Annals of Fuzzy Mathematics and Informatics
Volume 9, No. 1, (January 2015), pp. 43–63
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Semigroups characterized by the properties of $(\alpha, \beta)^*$-fuzzy ideals

Saleem Abdullah, Muhammad Aslam, Bijan Davvaz

Received 12 April 2014; Revised 26 May 2014; Accepted 7 July 2014

Abstract. In this paper, a generalization of fuzzy ideals in semigroups of type $(\alpha, \beta)^*$-fuzzy ideals are introduced, where $\alpha, \beta \in \{<, \gamma_k, < \vee\gamma_k, < \wedge\gamma_k\}$ with $\alpha \neq < \wedge\gamma_k$. We prove some fundamental results that determine the relation between these notions and ideals of semigroups. Moreover, we give characterizations of different classes of semigroups by the properties of $(\alpha, \beta)^*$-fuzzy ideals. We prove that a semigroup H is regular if and only if every $(<, < \vee\gamma_k)^*$-fuzzy ideal of H is k-idempotent. We also prove that a semigroup H is intra-regular if and only if every every $(<, < \vee\gamma_k)^*$-fuzzy ideal of H is k-idempotent. We give further characterizations of regular and intra-regular semigroups in term of $(<, < \vee\gamma_k)^*$-fuzzy right (resp. left) ideals.

2010 AMS Classification: 20N20, 20M19

Keywords: Semigroup, Regular semigroup, Intra-regular semigroup, Fuzzy set, (α, β)-fuzzy ideal.

Corresponding Author: Saleem Abdullah (saleemabdullah81@yahoo.com)

1. Introduction

The applications of fuzzy technology in information processing is already important and it will certainly increase the importance in the future. Our aim is to promote research and the development of fuzzy technology by studying the generalized fuzzy semigroups. The goal is to explain new methodological developments in fuzzy semigroups which will also be of growing importance in the future. This paper can be a bridge passing from the theory of semigroups to the theory of generalized fuzzy semigroups. According to this paper, fundamental notions of semigroups, like the notion of regular semigroups, can be expressed in terms of fuzzy sets, and the corresponding properties, like the important conditions which characterize the regular semigroups. On the other hand, the application of fuzzy semigroups in information processing, like in fuzzy coding, fuzzy languages and fuzzy finite state machines, have been proved to be useful.

So far as we know, the term "semigroup" first appeared in mathematical literature on page 8 of J.A. de Seguier's book, element de la Theorie des Groups Abstraits (Paris 1904), and the first paper about semigroups was a brief one by. Dickson in 1905 [3]. But theory realy began in 1928 with the publication of a paper of fundamental importance by A. K. Suschkewitsch [22]. In current terminology, he showed that every finite semigroup contains a "Kernal" (a simple ideal), and he completely determind the structure of finite semigroups.

The topic of these investigations belongs to the theoretical soft computing (fuzzy structures). Indeed, it is well known that semigroups are basic structures in many applicative branches of information sciences, like finite state machines or automata, formal languages, coding theory and others. To be more precise, structure theory of finite state machines (automata) is based on transformation semigroups. Algebraic automata theory provide powerful decomposition results for finite transformation semigroups by cascade simple parts. Any full transformation semigroup is regular. The class of completely regular semigroups is an important subclass of the class of regular semigroups. A completely regular semigroup is the one in which every element is in some subgroup of the semigroup. We know that l-semigroups appear in classical relevent logics, some non-classical logics, and multi model arrow logics. We note that complete l-semigroups appear in natural way in the theory of formal language and programing, the theory of automata, the theory of fuzzy automata and the theory of fuzzy sets. We know that some recent investigations of l-semigroups are nearly connected with algebraic logic and non classical logics.

Due to these possiblities of applications, semigroups are presently extensively investigated in fuzzy setting. The concept of fuzzy set was introduced by Zadeh of his paper [23] of 1965. The literature on fuzzy set theory and its applications have been growing rapidly by now to several papers. Application of fuzzy set theory have been studies in different areas, that is, artificial intelligence, computer sciences, control engineering, expert, robotics, automata theory, finite state machine, graph theory and others. Investigation of fuzzy groups and fuzzy semigroups have a long tradition, starting with the beginning of fuzzy era. In paper [18] by Rosenfeld as the first one about fuzzy groups, and the detail study of fuzzy groups can be found in [13]. The study of fuzzy semigroups was studied by Kuroki in his classical paper [5] and Kuroki initiated fuzzy ideals, bi-ideals, semi-prime ideals, quasi-ideals of semigroups [6, 7, 8, 9, 10, 11, 12]. A systematic exposition of fuzzy semigroups was given by Mordeson et.al. [14], and they have found theoretical results on fuzzy semigroups and their use in fuzzy finite state machines, fuzzy languages and fuzzy coding. Mordeson and Malik studied monograph in [15] deals with the applications of fuzzy approach to the concepts of formal languages and automata. Using the notions "*belongs to*" relation ($\in$) introduced by Pu and Lia [17]. In [16], Murali proposed the concept of a fuzzy point belonging to a fuzzy subset under natural equivalence on fuzzy subset. Bhakat and Das introduced the concepts of (α, β)-fuzzy subgroups by using the "*belongs to*" relation ($\in$) and "*quasi-coincident with*" relation (q) between a fuzzy point and a fuzzy subgroup, and defined an ($\in$, $\in \vee q$)-fuzzy subgroup of a group [1]. In [4], Khan et. al generalized the concept of fuzzy ideals of semigroups by using the notions of a fuzzy point and fuzzy set. In [19], Saeid and Jun initiated new definition of fuzzy BCK/BCI-algebras by considering two relations called besideness ($<$) and non-quasi-coincidence (γ) between an anti fuzzy point and a fuzzy subset. Saeid et al. in [20], generalized the concept of non-quasi-coincidence (γ). Saeid et al. introduced a new generalization of fuzzy BCK/BCI-algebras by considering besideness ($<$) and generalized non-quasi-coincidence (γ_k) for $k \in [-1, 0)$.

In this paper, we use the idea of Saeid et al. to semigroup and defined a new generalization of fuzzy semigroups. A generalization of fuzzy ideals in semigroups of type $(\alpha, \beta)^*$-fuzzy ideals are introduced, where $\alpha, \beta \in \{<, \gamma_k, < \vee\gamma_k, < \wedge\gamma_k\}$ with $\alpha \neq < \wedge\gamma_k$. We prove some fundamental results that determine the relation between these notions and ideals of semigroups. Moreover, we give characterizations of different classes of semigroups by the properties of $(\alpha, \beta)^*$-fuzzy ideals.We prove that a semigroup $\mathcal{H}$ is regular if and only if every $(<, < \vee\gamma_k)^*$-fuzzy ideal of $\mathcal{H}$ is k-idempotent. We also prove that a semigroup $\mathcal{H}$ is regular if and only if every $(<, < \vee\gamma_k)^*$-fuzzy ideal of $\mathcal{H}$ is k-idempotent. We give further characterization of regular and intra-regular semigroup in term of $(<, < \vee\gamma_k)^*$-fuzzy right (resp. left) ideals.

2. Preliminaries

A semigroup is an algebraic system $(\mathcal{H}, .)$ consisting of a non-empty set $\mathcal{H}$ together with an associative binary operation ".". By a subsemigroup of $\mathcal{H}$ we mean a non-empty subset A of $\mathcal{H}$ such that $A^2 \subseteq A$. A non-empty subset A of $\mathcal{H}$ is called a left (right) ideal of $\mathcal{H}$ if $\mathcal{H}A \subseteq A$ ($A\mathcal{H} \subseteq A$). A non-empty subset A of $\mathcal{H}$ is called a two-sided ideal or simply an ideal of $\mathcal{H}$ if it is both a left and a right ideal of $\mathcal{H}$. A non-empty subset Q of $\mathcal{H}$ is called a quasi-ideal of S if $Q\mathcal{H} \cap \mathcal{H}Q \subseteq Q$. A subsemigroup B of a semigroup $\mathcal{H}$ is called a bi-ideal of $\mathcal{H}$ if $B\mathcal{H}B \subseteq B$. A non-empty subset B of $\mathcal{H}$ is called a generalized bi-ideal of $\mathcal{H}$ if $B\mathcal{H}B \subseteq B$. A subsemigroup I of a semigroup $\mathcal{H}$ is called an interior ideal of $\mathcal{H}$ if $\mathcal{H}I\mathcal{H} \subseteq I$. Obviously every one-sided ideal of a semigroup $\mathcal{H}$ is a quasi-ideal, every quasi-ideal is a bi-ideal and every bi-ideal is a generalized bi-ideal. An element a of a semigroup $\mathcal{H}$ is called a regular element if there exists an element x in $\mathcal{H}$ such that $a = axa$. A semigroup $\mathcal{H}$ is called regular if every element of $\mathcal{H}$ is regular. It is well known that for a regular semigroup the concepts of quasi-ideal, bi-ideal and generalized bi-ideal coincide.

A fuzzy subset λ of a universe X is a function from X into the unit closed interval $[0, 1]$, i.e. $\lambda : X \to [0, 1]$. For any two fuzzy subsets λ and μ of X, $\lambda \leq \mu$ means that, for all $x \in X$, $\lambda(x) \leq \mu(x)$. The symbols $\lambda \wedge \mu$, and $\lambda \vee \mu$ will mean the following fuzzy subsets of X

$$(\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x),$$
$$(\lambda \vee \mu)(x) = \lambda(x) \vee \mu(x),$$

for all $x \in \mathcal{H}$.

Definition 2.1 ([21])**.** A fuzzy set $\mathcal{A}$ of a semigroup $\mathcal{H}$ is called an anti fuzzy subsemigroup if

$(\forall x, y \in \mathcal{H}) \, (\mathcal{A}(xy) \leq \max\{\mathcal{A}(x), \mathcal{A}(y)\})$.

Definition 2.2 ([21]). A fuzzy set $\mathcal{A}$ of a semigroup $\mathcal{H}$ is called an anti fuzzy left (resp. right) if

$(\forall x, y \in \mathcal{H})\,(\mathcal{A}\,(xy) \leq \mathcal{A}\,(y)$ (resp. $\mathcal{A}\,(xy) \leq \mathcal{A}\,(x)))$.

A fuzzy set $\mathcal{A}$ is called an anti fuzzy ideal of $\mathcal{H}$ if it is both anti fuzzy left and anti fuzzy right ideal of $\mathcal{H}$.

Proposition 2.3 ([21]). *Let $\mathcal{A}$ be a fuzzy set in $\mathcal{H}$. Then, $\mathcal{A}$ is an anti fuzzy subsemigroup (resp. left, right) ideal of $\mathcal{H}$ if and only if the set $L\,(\mathcal{A}:\, t) := \{x \in \mathcal{H} : \mathcal{A}\,(x) \leq t\}$ is a subsemigroup (resp. left, right) ideal of $\mathcal{H}$, $t \in [0, 1]$.*

In [17], Pu and Liu provided a concept of "*belongs to*" relation ($\in$) and "*quasi-coincident with* relation (q) between a fuzzy point and a fuzzy subset.

A fuzzy set $\mathcal{A}$ in $\mathcal{H}$ of the form

$$\mathcal{A}\,(x) = \begin{cases} t \in [0, 1) & \text{if } x = y, \\ 1 & \text{if } x \neq y \end{cases}$$

is called an anti fuzzy point with support x and value t and is denoted by x_t. A fuzzy set $\mathcal{A}$ in $\mathcal{H}$ is said to be non-unit if there exists $x \in \mathcal{H}$ such that $\mathcal{A}(x) < 1$.

Definition 2.4. An anti-fuzzy point x_t is said to beside (resp. be non-quasi-coincident with) a fuzzy set $\mathcal{A}$, denoted by $x_t {<} \mathcal{A}$ (resp. $x_t \gamma \mathcal{A}$) if $\mathcal{A}(x) \leq t$ (resp. $\mathcal{A}(x) + t < 1$). We say that $<$ (resp. γ) is a beside relation (resp. non-quasicoincident with relation) between anti-fuzzy points and fuzzy sets.

If $x_t {<} \mathcal{A}$ or $x_t \gamma \mathcal{A}$ (resp. $x_t {<}\, \mathcal{A}$ and $x_t \gamma \mathcal{A}$), we say that $x_t {<} \vee \gamma \mathcal{A}$ (resp. $x_t {<} \wedge \gamma A$).

The concept of a semigroup is needed in order to present the concept of regular fuzzy expressions to the various models of fuzzy automata.

Let $\mathcal{U}$ be a non-empty finite set and $\mathcal{U}^*$ be the free semigroup generated by $\mathcal{U}$, with identity $\curlywedge$. Let $\mathcal{H}$ be a semigroup. A function $f^A : \mathcal{U}^* \longrightarrow \mathcal{H}$ is called an $\mathcal{H}$-language (S–language) over $\mathcal{U}$. An $\mathcal{H}$-automation (S-automation) over $\mathcal{U}$ is a 4-tuple $A = (\mathcal{R}, p, h, g)$, where $\mathcal{R}$ is a finite non-empty set, p is a function from $\mathcal{R} \times \mathcal{U} \times \mathcal{R}$ into $\mathcal{H}$, h and g are functions from $\mathcal{R}$ into $\mathcal{H}$.

