Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 9, No. 1, (January 2015), pp. 65–76
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

On starplus nearly compact pseudo regular open fuzzy topology

A. Deb Ray, Pankaj Chettri

Received 31 December 2013; Revised 28 April 2014; Accepted 12 June 2014

Abstract. In this paper, using strong α-level topology, the concept of starplus nearly compact fuzzy sets is introduced and studied. A new fuzzy topology on function spaces, namely, starplus nearly compact pseudo regular open fuzzy topology is investigated and it is observed that such fuzzy topology is finer than pointwise fuzzy topology and weaker than any fuzzy topology that is pseudo δ-admissible on starplus near compacta.

2010 AMS Classification: 03E72, 54C35, 54A40

Keywords: Starplus nearly compact fuzzy set, Starplus nearly compact pseudo regular open fuzzy topology, Jointly pseudo fuzzy δ-continuous function, Pseudo δ-admissible fuzzy topology.

Corresponding Author: A. Deb Ray (atasi@hotmail.com)

1. Introduction

In earlier works, Peng [16] initiated the concept of fuzzy compact open topology on function spaces. In modification of this definition, taking into account another known form of fuzzy compactness, Jagar [8] has studied fuzzy function spaces. Following Peng, an article [9] on certain aspects of fuzzy function spaces was published by Kohli and Prasannan, where some questions were left open. In our paper [6], we have obtained some more results on fuzzy function spaces, following Jagar's definition, which partially answers Kohli and Prasannan's queries. Later, Kohli and Prasannan [10] introduced the notion of starplus compactness, based on strong α-level topology. In a previous communication [5], we have basically adopted their method of using functors between the category of fuzzy topological spaces and the category of topological spaces, to obtain interrelations between fuzzy regular open sets and regular open sets. Several mathematicians are doing fruitful researches in connection with fuzzy topology and intuitionistic fuzzy topology on function spaces, such as, [11], [1], [3], etc.

In the present paper, we have introduced the concept of starplus near compactness in terms of strong α-level topology. By utilizing starplus near compactness and pseudo regular open fuzzy sets [5], we have studied a new type of fuzzy topology on function spaces. It is observed that the starplus nearly compact pseudo regular open fuzzy topology is finer than pointwise fuzzy topology and it is fuzzy Hausdorff if the range space is so. Also, interrelations between starplus nearly compact pseudo regular open fuzzy topology and its strong α-level topology are investigated under various conditions. Defining pseudo δ-admissibility, we have shown that any pseudo δ-admissible fuzzy topology is finer than starplus nearly compact pseudo regular open fuzzy topology. Moreover, the concept of pseudo δ-admissibility ensures δ-admissibility of the corresponding strong α-level topology.

2. Preliminaries

To make this paper self sufficient, we state a few known definitions and results, that are required subsequently.
Let X be a non-empty set and I be the closed interval $[0, 1]$. A fuzzy set μ on X is a function on X into I. Let I^X denote the collection of all fuzzy sets on X. The support of a fuzzy set μ, denoted by $supp(\mu)$, is the crisp set $\{x \in X : \mu(x) > 0\}$. A fuzzy set is called a fuzzy point, denoted by x_α, where $0 < \alpha \leq 1$, and defined as
$$x_\alpha(z) = \begin{cases} \alpha, & \text{for } z = x \\ 0, & \text{otherwise.} \end{cases}$$
If f is a fuction from X into a set Y and A, B are fuzzy sets on X and Y respectively, then $f(A)$ and $f^{-1}(B)$ are fuzzy sets on Y and X respectively, defined by [19] $f(A)(y) = \begin{cases} \sup_{z \in f^{-1}(y)} A(z), when f^{-1}(y) \neq \varnothing \\ 0, \ otherwise \end{cases}$
and $f^{-1}(B)(x) = B(f(x))$.
A collection $\tau \subseteq I^X$ is called a *fuzzy topology* on X if (i) $0, 1 \in \tau$ (ii) $\forall\ \mu_1, \mu_2, ..., \mu_n \in \tau \Rightarrow \wedge_{i=1}^n \mu_i \in \tau$ (iii) $\mu_\alpha \in \tau$, $\forall\ \alpha \in \Lambda$ (where Λ is an index set) $\Rightarrow \vee \mu_\alpha \in \tau$. Then (X, τ) is called a fuzzy topological space (fts, for short) [4]. A collection $\tau \subseteq I^X$ is called a *fully stratified fuzzy topology* on X if (i) $\forall c \in I$, $\overline{c} \in \tau$ where $\overline{c}(x) = c$, for all $x \in X$ (ii) $\forall\ \mu_1, \mu_2, ..., \mu_n \in \tau \Rightarrow \wedge_{i=1}^n \mu_i \in \tau$; (iii) $\mu_\alpha \in \tau$, $\forall\ \alpha \in \Lambda$ (where Λ is an index set) $\Rightarrow \vee \mu_\alpha \in \tau$. Then (X, τ) is called a fully stratified fuzzy topological space [12]. A fuzzy point x_α is said to q-coincident with a fuzzy set A, denoted by $x_\alpha q A$ if $\alpha + A(x) > 1$. A fuzzy set A is said to be a q-neighbourhood (or, *q-nbd.*) of a fuzzy point x_α if there is a fuzzy open set B such that $x_\alpha q B \leq A$ [15]. A fuzzy set μ on a fts (X, τ) is called fuzzy regular open if $\mu = int(cl\mu)$ and complement of a fuzzy regular open set is fuzzy regular closed [2]. A fuzzy δ-open set is the union of fuzzy regular open sets [2]. A set μ is called fuzzy nearly compact if every fuzzy regular open cover has a finite subcover [2]. A fts is called fuzzy Hausdorff if for every pair of fuzzy points x_α, y_α on X with distinct supports there exist fuzzy open sets U and V on X such that $x_\alpha \in U$, $y_\alpha \in V$ and $U \wedge V = 0$ [10].

