde{\lambda}_n \to \widetilde{\lambda}$ we get, $\left|\widetilde{\lambda}_n - \widetilde{\lambda}\right| \to \overline{0}$ and $\|\tilde{x}_n - \tilde{x}\| \to \overline{0}$ as $n \to \infty$.
Now using (5.4) and (5.5) we get, $\left\|\widetilde{\lambda}_n.\tilde{x}_n - \widetilde{\lambda}.\tilde{x}\right\| \to \overline{0}$ as $n \to \infty$.
Hence $\widetilde{\lambda}_n.\tilde{x}_n \to \widetilde{\lambda}.\tilde{x}$.
(iii) Let $\{\tilde{x}_n\}$ and $\{\tilde{y}_n\}$ be Cauchy sequences in $\check{X}$, then for $\widetilde{\varepsilon}\tilde{>}\overline{0}$, there exists positive integers N_1, N_2 such that
$\|\tilde{x}_n - \tilde{x}_m\| \tilde{<} \frac{\tilde{\varepsilon}}{2}$, $\forall\ m, n \geq N_1$ and $\|\tilde{y}_n - \tilde{y}_n\| \tilde{<} \frac{\tilde{\varepsilon}}{2}$, $\forall\ m, n \geq N_2$.
Let $N = \max\{N_1, N_2\}$, then both the above relations hold for $m, n \geq N$.
Now, $\|(\tilde{x}_n + \tilde{y}_n) - (\tilde{x}_m + \tilde{y}_m)\| = \|(\tilde{x}_n - \tilde{x}_m) + (\tilde{y}_n - \tilde{y}_m)\|$
$\widetilde{\leq} \|\tilde{x}_n - \tilde{x}_m\| + \|\tilde{y}_n - \tilde{y}_m\| \tilde{<} \frac{\tilde{\varepsilon}}{2} + \frac{\tilde{\varepsilon}}{2} = \widetilde{\varepsilon}$,
$\forall\ m, n \geq N$. $\implies \{\tilde{x}_n + \tilde{y}_n\}$ is a Cauchy sequence in $\check{X}$.
Since $\{\tilde{x}_n\}$ is a Cauchy sequence in $\check{X}$, for $\widetilde{\varepsilon}\tilde{>}\overline{0}$, there exists positive integers N such that $\|\tilde{x}_n - \tilde{x}_m\| \tilde{<} \widetilde{\varepsilon}$, $\forall\ m, n \geq N$.
Taking in particular $n = m + 1$, $\|\tilde{x}_{m+1}\| \tilde{<} \widetilde{\varepsilon}$, $\forall\ m, n \geq N$, so $\{\|\tilde{x}_n\|\}$ is bounded. Now $\{\widetilde{\lambda}_n\}$ is bounded too.
Then, $\left\|\widetilde{\lambda}_n.\tilde{x}_n - \lambda_m.\tilde{x}_m\right\| = \left\|\widetilde{\lambda}_n.\tilde{x}_n - \widetilde{\lambda}_n.\tilde{x}_m + \widetilde{\lambda}_n.\tilde{x}_m - \widetilde{\lambda}_m.\tilde{x}_m\right\|$
$= \left\|\widetilde{\lambda}_n(\tilde{x}_n - \tilde{x}_m) + \tilde{x}_m(\widetilde{\lambda}_n - \widetilde{\lambda}_m)\right\| \widetilde{\leq} \left|\widetilde{\lambda}_n\right| \|(\tilde{x}_n - \tilde{x}_m)\| + \|\tilde{x}_m\| \left|(\widetilde{\lambda}_n - \widetilde{\lambda}_m)\right| \to \overline{0}$ as $n \to \infty$.
$\implies \{\widetilde{\lambda}_n.\tilde{x}_n\}$ is also a Cauchy sequence in $\check{X}$. □

Theorem 5.19. *If (M, A) is a soft subspace in a soft normed linear space $(\check{X}, \|.\|, A)$, then the closure of (M, A), $\overline{(M, A)}$ is also a soft subspace.*

