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On some algebraic structures of fuzzy multisets

T. K. Shinoj, Anagha Baby, Sunil Jacob John

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Abstract. As a beginning of the study of various algebraic structures of Fuzzy Multisets, in this paper the concept of Fuzzy Multi groups are introduced and its various properties are discussed.

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Corresponding Author: Sunil Jacob John (sunil@nitc.ac.in)

1. Introduction

Modern set theory formulated by George Cantor is fundamental for the whole Mathematics. One issue associated with the notion of a set is the concept of vagueness. Mathematics requires that all mathematical notions including set must be exact. This vagueness or the representation of imperfect knowledge has been a problem for a long time for philosophers, logicians and mathematicians. However, recently it became a crucial issue for computer scientists, particularly in the area of artificial intelligence. To handle situations like this, many tools were suggested. They include Fuzzy sets, Rough sets, Soft sets etc.

In particular, Lotfi Zadeh [12] proposed fuzzy sets as mathematical model of vagueness where elements belong to a given set to some degree that is typically a number that belongs to the unit interval [0,1].

Many fields of modern mathematics have been emerged by violating a basic principle of a given theory only because useful structures could be defined this way. Set is a well-defined collection of distinct objects, that is, the elements of a set are pair wise different. If we relax this restriction and allow repeated occurrences of any element, then we can get a mathematical structure that is known as Multisets or Bags. For example, the prime factorization of an integer $n > 0$ is a Multiset whose elements are primes. The number 120 has the prime factorization $120 = 2^3 3^1 5^1$ which gives the Multiset $\{2, 2, 2, 3, 5\}$. A complete account of the development of multiset theory can be seen in [1, 2, 9, 10] As a generalization of multiset, Yager [11] introduced the concept of Fuzzy Multiset (FMS). An element of a Fuzzy Multiset can occur more than once with possibly the same or different membership values.

2. Preliminaries

Definition 2.1 ([3]). Let X be a set. A multiset (mset) M drawn from X is represented by a function Count M or C_M defined as $C_M : X \to \{0, 1, 2, 3, \dots\}$.

For each $x \in X$, $C_M(x)$ is the characteristic value of x in M. Here $C_M(x)$ denotes the number of occurrences of x in M.

Definition 2.2 ([3]). Let M_1 and M_2 be two msets drawn from a set X. An mset M_1 is a submset of $M_2 (M_1 \subseteq M_2)$ if $C_{M_1}(x) \leq C_{M_2}(x)$ for all $x \in X$. M_1 is a proper sub mset of $M_2 (M_1 \subset M_2)$ if $C_{M_1}(x) \leq C_{M_2}(x)$ for all $x \in X$ and there exists at least one $x \in X$ such that $C_{M_1}(x) < C_{M_2}(x)$.

Definition 2.3 ([3]). Union of two msets M_1 and M_2 drawn from a set X is an mset M denoted by $M = M_1 \cup M_2$ such that for all $x \in X$, $C_M(x) = \max\{C_{M_1}(x), C_{M_2}(x)\}$.

Definition 2.4 ([3]). Intersection of two msets M_1 and M_2 drawn from a set X is an mset M denoted by $M = M_1 \cap M_2$ such that for all $x \in X$, $C_M(x) = \min\{C_{M_1}(x), C_{M_2}(x)\}$.

Definition 2.5. ([6]) Let X be a group. A multi set G over X is a multi group over X if the count of G satisfies the following two conditions

(1) $C_G(xy) \geq C_G(x) \wedge C_G(y)\ \forall x, y \in X$;
(2) $C_G(x^{-1}) \geq C_G(x)\ \forall x \in X$

Definition 2.6. ([8]) If X is a collection of objects, then a fuzzy set A in X is a set of ordered pairs: $A = \{(x, \mu_A(x)) : x \in X, \mu_A : X \to [0, 1]\}$ where μ_A is called the membership function of A, and is defined from X into [0, 1].

Definition 2.7. ([5]) Let G be a group and $\mu \in FP(G)$ (fuzzy power set of G), then μ is called fuzzy subgroup of G if

(1) $\mu(xy) \geq \mu(x) \wedge \mu(y)\ \forall x, y \in G$ and
(2) $\mu(x^{-1}) \geq \mu(x)\ \forall\ x \in G$

Definition 2.8. ([7]) Let X be a nonempty set. A Fuzzy Multiset (FMS) A drawn from X is characterized by a function, 'count membership' of A denoted by CM_A such that $CM_A : X \to Q$ where Q is the set of all crisp multisets drawn from the unit interval [0,1]. In particular, a fuzzy multiset A is characterized by a higher order function $A : X \to [0, 1] \to \mathbb{N}$, where of course $\mathbb{N}$ is the set of natural numbers.

Then for any $x \in X$, the value $CM_A(x)$ is a crisp multiset drawn from [0, 1]. For each $x \in X$, the membership sequence is defined as the decreasingly ordered sequence of elements in $CM_A(x)$. It is denoted by

$$\{\mu_A^1(x), \mu_A^2(x), \dots \mu_A^p(x)\};\ \mu_A^1(x) \geq \mu_A^2(x) \geq \dots \mu_A^p(x).$$

When every $x \in X$ is mapped to a finite multiset of Q under the count membership function CM_A, then A is called a finite fuzzy multiset of X. The collection of all finite multisets of X is denoted by $FM(X)$. Throughout this paper fuzzy multisets are taken from $FM(X)$.