A set S in a topological space (X, τ) is called regular open if $S = int(clS)$ [18]. Complement of a regular open set is regular closed and a δ-open set in X is the union of regular open sets in X [18]. A set S is called nearly compact [17] if every

regular open cover of S has a finite subcover. A topological space (X, τ) is nearly compact iff every family $\mathcal{F}$ of regular closed sets with $\bigcap\{f : f \in \mathcal{F}\} = \varnothing$, there exist a finite subfamily $\mathcal{F}_0$ of $\mathcal{F}$ such that $\bigcap\{f : f \in \mathcal{F}_0\} = \varnothing$ [17]. Every regular closed set in nearly compact space is nearly compact and in a Hausdorff space every nearly compact set is δ-closed [17].
A function $f : X \to Y$ is called δ-continuous [14] , if $f^{-1}(U)$ is δ-open in X for every δ-open set U of Y.
Let X, Y be two topological spaces and $Z \subset Y^X$. A topology on Z is said to be jointly δ-continuous or δ-admissible if the evaluation mapping $P : Z \times X \to Y$ given by $P(f, x) = f(x)$, is δ-continuous, where $Z \times X$ is endowed with the product topology. For any nearly compact set C in X and regular open set U in Y, $T(C, U) = \{f \in Y^X : f(C) \subset U\}$. Then the collection $\{T(C, U)\}$ forms a subbase for some topology on Y^X called nearly compact regular open topology [7]. Let us denote this topology by N_R topology.

For a fuzzy set μ in X, the set $\mu^\alpha = \{x \in X : \mu(x) > \alpha\}$ is called the strong α-level set of X. In a topological space (X, τ), $w(\tau)$ denotes the collection of all lower semicontinuous functions from X into I, i.e., $w(\tau) = \{\mu : \mu^{-1}(a, 1] \in \tau, \forall\ a \in [0, 1]\}$ is a fully stratified fuzzy topology on X. In an fuzzy topological space (X, τ), for each $\alpha \in I_1 = [0, 1)$, the collection $i_\alpha(\tau) = \{\mu^{-1}(\alpha, 1] : \mu \in \tau\}$ is a topology on X and is called strong α-level topology [10]. A fuzzy open set μ on a fts (X, τ) is said to be pseudo regular closed fuzzy set if the strong α-level set μ^α is regular closed in $(X, i_\alpha(\tau)), \forall \alpha \in I_1$. A fuzzy set μ on a fts (X, τ) is said to be pseudo δ-open (δ-closed) fuzzy set if the strong α-level set μ^α is δ-open (respectively, δ-closed) in $(X, i_\alpha(\tau)), \forall \alpha \in I_1$ [5]. A function f from a fts X to a fts Y is pseudo fuzzy δ-continuous iff $f^{-1}(U)$ is pseudo δ-open fuzzy set on X for each pseudo δ-open fuzzy set U in Y [5].

3. Starplus nearly compact fuzzy sets

Definition 3.1. A fuzzy set μ on a fts (X, τ) is said to be starplus nearly compact if μ^α is nearly compact on $(X, i_\alpha(\tau))$, $\forall\ \alpha \in I_1$. A fts (X, τ) is said to be starplus nearly compact fts if $(X, i_\alpha(\tau))$ is nearly compact, $\forall\ \alpha \in I_1$.

It is clear from the definition that starplus compact [10] implies starplus nearly compact.

Theorem 3.2. *The pseudo fuzzy δ-continuous image of a starplus nearly compact fuzzy set is also so.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be pseudo fuzzy δ-continuous and μ, a starplus nearly compact fuzzy set on X. Then $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$ is δ-continuous for each $\alpha \in I_1$ [5]. As μ^α is nearly compact on $(X, i_\alpha(\tau))$, $f(\mu^\alpha)$ is nearly compact on $(Y, i_\alpha(\sigma))$. Again since $(f(\mu))^\alpha = f(\mu^\alpha), f(\mu)$ is starplus nearly compact fuzzy set in X. □

Theorem 3.3. *Every pseudo regular closed fuzzy set is starplus nearly compact on a starplus nearly compact fts.*

Proof. Let (X, τ) be starplus nearly compact fts and μ be a pseudo regular closed fuzzy set on (X, τ). Hence, μ^α is regular closed in the nearly compact topological space $(X,\ i_\alpha(\tau))$ for each $\alpha \in I_1$. As every regular closed set on nearly compact space is nearly compact, μ^α is nearly compact in $(X,\ i_\alpha(\tau))$. Hence, μ is starplus nearly compact on (X, τ). □

Theorem 3.4. *Every pseudo δ-closed fuzzy set is starplus nearly compact fuzzy set on starplus nearly compact fts.*

Proof. Straightforward. □

Theorem 3.5. *The union of a finite number of starplus nearly compact fuzzy sets is starplus nearly compact.*

Proof. Let μ, γ be two starplus nearly compact fuzzy sets on a fts (X, τ). For all $\alpha \in I_1$, μ^α, γ^α and hence $\mu^\alpha \cup \gamma^\alpha$ is nearly compact on $(X, i_\alpha(\tau))$. As $(\mu \vee \gamma)^\alpha = \mu^\alpha \cup \gamma^\alpha$, $(\mu \vee \gamma)^\alpha$ is nearly compact on $(X, i_\alpha(\tau))$. Hence, $(\mu \vee \gamma)$ is starplus nearly compact fuzzy set on (X, τ). □

Theorem 3.6. *If μ is starplus nearly compact fuzzy set on a fts (X, τ), for any pseudo regular closed fuzzy set ϑ in $(X, \tau), \mu \wedge \vartheta$ is starplus nearly compact.*

Proof. Considering that every regular closed set on nearly compact space is nearly compact and $(\mu \wedge \vartheta)^\alpha = \mu^\alpha \cap \vartheta^\alpha$, the theorem follows. □

Theorem 3.7. *Every starplus nearly compact fuzzy set on a Hausdorff fts is pseudo δ-closed fuzzy set.*

Proof. Let μ be starplus nearly compact fuzzy set on a Hausdorff fts (X, τ). Then, $(X, i_\alpha(\tau))$ is Hausdorff and μ^α is nearly compact in $(X, i_\alpha(\tau)), \forall\ \alpha \in I_1$. So, μ^α is δ-closed in $(X, i_\alpha(\tau)), \forall\ \alpha \in I_1$. Hence, μ is pseudo δ-closed in (X, τ). □