Let $A = (\mathcal{R}, p, h, g)$ be an $\mathcal{H}$-automation (S-automation) over $\mathcal{U}$.

(1) Let p^* be the function from $\mathcal{R} \times \mathcal{U}^* \times \mathcal{R}$ into $\mathcal{H}$. i.e, $p^* : \mathcal{R} \times \mathcal{U}^* \times \mathcal{R} \longrightarrow [0, 1]$ defined respectively as follows:

$$\begin{aligned} p^*\,(r, \curlywedge, r') &= \begin{cases} 1 & \text{if } r = r' \\ 0 & \text{if } r \neq r' \end{cases} \\ p^*\,(r, au, r') &= \bigvee_{r'' \in \mathcal{R}} \{p\,(r, a, r'') \wedge p\,(r'', u, r)\}, \end{aligned}$$

for all $r, r' \in \mathcal{R}$ and $u \in \mathcal{U}^*$.

(2) Let f^A be the function from $\mathcal{U}^*$ into $\mathcal{H}$ defined by

$$f^A\,(x) = \bigvee_{r \in \mathcal{R}} \bigvee_{r \in \mathcal{R}} h\,(r)\,.p^*\,(r, x, r')\,.g\,(r'),$$

for all $x \in \mathcal{U}^*$.

In this definition, $\mathcal{U}$ is the set of input symbols, $\mathcal{R}$ is the set of states, $p\,(r, x, r')$ is the grade of membership that the next state is $r\prime$ given that the present state is r

and input a is applied, $h(r)$ is the grade of membership that r is the initial state and $g(r)$ is the grade of membership that r is an accepting state. Moreover, f^A is the S-language over $\mathcal{U}$ accepted by A. The above definition includes many of the various existing models of fuzzy automata. These models may be obtained by appropriate choice of S, *e.g.*, max–min automation: $S_M = (S, .)$, where S is a subset of the real number system, usually $[0, 1]$ and $ab = a \wedge b$.

We recall some well known results of semigroup theory.

Theorem 2.5. *For a semigroup $\mathcal{H}$ the following conditions are equivalent:*
(1) *$\mathcal{H}$ is regular,*
(2) *$A \cap B = AB$ for every right ideal A and every left ideal B of $\mathcal{H}$.*

Theorem 2.6. *For a semigroup $\mathcal{H}$ the following conditions are equivalent:*
(1) *$\mathcal{H}$ is regular,*
(2) *$A \cap B \subseteq AB$ for every right ideal A and every left ideal B of $\mathcal{H}$.*

Theorem 2.7. *For a semigroup $\mathcal{H}$ the following conditions are equivalent:*
(1) *$\mathcal{H}$ is regular and intra-regular,*
(2) *$A \cap B \subseteq AB \cap BA$ for every right ideal A and every left ideal B of $\mathcal{H}$.*

3. Major section

In what follows, let $\mathcal{H}$ denotes a semigroup, unless otherwise is stated. Now, we generalize the notion of non-quasi-coincidence. Let k denote an arbitrary element of $(-1, 0]$ unless otherwise specified. To say $x_t\gamma_k\mathcal{A}$, we mean that $\mathcal{A}(x) + t + k < 1$.

If $x_t{<}\mathcal{A}$ or $x_t\gamma_k\mathcal{A}$ (resp. $x_t{<}\,\mathcal{A}$ and $x_t\gamma_k\mathcal{A}$), we say that $x_t{<}\vee\gamma_k\mathcal{A}$ (resp. $x_t{<}\wedge\gamma_k A$).

Definition 3.1. A fuzzy set $\mathcal{A}$ in $\mathcal{H}$ is called $(\alpha, \beta)^*$-fuzzy subsemigroup of $\mathcal{H}$, if it satisfies the following implication:

$(\forall x, y \in \mathcal{H})(\forall t_1, t_2 \in [0, 1)(x_t\alpha\mathcal{A},\, y_t\alpha\mathcal{A} \Rightarrow (xy)_{t_1\vee t_2}\beta\mathcal{A})$.

Definition 3.2. A fuzzy set $\mathcal{A}$ in $\mathcal{H}$ is called $(\alpha, \beta)^*$-fuzzy left (resp. right) ideal of $\mathcal{H}$, if it satisfies the following implication:

$(\forall x, y \in \mathcal{H})(\forall t \in [0, 1)(x_t\alpha\mathcal{A},\, y \in \mathcal{H} \Rightarrow (xy)_t\,\beta\mathcal{A}$ (resp. $(yx)_t < \vee\gamma_k\mathcal{A}$).

Remark 3.3. We can esily construct twelve different types of fuzzy subsemigroups and fuzzy ideals of semigroups by replacing $\alpha, \beta \in \{<, \gamma_k, <\vee\gamma_k, <\wedge\gamma_k\}$ with $\alpha \neq\, <\wedge\gamma_k$ in definitions 3.1 and 3.2. Hence, we obtain the following types of fuzzy subsemigroups and fuzzy ideals of semigroups.

$(<,\ <)^*,\ (<,\ \gamma_k)^*,\ (<,\ <\vee\gamma_k)^*,\ (<,\ <\wedge\gamma_k)^*$;
$(\gamma_k, <)^*,\ (\gamma, \gamma_k)^*,\ (\gamma, <\vee\gamma_k)^*,\ (\gamma, <\wedge\gamma_k)^*$;
$(<\vee\gamma_k,\ <)^*,\ (<\vee\gamma_k,\ \gamma_k)^*,\ (<\vee\gamma_k,\ <\vee\gamma_k)^*,\ (<\vee\gamma_k,\ <\wedge\gamma_k)^*$.

Theorem 3.4. *Let $\mathcal{A}$ be a non-unit $(\alpha, \beta)^*$-fuzzy subsemigroup of $\mathcal{H}$. Then, $\mathcal{A}^* = \{x \in \mathcal{H} : \mathcal{A}(x) < 1\}$ is a subsemigroup of $\mathcal{H}$.*

Proof. Let $\mathcal{A}$ be a non-unit $(\alpha, \beta)^*$-fuzzy subsemigroup of $\mathcal{H}$ and let $x, y \in \mathcal{A}^*$. Then, $\mathcal{A}(x) < 1$ and $\mathcal{A}(y) < 1$. Suppose that $\mathcal{A}(xy) = 1$. If $\alpha \in \{<, <\vee\gamma_k\}$, then $x_{\mathcal{A}(x)}\alpha\mathcal{A}$ and $y_{\mathcal{A}(y)}\alpha\mathcal{A}$ but $(xy)_{\max\{\mathcal{A}(x),\mathcal{A}(y)\}}\overline{\beta}\mathcal{A}$, for every $\beta \in \{<,\ \gamma_k, <\vee\gamma_k, <\wedge\gamma_k\}$. Now, if $x_{0.5}\gamma_k\mathcal{A}$ and $y_0\gamma_k\mathcal{A}$ but $(xy)_{\max\{0.5,0.5\}} = (xy)_{0.5}\overline{\beta}\mathcal{A}$ for $k = -0.5$, which is a contradiction. Hence $\mathcal{A}(xy) < 1$. □

Theorem 3.5. *Let $\mathcal{A}$ be a non-unit $(\alpha,\beta)^*$-fuzzy left (resp. right) ideal of $\mathcal{H}$. Then, $\mathcal{A}^* = \{x \in \mathcal{H} : \mathcal{A}(x) < 1\}$ is a left (resp. right) ideal of $\mathcal{H}$.*

Proof. Let $\mathcal{A}$ be a non-unit $(\alpha,\beta)^*$-fuzzy left ideal of $\mathcal{H}$ and let $y \in \mathcal{A}^*$. Then, $\mathcal{A}(x) < 1$. Suppose that $\mathcal{A}(xy) = 1$. If $\alpha \in \{<, < \vee\gamma_k\}$, then $y_{\mathcal{A}(y)}\alpha\mathcal{A}$ but $(xy)_{\mathcal{A}(y)}\overline{\beta}\mathcal{A}$, for every $\beta \in \{<,\ \gamma_k,\ < \vee\gamma_k,\ < \wedge\gamma_k\}$. Now, if $y_{0.5}\gamma_k\mathcal{A}$ but $(xy)_{0.5}\overline{\beta}\mathcal{A}$ for $k = -0.5$, which is a contradiction. Hence $\mathcal{A}(xy) < 1$. □

Theorem 3.6. *Let $\mathcal{H}$ be a right (resp. left) zero semigroup and let $\mathcal{A}$ be a non-unit $(\gamma,\gamma_k)^*$-fuzzy subsemigroup of $\mathcal{H}$. Then, $\mathcal{A}$ is constant on $\mathcal{A}^*$.*

Proof. Let y be an element of $\mathcal{H}$ such that $\mathcal{A}(y) = \wedge\{\mathcal{A}(x) : x \in \mathcal{H}\}$. Then, $y \in \mathcal{A}^*$. Suppose that there exists $x \in \mathcal{A}^*$ such that $t_x = \mathcal{A}(x) \neq \mathcal{A}(y) = t_y$. Then, $t_x > t_y$. Choose $t_1, t_2 \in (0,1]$ and $k \in (-1,0]$ such that

$$1 - t_y - k > t_1 > 1 - t_x - k > t_2.$$

Then, $y_{t_1}\gamma_k\mathcal{A}$ and $x_{t_2}\gamma_k\mathcal{A}$ but $(yx)_{\max\{t_1,t_2\}} = x_{t_1}\overline{\gamma_k}\mathcal{A}$ (resp., $(xy)_{\max\{t_1,t_2\}} = x_{t_1}\overline{\gamma_k}\mathcal{A}$) because $\mathcal{H}$ is a right (resp., left) zero semigroup. This is a contradiction. Thus, $\mathcal{A}(x) = \mathcal{A}(e)$ for all $x \in \mathcal{A}^*$, and therefore $\mathcal{A}$ is a constant on $\mathcal{A}^*$. □

If we put $k = 0$ in the previous theorem, then we get the following corollary.

Corollary 3.7. *Let $\mathcal{H}$ be a right (resp. left) zero semigroup and let $\mathcal{A}$ be a non-unit $(\gamma,\gamma)^*$-fuzzy subsemigroup of $\mathcal{H}$. Then, $\mathcal{A}$ is constant on $\mathcal{A}^*$.*

Theorem 3.8. *Let $\mathcal{H}$ be a right (resp. left) zero semigroup and let $\mathcal{A}$ be a non-unit $(\gamma,\gamma_k)^*$-fuzzy left (resp. right) ideal of $\mathcal{H}$. Then, $\mathcal{A}$ is constant on $\mathcal{A}^*$.*

Proof. Let $\mathcal{H}$ be a right (resp. left) zero semigroup and $\mathcal{A}$ be a non-unit $(\gamma,\gamma_k)^*$-fuzzy left (resp. right) ideal of $\mathcal{H}$. Let $y \in \mathcal{H}$ such that $\mathcal{A}(y) = \wedge\{\mathcal{A}(x) : x \in \mathcal{H}\}$. Then, $y \in \mathcal{A}^*$. Suppose that there exists $x \in \mathcal{A}^*$ such that $t_x = \mathcal{A}(x) \neq \mathcal{A}(y) = t_y$. Then, $t_x > t_y$. Choose $t \in (0,1]$ and $k \in (-1,0]$ such that

$$1 - t_y - k > t > 1 - t_x - k.$$

Then, $y_t\gamma\mathcal{A}$ but $(yx)_t = x_t\overline{\gamma_k}\mathcal{A}$ (resp. $(xy)_t = x_t\overline{\gamma_k}\mathcal{A}$), which is a contradiction. Thus, $\mathcal{A}(x) = \mathcal{A}(e)$ for all $x \in \mathcal{A}^*$. This completes the proof. □

For $k = 0$.