Example 2.9. Assume $X = \{x, y, z\}$ is the set of three different breeds of goats. A cloning process was executed on them. A slight error in the process ended up giving clones which had exact physical features but different resistance to a particular disease. If the experts are not able to put tags to distinguish the clones of same goats, the natural representation of the situation is

$$A = \{(x, 0.3), (x, 0.3), (x, 0.9), (y, 0.4), (y, 0.7), (z, 0.7), (z, 0.6), (z, 0.6)\}$$

and we may write

$$A = \{\{0.9, 0.3, 0.3\}/x, \{0.7, 0.4\}/y, \{0.7, 0.6, 0.6\}/z\}$$

in which the msets of membership $\{0.9, 0.3, 0.3\}$, $\{0.7, 0.4\}$, $\{0.7, 0.6, 0.6\}$ correspond to the resistance to the disease.

Definition 2.10. ([4]) Let $A \in FM(X)$ and $x \in A$. Then $L(x; A) = \max\{j; \mu_A^j(x) \neq 0\}$.

When we define an operation between two fuzzy multisets, the length of their membership sequences should be set to equal. So if A and B are FMS at consideration, take $L(x; A, B) = \max\{L(x; A), L(x : B)\}$. When no ambiguity arises we denote the length of membership by $L(x)$.

Basic relations and operations, assuming that A and B are two fuzzy multisets of X is taken from [9] and is given below.

(1) **Inclusion**
$A \subseteq B \Leftrightarrow \mu_A^j(x) \leq \mu_B^j(x),\ j = 1, 2, \ldots, L(x)\ \forall\ x \in X$

(2) **Equality**
$A = B \Leftrightarrow \mu_A^j(x) = \mu_B^j(x),\ j = 1, 2, \ldots, L(x)\ \forall\ x \in X$

(3) **Union**
$\mu_{A\cup B}^j(x) = \mu_A^j(x) \vee \mu_B^j(x),\ j = 1, 2, \ldots, L(x)$ where $\vee$ is the maximum operation.

(4) **Intersection**
$\mu_{A\cap B}^j(x) = \mu_A^j(x) \wedge \mu_B^j(x),\ j = 1, 2, \ldots, L(x)$ where $\wedge$ is the minimum operation.

By $CM_A(x) \geq CM_A(y)$ it is taken that $\mu_A^i(x) \geq \mu_A^i(y)\ \forall i = 1, \ldots, \max\{L(x), L(y)\}$. And $CM_A(x) \wedge CM_A(y)$ means that $\{\mu_A^i(x) \wedge \mu_A^i(y)\}\ \forall\ i = 1, \ldots, \max\{L(x), L(y)\}$. And by $CM_A(x) \vee CM_A(y)$ we mean

$$\{\mu_A^i(x) \vee \mu_A^i(y)\}\ \forall i = 1, \ldots, \max\{L(x), L(y)\}.$$

Definition 2.11. Let X and Y be two nonempty sets and $f : X \to Y$ be a mapping. Then

(1) The image of the FMS $A \in FM(X)$ under the mapping f is denoted by $f(A)$ or

$$CM_{f[A]}(y) = \begin{cases} \vee_{f(x)=y} CM_A(x); & f^{-1}(y) \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

(2) The inverse image of the FMS $B \in FM(Y)$ under the mapping f is denoted by $f^{-1}(B)$ or $f^{-1}[B]$, where $CM_{f^{-1}[B]}(x) = CM_B f[x]$.

Proposition 2.12. *Let X, Y and Z be three nonempty sets and $f : X \to Y$ and $g : Y \to Z$ be two mappings. If $A, A_i \in FM(X)$, $B, B_i \in FM(Y)$, $C \in FM(Z)$; $i \in I$ then*
a) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$
b) $f[\cup_{i\in I} A_i] = \cup_{i\in I} f[A_i]$
c) $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$
d) $f^{-1}[\cup_{i\in I} B_i] = \cup_{i\in I} f^{-1}[B_i]$
e) $f^{-1}[\cap_{i\in I} B_i] = \cap_{i\in I} f^{-1}[B_i]$
f) $g[f(A_i)] = [gf](A_i)$ *and* $f^{-1}[g^{-1}(B_j)] = [gf]^{-1}(B_j)$

Proof. a) Let $A_1 \subseteq A_2$. So $CM_{A_1}(x) \leq CM_{A_2}(x)$.
Then $\vee_{f(x)=y} CM_{A_1}(x) \leq \vee_{f(x)=y} CM_{A_2}(x)$ $\forall x \in X$. Then by (2.11)

$$CM_{f(A_1)}(y) \leq CM_{f(A_2)}(y);\ f^{-1}(y) \neq \varnothing.$$

Hence the proof.
b) Let $A = \cup A_i$. So

$$\begin{aligned} CM_{f(\cup_{i\in I} A_i)}(y) &= CM_{f(A)}(y) \\ &= \vee_{f(x)=y} CM_A(x) \text{ by } 2.11 \\ &= \vee_{f(x)=y} \{\vee_{i\in I} CM_{f(A_i)}(x)\} \quad (2.1) \\ CM_{\cup_{i\in I}[f(A_i)]}(y) &= \vee_{i\in I} CM_{f(A_i)}(y) \\ &= \vee_{i\in I} \{\vee_{f(x)=y} CM_{A_i}(x)\};\ f^{-1}(y) \neq \varnothing \quad (2.2) \end{aligned}$$