Proof. Let $\tilde{x}, \tilde{y} \widetilde{\in} \overline{(M, A)}$, we must show that any linear combination of $\tilde{x}, \tilde{y}$ belongs to $\overline{(M, A)}$. Since $\tilde{x}, \tilde{y} \widetilde{\in} \overline{(M, A)}$, corresponding to $\widetilde{\varepsilon}\tilde{>}\overline{0}$, there exists soft elements $\tilde{x}_1, \tilde{y}_1 \widetilde{\in} (M, A)$ such that $\|\tilde{x} - \tilde{x}_1\| \tilde{<} \widetilde{\varepsilon}$, $\|\tilde{y} - \tilde{y}_1\| \tilde{<} \widetilde{\varepsilon}$.
For soft scalars $\widetilde{\alpha}, \widetilde{\beta}\tilde{>}\overline{0}$, $\left\|\left(\widetilde{\alpha}\tilde{x} + \widetilde{\beta}\tilde{y}\right) - (\widetilde{\alpha}\tilde{x}_1 + \widetilde{\beta}\tilde{y}_1)\right\|$

$\widetilde{\leq} |\widetilde{\alpha}| \, \|\tilde{x} - \tilde{x}_1\| + \left|\widetilde{\beta}\right| \|\tilde{y} - \tilde{y}_1\| \,\tilde{<}\, \widetilde{\varepsilon}\left(|\widetilde{\alpha}| + \left|\widetilde{\beta}\right|\right) = \widetilde{\varepsilon}' \text{(say)},$

The above inequality shows that $\widetilde{\alpha}\tilde{x}_1 + \widetilde{\beta}\tilde{y}_1$ belongs to the open ball $B(\widetilde{\alpha}\tilde{x} + \widetilde{\beta}\tilde{y}, \widetilde{\varepsilon}')$. As $\widetilde{\alpha}\tilde{x}_1 + \widetilde{\beta}\tilde{y}_1$ and $\widetilde{\varepsilon}' \tilde{>} \overline{0}$ are arbitrary, it follows that $\widetilde{\alpha}\tilde{x} + \widetilde{\beta}\tilde{y} \widetilde{\in} \overline{(M, A)}$.
Hence $\overline{(M, A)}$ is a soft subspace of $\check{X}$. □

Definition 5.20. A soft linear space $\check{X}$ is said to be of finite dimensional if there is a finite set of linearly independent soft vectors in $\check{X}$ which also generates $\check{X}$.

Lemma 5.21. *Let $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ be a linearly independent set of soft vectors in a soft normed linear space $\check{X}$. Then there is a soft real number $\tilde{c} \tilde{>} \overline{0}$ such that for every set of soft scalars $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n$ we have*

$$\|\widetilde{\alpha}_1\tilde{x}_1 + \widetilde{\alpha}_2\tilde{x}_2 + \cdots + \widetilde{\alpha}_n\tilde{x}_n\| \,\widetilde{\geq}\, \tilde{c}\left(|\widetilde{\alpha}_1| + |\widetilde{\alpha}_2| + \cdots + |\widetilde{\alpha}_n|\right).$$

Proof. The lemma will be proved if we can prove

$$\|\widetilde{\alpha}_1\tilde{x}_1 + \widetilde{\alpha}_2\tilde{x}_2 + \cdots + \widetilde{\alpha}_n\tilde{x}_n\|(\lambda) \geq \left[\tilde{c}\left(|\widetilde{\alpha}_1| + |\widetilde{\alpha}_2| + \cdots + |\widetilde{\alpha}_n|\right)\right](\lambda), \forall \lambda \in A.$$