Corollary 3.9. *Let $\mathcal{H}$ be a right (resp. left) zero semigroup and let $\mathcal{A}$ be a non-unit $(\gamma,\gamma)^*$-fuzzy left (resp. right) ideal of $\mathcal{H}$. Then, $\mathcal{A}$ is constant on $\mathcal{A}^*$.*

Theorem 3.10. *Let $\mathcal{A}$ be a fuzzy set in a semigroup $\mathcal{H}$. Then, $\mathcal{A}$ is a non-unit $(\gamma,\gamma_k)^*$-fuzzy subsemigroup of $\mathcal{H}$ if and only if there exists a subsemigroup S of $\mathcal{H}$ such that*

$$\mathcal{A}(x) := \begin{cases} t \in [0,1) & \text{if } x \in S, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Let $\mathcal{A}$ be a non-unit $(\gamma,\gamma_k)^*$-fuzzy subsemigroup of $\mathcal{H}$. Then, we have $\mathcal{A}(x) < 1$ for all $x \in \mathcal{H}$. In ordered to prove that there exists a subsemigroup of $\mathcal{H}$, we will show that $\mathcal{A}^*$ is a subsemigroup of $\mathcal{H}$. Since $\mathcal{A}$ is a non-unit $(\gamma,\gamma_k)^*$-fuzzy subsemigroup of $\mathcal{H}$. So, by Theorem 3.4, $\mathcal{A}^*$ is a subsemigroup of $\mathcal{H}$. It is clear that $\mathcal{A}(x) < 1$ and by Theorem 3.6, we have

$$\mathcal{A}(x) := \begin{cases} \mathcal{A}(x) & \text{if } x \in \mathcal{A}^* \\ 1 & \text{otherwise.} \end{cases}$$

Conversely, let S be a subsemigroup of $\mathcal{H}$ which satisfy

$$\mathcal{A}(x) := \begin{cases} t \in [0,1) & \text{if } x \in S, \\ 1 & \text{otherwise.} \end{cases}$$

Assume that $x_{t_1}\gamma\mathcal{A}$ and $y_{t_2}\gamma\mathcal{A}$ for some $t_1, t_2 \in [0,1)$. Then, $\mathcal{A}(x) + t_1 < 1$ and $\mathcal{A}(y) + t_2 < 1$, and so $\mathcal{A}(x) \neq 1$ and $\mathcal{A}(y) \neq 1$. Thus, $x, y \in S$ and so $xy \in S$. It follows that $\mathcal{A}(xy) + \max\{t_1, t_2\} + k < 1$ so that $(xy)_{\max\{t_1, t_2\}}\gamma_k\mathcal{A}$. Therefore, $\mathcal{A}$ is a non-unit $(\gamma, \gamma_k)^*$-fuzzy subsemigroup of $\mathcal{H}$. □

Theorem 3.11. *Let S be a subsemigroup of $\mathcal{H}$ and $\mathcal{A}$ a fuzzy set in $\mathcal{H}$ such that*
(*i*) $(\forall x \in \mathcal{H}\backslash S)(\mathcal{A}(x) = 1)$,
(*ii*) $(\forall x \in S)(\mathcal{A}(x) \leq \frac{1-k}{2})$.
Then, $\mathcal{A}$ is an $(<, < \vee\gamma_k)^*$-fuzzy subsemigroup, $(<, < \vee\gamma_k)^*$-fuzzy subsemigroup and $(< \vee\gamma_k, < \vee\gamma_k)^*$-fuzzy subsemigroup of $\mathcal{H}$.

Proof. (a) : Let $x, y \in \mathcal{H}$ and $t_1, t_2 \in [0,1)$ be such that $x_{t_1}\gamma\mathcal{A}$ and $y_{t_2}\gamma\mathcal{A}$, that is, $\mathcal{A}(x) + t_1 < 1$ and $\mathcal{A}(y) + t_2 < 1$. If $xy \notin S$, then $xy \in \mathcal{H}\backslash S$, i.e., $\mathcal{A}(x) = 1$ or $\mathcal{A}(y) = 1$. It follows that $t_1 + k < 0$ and $t_2 + k < 0$. This is a contradiction, and so $xy \in S$. Hence, $\mathcal{A}(xy) \leq \frac{1-k}{2}$. If $\max\{t_1, t_2\} < \frac{1-k}{2}$, then $\mathcal{A}(xy) + \max\{t_1, t_2\} + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and thus $(xy)_{\max\{t_1,t_2\}}\gamma_k\mathcal{A}$. If $\max\{t_1, t_2\} \geq \frac{1-k}{2}$, then $\mathcal{A}(xy) \leq \frac{1-k}{2} \leq \max\{t_1, t_2\}$ and so $(xy)_{\max\{t_1,t_2\}} < \mathcal{A}$. Therefore, $(xy)_{\max\{t_1,t_2\}} < \vee\gamma_k\mathcal{A}$.

(b) : Let $x, y \in \mathcal{H}$ and $t_1, t_2 \in [0,1)$ be such that $x_{t_1} < \mathcal{A}$ and $y_{t_2} < \mathcal{A}$, that is, $\mathcal{A}(x) \leq t_1$ and $\mathcal{A}(y) \leq t_2$. Then, $x, y \in S$ this implies that $xy \in S$. If $xy \notin S$, then $xy \in \mathcal{H}\backslash S$, i.e., $\mathcal{A}(x) = 1$ or $\mathcal{A}(y) = 1$. It follows that $t_1 + k < 0$ and $t_2 + k < 0$. This is a contradiction, and so $xy \in S$. Thus, $\mathcal{A}(xy) \leq \frac{1-k}{2}$. If $\max\{t_1, t_2\} < \frac{1-k}{2}$, then $\mathcal{A}(xy) + \max\{t_1, t_2\} + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and thus $(xy)_{\max\{t_1,t_2\}}\gamma_k\mathcal{A}$. If $\max\{t_1, t_2\} \geq \frac{1-k}{2}$, then $\mathcal{A}(xy) \leq \frac{1-k}{2} \leq \max\{t_1, t_2\}$ and so $(xy)_{\max\{t_1,t_2\}} < \mathcal{A}$. Therefore, $(xy)_{\max\{t_1,t_2\}} < \vee\gamma_k\mathcal{A}$.

(c) : It follows from (a) and (b).

Thus, $\mathcal{A}$ is an $(\alpha, < \vee\gamma_k)^*$-fuzzy subsemigroup of S. □

If we put $k = 0$ in the previous theorem, we get the following corollary.

Corollary 3.12. *Let S be a subsemigroup of $\mathcal{H}$ and $\mathcal{A}$ a fuzzy set in $\mathcal{H}$ such that*
(*i*) $(\forall x \in \mathcal{H}\backslash S)(\mathcal{A}(x) = 1)$,
(*ii*) $(\forall x \in S)(\mathcal{A}(x) \leq 0.5)$.
Then, $\mathcal{A}$ is an $(\alpha, < \vee\gamma)^*$-fuzzy subsemigroup of $\mathcal{H}$.

Theorem 3.13. *Let L be a left (resp. right) ideal of $\mathcal{H}$ and $\mathcal{A}$ a fuzzy set in $\mathcal{H}$ such that*
(*i*) $(\forall x \in \mathcal{H}\backslash L)(\mathcal{A}(x) = 1)$,
(*ii*) $(\forall x \in L)(\mathcal{A}(x) \leq \frac{1-k}{2})$.

Then, $\mathcal{A}$ is an $(\gamma, <\vee\gamma_k)^*$-fuzzy left (resp. right) ideal, $(<, <\vee\gamma_k)^*$-fuzzy left (resp. right) ideal and $(<\vee\gamma_k, <\vee\gamma_k)^*$-fuzzy left (resp. right) ideal of $\mathcal{H}$.

Proof. (a) : Let $x, y \in \mathcal{H}$ and $t \in [0,1)$ be such that $x_t\gamma\mathcal{A}$ and $y \in \mathcal{H}$. Then, $\mathcal{A}(x) + t < 1$ and so $x \in L$ and $y \in \mathcal{H}$ implies that $xy \in L$. Thus, $\mathcal{A}(xy) \leq \frac{1-k}{2}$. If $t < \frac{1-k}{2}$, then $\mathcal{A}(xy) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and thus $(xy)_t\gamma_k\mathcal{A}$. If $t \geq \frac{1-k}{2}$, then $\mathcal{A}(xy) \leq \frac{1-k}{2} \leq t$ and so $(xy)_t< \mathcal{A}$. Therefore, $(xy)_t< \vee\gamma_k\mathcal{A}$.

(b) : Let $x, y \in \mathcal{H}$ and $t \in [0,1)$ be such that $x_t< \mathcal{A}$ and $y \in \mathcal{H}$. Then, $\mathcal{A}(x) \leq t$ and so $x \in L$ and $y \in \mathcal{H}$ implies that $xy \in L$. Thus, $\mathcal{A}(xy) \leq \frac{1-k}{2}$. If $t \geq \frac{1-k}{2}$, $\mathcal{A}(xy) \leq \frac{1-k}{2} \leq t$ and so $(xy)_t< \mathcal{A}$. If $t \geq \frac{1-k}{2}$, then $\mathcal{A}(xy)+t+k < \frac{1-k}{2}+\frac{1-k}{2}+k = 1$ and thus $(xy)_t\gamma_k\mathcal{A}$. Therefore, $(xy)_t< \vee\gamma_k\mathcal{A}$.

(c) : It follows from (a) and (b). □

If we put $k = 0$ in the previous theorem, then we get the following corollary.

Corollary 3.14. *Let L be a left (resp. right) ideal of $\mathcal{H}$ and $\mathcal{A}$ a fuzzy set in $\mathcal{H}$ such that*

(i) $(\forall x \in \mathcal{H}\backslash L)(\mathcal{A}(x) = 1)$,

(ii) $(\forall x \in L)(\mathcal{A}(x) \leq 0.5)$.

Then, $\mathcal{A}$ is an $(\alpha, <\vee\gamma)^*$-fuzzy left (resp. right) ideal of $\mathcal{H}$.

Definition 3.15. A fuzzy set $\mathcal{A}$ in $\mathcal{H}$ is called $(<, <\vee\gamma_k)^*$-fuzzy subsemigroup of $\mathcal{H}$, if it satisfies the following implication:

$(\forall x, y \in \mathcal{H})(\forall t_1, t_2 \in [0,1)(x_t<\mathcal{A},\ y_t<\mathcal{A} \Rightarrow (xy)_{t_1\vee t_2} < \vee\gamma_k\mathcal{A})$.

Definition 3.16. A fuzzy set $\mathcal{A}$ in $\mathcal{H}$ is called $(<, <\vee\gamma_k)^*$-fuzzy left (resp. right) ideal of $\mathcal{H}$, if it satisfies the following implication:

$(\forall x, y \in \mathcal{H})(\forall t \in [0,1)(x_t<\mathcal{A},\ y \in \mathcal{H} \Rightarrow (xy)_t < \vee\gamma_k\mathcal{A}$ (resp. $(yx)_t < \vee\gamma_k\mathcal{A}$).

Example 3.17. Consider the semigroup $\mathcal{H} = \{a, b, c, d\}$ with the following Cayley table:

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let $\mathcal{A}_1$ be a fuzzy set in $\mathcal{H}$ defiend by $\mathcal{A}_1(a) = 0.4$, $\mathcal{A}_1(b) = 0.7$, $\mathcal{A}_1(c) = 0.5 = \mathcal{A}_1(d)$. For $k = -0.4$, it is routine calculation verify that $\mathcal{A}_1$ is an $(<, <\vee\gamma_k)^*$-fuzzy subsemigroup of $\mathcal{H}$. Now, let $\mathcal{A}_2$ be a fuzzy set in $\mathcal{H}$ defined by $\mathcal{A}_2(a) = 0.4$, $\mathcal{A}_2(b) = 0.6$, $\mathcal{A}_2(c) = 0.5$, $\mathcal{A}_2(d) = 0.45$. For $k = -0.3$, by routine calculation $\mathcal{A}_2$ is an $(<, <\vee\gamma_k)^*$-fuzzy left and right ideal of $\mathcal{H}$. It is also clear that $\mathcal{A}_1$ is not $(<, <)^*$-fuzzy subsemigroup,$(<, \gamma_k)^*$-fuzzy subsemigroup of $\mathcal{H}$. Similarly, $\mathcal{A}_2$ is not $(<, <)^*$-fuzzy subsemigroup,$(<, \gamma_k)^*$-fuzzy subsemigroup of $\mathcal{H}$.