From (2.1) and (2.2) the proof follows.
c) $B_1 \subseteq B_2$

$$\begin{aligned} &CM_{B_1}(f(x)) \leq CM_{B_2}(f(x)). \\ &\text{So by } CM_{f^{-1}(N)}(x) = CM_N(f(x)) \\ &CM_{f^{-1}(B_1)}(x) \leq CM_{f^{-1}(B_2)}(x). \\ &\text{Then } f^{-1}(B_1) \subseteq f^{-1}(B_2). \end{aligned}$$

Hence the proof.
d) Let $\cup_{i\in I} B_i = B$

$$\begin{aligned} CM_{f^{-1}(\cup_{i\in I} B_i)}(x) &= CM_{f^{-1}(B)}(x) \\ &= CM_B f(x) \quad \text{by } 2.11 \text{ (b)} \\ &= \vee_{i\in I} CM_{B_i} f(x) \\ &= \vee_{i\in I} CM_{f^{-1}[B_i]}(x) \\ &= CM_{\cup_{i\in I} f^{-1}(B_i)}(x) \end{aligned}$$

So $f^{-1}[\cup_{i\in I} B_i] = \cup_{i\in I} f^{-1}[B_i]$.

e) Let $\cap_{i\in I}B_i = B$.

$$\begin{aligned} CM_{f^{-1}(\cap_{i\in I}B_i)}(x) &= CM_{f^{-1}(B)}(x) \\ &= CM_B f(x) \quad \text{by (2.11) (b)} \\ &= \wedge_{i\in I} CM_{B_i} f(x) \\ &= \wedge_{i\in I} CM_{f^{-1}(B_i)}(x) \\ &= CM_{\cap_{i\in I} f^{-1}(B_i)}(x) \end{aligned}$$

So $f^{-1}[\cup_{i\in I}B_i] = \cap_{i\in I} f^{-1}[B_i]$.

f) Let $A \in FM(X)$ and $z \in Z$. Then

$$\begin{aligned} CM_{g[f(A)]}(z) &= \vee_{g(y)=z} CM_{f(A)}(y);\ y \in Y \\ &= \vee_{g(y)=z}\{\vee_{f(x)=y} CM_A(x)\};\ y \in Y \text{ and } x \in X \\ &= \vee\{\vee\{CM_A(x); x \in X \text{ and } f(x) = y\}; y \in Y \text{ and } g(y) = z\} \\ &= \vee\{CM_A(x); x \in X \text{ and } [gf](x) = z\} \\ &= \vee_{[gf](x)=z} CM_A(x); x \in X \\ &= CM_{[gf](A)}(z). \end{aligned}$$

Hence $g[f(A)] = [gf](A)$.

Now let $B \in FM(Z)$ and $x \in X$. Then

$$\begin{aligned} CM_{[gf]^{-1}(B)}(x) &= CM_B[gf](x) \text{ by 2.11 (b)} \\ &= CM_B g[f(x)] \text{ by the above part} \\ &= CM_{g^{-1}(B)} f(x) \text{ by 2.11(b)} \\ &= CM_{f^{-1}[g^{-1}(B)]}(x). \end{aligned}$$

Hence the proof.

□

Proposition 2.13. *Let X and Y be two non empty sets and $f : X \to Y$ be a mapping. If $A \in FM(X)$ then*
a) $A \subseteq f^{-1}[f(A)]$
b) $f^{-1}[f(A)] = A$, if f is injective.

Proof.
a)

$$\begin{aligned} CM_{f^{-1}[f(A)]}(x) &= CM_{f(A)} f(x) \\ &= \vee\{CM_A(x'); x' \in X, f(x') = f(x)\} \\ &\geq CM_A(x). \end{aligned}$$

Hence $A \subseteq f^{-1}[f(A)]$.
b) Let f be injective.

$$\begin{aligned} CM_{f^{-1}[f(A)]}(x) &= CM_{f(A)} f(x) \\ &= \vee\{CM_A(x'); x' \in X, f(x') = f(x)\} \\ &= CM_A(x) \end{aligned}$$

since f is injective. Hence the proof. □

Proposition 2.14. *Let X and Y be two non empty sets and $f : X \to Y$ be a mapping. If $B \in FM(Y)$ then*
a) $f[f^{-1}(B)] \subseteq B$
b) $f[f^{-1}(B)] = B$, if f is surjective.