i.e., $\|\widetilde{\alpha}_1(\lambda).\tilde{x}_1(\lambda) + \widetilde{\alpha}_2(\lambda).\tilde{x}_2(\lambda) + \cdots + \widetilde{\alpha}_n(\lambda).\tilde{x}_n(\lambda)\|_\lambda$
$\geq \left[\tilde{c}(\lambda).\left(|\widetilde{\alpha}_1(\lambda)| + |\widetilde{\alpha}_2(\lambda)| + \cdots + |\widetilde{\alpha}_n(\lambda)|\right)\right], \forall \lambda \in A.$
Now, $\tilde{x}_1, \tilde{x}_2, \ldots.., \tilde{x}_n$ being soft vectors in $\check{X}, \tilde{x}_1(\lambda), \tilde{x}_2(\lambda), \ldots, \tilde{x}_n(\lambda)$ are vectors in X and $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n$ being soft scalars $\widetilde{\alpha}_1(\lambda), \widetilde{\alpha}_2(\lambda), \ldots, \widetilde{\alpha}_n(\lambda)$ are scalars. Then using the property of normed linear space $(X, \|.\|_\lambda)$ we get a real number c_λ, such that the above relation holds for $\tilde{c}(\lambda) = c_\lambda, \forall \lambda \in A$. □

Theorem 5.22. *Every Cauchy sequence in $\mathcal{R}(A)$ with finite parameter set A is convergent, i.e., the set of all soft real numbers with its usual modulus soft norm as defined in Example 5.2, with finite parameter set A, is a soft Banach space.*

Proof. Let $\{\tilde{x}_n\}$ be any arbitrary Cauchy sequence in $\mathcal{R}(A)$. Then corresponding to every $\widetilde{\varepsilon} \tilde{>} \overline{0}$, there exists $m \in N$ such that $|\tilde{x}_i - \tilde{x}_j| \widetilde{\leq} \widetilde{\varepsilon}$, $\forall\, i, j \geq m$ i.e.,$|\tilde{x}_i - \tilde{x}_j|(\lambda) \leq \widetilde{\varepsilon}(\lambda)$, $\forall\, i, j \geq m$, $\forall \lambda \in A$ i.e.,$|\tilde{x}_i(\lambda) - \tilde{x}_j(\lambda)| \leq \widetilde{\varepsilon}(\lambda)$, $\forall\, i, j \geq m$, $\forall \lambda \in A$. Then $\{\tilde{x}_n(\lambda)\}$ is a Cauchy sequence of ordinary real numbers $\mathcal{R}$ for each $\lambda \in A$. By the Completeness of $\mathcal{R}$ and finiteness of A, it follows that $\{\tilde{x}_n(\lambda)\}$ is convergent for each $\lambda \in A$. Let $\tilde{x}_n(\lambda) \to x_\lambda$, for each $\lambda \in A$. Consider the soft element $\tilde{x}$ defined by $\tilde{x}(\lambda) = x_\lambda$, for each $\lambda \in A$. Then $\tilde{x}$ is a soft real number and it follows that the sequence $\{\tilde{x}_n\}$ of soft real numbers is convergent and it converges to the soft real number $\tilde{x}$. Hence $\mathcal{R}(A)$ is a soft Banach space. □

Theorem 5.23. *Every finite dimensional soft normed linear space over a finite parameter set A is complete.*

Proof. Let $\check{X}$ be a finite dimensional soft normed linear space over a finite parameter set A. Let $\{\tilde{y}_m\}$ be any arbitrary Cauchy sequence in $\check{X}$. We show that $\{\tilde{y}_m\}$ converges to some soft element $\tilde{y} \widetilde{\in} \check{X}$. Suppose that the dimension of $\check{X}$ is n, and let $\{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n\}$ be a basis for $\check{X}$. Then each $\tilde{y}_m$ has a unique representation $\tilde{y}_m = \widetilde{\alpha}_1^{(m)}\tilde{e}_1 + \widetilde{\alpha}_2^{(m)}\tilde{e}_2 + \cdots + \widetilde{\alpha}_n^{(m)}\tilde{e}_n$.
Because $\{\tilde{y}_m\}$ is a Cauchy sequence, for $\widetilde{\varepsilon} \tilde{>} \overline{0}$ arbitrary there exist a positive integer N such that $\|\tilde{y}_m - \tilde{y}_r\| \,\tilde{<}\, \widetilde{\varepsilon}$ for $m, r > N$.
From Lemma 5.21, it follows that there exists $\tilde{c} \tilde{>} \overline{0}$ such that