Theorem 3.18. *Let $\mathcal{A}$ be a fuzzy set in $\mathcal{H}$. Then, $\mathcal{A}$ is a $(<, <\vee\gamma_k)^*$-fuzzy subsemigroup of $\mathcal{H}$ if and only if it satisfies the following*

$$(\forall x, y \in \mathcal{H})\left(\mathcal{A}(xy) \leq \max\left\{\mathcal{A}(x),\ \mathcal{A}(y),\ \tfrac{1-k}{2}\right\}\right).$$

Proof. Suppose that $\mathcal{A}$ is a $(<, < \vee\gamma_k)^*$-fuzzy subsemigroup of $\mathcal{H}$ and let $x, y \in \mathcal{H}$ be such that $\mathcal{A}(xy) > \max\left\{\mathcal{A}(x), \mathcal{A}(y), \frac{1-k}{2}\right\}$. Then, for $t \in [0, 1)$, we have $\mathcal{A}(xy) > t > \max\left\{\mathcal{A}(x), \mathcal{A}(y), \frac{1-k}{2}\right\}$. If $\max\{\mathcal{A}(x), \mathcal{A}(y)\} > \frac{1-k}{2}$, then $\mathcal{A}(xy) > t > \max\{\mathcal{A}(x), \mathcal{A}(y)\}$. It follows that $x_t < \mathcal{A}$ and $y_t < \mathcal{A}$ but $(xy)_{t\vee t} = (xy)_t \overline{< \vee\gamma_k}\mathcal{A}$, which is a contradiction. If $\max\{\mathcal{A}(x), \mathcal{A}(y)\} \leq \frac{1-k}{2}$, then $x_{\frac{1-k}{2}} < \mathcal{A}$ and $y_{\frac{1-k}{2}} < \mathcal{A}$ but $(xy)_{\frac{1-k}{2}} \overline{<}\mathcal{A}$ and also $(xy)_{\frac{1-k}{2}} \overline{\gamma_k}\mathcal{A}$, which implies that $(xy)_{\frac{1-k}{2}} \overline{< \vee\gamma_k}\mathcal{A}$, which contradicts. Hence,

$$\mathcal{A}(xy) \leq \max\left\{\mathcal{A}(x), \mathcal{A}(y), \tfrac{1-k}{2}\right\} \text{ for all } x, y \in \mathcal{H}.$$

Conversely, suppose that $\mathcal{A}$ satisfies the given condition. Let $x, y \in \mathcal{H}$ and $t_1, t_2 \in [0, 1)$ be such that $x_{t_1} < \mathcal{A}$ and $y_{t_2} < \mathcal{A}$. Then, $\mathcal{A}(x) \leq t_1$ $\mathcal{A}(y) \leq t_2$. Since

$$\begin{aligned} \mathcal{A}(xy) &\leq \max\left\{\mathcal{A}(x), \mathcal{A}(y), \tfrac{1-k}{2}\right\}, \\ \mathcal{A}(xy) &\leq \max\left\{t_1, t_2, \tfrac{1-k}{2}\right\}. \end{aligned}$$

If $\max\{t_1, t_2\} \geq \frac{1-k}{2}$, then $\mathcal{A}(xy) \leq \max\{t_1, t_2\}$ and so $(xy)_{t_1\vee t_2} < \mathcal{A}$. If $\max\{t_1, t_2\} < \frac{1-k}{2}$, then $\mathcal{A}(xy) + \max\{t_1, t_2\} + k < \frac{1-k}{2} + \frac{1-k}{2} + k < 1$ and so $(xy)_{t_1\vee t_2}\gamma_k\mathcal{A}$. Hence, $(xy)_{t_1\vee t_2} < \vee\gamma_k\mathcal{A}$. This completes the proof. □

If we put $k = 0$ in the previous theorem, then we get the following corollary.

Corollary 3.19. *Let $\mathcal{A}$ be a fuzzy set in $\mathcal{H}$. Then, $\mathcal{A}$ is a $(<, < \vee\gamma)^*$-fuzzy subsemigroup of $\mathcal{H}$ if and only if it satisfies the following*

$$(\forall x, y \in \mathcal{H})\left(\mathcal{A}(xy) \leq \max\{\mathcal{A}(x), \mathcal{A}(y), 0.5\}\right).$$

Theorem 3.20. *Let $\mathcal{A}$ be a fuzzy set in $\mathcal{H}$. Then, $\mathcal{A}$ is a $(<, < \vee\gamma_k)^*$-fuzzy left (resp. right) ideal of $\mathcal{H}$ if and only if it satisfies the following*

$$(\forall x, y \in \mathcal{H})\left(\mathcal{A}(xy) \leq \max\left\{\mathcal{A}(y), \tfrac{1-k}{2}\right\} \ (\textit{resp. } \mathcal{A}(xy) \leq \max\left\{\mathcal{A}(x), \tfrac{1-k}{2}\right\}\right).$$

Proof. Suppose that $\mathcal{A}$ is a $(<, < \vee\gamma_k)^*$-fuzzy left ideal of $\mathcal{H}$ and let $x, y \in \mathcal{H}$ be such that $\mathcal{A}(xy) > \max\left\{\mathcal{A}(y), \frac{1-k}{2}\right\}$. Then, for $t \in [0, 1)$, we have $\mathcal{A}(xy) > t > \max\left\{\mathcal{A}(y), \frac{1-k}{2}\right\}$. If $\mathcal{A}(y) \geq \frac{1-k}{2}$, then $y_t < \mathcal{A}$ but $(xy)_t \overline{< \vee\gamma_k}\mathcal{A}$, which is a contradiction. If $\mathcal{A}(y) < \frac{1-k}{2}$, then $y_{\frac{1-k}{2}} < \mathcal{A}$ but $(xy)_{\frac{1-k}{2}} \overline{< \vee\gamma_k}\mathcal{A}$, which contradicts to our hypothesis. Hence, $\mathcal{A}(xy) \leq \max\left\{\mathcal{A}(y), \frac{1-k}{2}\right\}$ for all $x, y \in \mathcal{H}$.

Conversely, suppose that $\mathcal{A}$ satisfies the given condition. Let $x, y \in \mathcal{H}$ and $t \in [0, 1)$ be such that $y_t < \mathcal{A}$. Then, $\mathcal{A}(y) \leq t$. Since

$$\begin{aligned} \mathcal{A}(xy) &\leq \max\left\{\mathcal{A}(y), \tfrac{1-k}{2}\right\} \\ \mathcal{A}(xy) &\leq \max\left\{t, \tfrac{1-k}{2}\right\} \end{aligned}$$

If $t \geq \frac{1-k}{2}$, then $\mathcal{A}(xy) \leq t$ and so $(xy)_t < \mathcal{A}$. If $t < \frac{1-k}{2}$, then $\mathcal{A}(xy) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k < 1$ and so $(xy)_t\gamma_k\mathcal{A}$. Hence, $(xy)_t < \vee\gamma_k\mathcal{A}$. This completes the proof. □

If we put $k = 0$ in the previous theorem, then we get the following corollary.

Corollary 3.21. *Let $\mathcal{A}$ be a fuzzy set in $\mathcal{H}$. Then, $\mathcal{A}$ is an $(<, < \vee\gamma)^*$-fuzzy left (resp. right) ideal of $\mathcal{H}$ if and only if it satisfies the following*

$$(\forall x, y \in \mathcal{H}) \left(\mathcal{A}(xy) \leq \max\left\{\mathcal{A}(y), 0.5\right\} \text{ (resp. } \mathcal{A}(xy) \leq \max\left\{\mathcal{A}(x), 0.5\right\}\right).$$

Theorem 3.22. *For any subset of $\mathcal{H}$, let $\mathfrak{X}_S$ denote the characteristic function of $\mathcal{H}$. Then, the function $\mathfrak{X}_{S^c} : \mathcal{H} \longrightarrow \{0, 1\}$ defined by $\mathfrak{X}_{S^c}(x) = 1 - \mathfrak{X}_S(x)$ for all $x \in \mathcal{H}$ is an $(<, < \vee\gamma_k)^*$-fuzzy subsemigroup of $\mathcal{H}$ if and only if S is a subsemigroup of $\mathcal{H}$.*

Proof. Assume that $\mathfrak{X}_{S^c}$ is an $(<, < \vee\gamma_k)^*$-fuzzy subsemigroup of $\mathcal{H}$ and let $x, y \in S$, Then, $\mathfrak{X}_{S^c}(x) = 1 - \mathfrak{X}_S(x) = 0$ and $\mathfrak{X}_{S^c}(y) = 1 - \mathfrak{X}_S(y) = 0$. Hence, $x_0 < \mathfrak{X}_{S^c}$ and $y_0 < \mathfrak{X}_{S^c}$, which imply that $(xy)_0 = (xy)_{\max\{0;0\}} < \vee\gamma_k \mathfrak{X}_{S^c}$. Thus, $\mathfrak{X}_{S^c}(xy) \leq 0$ or $\mathfrak{X}_{S^c}(xy) + 0 < 1$. If $\mathfrak{X}_{S^c}(xy) \leq 0$, then $1 - \mathfrak{X}_S(xy) = 0$, i.e., $\mathfrak{X}_S(xy) = 0$. Hence $xy \in S$. If $\mathfrak{X}_{S^c}(xy) + 0 < 1$, then $\mathfrak{X}_S(xy) > 0$. Thus, $\mathfrak{X}_S(xy) = 1$, and so $xy \in S$. Therefore, S is a subsemigroup of $\mathcal{H}$.

Conversely, suppose that S is a subsemigroup of $\mathcal{H}$. Let $x, y \in \mathcal{H}$. If $x, y \in S$, then $xy \in S$, and thus

$$\begin{aligned} \mathfrak{X}_{S^c}(xy) &= \max\left\{\mathfrak{X}_{S^c}(x), \mathfrak{X}_{S^c}(y)\right\} \\ &\leq \max\left\{\mathfrak{X}_{S^c}(x), \mathfrak{X}_{S^c}(y), \tfrac{1-k}{2}\right\}. \end{aligned}$$

If any one of x and y does not belong to S, then $\mathfrak{X}_{S^c}(x) = 1$ or $\mathfrak{X}_{S^c}(y) = 1$. Hence

$$\begin{aligned} \mathfrak{X}_{S^c}(xy) &\leq \max\left\{\mathfrak{X}_{S^c}(x), \mathfrak{X}_{S^c}(y)\right\} \\ &\leq \max\left\{\mathfrak{X}_{S^c}(x), \mathfrak{X}_{S^c}(y), \tfrac{1-k}{2}\right\}. \end{aligned}$$

By Theorem 3.18, $\mathfrak{X}_{S^c}$ is an $(<, < \vee\gamma_k)^*$-fuzzy subsemigroup of $\mathcal{H}$. □

If we put $k = 0$ in the previous theorem, then we get the following corollary.

Corollary 3.23. *For any subset of $\mathcal{H}$, let $\mathfrak{X}_S$ denotes the characteristic function of $\mathcal{H}$. Then, the function $\mathfrak{X}_{S^c} : \mathcal{H} \longrightarrow \{0, 1\}$ defined by $\mathfrak{X}_{S^c}(x) = 1 - \mathfrak{X}_S(x)$ for all $x \in \mathcal{H}$ is an $(<, < \vee\gamma)^*$-fuzzy subsemigroup of $\mathcal{H}$ if and only if S is a subsemigroup of $\mathcal{H}$.*

Theorem 3.24. *For any subset of $\mathcal{H}$, let $\mathfrak{X}_L$ denote the characteristic function of $\mathcal{H}$. Then, the function $\mathfrak{X}_{L^c} : \mathcal{H} \longrightarrow \{0, 1\}$ defined by $\mathfrak{X}_{L^c}(x) = 1 - \mathfrak{X}_L(x)$ for all $x \in \mathcal{H}$ is an $(<, < \vee\gamma_k)^*$-fuzzy left (resp. right) ideal of $\mathcal{H}$ if and only if L is a left (resp. right) ideal of $\mathcal{H}$.*

Proof. Assume that $\mathfrak{X}_{L^c}$ is an $(<, < \vee\gamma_k)^*$-fuzzy left ideal of $\mathcal{H}$ and let $y \in L$ and $x \in \mathcal{H}$, Then, $\mathfrak{X}_{L^c}(y) = 1 - \mathfrak{X}_L(y) = 0$. Hence, $y_0 < \mathfrak{X}_{L^c}$, which imply that $(xy)_0 < \vee\gamma_k \mathfrak{X}_{L^c}$. Thus, $\mathfrak{X}_{L^c}(xy) \leq 0$ or $\mathfrak{X}_{L^c}(xy) + 0 < 1$. If $\mathfrak{X}_{L^c}(xy) \leq 0$, then $1 - \mathfrak{X}_L(xy) = 0$, i.e., $\mathfrak{X}_L(xy) = 0$. Hence, $xy \in L$. If $\mathfrak{X}_{L^c}(xy) + 0 < 1$, then $\mathfrak{X}_L(xy) > 0$. Thus, $\mathfrak{X}_L(xy) = 1$, and so $xy \in L$. Therefore, L is a left ideal of $\mathcal{H}$.

Conversely, suppose that L is a left ideal of $\mathcal{H}$. Let $x, y \in \mathcal{H}$. If $y \in L$, then $xy \in L$, and thus

$$\begin{aligned} \mathfrak{X}_{L^c}(xy) &= \mathfrak{X}_{L^c}(y) \\ &\leq \max\left\{\mathfrak{X}_{L^c}(y), \tfrac{1-k}{2}\right\}. \end{aligned}$$

If y does not belong to L, then $\mathfrak{X}_{L^c}(y) = 1$. Hence

$$\begin{aligned}\mathfrak{X}_{L^c}(xy) &\leq \mathfrak{X}_{L^c}(y) \\ &\leq \max\left\{\mathfrak{X}_{L^c}(y), \tfrac{1-k}{2}\right\}.\end{aligned}$$

By Theorem 3.20, $\mathfrak{X}_{L^c}$ is an $(<, < \vee\gamma_k)^*$-fuzzy left ideal of $\mathcal{H}$. This completes the proof. □

For $k = 0$ in the previous theorem, then we get the following corollary.