Proof. a)

$$\begin{aligned} CM_{f[f^{-1}(B)]}(y) &= \vee_{f(x)=y}\{CM_{f^{-1}(B)}(x)\};\ x \in X \\ &= \vee_{f(x)=y}\{CM_B f(x)\};\ x \in X \\ &= \begin{cases} \vee_{f(x)=y}\{CM_B f(x)\}; & y \in f(X) \\ \{0\}/y & y \notin f(X) \end{cases} \\ &\leq CM_B(y). \end{aligned}$$

b)

$$\begin{aligned} CM_{f[f^{-1}(B)]}(y) &= \vee_{f(x)=y}\{CM_{f^{-1}(B)}(x)\};\ x \in X \\ &= \vee_{f(x)=y}\{CM_B f(x)\};\ x \in X \\ &= \begin{cases} \vee_{f(x)=y}\{CM_B f(x)\}; & y \in f(X) \\ \{0\}/y & y \notin f(X) \end{cases} \\ &= CM_B(y). \end{aligned}$$

Since f is surjective. □

3. Fuzzy multigroups

Throughout this section, let X be a group with a binary operation and the identity element is e. Also through the rest of the paper we assume that the fuzzy multisets are taken from the $FM(X)$ and $FMG(X)$ denote the set of all fuzzy multi groups (FMG) over the group X. We introduce some operations on a fuzzy multi subset of a group X in terms of the group operations.

Definition 3.1. Let $A \in FM(X)$. Then A^{-1} is defined as $CM_{A^{-1}}(x) = CM_A(x^{-1})$.

Definition 3.2. Let $A, B \in FM(X)$. Then define $A \circ B$ as

$$CM_{A\circ B}(x) = \vee\{CM_A(y) \wedge CM_B(z); y, z \in X \text{ and } yz = x\}.$$

Proposition 3.3. *Let $A, B, A_i \in FM(X)$, then the following hold*
a) $[A^{-1}]^{-1} = A$
b) $A \subseteq B \Rightarrow A^{-1} \subseteq B^{-1}$.
c) $[\bigcup_{i=1}^n A_i]^{-1} = \bigcup_{i=1}^n [A_i^{-1}]$
d) $[\bigcap_{i=1}^n A_i]^{-1} = \bigcap_{i=1}^n [A_i^{-1}]$
e) $(A \circ B)^{-1} = B^{-1} \circ A^{-1}$
f)

$$\begin{aligned} CM_{A\circ B}(x) &= \vee_{y\in X}\{CM_A(y) \wedge CM_B(y^{-1}x)\} \quad \forall x \in X \\ &= \vee_{y\in X}\{CM_A(xy^{-1}) \wedge CM_B(y)\} \quad \forall x \in X. \end{aligned}$$

Proof. a)

$$\begin{aligned} CM_{(A^{-1})^{-1}}(x) &= CM_{(A^{-1})}(x^{-1}) \text{ by } 3.1 \\ &= CM_A((x^{-1})^{-1}) \\ &= CM_A(x)\ \forall x \in X. \end{aligned}$$

Since X is a group $((x^{-1})^{-1}) = x \Rightarrow A = (A^{-1})^{-1}$.

b) Given $A \subseteq B$

$$\begin{aligned} &\Rightarrow\ CM_A(x^{-1}) \leq CM_B(x^{-1})\ \forall x \in X \\ &CM_{(A^{-1})}(x) \leq CM_{(B^{-1})}(x) \text{ by } 3.1 \\ &\Rightarrow\ A^{-1} \subseteq B^{-1}. \end{aligned}$$

c)

$$\begin{aligned} CM_{(\cup_{i=1}^n A_i)^{-1}}(x) &= CM_{(\cup_{i=1}^n A_i)}(x^{-1}) \text{ by } 3.1 \\ &= \vee\{CM_{A_i}(x^{-1}); i = 1, \dots, n\} \text{ by definition of union} \\ &= \vee\{CM_{A_i^{-1}}(x); i = 1, \dots, n\} \text{ by } 3.1 \\ &= CM_{\cup_{i=1}^n A_i^{-1}}(x) \text{ by definition of union} \\ \Rightarrow \left[\bigcup_{i=1}^n A_i\right]^{-1} &= \bigcup_{i=1}^n (A_i^{-1}) \end{aligned}$$

d)

$$\begin{aligned} CM_{(\cap_{i=1}^n A_i)^{-1}}(x) &= CM_{(\cap_{i=1}^n A_i)}(x^{-1}) \text{ by } 3.1 \\ &= \wedge\{CM_{A_i}(x^{-1}); i = 1, \dots, n\} \\ &= \wedge\{CM_{A_i^{-1}}(x); i = 1, \dots, n\} \text{ by } 3.1 \\ &= CM_{\cap_{i=1}^n A_i^{-1}}(x) \text{ by definition of intersection} \\ \Rightarrow \left[\bigcap_{i=1}^n A_i\right]^{-1} &= \bigcap_{i=1}^n (A_i^{-1}) \end{aligned}$$

e)

$$\begin{aligned} CM_{(A\circ B)^{-1}}(x) &= CM_{(A\circ B)}(x^{-1}) \\ &= \vee\{CM_A(y) \wedge CM_B(z); y, z \in X \text{ and } yz = x^{-1}\} \\ &= \vee\{CM_B(z) \wedge CM_A(y); y, z \in X \text{ and } (yz)^{-1} = x\} \\ &= \vee\{CM_B(z^{-1})^{-1} \wedge CM_A(y^{-1})^{-1}; y, z \in X \text{ and } (yz)^{-1} = x\} \\ &\quad\ \text{(Since } X \text{ is a group)} \\ &= \vee\{CM_{B^{-1}}(z^{-1}) \wedge CM_{A^{-1}}(y^{-1}); y^{-1}, z^{-1} \in X \text{ and } z^{-1}y^{-1} = x\} \\ &= CM_{B^{-1}\circ A^{-1}}(x) \quad x \in X. \end{aligned}$$