$\widetilde{\varepsilon}\tilde{>}\|\tilde{y}_m-\tilde{y}_r\|=\left\|\sum_{j=1}^{n}(\widetilde{\alpha}_j^{(m)}-\widetilde{\alpha}_j^{(r)})\tilde{e}_j\right\|\tilde{\geq}\tilde{c}\sum_{j=1}^{n}\left|\widetilde{\alpha}_j^{(m)}-\widetilde{\alpha}_j^{(r)}\right|$, for $m,r>N$.
Consequently, $\left|\widetilde{\alpha}_j^{(m)}-\widetilde{\alpha}_j^{(r)}\right|\tilde{\leq}\sum_{j=1}^{n}\left|\widetilde{\alpha}_j^{(m)}-\widetilde{\alpha}_j^{(r)}\right|\tilde{<}\widetilde{\varepsilon}/\tilde{c}$
shows that each of the n sequences $\left\{\widetilde{\alpha}_j^{(m)}\right\}=\left\{\widetilde{\alpha}_j^{(1)},\widetilde{\alpha}_j^{(2)},\widetilde{\alpha}_j^{(3)},\dots\right\}$, $j=1,2,..,n$ is Cauchy in $\mathcal{R}(A)$ and A is finite, converges to $\widetilde{\alpha}_j$, (say), $j=1,2,\dots,n$.
We now define the soft element $\tilde{y}=\widetilde{\alpha}_1\tilde{e}_1+\widetilde{\alpha}_2\tilde{e}_2+\cdots+\widetilde{\alpha}_n\tilde{e}_n$ which is clearly a soft element of $\check{X}$. Moreover, since $\widetilde{\alpha}_j^{(m)}\to\widetilde{\alpha}_j$ as $m\to\infty$ and $j=1,2,\dots,n$; we have $\|\tilde{y}_m-\tilde{y}\|=\left\|\sum_{j=1}^{n}(\widetilde{\alpha}_j^{(m)}-\widetilde{\alpha}_j)\tilde{e}_j\right\|\tilde{\leq}\sum_{j=1}^{n}\left|\widetilde{\alpha}_j^{(m)}-\widetilde{\alpha}_j\right|\|\tilde{e}_j\|\to\overline{0}$ as $m\to\infty$.
i.e., $\tilde{y}_m\to\tilde{y}$ as $m\to\infty$. □

5.3. **Equivalent soft norms.**

Definition 5.24. Let $\check{X}$ be a soft linear (vector) space. A soft norm $\|.\|_1$ on $\check{X}$ is said to be equivalent to a soft norm $\|.\|_2$ on $\check{X}$ if there are positive soft real numbers $\tilde{a}$ and $\tilde{b}$ such that for all $\tilde{x}\tilde{\in}\check{X}$ we have

$$\tilde{a}\|\tilde{x}\|_2\tilde{\leq}\|\tilde{x}\|_1\tilde{\leq}\tilde{b}\|\tilde{x}\|_2 \tag{5.6}$$

Theorem 5.25. *On a finite dimensional soft linear space $\check{X}$, any soft norm $\|.\|_1$ is equivalent to any other norm $\|.\|_2$.*

Proof. Let n be the dimension of $\check{X}$ and $\{\tilde{e}_1,\tilde{e}_2,\dots..,\tilde{e}_n\}$ be a basis for $\check{X}$. If $\tilde{x}\tilde{\in}\check{X}$, then $\tilde{x}$ has the representation $\tilde{x}=\widetilde{\alpha}_1\tilde{e}_1+\widetilde{\alpha}_2\tilde{e}_2+\dots.+\widetilde{\alpha}_n\tilde{e}_n$.
By Lemma 5.21, there is a soft real number $\tilde{c}\tilde{>}\overline{0}$ such that,
$\|\tilde{x}\|_1\tilde{\geq}\tilde{c}\,(|\widetilde{\alpha}_1|+|\widetilde{\alpha}_2|+\cdots+|\widetilde{\alpha}_n|)$.
If $\tilde{R}(\lambda)=\max_j\{\|\tilde{e}_j\|_2\,(\lambda)\},\forall\lambda\in A$. Then soft norm axioms give,