Corollary 3.25. *For any subset of $\mathcal{H}$, let $\mathfrak{X}_L$ denote the characteristic function of $\mathcal{H}$. Then, the function $\mathfrak{X}_{L^c} : \mathcal{H} \longrightarrow \{0, 1\}$ defined by $\mathfrak{X}_{L^c}(x) = 1 - \mathfrak{X}_L(x)$ for all $x \in \mathcal{H}$ is an $(<, < \vee\gamma)^*$-fuzzy left (resp. right) ideal of $\mathcal{H}$ if and only if L is a left (resp. right) ideal of $\mathcal{H}$.*

Theorem 3.26. *A fuzzy set $\mathcal{A}$ in a $\mathcal{H}$ is an $(<, < \vee\gamma_k)^*$-fuzzy subsemigroup of $\mathcal{H}$ if and only if the set $L(\mathcal{A}; t) = \left\{x \in \mathcal{H} : \mathcal{A}(x) \leq t,\ t \in \left[\frac{1-k}{2}, 1\right)\right\}$ is a subsemigroup of $\mathcal{H}$.*

Proof. Assume that $\mathcal{A}$ is an $(<, < \vee\gamma_k)^*$-fuzzy subsemigroup of $\mathcal{H}$. Let $x, y \in L(\mathcal{A}; t)$. Then, $\mathcal{A}(x) \leq t$ and $\mathcal{A}(y) \leq t$, which follows from Theorem 3.18

$$\mathcal{A}(xy) \leq \max\left\{\mathcal{A}(x), \mathcal{A}(y), \tfrac{1-k}{2}\right\} \leq \left\{t, \tfrac{1-k}{2}\right\} = t,$$

so that $xy \in L(\mathcal{A}; t)$. Hence, $L(\mathcal{A}; t)$ is a subsemigroup of $\mathcal{H}$.

Conversely, let $\mathcal{A}$ be fuzzy set in $\mathcal{H}$ such that the set

$$L(\mathcal{A}; t) = \left\{x \in \mathcal{H} : \mathcal{A}(x) \leq t,\ t \in \left[\tfrac{1-k}{2}, 1\right)\right\}$$

is a subsemigroup of $\mathcal{H}$. Let us suppose that there exist $x, y \in \mathcal{H}$ such that

$$\mathcal{A}(xy) > \max\left\{\mathcal{A}(x), \mathcal{A}(y), \tfrac{1-k}{2}\right\}.$$

Then, we take $t \in (0, 1)$ such that

$$\mathcal{A}(xy) > t > \max\left\{\mathcal{A}(x), \mathcal{A}(y), \tfrac{1-k}{2}\right\}.$$

Thus, $x, y \in L(\mathcal{A}; t)$ and $t > \frac{1-k}{2}$, and so $xy \in L(\mathcal{A}; t)$, i.e., $\mathcal{A}(xy) \leq t$, a contradicts to our hypothesis. Therefore, $\mathcal{A}(xy) \leq \max\left\{\mathcal{A}(x), \mathcal{A}(y), \frac{1-k}{2}\right\}$, for all $x, y \in \mathcal{H}$. By Theorem 3.18, $\mathcal{A}$ is an $(<, < \vee\gamma_k)^*$-fuzzy subsemigroup of $\mathcal{H}$. □

For $k = 0$, we have the following corollary.

Corollary 3.27. *A fuzzy set $\mathcal{A}$ in a $\mathcal{H}$ is an $(<, < \vee\gamma)^*$-fuzzy subsemigroup of $\mathcal{H}$ if and only if the set $L(\mathcal{A}; t) = \{x \in \mathcal{H} : \mathcal{A}(x) \leq t,\ t \in [0.5, 1)\}$ is a subsemigroup of $\mathcal{H}$.*

Theorem 3.28. *A fuzzy set $\mathcal{A}$ in a $\mathcal{H}$ is an $(<, < \vee\gamma_k)^*$-fuzzy left (resp. right) ideal of $\mathcal{H}$ if and only if the set $L(\mathcal{A}; t) = \left\{x \in \mathcal{H} : \mathcal{A}(x) \leq t,\ t \in \left[\frac{1-k}{2}, 1\right)\right\}$ is a left (resp. right) ideal of $\mathcal{H}$.*

Proof. Assume that $\mathcal{A}$ is an $(<, < \vee\gamma_k)^*$-fuzzy left ideal of $\mathcal{H}$. Let $y \in L(\mathcal{A}; t)$ and $x \in \mathcal{H}$. Then, $\mathcal{A}(y) \leq t$. It follows from Theorem 3.20,

$$\mathcal{A}(xy) \leq \max\left\{\mathcal{A}(y), \tfrac{1-k}{2}\right\} \leq \max\left\{t, \tfrac{1-k}{2}\right\} = t,$$

so that $xy \in L(\mathcal{A}; t)$. Hence, $L(\mathcal{A}; t)$ is a left ideal of $\mathcal{H}$.

Conversely, let $\mathcal{A}$ be fuzzy set in $\mathcal{H}$ such that the set

$$L(\mathcal{A};t) = \left\{x \in \mathcal{H} : \mathcal{A}(x) \leq t,\ t \in \left[\tfrac{1-k}{2}, 1\right)\right\}$$

is a left ideal of $\mathcal{H}$. Let us suppose that there exist $x, y \in \mathcal{H}$ such that

$$\mathcal{A}(xy) > \max\left\{\mathcal{A}(y),\ \tfrac{1-k}{2}\right\}.$$

Then, we take $t \in (0,1)$ such that

$$\mathcal{A}(xy) > t > \max\left\{\mathcal{A}(y),\ \tfrac{1-k}{2}\right\}.$$

Thus, $y \in L(\mathcal{A};t)$ and $t > \frac{1-k}{2}$, and so $xy \in L(\mathcal{A};t)$, i.e., $\mathcal{A}(xy) \leq t$, a contradicts to our hypothesis. Therefore, $\mathcal{A}(xy) \leq \max\left\{\mathcal{A}(y),\ \frac{1-k}{2}\right\}$, for all $x, y \in \mathcal{H}$. By Theorem 3.20, $\mathcal{A}$ is an $(<, < \vee\gamma_k)^*$-fuzzy left ideal of $\mathcal{H}$. □

For $k = 0$, we have the following corollary.

Corollary 3.29. *A fuzzy set $\mathcal{A}$ in a $\mathcal{H}$ is an $(<, < \vee\gamma)^*$-fuzzy left (resp. right) ideal of $\mathcal{H}$ if and only if the set $L(\mathcal{A};t) = \{x \in \mathcal{H} : \mathcal{A}(x) \leq t,\ t \in [0.5, 1)\}$ is a left (resp. right) ideal of $\mathcal{H}$.*

In what follows, let us consider some of the relationships among $(\alpha,\beta)^*$-fuzzy subsemigroup (left, right) ideals of $\mathcal{H}$.

Lemma 3.30. (1) *Every $(<,\ <)^*$-fuzzy subsemigroup (resp., left, right) ideal of a semigroup S is an $(<,\ < \vee\gamma_k)^*$-fuzzy subsemigroup (resp., left, right) ideal.*

(2) *Every $(< \vee\gamma_k,\ < \vee\gamma_k)^*$-fuzzy subsemigroup (resp., left, right) ideal of a semigroup S is an $(<,\ < \vee\gamma_k)^*$-fuzzy subsemigroup (resp., left, right) ideal.*

Proposition 3.31. *Let $\mathcal{A}$ be an $(<,\ < \vee\gamma_k)^*$-fuzzy subsemigroup of $\mathcal{H}$ such that $\mathcal{A}(x) > \frac{1-k}{2}$, for all $x \in \mathcal{H}$. Then, $\mathcal{A}$ is an $(<,\ <)^*$-fuzzy subsemigroup of $\mathcal{H}$.*

Proof. Let $x_{t_1}, y_{t_2} < \mathcal{A}$. Then, $\mathcal{A}(x) \leq t_1$, $\mathcal{A}(y) \leq t_2$. Since $\mathcal{A}$ is an $(<,\ < \vee\gamma_k)^*$-fuzzy subsemigroup of $\mathcal{H}$, so by Theorem 3.18, we have

$$\mathcal{A}(xy) \leq \max\left\{\mathcal{A}(x),\ \mathcal{A}(y),\ \tfrac{1-k}{2}\right\}.$$

Since $\mathcal{A}(x) > \frac{1-k}{2}$, for all $x \in \mathcal{H}$. So, that is, $\mathcal{A}(xy) \leq \max\{\mathcal{A}(x), \mathcal{A}(y)\} \leq \max\{t_1, t_2\}$, this imply $\mathcal{A}(xy) \leq \max\{t_1, t_2\}$. Thus, $(xy)_{\max\{t_1,t_2\}} < \mathcal{A}$. Hence, $\mathcal{A}$ is an $(<,\ <)^*$-fuzzy subsemigroup of $\mathcal{H}$. □

Proposition 3.32. *Let $\mathcal{A}$ be an $(<,\ < \vee\gamma_k)^*$-fuzzy left (resp. right) of $\mathcal{H}$ such that $\mathcal{A}(x) > \frac{1-k}{2}$, for all $x \in \mathcal{H}$. Then, $\mathcal{A}$ is an $(<,\ <)^*$-fuzzy left (resp. right) of $\mathcal{H}$.*

Proof. Suppose that $\mathcal{A}$ is an $(<,\ <)^*$-fuzzy left (resp. right). Take $y_t < \mathcal{A}$ and $x \in \mathcal{H}$. Then, $\mathcal{A}(x) \leq t$. Since $\mathcal{A}$ is an $(<,\ < \vee\gamma_k)^*$-fuzzy left (resp. right) of $\mathcal{H}$, so by Theorem 3.20, we have $\mathcal{A}(xy) \leq \max\left\{\mathcal{A}(y),\ \frac{1-k}{2}\right\}$ $\left(\text{resp. } \mathcal{A}(yx) \leq \max\left\{\mathcal{A}(y),\ \frac{1-k}{2}\right\}\right)$. Since $\mathcal{A}(x) > \frac{1-k}{2}$, for all $x \in \mathcal{H}$. So, that is, $\mathcal{A}(xy) \leq \mathcal{A}(x) \leq t$ (resp. $\mathcal{A}(yx) \leq \mathcal{A}(y) \leq t$), this imply $\mathcal{A}(xy) \leq t$ (resp. $\mathcal{A}(yx) \leq t$. Thus, $(xy)_t < \mathcal{A}$ (resp. $(yx)_t < \mathcal{A}$. Hence, $\mathcal{A}$ is an $(<,\ < \vee\gamma_k)^*$-fuzzy left (resp. right) ideal of $\mathcal{H}$. □

Theorem 3.33. *Let $\psi : \mathcal{H} \longrightarrow \mathcal{H}^{'}$ be semigroup homomorphism and let $\mathcal{A}$ and $\mathcal{B}$ be $(<, <\vee\gamma_k)^{*}$-fuzzy subsemigroups of $\mathcal{H}$ and $\mathcal{H}^{'}$, respectively. Then,*

(i) $\psi^{-1}(\mathcal{B})$ is an $(<, <\vee\gamma_k)^{*}$-fuzzy subsemigroup of $\mathcal{H}$ and

(ii) If $\mathcal{A}$ satisfies the sup property i.e., for any subset T of $\mathcal{H}$ there exist $x_{\circ} \in T$ such that $\mathcal{A}(x_{\circ}) = \bigvee\{\mathcal{A}(x) : x \in T\}$, then $\psi(\mathcal{A})$ is an $(<, <\vee\gamma_k)^{*}$-fuzzy subsemigroup of $\mathcal{H}^{'}$, when ψ is onto.

Proof. (i) Let $x, y \in \mathcal{H}$ and $t_1, t_2 \in [0,1)$ be such that $x_{t_1} < \psi^{-1}(\mathcal{B})$ and $y_{t_2} < \psi^{-1}(\mathcal{B})$. Then, $(\psi(x))_{t_1} < \mathcal{B}$ and $(\psi(y))_{t_2} < \mathcal{B}$. Since $\mathcal{B}$ is an $(<, <\vee\gamma_k)^{*}$-fuzzy subsemigroup of $\mathcal{H}$, so $(\psi(xy))_{t_1\vee t_2} = (\psi(x)\psi(y))_{t_1\vee t_2} < \vee\gamma_k\mathcal{B}$. This implies that

$$(xy)_{t_1\vee t_2} < \vee\gamma_k\psi^{-1}(\mathcal{B}).$$

Thus, $\psi^{-1}(\mathcal{B})$ is an $(<, <\vee\gamma_k)^{*}$-fuzzy subsemigroup of $\mathcal{H}$.