Hence the proof.

f) Since X is a group, it follows that for each $x, y \in X$, $\exists$ a unique $z\ (= y^{-1}x) \in X$, such that $yz = x$. Then

$$CM_{A\circ B}(x) = \vee_{y\in X}\{CM_A(y) \wedge CM_B(y^{-1}x)\} \quad \forall x \in X.$$

Also

$$CM_{A\circ B}(x) = \vee\{CM_B(z) \wedge CM_A(y); y, z \in X \text{ and } yz = x\}.$$

(By commutative property of minimum). Since X is a group, it follows that for each $x, y \in X$, $\exists$ a unique $z\ (= xy^{-1}) \in X$, such that $zy = x$. Then

$$CM_{A\circ B}(x) = \vee_{y\in X}\{CM_B(xy^{-1}) \wedge CM_A(y)\} \quad \forall x \in X.$$

□

Note. Similarly we could define $CM_{A\circ B}(x)$ w.r.t z.

Definition 3.4. Let X be a group. A fuzzy multiset G over X is a fuzzy multi group (FMG) over X if the count (count membership) of G satisfies the following two conditions.

(1) $CM_G(xy) \geq CM_G(x) \wedge CM_G(y)\ \forall\ x, y \in X$
(2) $CM_G(x^{-1}) = CM_G(x)\ \forall\ x \in X$.

Example 3.5. $(Z_4, +_4)$ is a group. Then
$A = \{(.8, .7, .7, .5, .1, .1)/2, (.6, .4, .3, .1)/1, (.6, .4, .3, .1)/3, (.9, .8, .7, .5, .1, .1)/0\}$ is a fuzzy multi group. But
$B = \{(.6, .4, .3, .1)/2, (.9, .7, .7, .5, .1, .1)/1, (.8, .7, .7, .5, .1, .1)/3, (.9, .8, .7, .5, .1, .1)/0\}$ is not a FMG. Because $CM_B(1) \neq CM_B(3)$.

From the definition and above example it is clear that FMG is a generalized case of fuzzy group. Also it is a general case of multi group (when $\mu_G^i(x) = 1 \forall x \in X$; $i = 1, 2, \ldots, L(x)$).

Proposition 3.6. *Let $A \in FM(X)$ and $CM_A(x^{-1}) \geq CM_A(x)$. Then $CM_A(x^{-1}) = CM_A(x)$.*

Proof.

$$\begin{aligned} CM_A(x^{-1}) &\geq CM_A(x) \text{ (given)} \\ \text{Now } CM_A(x) &= CM_A((x^{-1})^{-1}) \geq CM_A(x^{-1}) \\ \text{Then } CM_A(x) &= CM_A(x^{-1}). \end{aligned}$$

□

Proposition 3.7. *Let $A \in FMG(X)$. Then*
a) $CM_A(e) \geq CM_A(x)\ \forall x \in X$
b) $CM_A(x^n) \geq CM_A(x) \forall x \in X$
c) $A^{-1} = A$.

Proof. Let $x, y \in X$.

a)

$$\begin{aligned} CM_A(e) &= CM_A(xx^{-1}) \\ &\geq CM_A(x) \wedge CM_A(x^{-1}) \text{ by } 3.4 \\ &= CM_A(x) \wedge CM_A(x) \text{ by } 3.4 \\ &= CM_A(x). \end{aligned}$$

b)

$$\begin{aligned} CM_A(x^n) &\geq CM_A(x^{n-1}) \wedge CM_A(x) \text{ by } 3.4 \text{ and since } x^{n-1}x = x^n \\ &\geq CM_A(x) \wedge CM_A(x) \wedge \cdots \wedge CM_A(x) \text{ by recursion} \\ &= CM_A(x). \end{aligned}$$

c)

$$\begin{aligned} CM_A^{-1}(x) &= CM_A(x^{-1}) \text{ by } 3.1 \\ &= CM_A(x) \text{ by } 3.4 \\ \Rightarrow A^{-1} &= A. \end{aligned}$$

□

Proposition 3.8. *Let $A \in FM(X)$. Then $A \in FMG(X)$ iff $CM_A(xy^{-1}) \geq CM_A(x) \wedge CM_A(y)$ $\forall x, y \in X$.*

Proof. Let $A \in FMG(X)$. Then

$$\begin{aligned} CM_A(xy^{-1}) &\geq CM_A(x) \wedge CM_A(y^{-1}) \\ &= CM_A(x) \wedge CM_A(y) \quad \forall x, y \in X \end{aligned}$$

by 3.4.