Theorem 3.8. *A fuzzy set μ on a fts (X, τ) is starplus nearly compact, if one of the following holds:*
(i) *τ is a finite fuzzy topology;*
(ii) *μ has finite support.*

Proof. Straightforward and hence omitted. □

Lemma 3.9 ([13])**.** *Let (X, τ) be a topological space and let $(X, w(\tau))$ be the fully stratified fts. Then for each $\alpha \in I_1, i_\alpha(w(\tau)) = \tau$.*

Theorem 3.10. *Let (X, τ) be a topological space and let $(X, w(\tau))$ be the corresponding fully stratified fts. Then for any starplus nearly compact fuzzy set μ in fts $(X, w(\tau))$, supp(μ) is nearly compact in (X, τ).*

Proof. As μ is starplus nearly compact fuzzy set on fts $(X, w(\tau)), \mu^\alpha$ is nearly compact in $i_\alpha(w(\tau)), \forall\ \alpha \in I_1$. By Lemma 3.9, μ^α is nearly compact in $\tau, \forall\ \alpha \in I_1$. In particular, $supp(\mu) = \mu^0$ is nearly compact in τ. □

We conclude this section with a couple of necessary conditions for starplus nearly compact fts, which play important role in determining when (X, τ) is not starplus nearly compact.

Theorem 3.11. *If a fts (X,τ) is starplus nearly compact then*
(i) every collection of pseudo regular open fuzzy sets $\{\mu_i\}$ with $\vee\mu_i = 1$, implies, there exist a finite subcollection $\{\mu_i : i = 1, 2, .., n\}$ such that, $\bigvee_{i=1}^{n} \mu_i = 1$.
(ii) every family $\mathcal{F}$ of pseudo regular closed fuzzy sets with $\wedge\{\mu_i : \mu_i \in \mathcal{F}\} = 0$ implies for each $\alpha \in I_1 - \{0\}$ there exist a finite subfamily $\mathcal{F}_0$ of $\mathcal{F}$ such that $\wedge\{\mu_i : \mu_i \in \mathcal{F}_0\} \leq \alpha$.

Proof. (i) Let (X,τ) be a starplus nearly compact fts and $\{\mu_i\}$ be a collection of pseudo regular open fuzzy sets with $\vee\mu_i = 1$. $(X, i_\alpha(\tau))$ is nearly compact $\forall\ \alpha \in I_1$. Now, $X = 1^\alpha = (\vee\mu_i)^\alpha = \cup\mu_i^\alpha$, which shows that $\{\mu_i^\alpha\}$ is a regular open cover of X. Since X is nearly compact, $\{\mu_i^\alpha\}$ has a finite subcover. i.e., there exist $i = 1, 2, .., n$ such that $X = \bigcup_{i=1}^{n} {\mu_i}^\alpha = (\bigvee_{i=1}^{n} \mu_i)^\alpha$. Hence, for each $\forall\ \alpha \in I_1$, $1^\alpha = (\bigvee_{i=1}^{n} \mu_i)^\alpha \Rightarrow 1 = \bigvee_{i=1}^{n} \mu_i$.
(ii) Let (X,τ) be a starplus nearly compact fts and $\mathcal{F}$ be a family of pseudo regular closed fuzzy sets with $\wedge\{\mu_i : \mu_i \in \mathcal{F}\} = 0$.
Hence, $\{{\mu_i}^\alpha\}$ is a collection of regular closed sets in $(X, i_\alpha(\tau)), \forall\ \alpha \in I_1 - \{0\}$. We claim that $\cap{\mu_i}^\alpha = \varnothing$. If not, let $x \in \cap{\mu_i}^\alpha$, then for all i, $x \in {\mu_i}^\alpha \Rightarrow \mu_i(x) > \alpha$. But, $\alpha < \mu_i(x) < 1 \Rightarrow \wedge\mu_i \neq 0$, which is a contradiction. Hence, $\cap{\mu_i}^\alpha = \varnothing$. As (X,τ) is starplus nearly compact fts, $(X, i_\alpha(\tau))$, $\forall\ \alpha \in I_1 - \{0\}$ is nearly compact. So, for each $\alpha \in I_1 - \{0\}$ there is a finite sub family $\mathcal{F}_0 = \{{\mu_i}^\alpha : i = 1, 2, ..., n\}$ of $\mathcal{F}$ such that $\cap\{{\mu_i}^\alpha : {\mu_i}^\alpha \in \mathcal{F}_0\} = \varnothing$. We claim that $\wedge\{\mu_i : i = 1, 2, ..., n\} \leq \alpha$. If possible let for $i = 1, 2, ..., n$, $\inf\{\mu_i(x)\} > \alpha$. So, $\mu_i(x) > \alpha$, for all $i = 1, 2, ..., n$. $\Rightarrow \cap{\mu_i}^\alpha \neq \varnothing$, which is a contradiction. Hence, $\wedge\{\mu_i : i = 1, 2, ..., n\} \leq \alpha$ □

4. Starplus nearly compact pseudo regular open fuzzy topology

Definition 4.1. Let (X,τ) and (Y,σ) be two fts and $\mathcal{F}$ be a nonempty collection of functions from X to Y. For each starplus nearly compact fuzzy set K on X and each pseudo regular open fuzzy set μ on Y, a fuzzy set K_μ on $\mathcal{F}$ is given by

$$K_\mu(g) = \inf_{x \in supp(K)} \mu(g(x))$$

The collection of all such K_μ forms a subbase for some fuzzy topology on $\mathcal{F}$, called starplus nearly compact pseudo regular open fuzzy topology and it is denoted by τ_{*NC}.