$$\|\tilde{x}\|_2\tilde{\leq}\sum_{j=1}^{n}|\widetilde{\alpha}_j|\|\tilde{e}_j\|_2\tilde{\leq}\tilde{R}.\sum_{j=1}^{n}|\widetilde{\alpha}_j|\tilde{\leq}\left(\frac{\tilde{R}}{\tilde{c}}\right).\|\tilde{x}\|_1$$

or, $\left(\frac{\tilde{c}}{\tilde{R}}\right).\|\tilde{x}\|_2\tilde{\leq}\|\tilde{x}\|_1$
The other side inequality in (5.6) is obtained by interchanging the roles of $\|.\|_1$ and $\|.\|_2$ in the above argument. □

Lemma 5.26. *(Riesz's Lemma) Let $\tilde{L}$ be a proper soft closed subspace of a soft normed linear space $\check{X}$. Then for $\widetilde{\varepsilon}\tilde{>}\overline{0}$, there exists $\tilde{y}\tilde{\in}\check{X}-\tilde{L}$ with $\|\tilde{y}\|=\overline{1}$ such that for all $\tilde{x}\tilde{\in}\tilde{L}$, the inequality $\|\tilde{x}-\tilde{y}\|\tilde{>}\overline{1}-\widetilde{\varepsilon}$ is satisfied.*

Proof. Let $\widetilde{\varepsilon}\tilde{>}\overline{0}$, then $\widetilde{\varepsilon}(\lambda)=\varepsilon_\lambda>0,\forall\lambda\in A$.
Also, $\tilde{L}(\lambda)=L_\lambda$ is a proper closed subspace of the normed linear space $\check{X}(\lambda)=X$, for each $\lambda\in A$. Thus by Riesz's Lemma for normed linear space $(X,\|.\|_\lambda)$, there exists $\tilde{y}(\lambda)\in X-L_\lambda$ with $\|\tilde{y}(\lambda)\|_\lambda=1$ such that for all $\tilde{x}(\lambda)\in L_\lambda$ the inequality $\|\tilde{x}(\lambda)-\tilde{y}(\lambda)\|_\lambda>1-\varepsilon_\lambda$ is satisfied.
Then for $\widetilde{\varepsilon}\tilde{>}\overline{0}$, there exists $\tilde{y}\tilde{\in}\check{X}-\tilde{L}$ with $\|\tilde{y}\|=\overline{1}$ such that for all $\tilde{x}\tilde{\in}\tilde{L}$, the inequality $\|\tilde{x}-\tilde{y}\|\tilde{>}\overline{1}-\widetilde{\varepsilon}$ is satisfied. □

5.4. **Convex sets in soft normed linear spaces.**

Definition 5.27. Let $\check{X}$ be a soft normed linear space and $\tilde{x}_1,\ \tilde{x}_2\tilde{\in}\check{X}$. The set of all soft elements of the form $\tilde{y} = \tilde{t}\tilde{x}_1 + \left(\bar{1} - \tilde{t}\right)\tilde{x}_2$, where $\tilde{t}$ assumes all soft real numbers such that $\tilde{t}\,(\lambda) \in [0,\ 1],\ \forall\lambda \in A$, is called the segment joining the soft elements $\tilde{x}_1$ and $\tilde{x}_2$. A soft set $\tilde{K}\tilde{\subset}\check{X}$ is called convex if all segments joining any two soft elements of $\tilde{K}$ are contained in $\tilde{K}$. Clearly, every soft subspace $\tilde{M}$ of $\check{X}$ is a convex soft set.