Conversely, let the given condition be satisfied.

$$\text{i.e., } CM_A(xy^{-1}) \geq CM_A(x) \wedge CM_A(y) \tag{3.1}$$

$$\begin{aligned} \text{Now} \quad CM_A(e) &= CM_A(xx^{-1}) \\ &\geq CM_A(x) \wedge CM_A(x) \quad \text{by (3.1)} \\ &= CM_A(x) \end{aligned} \tag{3.2}$$

$$\begin{aligned} \text{Also} \quad CM_A(x^{-1}) &= CM_A(ex^{-1}) \\ &\geq CM_A(e) \wedge CM_A(x) \quad \text{by (3.1)} \\ &= CM_A(x) \quad \text{by (3.2)} \end{aligned}$$

$$\text{i.e.,} \quad CM_A(x^{-1}) = CM_A(x). \tag{3.3}$$

$$\begin{aligned} \text{Now } CM_A(xy) &\geq CM_A(x) \wedge CM_A(y^{-1}) \text{ by (3.1)} \\ &= CM_A(x) \wedge CM_A(y) \text{ by (3.3)} \end{aligned}$$

Hence the proof. □

Definition 3.9. Let $A \in FM(X)$. Then $A[\alpha, n] = \{x \in X : \mu_A^j(x) \geq \alpha; L(x) \geq j \geq n \text{ and } j, n \in \mathbb{N}\}$. This is called n-α level set of A.

Proposition 3.10. *Let $A \in FMG(X)$. Then $A[\alpha, n]$ are subgroups of X.*

Proof. Let $x, y \in A[\alpha, n]$. It implies that

$$\mu_A^j(x) \geq \alpha \text{ and } \mu_A^j(y) \geq \alpha;\ j \geq n.$$

Then $\mu_A^j(xy^{-1}) \geq \alpha$ by 3.8.

This $\Rightarrow$ if $x, yA[\alpha, n]$ then $xy^{-1} \in A[\alpha, n]$. Then $A[\alpha, n]$ is a subgroup of X. Hence the proof. □

Definition 3.11. Let $A \in FMG(X)$. Then define $A^* = \{x \in X : CM_A(x) = CM_A(e)\}$.

Proposition 3.12. *Let $A \in FMG(X)$. Then A^* is a subgroup of X.*

Proof. Let $x, y \in A^*$. Then

$$CM_A(x) = CM_A(y) = CM_A(e) \tag{3.4}$$

Then

$$\begin{aligned} CM_A(xy^{-1}) &\geq CM_A(x) \wedge CM_A(y) \quad \text{by } (3.8)\\ &= CM_A(e) \wedge CM_A(e) \quad \text{by } (3.4)\\ &= CM_A(e). \end{aligned}$$

$$\text{But} \quad CM_A(xy^{-1}) \leq CM_A(e) \quad \text{by 3.7(a)}$$

$$\text{i.e.} \quad CM_A(xy^{-1}) = CM_A(e)$$

$\Rightarrow xy^{-1} \in A^*$. Then A^* is a subgroup of X. Hence the proof. □

Definition 3.13. Let $A \in FM(X)$. Let $j \in \mathbb{N}$. Then define

$$A^j = \{x \in X : \mu_A^j(x)0 \text{ and } \mu_A^{j+1}(x) = 0\}.$$

Proposition 3.14. *Let $A \in FMG(X)$. Then A^j is a subgroup of X iff*

$$\mu_A^{j+1}(xy^{-1}) = 0 \quad \forall x, y \in A^j.$$

Proof. Let $x, y \in A^j$. Then it implies that $\mu_A^j(x) > 0$ and $\mu_A^j(y) > 0$. Also $\mu_A^{j+1}(x) = 0$ and $\mu_A^{j+1}(y) = 0$. Then $\mu_A^j(xy^{-1}) > 0$ (Since $x, y \in A$, by 3.8). Also $\mu_A^{j+1}(xy^{-1}) = 0$ (given) $\Rightarrow xy^{-1} \in A^j$. Then A^j is a subgroup of X. Hence the proof.

Conversely, A^j is a subgroup of X. Then

$$x, y \in A^j \Rightarrow xy^{-1} \in A^j \Rightarrow \mu_A^{j+1}(xy^{-1}) = 0$$

by 3.13. Hence the proof. □

Proposition 3.15. *Let $A \in FM(X)$. Then $A \in FMG(X)$ iff $A \circ A \subseteq A$ and $A^{-1} = A$.*

Proof. Let $A \in FMG(X)$ and $x, y, z \in X$.

$$\begin{aligned} &\Rightarrow CM_A(xy) \geq CM_A(x) \wedge CM_A(y) \text{ by } 3.4\\ &\Rightarrow CM_A(z) \geq \vee\{CM_A(x) \wedge CM_A(y); xy = z\} \text{since } X \text{ is a group, by } 3.4\\ &\qquad = C_{A\circ A}(z) \quad \text{by } 3.2\\ &\Rightarrow A \circ A \subseteq A. \end{aligned}$$

Now by 3.7(c) we get the 2nd condition.