Example 4.2. Let $X = \{a, b\}$ and Y be any set. Define fuzzy topologies on X and Y respectively by the following collections τ and σ :
$\tau = \{0, 1, U\}$ where $U(a) = U(b) = 1/2$ and $\sigma = \{0, 1, V\}$ where $V(y) = 1/3$ for all $y \in Y$. Let $\mathcal{F}$ be the collection of all functions from X to Y. It is easy to see that $i_\alpha(\tau)$ is an indiscrete topology on X and hence every fuzzy set on X is starplus nearly compact. On the otherhand, $i_\alpha(\sigma)$ is also an indiscrete topology on Y and so, each of $0, 1$ and V are pseudo regular open in Y. Let K be any fuzzy set on X with *supp*(K) = singleton. Then for any $g \in \mathcal{F}$, $K_0(g) = 0$, $K_1(g) = 1$ and $K_V(g) = V(g(x)) = 1/3$ (*supp*$(K) = \{x\}$). If L is any fuzzy set on X with *supp*$(L) = X$, then $L_0(g) = 0$, $L_1(g) = 1$ and $L_V(g) = min(V(g(a)), V(g(b))) = 1/3$. Hence, the subbase for starplus nearly compact pseudo regular open fuzzy topology on $\mathcal{F}$ is $\{0, 1, W\}$, where $W(g) = 1/3$ for each $g \in \mathcal{F}$.

Definition 4.3 ([9])**.** Let (X,τ) and (Y,σ) be two fts and $\mathcal{F}$ be a nonempty collection of functions from X to Y. For each $x \in X$, define a map $e_x : \mathcal{F} \to Y$ by $e_x(g) = g(x)$. The map e_x is called the evaluation map at the point x. The initial fuzzy topology τ_p generated by the collection of maps $\{e_x : x \in X\}$ is called the pointwise fuzzy topology on $\mathcal{F}$.

Theorem 4.4 ([9])**.** *If Y is a fuzzy Hausdorff fts, then the fts $(\mathcal{F},\tau_p)$ is fuzzy Hausdorff.*

Remark 4.5. For all $x \in X$, $g \in \mathcal{F}$ and any fuzzy set μ on Y, $({e_x}^{-1}(\mu))(g) = \mu(e_x(g)) = \mu(g(x))$. So,

$$\begin{aligned} K_\mu(g) &= \inf\{\mu(g(x)) : x \in supp(K)\} \\ &= \inf\{({e_x}^{-1}(\mu))(g) : x \in supp(K)\} \\ &= (\inf\{{e_x}^{-1}(\mu) : x \in supp(K)\})(g) \end{aligned}$$

Hence, $K_\mu = \inf\{e_x^{-1}(\mu) : x \in supp(K)\} = \bigwedge_{x\in supp(K)} e_x^{-1}(\mu)$.

Theorem 4.6. *The starplus nearly compact pseudo regular open fuzzy topology τ_{*NC} is finer than the pointwise fuzzy topology τ_p on $\mathcal{F}$.*

Proof. As every fuzzy set with finite support is starplus compact, it is starplus nearly compact, and so the theorem follows. □

Theorem 4.7. *Let (X,τ) and (Y,σ) be two fts and $\mathcal{F}$ be endowed with starplus nearly compact pseudo regular open fuzzy topology. Then $(\mathcal{F},\tau_{*NC})$ is fuzzy Hausdorff when (Y,σ) is fuzzy Hausdorff.*

Proof. By Theorems 4.3 and 4.4, the theorem follows. □

Remark 4.8. If $\mathcal{F}$ is a collection of functions from X to Y, then we shall denote the set $\{f \in \mathcal{F} : f(T) \subset U\}$, by $[T,U]$, where $T \subseteq X$ and $U \subseteq Y$.

Theorem 4.9. *If K_μ is a subbasic open fuzzy set on τ_{*NC}, then $K_\mu^\alpha = [supp(K),\mu^\alpha]$.*

Proof. In view of Remark 4.5,

$$\begin{aligned} K_\mu^\alpha &= \bigwedge_{x\in supp(K)} (e_x^{-1}(\mu))^\alpha \\ &= \{f \in \mathcal{F} : \bigwedge_{x\in supp(K)} e_x^{-1}(\mu)(f) > \alpha\} \\ &= \{f \in \mathcal{F} : \bigwedge_{x\in supp(K)} \mu(f(x)) > \alpha\} \\ &= \{f \in \mathcal{F} : f(supp(K)) \subseteq \mu^\alpha\} \\ &= [supp(K),\mu^\alpha] \end{aligned}$$

□

Remark 4.10. Let (X,τ) and (Y,σ) be two fts and $\mathcal{F}$ be a nonempty collection of functions from X to Y. Let us denote by N_R^α the ordinary nearly compact regular open topology [7] when X is endowed with 0-level topology $i_0(\tau)$ and Y with $i_\alpha(\sigma), \forall\ \alpha \in I_1$.

Theorem 4.11. *The strong α-level topology $i_\alpha(\tau_{*NC})$ on $\mathcal{F}$, where $\alpha \in I_1$, is coarser than N_R^α topology on $\mathcal{F}$.*

Proof. Let $\beta = \bigwedge_{i=1}^n K^i_{\mu_i}$ be a basic fuzzy open set in τ_{*NC}. The strong α-level set β^α is given by $\bigcap_{i=1}^n [supp(K^i), \mu_i^\alpha]$. As each K^i is starplus nearly compact, $supp(K^i)$ is nearly compact in $i_0(\tau)$. Again $\forall\ \alpha \in I_1$, μ_i^α being regular open in $i_\alpha(\sigma)$, it follows that β^α is a basic open set in N_R^α. This proves the theorem. □

We shall take the same example as discussed in [10] to show that in general, two topologies $i_\alpha(\tau_{*NC})$ and N_R^α on $\mathcal{F}$ are not same.

Example 4.12. *Let X be an infinite set and τ be the fuzzy topology on X generated by the collection*
$\{(\frac{1}{2}\chi_U) \vee \chi_{X-U} : U \subseteq X$ *and* $(X-U)$ *is finite*$\}$. *Then*

$$i_\alpha(\tau) = \begin{cases} \text{the discrete topology on } X, & \text{for } \alpha \geq \frac{1}{2} \\ \text{the indiscrete topology on } X, & \text{for } \alpha < \frac{1}{2} \end{cases}$$

For an infinite subset T of X, $\chi_T = K$ (say), is a fuzzy set on X. Now, $supp(K) = T$ is compact and hence nearly compact in $i_0(\tau)$ but K^α is not nearly compact in $i_\alpha(\tau)$, for $\alpha \geq \frac{1}{2}$. Hence, K^α is not starplus nearly compact on (X,τ) and hence the fuzzy set $K_\mu \notin \tau_{*NC}$, for any pseudo regular open fuzzy set on Y. In fact, $K_\mu^\alpha = [supp(K), \mu^\alpha] = [T, \mu^\alpha] \in N_R^\alpha$. Hence, $i_\alpha(\tau_{*NC}) \neq N_R^\alpha$.