Example 5.28. Let $\mathcal{R}$ be the set of all real numbers and A be a set of parameters. Consider a soft set (F, A) over the Euclidean space $\mathcal{R}^2$ such that $F(\lambda)$ is a circle with center at (a_λ, b_λ) and radius r_λ for each $\lambda \in A$; where a_λ, b_λ and r_λ are real numbers.
Let $\tilde{x}_1,\ \tilde{x}_2$ be any two soft elements of (F, A) and $\tilde{t}$ be a soft real number such that $\tilde{t}\,(\lambda) \in [0,\ 1],\ \forall\lambda \in A$. Consider the segment $\tilde{t}\tilde{x}_1 + \left(\bar{1} - \tilde{t}\right)\tilde{x}_2$ joining the soft elements $\tilde{x}_1$ and $\tilde{x}_2$. Then for each $\lambda \in A$, $(\tilde{t}\tilde{x}_1 + \left(\bar{1} - \tilde{t}\right)\tilde{x}_2)(\lambda) = \tilde{t}(\lambda)\tilde{x}_1(\lambda) + \left(1 - \tilde{t}(\lambda)\right)\tilde{x}_2(\lambda)$ and the right hand side is the segment joining two crisp points $\tilde{x}_1(\lambda)$ and $\tilde{x}_2(\lambda)$ of a circle with center at (a_λ, b_λ) and radius r_λ. So, $(\tilde{t}\tilde{x}_1 + \left(\bar{1} - \tilde{t}\right)\tilde{x}_2)(\lambda)$ is an element of the circle with center at (a_λ, b_λ) and radius r_λ. Hence $\tilde{t}\tilde{x}_1 + \left(\bar{1} - \tilde{t}\right)\tilde{x}_2$ is a soft element of (F, A); proving that (F, A) is a convex soft set.

Theorem 5.29. *A soft sphere in a soft normed linear space, is a convex soft set.*

Proof. We prove the theorem for a soft closed sphere. The proof for the soft open sphere is analogous. Let $\tilde{x}_1,\ \tilde{x}_2\tilde{\in}SS\left(\overline{B}(\tilde{a}, \tilde{r})\right)$, so that $\|\tilde{x}_1 - \tilde{a}\|\tilde{\leq}\tilde{r}$ and $\|\tilde{x}_2 - \tilde{a}\|\tilde{\leq}\tilde{r}$.
Let $\tilde{y} = \tilde{t}\tilde{x}_1 + \left(\bar{1} - \tilde{t}\right)\tilde{x}_2,\ \tilde{t}\,(\lambda) \in [0,\ 1],\ \forall\lambda \in A$. Then we have, $\|\tilde{y} - \tilde{a}\| = \left\|\tilde{t}\tilde{x}_1 + \left(\bar{1} - \tilde{t}\right)\tilde{x}_2 - \tilde{a}\right\| = \left\|\tilde{t}\tilde{x}_1 + \left(\bar{1} - \tilde{t}\right)\tilde{x}_2 - \tilde{t}\tilde{a} - \left(\bar{1} - \tilde{t}\right)\tilde{a}\right\|$
$= \left\|\tilde{t}\tilde{x}_1 - \tilde{t}\tilde{a}\right\| + \left\|\left(\bar{1} - \tilde{t}\right)\tilde{x}_2 - \left(\bar{1} - \tilde{t}\right)\tilde{a}\right\| = \tilde{t}\,\|\tilde{x}_1 - \tilde{a}\| + \left(\bar{1} - \tilde{t}\right)\|\tilde{x}_2 - \tilde{a}\|$
$\tilde{\leq}\tilde{t}\tilde{r} + \left(\bar{1} - \tilde{t}\right)\tilde{r} = \tilde{r}$. So, $\tilde{y}\tilde{\in}SS\left(\overline{B}(\tilde{a}, \tilde{r})\right)$. This proves the theorem. □

Theorem 5.30. *Let $\check{X}$ be a soft normed linear space and $\tilde{K}$ be a convex soft subset of $\check{X}$. Then the closure of $\tilde{K}$, $\overline{\tilde{K}}$ is convex.*