Conversely, assume

$$A \circ A \subseteq A \tag{3.5}$$

and $A^{-1} = A$

$$\Rightarrow CM_A^{-1}(x) = CM_A(x)$$

But $CM_A^{-1}(x) = CM_A(x^{-1})$ by 3.1

$$\Rightarrow CM_A(x) = CM_A(x^{-1}). \tag{3.6}$$

Since $A \in FM(X)$, then to prove $A \in FMG(X)$ it is enough to prove that

$$\begin{aligned} CM_A(xy^{-1}) &\geq CM_A(x) \wedge CM_A(y) \quad \forall x, y \in X \text{ by } 3.8 \\ CM_A(xy^{-1}) &\geq CM_{A\circ A}(xy^{-1}) \quad \text{by } (3.5) \\ &= \vee_{z\in X}\{CM_A(z) \wedge CM_A(z^{-1}xy^{-1})\} \quad \text{by } 3.3 \\ &\geq \{CM_A(x) \wedge CM_A(y^{-1})\};\ z = x \\ &\quad \text{(as it is only one case out of all possibilities)} \\ &= CM_A(x) \wedge CM_A(y) \text{by } (3.6). \end{aligned}$$

Hence the proof. □

Corollary 3.16. *Let $A \in FM(X)$. Then $A \in FMG(X)$ iff $A \circ A = A$ and $A^{-1} = A$.*

Proof. Let $A \in FMG(X)$. Then

$$\begin{aligned} CM_{A\circ A}(x) &= \vee\{CM_A(y) \wedge CM_A(z); y, z \in X \text{ and } yz = x\} \\ &\geq \{CM_A(e) \wedge CM_A(e^{-1}x)\} \\ &= CM_A(x) \end{aligned}$$

So $A \subseteq A \circ A$. Hence the proof by 3.15. □

Proposition 3.17. *Let $A \in FM(X)$. Then $A \in FMG(X)$ iff $A \circ A^{-1} \subseteq A$.*

Proof. Let $A \in FMG(X)$. Then

$$A \circ A \subseteq A \text{ and } A^{-1} = A \text{ by } 3.15$$
$$\Rightarrow A \circ A^{-1} = A \circ A \subseteq A.$$

Conversely, assume

$$A \circ A^{-1} \subseteq A \tag{3.7}$$

Since $A \in FM(X)$, then to prove $A \in FMG(X)$ it is enough to prove that

$$\begin{aligned} CM_A(xy^{-1}) &\geq CM_A(x) \wedge CM_A(y)\ \forall x, y \in X \text{ by } 3.8 \\ CM_A(xy^{-1}) &\geq CM_{A\circ A^{-1}}(xy^{-1}) \text{ by } (3.7) \\ &= \vee_{z\in X}\{CM_A(z) \wedge CM_{A^{-1}}(z^{-1}xy^{-1})\} \text{ by } 3.3 \\ &\geq \{CM_A(x) \wedge CM_{A^{-1}}(y^{-1})\}; z = x \\ &\quad \text{(as it is only one case out of all possibilities)} \\ &= CM_A(x) \wedge CM_A(y). \end{aligned}$$

Hence the proof. □

Corollary 3.18. *If $A \in FM(X)$. Then $A \in FMG(X)$ iff $A \circ A^{-1} = A$.*

Proof. Let $A \in FMG(X)$.

$$\begin{aligned} CM_{A\circ A^{-1}}(x) &= \vee\{CM_A(y) \wedge CM_{A^{-1}}(z); y, z \in X \text{ and } yz = x\} \\ &\geq \{CM_A(e) \wedge CM_{A^{-1}}(e^{-1}x)\} \\ &= CM_{A^{-1}}(x) = CM_A(x) \quad \text{ by } (A \in FMG(X)) \end{aligned}$$

So $A \subseteq A \circ A^{-1}$.
Then by 3.17 we get the proof. □

Proposition 3.19. *Let $A, B \in FMG(X)$. Then $A \cap B \in FMG(X)$.*

Proof. Let $x, y \in A \cap B \in FM(X)$
$\Rightarrow x, y \in A$ and $x, y \in B$
$\Rightarrow CM_A(xy^{-1}) \geq CM_A(x) \wedge CM_A(y^{-1})$ and $CM_B(xy^{-1}) \geq CM_B(x) \wedge CM_B(y^{-1})$.
Now

$$\begin{aligned} CM_{A\cap B}(xy^{-1}) &= CM_A(xy^{-1}) \wedge CM_B(xy^{-1}) \\ &\quad \text{by definition of intersection} \\ &\geq [CM_A(x) \wedge CM_A(y^{-1})] \wedge [CM_B(x) \wedge CM_B(y^{-1})] \\ &= [CM_A(x) \wedge CM_B(x)] \wedge [CM_A(y^{-1}) \wedge CM_B(y^{-1})] \\ &\quad \text{by commutative property of minimum} \\ &= [CM_A(x) \wedge CM_B(x)] \wedge [CM_A(y) \wedge CM_B(y)] \\ &\quad \text{ since } A, B \in FMG(X) \\ &= CM_{A\cap B}(x) \wedge CM_{A\cap B}(y) \\ &\quad \text{by definition of intersection} \end{aligned}$$

$$\Rightarrow CM_{A\cap B}(xy^{-1}) \geq CM_{A\cap B}(x) \wedge CM_{A\cap B}(y).$$

Hence by 3.8 $A \cap B \in FMG(X)$. Hence the proof. □

Remark 3.20. If $\{A_i; iI\}$ is a family of FMG over X, then their intersection $\cap_{i\in I} A_i$ is also a FMG over X.