Theorem 4.13. *Let (X,T) and (Y,U) be topological spaces and let (X,T_f) and (Y,U_f) denote corresponding characteristic fts, respectively. Then for each $\alpha \in I_1$, $i_\alpha(\tau_{*NC}) = N_R$, where N_R is the ordinary nearly compact regular open topology on $\mathcal{F}$.*

Proof. $\forall\ \alpha \in I_1$, $i_\alpha(T_f) = T$ and $i_\alpha(U_f) = U, N_R^\alpha = N_R$. By Theorem 4.11 $i_\alpha(\tau_{*NC}) \subseteq N_R$. Now, let $[K,V] \in N_R$, where K is nearly compact in X and V is regular open in Y. Then the fuzzy set $S = \chi_K$ is starplus nearly compact in X and $\mu = \chi_V$ is pseudo regular open fuzzy set on Y, and $S_\mu^\alpha = [K,V]$, for each $\alpha \in I_1$. So, $[K,V] \in i_\alpha(\tau_{*NC})$. Hence, for each $\alpha \in I_1$, $i_\alpha(\tau_{*NC}) = N_R$. □

Theorem 4.14. *Let (X,T_X) and (Y,T_Y) be topological spaces and let N_R denote nearly compact regular open topology on $\mathcal{F}$. Then $\tau_{*NC} = w(N_R)$, where X and Y are endowed with the fuzzy topologies $w(T_X)$ and $w(T_Y)$, respectively.*

Proof. Let N_R be the nearly compact regular open topology on $\mathcal{F}$, where X and Y are endowed with the fuzzy topologies $w(T_X)$ and $w(T_Y)$, respectively. Let $K_\mu \in \tau_{*NC}$, where K is starplus nearly compact in $w(T_X)$ and μ is pseudo regular open fuzzy set on $w(T_Y)$. By Theorem 4.9, $K_\mu^\alpha = [supp(K), \mu^\alpha]$. Using Theorem 3.10, $supp(K)$ is nearly compact in T_X. As μ^α is regular open in T_Y, $[supp(K), \mu^\alpha]$ is a subbasic open set in N_R and so K_μ is lower semicontinuous. Hence, $K_\mu \in w(N_R)$. Consequently, $\tau_{*NC} \subseteq w(N_R)$.

Conversely, let $v \in w(N_R)$. Then $\forall\ \alpha \in I_1, v^{-1}(\alpha,1] \in N_R$. We will show that v is a τ_{*NC} neighborhood of each of its points. Let $g_\lambda \in v$. Then for $\alpha < \lambda$, $g \in v^{-1}(\alpha,1]$. Since $v^{-1}(\alpha,1] \in N_R$, there exist nearly compact sets $K_1, K_2, ...K_n$ in X and regular open sets $U_1, U_2, ...U_n$ in Y such that $g \in \cap_{i=1}^n [K_i, U_i] \subset v^{-1}(\alpha,1]$.

Now, the fuzzy set $S^i = \chi_{K_i}$ is starplus nearly compact in $w(T_X)$ with support K_i. Since for each $i = 1, 2..., n$, $U_i \in T_Y$ and $g(K_i) \subset U_i$, then the fuzzy set $\mu_i = (\chi_{U_i} \wedge \lambda) \in w(T_Y)$ such that $\mu_i(g(x)) = \lambda$, for each $x \in K_i$ and $\mu_i^\alpha = U_i$ for $\alpha < \lambda$. Hence, $g \in \cap_{i=1}^n [supp(S^i), \mu_i^\alpha] \subset v^{-1}(\alpha, 1]$. Now, using Theorem 4.11, $g_\lambda \in \wedge_{i=1}^n S^i_{\mu_i} \leq v$. Hence, v is a τ_{*NC} neighborhood of each of its points. □

5. Pseudo δ-admissible fuzzy topology

Definition 5.1. Let (X, τ) and (Y, σ) be two fts and $\mathcal{F}$ be a nonempty collection of functions from X to Y. A fuzzy topology T on $\mathcal{F}$ is said to be pseudo δ-admissible (pseudo δ-admissible on starplus near compacta) if a function $P : \mathcal{F} \times X \to Y$ given by $P(f, x) = f(x)$ is pseudo fuzzy δ-continuous (respectively, $P \mid_{\mathcal{F} \times supp(K)}$ is pseudo fuzzy δ-continuous for each starplus nearly compact set K on X), where $\mathcal{F} \times X$ is endowed with the product fuzzy topology.

Before we cite an example of pseudo δ-admissible fuzzy topology, we state a known result as follows :

Theorem 5.2 ([5]). *A function $f : (X, \tau) \to (Y\sigma)$ is pseudo fuzzy δ-continuous if and only if for every pseudo δ-open fuzzy set μ in Y with $(f(x))_\alpha q\mu$, there exists a pseudo δ-open fuzzy set ν in X, such that $x_\alpha q\nu$ and $f(\nu) \leq \mu$.*

We also observe the following:

Observation 5.3. (a) If μ and ν are pseudo δ-open then $\mu \times \nu$ is pseudo δ-open in the product space.
(b) If (Z, τ) is a fuzzy topological space where τ is generated by $\{\chi_U : U \subseteq Z\}$ then $(Z, i_\alpha(\tau))$ is a discrete space for all $\alpha \in I_1$.

Example 5.4. Let X and Y be two fts. The fuzzy topology generated by $\{\chi_U : U \subseteq Z\}$ on the collection Z of all pseudo fuzzy δ-continuous functions from X to Y, is pseudo δ-admissible.