(ii) Let $x, y \in \mathcal{H}^{'}$ and $t_1, t_2 \in [0,1)$ be such that $x_{t_1} < \psi(\mathcal{A})$ and $y_{t_2} < \psi(\mathcal{A})$. Then, $(\psi(\mathcal{A}))(x) < t_1$ and $(\psi(\mathcal{A}))(y) < t_2$. Since $\mathcal{A}$ has the sup property, so there exists $x \in \psi^{-1}(a)$ and $y \in \psi^{-1}(b)$ such that

$$\mathcal{A}(x) = \bigvee\left\{\mathcal{A}(z) : z \in \psi^{-1}(a)\right\}$$

and

$$\mathcal{A}(y) = \bigvee\left\{\mathcal{A}(w) : w \in \psi^{-1}(b)\right\}.$$

Then, $x_{t_1} < \mathcal{A}$ and $y_{t_2} < \mathcal{A}$. Since $\mathcal{A}$ is an $(<, <\vee\gamma_k)^{*}$-fuzzy subsemigroup of $\mathcal{H}$,we have $(xy)_{t_1\vee t_2} < \vee\gamma_k\mathcal{A}$. Now, $xy \in \psi^{-1}(a)$ and so $(\psi(\mathcal{A}))(xy) \leq \mathcal{A}(xy)$. Thus, $(\psi(\mathcal{A}))(xy) \leq \max\{t_1, t_2\}$ or $(\psi(\mathcal{A}))(xy) + \max\{t_1, t_2\} + k < 1$, which means that $(xy)_{t_1\vee t_2} < \vee\gamma_k\psi(\mathcal{A})$. This completes the proof. □

Theorem 3.34. *Let $\psi : \mathcal{H} \longrightarrow \mathcal{H}^{'}$ be semigroup homomorphism and let $\mathcal{A}$ and $\mathcal{B}$ be $(<, <\vee\gamma_k)^{*}$-fuzzy left (resp. right) ideals of $\mathcal{H}$ and $\mathcal{H}^{'}$, respectively. Then,*

(i) $\psi^{-1}(\mathcal{B})$ is an $(<, <\vee\gamma_k)^{*}$-fuzzy left (resp. right) ideal of $\mathcal{H}$ and

(ii) If $\mathcal{A}$ satisfies the sup property i.e., for any subset N of $\mathcal{H}$ there exist $x_{\circ} \in T$ such that $\mathcal{A}(x_{\circ}) = \bigvee\{\mathcal{A}(x) : x \in T\}$, then $\psi(\mathcal{A})$ is an $(<, <\vee\gamma_k)^{*}$-fuzzy left (resp. right) ideal of $\mathcal{H}^{'}$, when ψ is onto.

Proof. (i) Let $x, y \in \mathcal{H}$ and $t[0,1)$ be such that $y_t < \psi^{-1}(\mathcal{B})$. Then, $(\psi(y))_t < \mathcal{B}$. Since $\mathcal{B}$ is an $(<, <\vee\gamma_k)^{*}$-fuzzy left (resp. right) ideal of $\mathcal{H}$, so $(\psi(xy))_t = (\psi(x)\psi(y))_{t_1\vee t_2} < \vee\gamma_k\mathcal{B}$ (reps. $(\psi(yx))_t = (\psi(y)\psi(x))_t < \vee\gamma_k\mathcal{B}$). This implies $(xy)_t < \vee\gamma_k\psi^{-1}(\mathcal{B})$ (resp. $(yx)_t < \vee\gamma_k\psi^{-1}(\mathcal{B})$). Therefore, $\psi^{-1}(\mathcal{B})$ is an $(<, <\vee\gamma_k)^{*}$-fuzzy left (resp. right) ideal of $\mathcal{H}$.

(ii) Let $x, y \in \mathcal{H}^{'}$ and $t \in [0,1)$ be such that $y_{t_2} < \psi(\mathcal{A})$. Then, $(\psi(\mathcal{A}))(y) < t$. Since $\mathcal{A}$ has the sup property, so there exists $y \in \psi^{-1}(b)$ such that

$$\mathcal{A}(y) = \bigvee\left\{\mathcal{A}(w) : w \in \psi^{-1}(b)\right\}.$$

Then, $y_t < \mathcal{A}$. Since $\mathcal{A}$ is an $(<, <\vee\gamma_k)^{*}$-fuzzy subsemigroup of $\mathcal{H}$, so we have $(xy)_t < \vee\gamma_k\mathcal{A}$ (resp. $(yx)_t < \vee\gamma_k\mathcal{A}$). Now, $xy \in \psi^{-1}(b)$, then $\psi(\mathcal{A})(yx) \leq t$ (resp. $(\psi(\mathcal{A}))(yx) \leq t$) or $(\psi(\mathcal{A}))(xy) + t + k < 1$ (resp. $(\psi(\mathcal{A}))(yx) + t + k < 1$), which means that $(xy)_t < \vee\gamma_k\psi(\mathcal{A})$ (resp. $(\psi(\mathcal{A}))(xy) + t + k < 1$. This completes the proof. □

4. Regular semigroups

In this section, we characterize regular semigroups by their $(<,<\vee\gamma_k)^*$-fuzzy ideals.

Definition 4.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two fuzzy subset of a semigroup $\mathcal{H}$. Then, the k-product of $\mathcal{A}$ and $\mathcal{B}$ denoted by $\mathcal{A}\circ^k\mathcal{B}$ and defined by

$$\left(\mathcal{A}\circ^k\mathcal{B}\right)(x)=\begin{cases}\bigwedge_{x=yz}\left\{\mathcal{A}(y)\vee\mathcal{B}(z)\vee\frac{1-k}{2}\right\} & \text{if } x=yz\\ 1 & \text{if } x\neq yz,\end{cases}$$

where $k\in[-1,0)$.

Definition 4.2. Let $\mathcal{A}$ and $\mathcal{B}$ be fuzzy subset of a semigroup $\mathcal{H}$. Then, we define the fuzzy subset $\mathcal{A}^k,\mathcal{A}\vee_k\mathcal{B}$ and $\mathcal{A}\wedge_k\mathcal{B}$, for $k\in[-1,0)$, of $\mathcal{H}$ as follows:
(1) $\mathcal{A}^k(x)=\mathcal{A}\vee\frac{1-k}{2}$,
(2) $(\mathcal{A}\vee\mathcal{B})^k(x)=(\mathcal{A}\vee\mathcal{B})\vee\frac{1-k}{2}$,
(2) $(\mathcal{A}\wedge\mathcal{B})^k(x)=(\mathcal{A}\wedge\mathcal{B})\wedge\frac{1-k}{2}$.

Lemma 4.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be fuzzy subset of a semigroup $\mathcal{H}$. Then, the following conditions hold.*
(a) $(\mathcal{A}\vee\mathcal{B})^k=\mathcal{A}^k\vee\mathcal{B}^k$,
(b) $(\mathcal{A}\wedge\mathcal{B})^k=\mathcal{A}^k\wedge\mathcal{B}^k$,
(c) $\mathcal{A}\circ^k\mathcal{B}=\mathcal{A}^k\circ\mathcal{B}^k$.

Proof. The proofs of (a) and (b) are straightforward.

(c) : If $x\neq yz$ for all $y,z\in\mathcal{H}$, then clearly $\left(\mathcal{A}\circ^k\mathcal{B}\right)(x)=0=\left(\mathcal{A}^k\circ\mathcal{B}^k\right)(x)$. If $x=yz$ for some $y,z\in\mathcal{H}$, then we have

$$\begin{aligned}\left(\mathcal{A}\circ^k\mathcal{B}\right)(x) &= \bigwedge_{x=yz}\left\{\mathcal{A}(y)\vee\mathcal{B}(z)\vee\tfrac{1-k}{2}\right\}\\ &= \bigwedge_{x=yz}\left\{\left(\mathcal{A}(y)\vee\tfrac{1-k}{2}\right)\vee\left(\mathcal{B}(z)\vee\tfrac{1-k}{2}\right)\right\}\\ &= \bigwedge_{x=yz}\left\{\mathcal{A}^k(y)\vee\mathcal{B}^k(z)\right\}.\end{aligned}$$

Thus, $\mathcal{A}\circ^k\mathcal{B}=\mathcal{A}^k\circ\mathcal{B}^k$. □

Lemma 4.4. *Let A and B be non-empty subsets of a semigroup $\mathcal{H}$. Then, the following conditions hold.*
(a) $\left(\mathfrak{X}_{(A\cup B)^c}\right)^k=(\mathfrak{X}_{A^c}\wedge\mathfrak{X}_{B^c})^k$,
(b) $\left(\mathfrak{X}_{(A\cap B)^c}\right)^k=(\mathfrak{X}_{A^c}\vee\mathfrak{X}_{B^c})^k$,
(c) $\left(\mathfrak{X}_{(AB)^c}\right)^k=\left(\mathfrak{X}_{A^c}\circ^k\mathfrak{X}_{B^c}\right)$, *where* $(\mathfrak{X}_{A^c})^k$ *is defined as:*

$$(\mathfrak{X}_{A^c})^k(x)=\begin{cases}\frac{1-k}{2} & \text{if } x\in A,\\ 1 & \text{if } x\notin A.\end{cases}$$

Proposition 4.5. *A non-empty subset L of $\mathcal{H}$ is a left (right) ideal of $\mathcal{H}$ if and only if $(\mathfrak{X}_{L^c})^k$ is an $(<,<\vee\gamma_k)^*$-fuzzy left (right) ideal of $\mathcal{H}$.*

Proof. Let L be a left (right) ideal of $\mathcal{H}$. Then, by Theorem 3.13, $(\mathfrak{X}_{L^c})^k$ is an $(<,<\vee\gamma_k)^*$-fuzzy left (right) ideal of $\mathcal{H}$.

Conversely, take $(\mathfrak{X}_{L^c})^k$ is an $(<,<\vee\gamma_k)^*$-fuzzy left (right) ideal of $\mathcal{H}$. Let $x\in L$. Then, $(\mathfrak{X}_{L^c})^k(x)=\frac{1-k}{2}$. So $x_{\frac{1-k}{2}}<(\mathfrak{X}_{L^c})^k$. Since $(\mathfrak{X}_{L^c})^k$ is an $(<,<\vee\gamma_k)^*$-fuzzy left (right) ideal of $\mathcal{H}$, so $(yx)_{\frac{1-k}{2}}<\vee\gamma_k(\mathfrak{X}_{L^c})^k$ $((xy)_{\frac{1-k}{2}}<\vee\gamma_k(\mathfrak{X}_{L^c})^k)$, which implies that

$$(yx)_{\frac{1-k}{2}}<(\mathfrak{X}_{L^c})^k \text{ or } (yx)_{\frac{1-k}{2}}\,\gamma_k\,(\mathfrak{X}_{L^c})^k\ \ ((xy)_{\frac{1-k}{2}}<(\mathfrak{X}_{L^c})^k \text{ or } (xy)_{\frac{1-k}{2}}\,\gamma_k\,(\mathfrak{X}_{L^c})^k).$$

Hence, $(\mathfrak{X}_{L^c})^k(yx)\le\frac{1-k}{2}$ or $(\mathfrak{X}_{L^c})^k(yx)+\frac{1-k}{2}+k<1$ $((\mathfrak{X}_{L^c})^k(xy)\le\frac{1-k}{2}$ or $(\mathfrak{X}_{L^c})^k(xy)+\frac{1-k}{2}+k<1)$.. If $(\mathfrak{X}_{L^c})^k(yx)+\frac{1-k}{2}+k<1$ $((\mathfrak{X}_{L^c})^k(xy)+\frac{1-k}{2}+k<1)$, then $(\mathfrak{X}_{L^c})^k(xy)<\frac{1-k}{2}$ $((\mathfrak{X}_{L^c})^k(xy)<\frac{1-k}{2})$, this imply $(\mathfrak{X}_{L^c})^k(xy)\le\frac{1-k}{2}$ $((\mathfrak{X}_{L^c})^k(xy)\le\frac{1-k}{2})$. Hence $yx\in L$ $(xy\in L)$. This completes the proof. □

Proposition 4.6. *Let $\mathcal{A}$ be an $(<,<\vee\gamma_k)^*$-fuzzy left (right) ideal of $\mathcal{H}$. Then, $\mathcal{A}^k$ is an anti fuzzy left (right) ideal of $\mathcal{H}$.*

Proof. Let $\mathcal{A}$ be an $(<,<\vee\gamma_k)^*$-fuzzy left ideal of $\mathcal{H}$ and let $x,y\in\mathcal{H}$. Then,

$$\begin{aligned}\mathcal{A}^k(xy) &= \mathcal{A}(xy)\vee\tfrac{1-k}{2}\\ &\le \left(\mathcal{A}(y)\vee\tfrac{1-k}{2}\right)\vee\tfrac{1-k}{2},\\ \mathcal{A}^k(xy) &\le \mathcal{A}(y)\vee\tfrac{1-k}{2}=\mathcal{A}^k(y).\end{aligned}$$

Thus, $\mathcal{A}^k$ is an anti fuzzy left ideal of $\mathcal{H}$. □

Next we show that every anti fuzzy left ideal of $\mathcal{H}$ is not of the form $\mathcal{A}^k$ for some $(<,<\vee\gamma_k)^*$-fuzzy left ideal of $\mathcal{H}$.

Example 4.7. Take the semigroup of Example 3.17, The fuzzy subset $\mathcal{A}(a)=0.1$, $\mathcal{A}(a)=0.2$, $\mathcal{A}(a)=0.3=\mathcal{A}(a)$ is an anti fuzzy ideal of $\mathcal{H}$ but this is not of the form $\mathcal{B}^k$ for some $(<,<\vee\gamma_k)^*$-fuzzy left ideal of $\mathcal{H}$ for $k=-0.2$.