Proposition 3.21. *Let $A, B \in FMG(X)$. Then $CM_{A\cup B}(x) = CM_{A\cup B}(x^{-1})$.*

Proof.

$$\begin{aligned} CM_{A\cup B}(x^{-1}) &= \vee\{CM_A(x^{-1}), CM_B(x^{-1})\} \\ &= \vee\{CM_A(x), CM_B(x)\} \text{ since } A, B \in FMG(X) \\ &= CM_{A\cup B}(x). \end{aligned}$$

Hence the proof.
From this it is clear that, if $A, B \in FMG(X)$ then $A \cup B \in FMG(X)$ iff

$$CM_{A\cup B}(xy) \geq CM_{A\cup B}(x) \wedge CM_{A\cup B}(y).$$

□

Corollary 3.22. *Let $A, B \in FMG(X)$. Then $A \cup B$ need not be an element of $FMG(X)$.*

Proof. $X = \{a, b, c, e\}$ is Klein's 4 group. Then

$$A = \{(.6, .4, .3, .1)/a, (.9, .8, .7, .5, .1, .1)/e\}$$
$$\text{and } B = \{(.8, .8, .5, .5, .1, .1)/b, (.9, .8, .7, .5, .1, .1)/e\}$$

are fuzzy multi groups.

$$A \cup B = \{(.6, .4, .3, .1)/a, (.8, .8, .5, .5, .1, .1)/b, (.9, .8, .7, .5, .1, .1)/e\}.$$

But $CM_{A\cup B}(c) < CM_{A\cup B}(a) \wedge CM_{A\cup B}(b)$ as $ab = c$ in Klein's 4 group. Then $A \cup B \notin FMG(X)$. □

Proposition 3.23. *Let $A \in FMG(X)$. Then $CM_A(xy^{-1}) \geq CM_{A\circ A}(xy)$.*

Proof.

$$\begin{aligned} CM_A(xy^{-1}) &\geq CM_{A\circ A^{-1}}(xy^{-1}) \text{ by } 3.16 \\ &= \vee_{z\in X}\{CM_A(x) \wedge CM_{A^{-1}}(z^{-1}xy^{-1})\} \text{ by } 3.3 \\ &= \{CM_A(x) \wedge CM_{A^{-1}}(y^{-1})\} \text{ (when } z = x) \\ &= \{CM_A(x) \wedge CM_A(y)\} \text{ by } 3.1 \\ &= CM_{A\circ A}(xy). \end{aligned}$$

□

Definition 3.24. Let $A, B \in FMG(X)$. Then A is said to be a sub-fuzzy multi group of B if $A \subseteq B$.

Example 3.25. $(Z_4, +_4)$ is a group. Then $A = \{(.6, .4, .3, .1)/2$, $(.8, .7, .7, .5, .1, .1)/1$, $(.8, .7, .7, .5, .1, .1)/3$, $(.9, .8, .7, .5, .1, .1)/0\}$ is a fuzzy multi group. And $B = \{(.6, .4, .3, .1)/2$, $(.7, .6, .5, .5, .1, .1)/1$, $(.7, .6, .5, .5, .1, .1)/3$, $(.9, .8, .7, .5, .1, .1)/0\}$ is a sub-fuzzy multi group of A.

Definition 3.26. Let $A \in FM(X)$. Then $\langle A\rangle = \{\wedge A_i : A \subseteq A_i \in FMG(X)\}$ is called the fuzzy muli-subgroup of X generated by A.

Remark 3.27. $\langle A\rangle$ is the smallest fuzzy multi-subgroup of X that contains A.

Proposition 3.28. *If $A \in FMG(X)$, and H is a subgroup of X, then $A|H$ (i.e., A restricted to H) $\in FMG(H)$ and is a fuzzy multi-subgroup of A.*

Proof. Let $x, y \in H$. Then $xy^{-1} \in H$. Now

$$CM_{A|H}(xy^{-1}) = CM_A(xy^{-1}) \geq CM_A(x) \wedge CM_A(y) = CM_{A|H}(x) \wedge CM_{A|H}(y).$$

Hence the proof by 3.18. The second part is trivial. □

4. Conclusion

In this paper the algebraic structure of Fuzzy multiset is introduced as Fuzzy Multigroup. Fuzzy Multigroup is a generalized case of fuzzy group and multi group. The various basic operations, definitions and theorems related to Fuzzy Multigroup have been discussed. The foundations which we made through this paper can be used to get an insight into the higher order structures of Group theory. In the applications point of view, since the concept of Fuzzy Multiset theory is well established in dealing with the problems in Information retrieval and Flexible querying, the algebraic structures defined on this will help to approach the problem with a different perspective. In this way the structure which we introduced is useful in higher Mathematics as well as Computer science, in both theoretical and application points of view.

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T. K. Shinoj (shinojthrissur@gmail.com)
Department of Mathematics, National Institute of Technology, Calicut-673 601, Kerala, India

Anagha Baby (anaghababy26@gmail.com)
Department of Mathematics, National Institute of Technology, Calicut-673 601, Kerala, India

Sunil Jacob John (sunil@nitc.ac.in)
Department of Mathematics, National Institute of Technology, Calicut-673 601, Kerala, India