Proof. Let $\tilde{x}, \tilde{y}\tilde{\in}\overline{\tilde{K}}$ and $\tilde{\varepsilon}\tilde{>}\bar{0}$, there exists $\tilde{x}_1, \tilde{y}_1\tilde{\in}K$ such that
$\|\tilde{x} - \tilde{x}_1\|\,\tilde{<}\tilde{\varepsilon}, \|\tilde{y} - \tilde{y}_1\|\,\tilde{<}\tilde{\varepsilon}$.
Let $\tilde{t}\,(\lambda) \in [0,\ 1],\ \forall\lambda \in A$. Then $\tilde{t}$ is a non-negative soft real number.
Then, $\left\|\tilde{t}\tilde{x} + \left(\bar{1} - \tilde{t}\right)\tilde{y} - \left\{\tilde{t}\tilde{x}_1 + \left(\bar{1} - \tilde{t}\right)\tilde{y}_1\right\}\right\|\tilde{\leq}\tilde{t}\,\|\tilde{x} - \tilde{x}_1\| + \left(\bar{1} - \tilde{t}\right)\|\tilde{y} - \tilde{y}_1\|$
$\tilde{\leq}\tilde{t}\tilde{\varepsilon} + \left(\bar{1} - \tilde{t}\right)\tilde{\varepsilon} = \tilde{\varepsilon}$;
Since $\tilde{K}$ is convex, $\tilde{t}\tilde{x}_1 + \left(\bar{1} - \tilde{t}\right)\tilde{y}_1\tilde{\in}\tilde{K}$. Again since $\tilde{\varepsilon}\tilde{>}\bar{0}$, is arbitrary,
$\tilde{t}\tilde{x} + \left(\bar{1} - \tilde{t}\right)\tilde{y}\tilde{\in}\overline{\tilde{K}}$. This implies $\overline{\tilde{K}}$ is convex. □

Theorem 5.31. *The intersection of an arbitrary number of convex soft sets is a convex soft set.*

Proof. Let $\tilde{M} = \bigcap_a \tilde{M}_a$, where each $\tilde{M}_a$ is a convex soft set. If $\tilde{x}, \tilde{y}\tilde{\in}\tilde{M}$, then $\tilde{x}, \tilde{y}$ belongs to all $\tilde{M}_a$ and because each $\tilde{M}_a$ is a convex, $\tilde{t}\tilde{x} + \left(\bar{1} - \tilde{t}\right)\tilde{y}\tilde{\in}\tilde{M}_a$, where $\tilde{t}\,(\lambda) \in [0,\ 1],\ \forall\lambda \in A$. So, $\tilde{t}\tilde{x} + \left(\bar{1} - \tilde{t}\right)\tilde{y}\tilde{\in}\tilde{M}$ and $\tilde{M}$ is convex. □

Remark 5.32. However, the inverse of the above theorem may not be true. For example let us consider two soft sets (F, A) and (G, A) over $\mathcal{R}^2$ such that $F(\lambda)$ is the circle with center at $(1, 0)$ and radius 2 units and $G(\lambda)$ is the circle with center at $(9, 0)$ and radius 4 units; for each $\lambda \in A$. Then by Example 5.28, (F, A) and (G, A) are convex soft sets. Let us choose soft elements $\tilde{x}_1,\ \tilde{x}_2$ of (F, A) and (G, A) respectively, such that $\tilde{x}_1(\lambda) = (2, 0)$ and $\tilde{x}_2(\lambda) = (6, 0)$, for each $\lambda \in A$. We consider $\tilde{t} = \overline{1/2}$. Then we have $(\tilde{t}\tilde{x}_1 + \left(\overline{1} - \tilde{t}\right)\tilde{x}_2)(\lambda) = (1/2)(2, 0) + (1/2)(6, 0) = (1, 0) + (3, 0) = (4, 0)$; which is not a element of $F(\lambda) \cup G(\lambda)$. Consequently, $\tilde{t}\tilde{x}_1 + \left(\overline{1} - \tilde{t}\right)\tilde{x}_2$ is not a soft element of $(F, A)\widetilde{\cup}(G, A)$. Hence $(F, A)\widetilde{\cup}(G, A)$ is not a convex soft set.

6. Conclusions

In this paper we have introduced a concept of soft linear space and soft norm on a soft linear space. Completeness of soft normed linear spaces, convex soft sets and equivalent soft norms are introduced and some basic properties are investigated. There is an ample scope for further research on soft linear spaces, soft normed linear spaces and operators on soft normed linear spaces.

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