Proof. Let μ be a pseudo δ-open fuzzy set with $(P(f, x))_\alpha = (f(x))_\alpha q\mu$. Then as f is pseudo fuzzy δ-continuous, there exists a pseudo δ-open fuzzy set ν in X, such that $x_\alpha q\nu$ and $f(\nu) \leq \mu$. Choose $\beta \in I_1$ such that $\beta + \alpha > 1$. It is easy to see that

$$(f_\beta)^\gamma = \begin{cases} \{f\}, & if\ \beta > \gamma \\ \varnothing, & otherwise \end{cases}.$$

Consequently, f_β is pseudo δ-open in Z. Then $f_\beta \times \nu$ is a pseudo δ-open fuzzy set such that $P(f_\beta \times \nu) \leq \mu$ and $(f, x)_\alpha q(f_\beta \times \nu)$. □

Theorem 5.5. *If T is a pseudo δ-admissible fuzzy topology on $\mathcal{F}$, then for each $\alpha \in I_1$, the strong α-level topology $i_\alpha(T)$ is jointly δ-continuous.*

Proof. Let T be a pseudo δ-admissible fuzzy topology on $\mathcal{F}$. So, $P : \mathcal{F} \times X \to Y$ given by $P(f, x) = f(x)$ is pseudo fuzzy δ-continuous. Hence, $P : (\mathcal{F}, i_\alpha(T)) \times (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma)$ is δ-continuous, for all $\alpha \in I_1$. This shows that $i_\alpha(T)$ is jointly δ-continuous for all $\alpha \in I_1$. □

Definition 5.6. Let μ be a fuzzy set on a fts (X, τ). The subspace fuzzy topology on μ is given by $\{v \mid_{supp(\mu)}: v \in \tau\}$ and is denoted by τ_μ. The pair $(supp(\mu), \tau_\mu)$ is called the subspace fts of μ.

Definition 5.7. Let (X, τ) and (Y, σ) be two fts. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be pseudo fuzzy δ-continuous function on a fuzzy set μ on X, if $f \mid_{supp(\mu)}$ is pseudo fuzzy δ-continuous, where $supp(\mu)$ is endowed with the subspace fuzzy topology τ_μ.

Theorem 5.8. *Let (X, τ) and (Y, σ) be two fts and $\mathcal{F} \subset Y^X$. Then every fuzzy topology on $\mathcal{F}$ which is pseudo δ-admissible on starplus near compacta is finer than the starplus nearly compact pseudo regular open fuzzy topology τ_{*NC} on $\mathcal{F}$.*

Proof. Let $(\mathcal{F}, T)$ be pseudo δ-admissible on starplus near compacta. Let K_μ be any subbasic fuzzy open set on τ_{*NC} where K is starplus near compact on X and μ is pseudo regular open fuzzy set on Y. The function $P \mid_{\mathcal{F}\times supp(K)}: \mathcal{F} \times supp(K) \to Y$ given by $P(f, x) = f(x)$ is pseudo fuzzy δ-continuous. So, $P \mid^{-1}_{\mathcal{F}\times supp(K)} (\mu)$ is pseudo δ-open fuzzy set on $\mathcal{F} \times supp(K)$. For simplicity of notation, instead of $P \mid_{\mathcal{F}\times supp(K)}$ we shall use the symbol P only. Let f_α be any fuzzy point in K_μ. i.e., $K_\mu(f) \geq \alpha \Rightarrow \inf\{\mu(f(x)) : x \in supp(K)\} \geq \alpha$. We now prove, $f_\alpha \times \chi_{supp(K)} \leq P^{-1}(\mu)$. Now,

$$\begin{aligned}(f_\alpha \times \chi_{supp(K)})(g, t) &= f_\alpha(g) \wedge \chi_{supp(K)}(t) \\ &= \begin{cases} \alpha, & iff = g \ and \ t \in supp(K) \\ 0, & otherwise. \end{cases}\end{aligned}$$

Again, $P^{-1}(\mu)(g, t) = \mu(P(g, t)) = \mu(g(t)) \geq \alpha$. Hence, $f_\alpha \times \chi_{supp(K)} \leq P^{-1}(\mu)$. Consider the first projection $\Pi_1 : (\mathcal{F}, i_\alpha(T)) \times (X, i_\alpha(\tau)) \to (\mathcal{F}, i_\alpha(T))$. Now,

$$\begin{aligned}(\Pi_1(P^{-1}(\mu)))^\alpha &= \{f : \Pi_1(P^{-1}(\mu))(f) > \alpha\} \\ &= \{f : \sup_{\Pi_1(g,t)=f} [P^{-1}(\mu)(g, t) > \alpha]\} \\ &= \{f : \sup_{g=f} P^{-1}(\mu)(g, t) > \alpha\} \\ &= \{f : P^{-1}(\mu)(f, t) > \alpha\} \\ &= \{\Pi_1(f, t) : (f, t) \in (P^{-1}(\mu))^\alpha\} \\ &= \Pi_1(P^{-1}(\mu))^\alpha\end{aligned}$$

So, $(\Pi_1(P^{-1}(\mu)))^\alpha = \Pi_1(P^{-1}(\mu))^\alpha$. As $P^{-1}(\mu)$ is pseudo δ-open fuzzy set, $(P^{-1}(\mu))^\alpha$ is δ-open. Π_1 being projection mapping,
$(\Pi_1(P^{-1}(\mu)))^\alpha = \Pi_1(P^{-1}(\mu))^\alpha$ is δ-open set. So, $\Pi_1(P^{-1}(\mu))$ is pseudo δ-open fuzzy set. As, $f_\alpha \in K_\mu$ we have, $K_\mu(f) \geq \alpha \Rightarrow \inf\{\mu(f(x)) : x \in supp(K)\} \geq \alpha$. So, $\mu(f(s)) \geq \alpha, \forall\ s \in supp(K)$.