Proposition 4.8. *Let $\mathcal{H}$ be a semigroup, $\mathcal{A}$ an $(<,<\vee\gamma_k)^*$-fuzzy right ideal and $\mathcal{B}$ an $(<,<\vee\gamma_k)^*$-fuzzy left ideal of $\mathcal{H}$. Then, $\mathcal{A}\circ^k\mathcal{B}\ge(\mathcal{A}\vee\mathcal{B})^k$.*

Proof. Let $x\in\mathcal{H}$. Then, $\mathcal{A}\circ^k\mathcal{B}\ge(\mathcal{A}\vee\mathcal{B})^k$. Indeed, if $x\neq yz$ for any $y,z\in\mathcal{H}$, then $\left(\mathcal{A}\circ^k\mathcal{B}\right)(x)=1\ge(\mathcal{A}\vee\mathcal{B})^k(x)$. Thus, $\mathcal{A}\circ^k\mathcal{B}\le(\mathcal{A}\vee\mathcal{B})^k$. If $x=yz$ for some

$y, z \in \mathcal{H}$, then

$$\begin{aligned}\left(\mathcal{A} \circ^k \mathcal{B}\right)(x) &= \bigwedge_{x=yz} \left\{\mathcal{A}(y) \vee \mathcal{B}(z) \vee \tfrac{1-k}{2}\right\} \\ &= \bigwedge_{x=yz} \left\{\left(\mathcal{A}(y) \vee \tfrac{1-k}{2}\right) \vee \left(\mathcal{B}(z) \vee \tfrac{1-k}{2}\right) \vee \tfrac{1-k}{2}\right\} \\ &\geq \bigwedge_{x=yz} \left\{\mathcal{A}(yz) \vee \mathcal{B}(yz) \vee \tfrac{1-k}{2}\right\} \\ &\left(\text{Since } \mathcal{A},\ \mathcal{B} \text{ are } (<,< \vee\gamma_k)^* \text{-fuzzy right ideal and left ideal of } \mathcal{H}\right) \\ &= \bigwedge_{x=yz} \left\{(\mathcal{A} \vee \mathcal{B})(yz) \vee \tfrac{1-k}{2}\right\} = (\mathcal{A} \vee \mathcal{B})^k (x).\end{aligned}$$

Thus, $\mathcal{A} \circ^k \mathcal{B} \geq (\mathcal{A} \vee \mathcal{B})^k$. □

Theorem 4.9. *Let $\mathcal{H}$ be a semigroup, $\mathcal{A}$ an $(<,< \vee\gamma_k)^*$-fuzzy right ideal and $\mathcal{B}$ an $(<,< \vee\gamma_k)^*$-fuzzy left ideal of $\mathcal{H}$. Then, $\mathcal{H}$ is a regular if and only if $\mathcal{A} \circ^k \mathcal{B} = (\mathcal{A} \vee \mathcal{B})^k$.*

Proof. **Necessity.** Take $\mathcal{H}$ is a regular semigroup. Let$\mathcal{A}$ be an $(<,< \vee\gamma_k)^*$-fuzzy right ideal and $\mathcal{B}$ an $(<,< \vee\gamma_k)^*$-fuzzy left ideal of $\mathcal{H}$. In ordered to show that $\mathcal{A} \circ^k \mathcal{B} = (\mathcal{A} \vee \mathcal{B})^k$, it is enough to prove that $\mathcal{A} \circ^k \mathcal{B} \leq (\mathcal{A} \vee \mathcal{B})^k$. Let $a \in \mathcal{H}$. Since $\mathcal{H}$ is regular semigroup, so there exists $x \in \mathcal{H}$ such that $a = axa$. Then,

$$\begin{aligned}\left(\mathcal{A} \circ^k \mathcal{B}\right)(x) &= \bigwedge_{a=pq} \left\{\mathcal{A}(p) \vee \mathcal{B}(q) \vee \tfrac{1-k}{2}\right\} \\ &\leq \left\{\mathcal{A}(ax) \vee \mathcal{B}(a) \vee \tfrac{1-k}{2}\right\} \\ &\leq \left\{\mathcal{A}(a) \vee \tfrac{1-k}{2} \vee \mathcal{B}(a) \vee \tfrac{1-k}{2}\right\} \\ &\quad \left(\text{Since } \mathcal{A} \text{ is } an\, (<,< \vee\gamma_k)^* \text{-fuzzy right ideal.}\right) \\ &= \left\{(\mathcal{A} \vee \mathcal{B})(a) \vee \tfrac{1-k}{2}\right\} = (\mathcal{A} \vee \mathcal{B})^k (x).\end{aligned}$$

Hence, $\mathcal{A} \circ^k \mathcal{B} \leq (\mathcal{A} \vee \mathcal{B})^k$ and by using Proposition 4.8, we get $\mathcal{A} \circ^k \mathcal{B} = (\mathcal{A} \vee \mathcal{B})^k$.

Conversely, suppose that $\mathcal{A} \circ^k \mathcal{B} = (\mathcal{A} \vee \mathcal{B})^k$ for every $(<,< \vee\gamma_k)^*$-fuzzy right ideal $\mathcal{A}$ and every $(<,< \vee\gamma_k)^*$-fuzzy left ideal $\mathcal{B}$ of $\mathcal{H}$. Then, S is regular. Indeed, it is enough to prove that $A \cap B = AB$. for every right ideal A and for every left ideal B of $\mathcal{H}$.

Let A and B be right and left ideals of $\mathcal{H}$. Then by Proposition 4.5, $(\mathfrak{X}_{A^c})^k$ and $(\mathfrak{X}_{B^c})^k$ are $(<,< \vee\gamma_k)^*$-fuzzy right and $(<,< \vee\gamma_k)^*$-fuzzy left ideals of $\mathcal{H}$, respectively. Thus, we have

$$\begin{aligned}1 - \left(\mathfrak{X}_{(AB)}\right)^k (x) &= \left(\mathfrak{X}_{(AB)^c}\right)^k (x) = \left(\mathfrak{X}_{A^c} \circ^k \mathfrak{X}_{B^c}\right)(x) \\ &= \left(\mathfrak{X}_{A^c} \vee_k \mathfrak{X}_{B^c}\right)(x) = \left(\mathfrak{X}_{A^c \cup B^c}\right)^k (x) \\ &= \left(\mathfrak{X}_{(A \cap B)^c}\right)^k (x) = 1 - \left(\mathfrak{X}_{(A \cap B)}\right)^k (x) \\ 1 - \left(\mathfrak{X}_{(AB)}\right)^k (x) &= 1 - \left(\mathfrak{X}_{(A \cap B)}\right)^k (x).\end{aligned}$$

Thus, $\left(\mathfrak{X}_{(AB)}\right)^k(x) = \left(\mathfrak{X}_{(A\cap B)}\right)^k(x)$, this imply $A \cap B = AB$. Hence, by Theorem 2.5, $\mathcal{H}$ is regular. □

Proposition 4.10. *Let $\mathcal{H}$ be a semigroup and let $\mathcal{A}$ an $(<, < \vee\gamma_k)^*$-fuzzy right ideal ideal of $\mathcal{H}$. Then, $\mathcal{A} \circ^k \theta \supseteq \mathcal{A}$, where $\theta(x) = 0$ for all $x \in \mathcal{H}$.*

Proof. Let $x \in \mathcal{H}$. Then, either $x \neq yz$ for all $y, z \in \mathcal{H}$ or $x = yz$ for some $y, z \in \mathcal{H}$. If $x \neq yz$ for all $y, z \in \mathcal{H}$, then $\left(\mathcal{A} \circ^k \theta\right)(x) = 1 \geq \mathcal{A}(x)$. If $x = yz$ for some $y, z \in \mathcal{H}$, then

$$\begin{aligned} \left(\mathcal{A} \circ^k \theta\right)(x) &= \bigwedge_{x=pq} \left\{\mathcal{A}(p) \vee \theta(q) \vee \tfrac{1-k}{2}\right\} \\ &= \bigwedge_{x=pq} \left\{\mathcal{A}(p) \vee 0 \vee \tfrac{1-k}{2}\right\} \\ &\geq \bigwedge_{x=pq} \left\{\mathcal{A}(pq)\right\} = \mathcal{A}(x). \end{aligned}$$

Thus, $\left(\mathcal{A} \circ^k \theta\right)(x) \geq \mathcal{A}(x)$. Therefore, $\mathcal{A} \circ^k \theta \supseteq \mathcal{A}$. □

Proposition 4.11. *Let $\mathcal{H}$ be a semigroup and let $\mathcal{A}$ an $(<, < \vee\gamma_k)^*$-fuzzy left ideal of $\mathcal{H}$. Then, $\theta \circ^k \mathcal{A} \supseteq \mathcal{A}$, where $\theta(x) = 0$ for all $x \in \mathcal{H}$.*

Definition 4.12. Let $\mathcal{A}$ be an $(<, < \vee\gamma_k)^*$-fuzzy ideal of a semigroup $\mathcal{H}$. Then, $\mathcal{A}$ is called k-idempotent if $\mathcal{A} \circ_k \mathcal{A} = \mathcal{A}^k$

Theorem 4.13. *Every $(<, < \vee\gamma_k)^*$-fuzzy two-sided ideal of a regular semigroup $\mathcal{H}$ is k-idempotent.*

Proof. Let $\mathcal{A}$ be an $(<, < \vee\gamma_k)^*$-fuzzy two sided ideal of a regular semigroup $\mathcal{H}$ and let $x \in \mathcal{H}$. Then, $x = yz$ for some $y, z \in \mathcal{H}$. So,

$$\begin{aligned} (\mathcal{A} \circ_k \mathcal{A})(x) &= \bigwedge_{x=pq} \left\{\mathcal{A}(p) \vee \mathcal{A}(q) \vee \tfrac{1-k}{2}\right\} \\ &= \bigwedge_{x=pq} \left\{\left(\mathcal{A}(p) \vee \tfrac{1-k}{2}\right) \vee \left(\mathcal{A}(q) \vee \tfrac{1-k}{2}\right) \vee \tfrac{1-k}{2}\right\} \\ &\geq \bigwedge_{x=pq} \left\{\mathcal{A}(pq) \vee \mathcal{A}(pq) \vee \tfrac{1-k}{2}\right\} \\ &\quad \left(\text{Since } \mathcal{A}, \text{ is an } (<, < \vee\gamma_k)^* \text{-fuzzy two sided ideal of } \mathcal{H}.\right) \\ &= (\mathcal{A} \vee \mathcal{A})^k(x) = \mathcal{A}^k(x) \end{aligned}$$

Now,

$$\begin{aligned} \mathcal{A}^k(x) &= \mathcal{A}(x) \vee \tfrac{1-k}{2} = \mathcal{A}(x) \vee \mathcal{A}(x) \vee \tfrac{1-k}{2} \\ &\geq \mathcal{A}(xa) \vee \mathcal{A}(x) \vee \tfrac{1-k}{2} \left(\begin{array}{c}\text{Since } \mathcal{A}, \text{ is an } (<, < \vee\gamma_k)^* \text{-fuzzy} \\ \text{two sided ideal of } \mathcal{H}.\end{array}\right) \\ &\geq \bigwedge_{x=xax} \left\{\mathcal{A}(xa) \vee \mathcal{A}(x) \vee \tfrac{1-k}{2}\right\} \ (\text{Since } \mathcal{H} \text{ is regular})) \\ &= (\mathcal{A} \circ_k \mathcal{A})(x). \end{aligned}$$

Hence, $\mathcal{A}^k(x) = (\mathcal{A} \circ_k \mathcal{A})(x)$. This completes the proof. □

Theorem 4.14. *A semigroup $\mathcal{H}$ is a regular if and only if every $(<, < \vee\gamma_k)^*$-fuzzy two-sided ideal of $\mathcal{H}$ is k-idempotent.*

5. Intra-regular

In this section, we characterize intra-regular semigroup by the properties of $(<, < \vee\gamma_k)^*$-fuzzy left (right, two sided) ideals. Recall that an element a of a semigroup $\mathcal{H}$ is intra-regular if there exist $x, y \in \mathcal{H}$ such that $a = xa^2y$. If every element of a semigroup is intra-regular, then it is called intra-regular semigroup. In general, neither intra-regular semigroups are regular semigroups nor regular semigroups are intra-regular semigroups. However, in commutative semigroups both the concepts coincide. The following examples clarify the difference between regular semigroup and intra-regular semigroup.

Example 5.1. Let M be a countably infinite set and let $\mathcal{H}$ be the set of one-one maps $\pi : M \longrightarrow M$ with the property that $M - \pi(M)$ is infinite. Then, $\mathcal{H}$ is a semigroup with respect to the composition of functions and is called Baer Levi semigroup [2].

This semigroup $\mathcal{H}$ is a right cancellative, right simple semigroup without idempotents [2, Th. 8.2]. Thus, S is not regular but intra regular.

Example 5.2. Take $\mathcal{H} = \{a, b, c, d, e\}$ is a semigroup with the following Cayley table:

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	b	c
c	a	b	c	a	a
d	a	a	a	d	e
e	a	d	e	a	a

By routine computation $\mathcal{H}$ is a regular semigroup but not an intra regular semigroup.