$$\begin{aligned}\Pi_1(P^{-1}(\mu))(g, s) &= \sup_{\Pi_1(g,s)=f} [P^{-1}(\mu)(g, s)] \\ &= P^{-1}(\mu)(f, s) \\ &= \mu(f(s)) \\ &\geq \alpha.\end{aligned}$$

i.e., $f_\alpha \leq \Pi_1(P^{-1}(\mu))$. Now,

$$\begin{aligned}
\Pi_1(P^{-1}(\mu)) \times \chi_{supp(K)})(g,t) &= \Pi_1(P^{-1}(\mu))(g) \wedge \chi_{supp(K)}(t) \\
&= \sup_{\Pi_1(f,t)=g} [P^{-1}(\mu)(g,t)] \wedge \chi_{supp(K)}(t) \\
&= P^{-1}(\mu)(g,t) \wedge \chi_{supp(K)}(t) \\
&= \begin{cases} \mu(g(t)), & if\ t \in supp(K) \\ 0, & otherwise. \end{cases} \\
&\leq P^{-1}(\mu)(g,t).
\end{aligned}$$

Hence, $\Pi_1(P^{-1}(\mu)) \times \chi_{supp(K)} \leq P^{-1}(\mu)$. Now,

$$\begin{aligned}
\Pi_1(P^{-1}(\mu))(g) &= \sup_{\Pi_1(h,s)=g} [P^{-1}(\mu)(h,s)\ : s \in supp(K)] \\
&= P^{-1}(\mu)(g,s) \\
&= \mu(g(s)), \forall\ s \in supp(K) \\
&= \inf_{t \in supp(K)} \mu(g(t)) \\
&= K_\mu(g)
\end{aligned}$$

So, for any fuzzy point g_λ on $\Pi_1(P^{-1}(\mu)) \in T$, $g_\lambda \in K_\mu$, which proves the theorem. □

We now prove a lemma that is required for the final result of this section.

Lemma 5.9. *A function $f : (X, \tau) \to (Y, \sigma)$ is pseudo fuzzy δ-continuous iff for any pseudo δ-open fuzzy set μ on Y with $(f(x))_\alpha q\mu$ there exists a pseudo δ-open fuzzy set ν on X with $x_\alpha q\nu$ and $f(\nu) \leq \mu$.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be pseudo fuzzy δ-continuous and μ be any pseudo δ-open fuzzy set on Y with $(f(x))_\alpha q\mu$. Then $\mu(f(x))+\alpha > 1$. i.e., $(f^{-1}(\mu))(x)+\alpha > 1$. So, $x_\alpha q f^{-1}(\mu)$. Since f pseudo fuzzy δ-continuous, $f^{-1}(\mu)$ is pseudo δ-open in X. Now, $f(f^{-1}(\mu)) \leq \mu$ is always true, which proves the result.
Conversely, let the condition hold. We shall prove $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$ is δ-continuous, $\forall\ \alpha \in I_1$, which is sufficient to prove $f : (X, \tau) \to (Y, \sigma)$ is pseudo fuzzy δ-continuous. Let μ^α be δ-open in Y with $f(x) \in \mu^\alpha$. i.e., $\mu f(x) > \alpha$. Let us consider a fuzzy set ζ on Y by $\zeta(z) = \begin{cases} 1, & \text{if } \mu(t) > \alpha \\ \alpha, & \text{otherwise.} \end{cases}$ For $\beta > \alpha$, $y \in \zeta^\beta \Rightarrow \zeta(y) > \beta$ $\Rightarrow \zeta(y) > \alpha \Rightarrow \zeta(y) = 1 \Rightarrow \mu(y) > \alpha$. So, $(\zeta)^\beta \subseteq (\mu)^\alpha$. Similarly, we have $(\mu)^\alpha \subseteq (\zeta)^\beta$. So, $(\mu)^\alpha = (\zeta)^\beta$. For $\alpha > \beta, \forall\ y \in Y$, by definition of ζ, $\zeta(y) > \beta \Rightarrow y \in \zeta^\beta$. i.e, $Y \subseteq \zeta^\beta$. So, $Y = \zeta^\beta$. Also, for $\alpha = \beta$, $(\mu)^\alpha = (\zeta)^\beta$. So, $\zeta^\beta = \begin{cases} \mu^\alpha, & \text{if } \beta \geq \alpha \\ Y, & \text{if } \beta < \alpha. \end{cases}$
Hence, ζ^β is δ-open, $\forall\ \beta \in I_1$. Thus, ζ is pseudo δ-open fuzzy set on Y. As $\mu f(x) > \alpha$, $\zeta(f(x)) = 1 > \alpha \Rightarrow \zeta(f(x)) + (1 - \alpha) > 1 \Rightarrow (f(x))_{1-\alpha} q\zeta$. By the given condition there exist a pseudo δ-open fuzzy set ν on X with $x_{1-\alpha} q\nu$ and $f(\nu) \leq \zeta$. i.e., with $1 - \alpha + \nu(x) > 1 \Rightarrow \nu(x) > \alpha \Rightarrow x \in \nu^\alpha$ and $(f(\nu))^\alpha \subseteq \zeta^\alpha$,

as $f(\nu) \leq \zeta \Rightarrow (f(\nu))^\alpha \subseteq \zeta^\alpha$. Hence, $f : (X, i_\alpha(\tau)) \to (Y, i_\alpha(\sigma))$ is δ-continuous, $\forall\ \alpha \in I_1$ and so, $f : (X, \tau) \to (Y, \sigma)$ is pseudo fuzzy δ-continuous. □

Theorem 5.10. *In a fully stratified Hausdorff fts (X, τ), if each member of $\mathcal{F} \subset Y^X$ is pseudo fuzzy δ-continuous on every starplus nearly compact fuzzy set of X, then the starplus nearly compact pseudo regular open fuzzy topology τ_{*NC} on $\mathcal{F}$ is pseudo δ-admissible on starplus near compacta.*

Proof. Let (X, τ) be a fully stratified Hausdorff fts. Let $f \in \mathcal{F}$ and $(f(x))_\alpha q\mu$, for any pseudo δ-open fuzzy set on Y. Since K is starplus nearly compact in $(supp(K), \tau_K)$, where τ_K is the subspace fuzzy topology on $supp(K)$, K^α is nearly compact on $(supp(K), i_\alpha(\tau_K)), \forall\ \alpha \in I_1$. Again $f : (supp(K), \tau_K) \to (Y, \sigma)$ is pseudo fuzzy δ-continuous, $f : (supp(K), i_\alpha(\tau_K)) \to (Y, i_\alpha(\sigma))$ is δ-continuous, $\forall\ \alpha \in I_1$. (X, τ) being Hausdorff fts, $(supp(K), \tau_K)$ is Hausdorff, $\forall\ \alpha \in I_1$. Hence, there exist a nearly compact *nbd.* M_α of x in $(supp(K), i_\alpha(\tau_K))$ and $f(M_\alpha) \subset \mu^\alpha$, as μ^α is open in $i_\alpha(\tau_K), \forall\ \alpha \in I_1$. Now, we choose $\beta > 1 - \lambda$. Let $K^* = (\chi_{M_\beta} \wedge \beta)$. As,

$$\begin{aligned}(K^*)^\alpha &= (\chi_{M_\beta} \wedge \beta)^\alpha \\ &= \{x : (\chi_{M_\beta} \wedge \beta)(x) > \alpha\} \\ &= \{x : \beta > \alpha \textit{ and } x \in M_\beta\} \\ &= \begin{cases} M_\beta, & if\ \beta > \alpha \\ \varnothing, & if\ \beta \leq \alpha \end{cases}\end{aligned}$$

K^* is starplus nearly compact in the subspace fts $(supp(K), \tau_K)$ such that $f(K^*) \leq \mu$. Now,

$$\begin{aligned}(K^*_\mu \times \chi_{supp(K)})(f, x) &= K^*_\mu(f) \wedge \chi_{supp(K)}(x) \\ &= \begin{cases} K^*_\mu(f), & if\ x \in supp(K) \\ 0, & otherwise \end{cases} \\ &= \inf\{\mu f(z) : z \in supp(K^*)\}, x \in supp(K) \\ &> 1 - \lambda,\ as\ (f(x))_\lambda q\mu \textit{ and } z \in supp(K^*) \Rightarrow \mu f(z) > \beta.\end{aligned}$$

Hence, $(K^*_\mu \times \chi_{supp(K)}) \mid_{(\mathcal{F} \times supp(K))}$ is a *q-nbd.* of the fuzzy point $(f, x)_\lambda$ on $(\mathcal{F} \times supp(K))$. Also, it can be seen that $(K^*_\mu \times \chi_{supp(K)}) \mid_{(\mathcal{F} \times supp(K))} \leq P^{-1}(\mu)$. Hence the theorem. □

6. Conclusion and future direction

In this paper we have shown that starplus nearly compact pseudo regular open fuzzy topology, denoted by τ_{*NC}, on function spaces is finer than pointwise fuzzy topology. We have also observed the interplay between the strong α-level topology obtained from τ_{*NC} and the nearly compact-regular open topology (i.e., N_R-topology) on the function space. It is worth mentioning that strong α-level topology of τ_{*NC} is in general coarser than N_R-topology. So, the question remains open that under what condition (or, conditions) strong α-level topology of τ_{*NC} coincide with N_R-topology?. A sufficient condition for τ_{*NC} to be δ-admissible is also established here. It is also an open problem to figure out precisely when τ_{*NC} becomes δ-admissible.

Acknowledgements. The authors are grateful to the referee for valuable comments and suggestions.

References

[1] A. I. Aggour and F. E. Attounsi, Fuzzy topological properties on fuzzy function spaces, Appl. Math. Inf. Sci. Lett. 1(2) (2013) 1–5.
[2] K. K. Azad, On fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl. 82 (1981) 14–32.
[3] S. Bayramov and C. Gunduz, Intuitionistic fuzzy topology on function spaces, Ann. Fuzzy Math. Inform. 3(1) (2012) 19–30.
[4] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.
[5] A. Deb Ray and P. Chettri, On pseudo δ-open fuzzy sets and pseudo fuzzy δ-continuous functions, Int. J. Contemp. Math. Sci. 5(29) (2010) 1403–1411.
[6] A. Deb Ray and P. Chettri, A note on fuzzy compact open topology, NBU J. Math. 2(2) (2009) 119–130.
[7] S. Ganguly and K. Dutta, A new topology on function spaces, J. Pure Math. 17 (2000) 109–118
[8] G. Jagar, On fuzzy function spaces, Internat J. Math. Math Sc., 22(4)(1999),727-737.
[9] J. K. Kohli and A. R. Prasannan, Fuzzy topologies on function spaces, Fuzzy Sets and Systems 116(3) (2000) 415–420.
[10] J. K. Kohli and A. R. Prasannan, Starplus-compactness and starplus-compact open fuzzy topologies on function spaces, J. Math. Anal. Appl. 254 (2001) 87–100.
[11] J. K. Kohli and A. R. Prasannan, Fuzzy uniformities on function spaces, Appl. Gen. Topol. 7(2) (2006) 177–189.
[12] M. J. Liang and Y. C. Chun, Fuzzy topological groups, Fuzzy Sets and Systems 12 (1984) 289–299.
[13] R. Lowen, Initial and final fuzzy topologies and the fuzzy Tychonoff theorem, J. Math. Anal. Appl. 58 (1977) 11–21.
[14] T. Noiri, On δ-continuous functions, J. Korean Math. Soc. 16(2) (1980) 161–166.
[15] P. M. Pu and Y. M. Liu, Fuzzy topology I. Neighbourhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl. 76 (1980) 571–599.
[16] Y. Peng, Topological structure of a fuzzy function space-the point wise convergent topology and compact-open topology, Kexue Tongbao(English Ed.) 29(3) (1984) 289–292.
[17] M. K. Singal and A. Mathur, On nearly compact spaces, Boll. Un. Mat. Ital. 4(6)(1969) 702–710.
[18] N. Velicko, H-closed topological spaces, Mat. Sb. (N.S.) 70(112) (1966) 98–112.
[19] L. A. Zadeh, Fuzzy Sets, Information and Control 8 (1965) 338–353.

A. Deb Ray (atasi@hotmail.com)
Department of Mathematics, West Bengal State University, Berunanpukuria, Malikapur, North 24 Parganas- 700126, India

Pankaj Chettri (pankajct@gmail.com)
Department of Mathematics, Sikkim Manipal Institute of Technology, Majitar, Rangpoo, East Sikkim, 737136, India