Theorem 5.3. *A semigroup $\mathcal{H}$ is intra-regular if and only if $\mathcal{A} \circ^k \mathcal{B} \subseteq (\mathcal{A} \vee \mathcal{B})^k$ for every $(<, < \vee\gamma_k)^*$-fuzzy right ideal $\mathcal{A}$ and every $(<, < \vee\gamma_k)^*$-fuzzy left ideal $\mathcal{B}$ of the semigroup $\mathcal{H}$.*

Proof. Assume that $\mathcal{H}$ is intra-regular and let $\mathcal{A}$ be an $(<, < \vee\gamma_k)^*$-fuzzy right ideal and $\mathcal{B}$ an $(<, < \vee\gamma_k)^*$-fuzzy left ideal $\mathcal{B}$ of $\mathcal{H}$. For $x \in \mathcal{H}$, there exist $y, z \in \mathcal{H}$ such that $x = yx^2z$. Thus,

$$\begin{aligned}
\left(\mathcal{A} \circ^k \mathcal{B}\right)(x) &= \bigwedge_{x=yz} \left\{\mathcal{A}(y) \vee \mathcal{B}(z) \vee \tfrac{1-k}{2}\right\} \\
&\leq \left\{\mathcal{A}(yx) \vee \mathcal{B}(xz) \vee \tfrac{1-k}{2}\right\} \\
&\leq \left\{\left(\mathcal{A}(x) \vee \tfrac{1-k}{2}\right) \vee \left(\mathcal{B}(x) \vee \tfrac{1-k}{2}\right) \vee \tfrac{1-k}{2}\right\} \\
&\quad \text{(Since } \mathcal{A} \text{ is an } (<, < \vee\gamma_k)^* \text{-fuzzy right ideal.)} \\
&= \left\{(\mathcal{A} \vee \mathcal{B})(x) \vee \tfrac{1-k}{2}\right\} = (\mathcal{A} \cup \mathcal{B})^k(x) \\
\left(\mathcal{A} \circ^k \mathcal{B}\right)(x) &\leq (\mathcal{A} \vee \mathcal{B})^k(x).
\end{aligned}$$

Thus, $\mathcal{A} \circ^k \mathcal{B} \subseteq (\mathcal{A} \vee \mathcal{B})^k$.

Conversely, suppose that $\mathcal{A} \circ^k \mathcal{B} \subseteq (\mathcal{A} \vee \mathcal{B})^k$ for $(<, < \vee\gamma_k)^*$-fuzzy right ideal $\mathcal{A}$ and $(<, < \vee\gamma_k)^*$-fuzzy left ideal $\mathcal{B}$ of $\mathcal{H}$. Then, S is intra-regular . Indeed: It is enough to prove that $A \cap B \subseteq AB$ for every right ideal A and for every left ideal B of $\mathcal{H}$.

Let A and B be right and left ideals of $\mathcal{H}$. Then by Proposition 4.5, $(\mathfrak{X}_{A^c})^k$ and $(\mathfrak{X}_{B^c})^k$ are $(<, < \vee\gamma_k)^*$-fuzzy right and .$(<, < \vee\gamma_k)^*$-fuzzy left ideals of $\mathcal{H}$, respectively. Thus, we have

$$\begin{aligned} 1 - \left(\mathfrak{X}_{(AB)^c}\right)^k(x) &= \left(\mathfrak{X}_{(AB)^c}\right)^k(x) = \left(\mathfrak{X}_{A^c} \circ^k \mathfrak{X}_{B^c}\right)(x) \\ &\leq (\mathfrak{X}_{A^c} \vee \mathfrak{X}_{B^c})^k(x) = (\mathfrak{X}_{A^c \cup B^c})^k(x) \\ &= \left(\mathfrak{X}_{(A\cap B)^c}\right)^k(x) = 1 - \left(\mathfrak{X}_{(A\cap B)}\right)^k(x). \end{aligned}$$

Thus, $\left(\mathfrak{X}_{(AB)}\right)^k(x) \leq \left(\mathfrak{X}_{(A\cap B)}\right)^k(x)$, this implies that $A \cap B \subseteq AB$. Therefore, by using Theorem 2.6, $\mathcal{H}$ is intra-regular. □

Theorem 5.4. *Let $\mathcal{H}$ be an intra-regular semigroup. Then, every $(<, < \vee\gamma_k)^*$-fuzzy ideal of the semigroup $\mathcal{H}$ is k-idempotent.*

Proof. Let $\mathcal{A}$ be an $(<, < \vee\gamma_k)^*$-fuzzy ideal of the semigroup $\mathcal{H}$ and let $x \in \mathcal{H}$. Then, $\mathcal{A} \circ_k \mathcal{A} = \mathcal{A}^k$. Indeed, since $\mathcal{H}$ is an intra-regular, so there exist $a, b \in \mathcal{H}$ such that $x = ax^2b = (ax)(xb)$. Thus, $x = yz$ for some $y, z \in \mathcal{H}$.

$$\begin{aligned} (\mathcal{A} \circ_k \mathcal{A})(x) &= \bigwedge_{x=yz} \left\{\mathcal{A}(y) \vee \mathcal{A}(z) \vee \tfrac{1-k}{2}\right\} \\ &\leq \bigwedge_{x=yz} \left\{\mathcal{A}(ax) \vee \mathcal{A}(xb) \vee \tfrac{1-k}{2}\right\} \\ &= \left\{\left(\mathcal{A}(ax) \vee \tfrac{1-k}{2}\right) \vee \left(\mathcal{A}(xb) \vee \tfrac{1-k}{2}\right) \vee \tfrac{1-k}{2}\right\} \\ &\leq \left\{\mathcal{A}(x) \vee \mathcal{A}(x) \vee \tfrac{1-k}{2}\right\} = \mathcal{A}(x) \vee \tfrac{1-k}{2} \\ (\mathcal{A} \circ_k \mathcal{A})(x) &\leq \mathcal{A}^k(x) \end{aligned}$$

Consequently, $(\mathcal{A} \circ_k \mathcal{A})(x) \geq \mathcal{A}^k(x)$. Thus, $(\mathcal{A} \circ_k \mathcal{A})(x) = \mathcal{A}^k(x)$. This complete the proof. □

Theorem 5.5. *A semigroup $\mathcal{H}$ is both regular and intr-regular if and only if*

$$(\mathcal{A} \vee \mathcal{B})^k \supseteq (\mathcal{A} \circ_k \mathcal{B}) \vee (\mathcal{B} \circ_k \mathcal{A})$$

for every $(<, < \vee\gamma_k)^*$-fuzzy right ideal $\mathcal{A}$ and every $(<, < \vee\gamma_k)^*$-fuzzy left ideal of a semigroup $\mathcal{H}$.

Proof. Let $\mathcal{H}$ be an regular and intra-regular semigroup and let $x \in \mathcal{H}$. Then, $(\mathcal{A} \vee \mathcal{B})^k(x) \supseteq (\mathcal{A} \circ_k \mathcal{B}) \vee (\mathcal{B} \circ_k \mathcal{A})(x)$ for every $(<, < \vee\gamma_k)^*$-fuzzy right ideal $\mathcal{A}$ and every $(<, < \vee\gamma_k)^*$-fuzzy left ideal of $\mathcal{H}$. Indeed: Since $\mathcal{H}$ is a regular, so by Theorem 4.9, it follows that $(\mathcal{A} \vee \mathcal{B})^k(x) = (\mathcal{A} \circ_k \mathcal{B})(x)$. Also, since $\mathcal{H}$ is an intra-regular, so by Theorem 5.4, this imply $(\mathcal{A} \vee \mathcal{B})^k(x) \supseteq (\mathcal{B} \circ_k \mathcal{A})(x)$. Therefore, we have $(\mathcal{A} \vee \mathcal{B})^k(x) \supseteq (\mathcal{A} \circ_k \mathcal{B}) \vee (\mathcal{B} \circ_k \mathcal{A})(x)$.

Conversely, let $(\mathcal{A} \vee \mathcal{B})^k \supseteq (\mathcal{A} \circ_k \mathcal{B}) \vee (\mathcal{B} \circ_k \mathcal{A})$ for every $(<, < \vee\gamma_k)^*$-fuzzy right ideal $\mathcal{A}$ and every $(<, < \vee\gamma_k)^*$-fuzzy left ideal of a semigroup $\mathcal{H}$. Then, $\mathcal{H}$ is both regular and intra-regular semigroup. Indeed: By Theorem 2.7 Its enough to prove that for every right ideal R and left ideal L of a semigroup $\mathcal{H}$

$$R \cap L \subseteq RL \cap LR.$$

Take $x \in R \cap L$. So, $x \in RL \cap LR$. Indeed: Since R is a right ideal and L is a left ideal, so by Proposition 4.5, $(\mathfrak{X}_{R^c})^k$ and $(\mathfrak{X}_{L^c})^k$ are $(<, < \vee\gamma_k)^*$-fuzzy right and .$(<, < \vee\gamma_k)^*$-fuzzy left ideals of $\mathcal{H}$, respectively. By given hypothesis $(\mathfrak{X}_{R^c} \vee \mathfrak{X}_{L^c})^k (x) \geq \left(\left(\mathfrak{X}_{R^c} \circ^k \mathfrak{X}_{L^c}\right) \vee \left(\mathfrak{X}_{L^c} \circ^k \mathfrak{X}_{R^c}\right)\right)(x)$, i.e., since $x \in R$ and $x \in L$, so $(\mathfrak{X}_{R^c} \vee \mathfrak{X}_{L^c})^k (x) = 0$. This implies that

$$\left(\left(\mathfrak{X}_{R^c} \circ^k \mathfrak{X}_{L^c}\right) \vee \left(\mathfrak{X}_{L^c} \circ^k \mathfrak{X}_{R^c}\right)\right)(x) = 0.$$

Aslo, by Lemma 4.4, we have

$$\begin{aligned} 0 &= \left(\left(\mathfrak{X}_{(RL)^c}\right)^k \vee \left(\mathfrak{X}_{(LR)^c}\right)^k\right)(x) \\ &= \left(\left(\mathfrak{X}_{(RL)^c}\right) \vee \left(\mathfrak{X}_{(LR)^c}\right)\right)^k (x) \\ &= \left(\mathfrak{X}_{((LR)\cap(LR))^c}\right)^k (x). \end{aligned}$$

Thus, $\left(\mathfrak{X}_{((LR)\cap(LR))^c}\right)^k (x) = 0 \Rightarrow x \in RL \cap LR$. This completes the proof. □

6. Conclusions and applications

The application of fuzzy technology in information precessing is already important and it will certainly increase in importance in the future. Our aim is to promote research and the development of fuzzy technology by studying the generalized fuzzy semigroups. The goal is to explain new methodological developments in fuzzy semigroups which will also be of growing importance in the future. Since the notion of fuzzy ideal of a semigroup play an important role in the study of semigroup structures, by using the idea of a quasi-coincidence of a fuzzy point with a fuzzy set, we use the idea of Saeid et al. to semigroup and defined a new generalization of fuzzy semigroups. A generalization of fuzzy ideals in semigroups of type $(\alpha, \beta)^*$-fuzzy ideals are introduced, where $\alpha, \beta \in \{<, \gamma_k, < \vee\gamma_k, < \wedge\gamma_k\}$ with $\alpha \neq < \wedge\gamma_k$. We have proved some fundamental results that determine the relation between these notions and ideals of semigroups. We hope that the research along this direction can be continued, and in fact, this work would serve as a foundation for further study of the theory of semigroups.

On the other hand, we know that $\mathcal{H}$-language (S-language) and $\mathcal{H}$-automation (S-automation) over a finite non-empty set $\mathcal{U}$, we need a semigroup. Indeed: Let $\mathcal{U}$ be a non-empty finite set and $\mathcal{U}^*$ be the free semigroup generated by $\mathcal{U}$, with identity $\curlywedge$. Let $\mathcal{H}$ be a semigroup. A function $f^A : \mathcal{U}^* \longrightarrow \mathcal{H}$ is called an $\mathcal{H}$-language (S–language) over $\mathcal{U}$. An $\mathcal{H}$-automation (S-automation) over $\mathcal{U}$ is a 4-tuple $A = (\mathcal{R}, p, h, g)$, where $\mathcal{R}$ is a finite non-empty set, p is a function from $\mathcal{R} \times \mathcal{U} \times \mathcal{R}$ into $\mathcal{H}$, h and g are functions from $\mathcal{R}$ into $\mathcal{H}$. We will focus on the following topic in future: 1) We apply the present concept to other algebraic structures, i.e Ring, Hemiring, Nearring etc. 2) We will define (α, β)-fuzzy soft ideals in semigroups. 3) $(\alpha, \beta)^*$-fuzzy soft ideals in semigroups.

References

[1] S. K. Bhakat and P. Das, $(\in, \in \vee q)$-fuzzy subgroup, Fuzzy Sets and Systems 80 (1996) 359–368.

[2] A. H. Clifford and G. B. Preston, The Algebraic Theory of Semigroups, Volume II, Ammerican Mathematical Socity, 1967.

[3] L. E. Dickson, On semi-groups and the general isomorphism between infinite groups, Trans. Amer. Math. Soc. 6 (1905) 205–208.

[4] A. Khan, Y. B. Jun and M. Shabir, A study of generalized fuzzy ideals in ordered semigroups, Neural Comput. & Applic. 21(Suppl 1) (2012) S69–S78.

[5] N. Kuroki, On fuzzy semigroups, Inform. Sci. 53 (1991) 203–236.

[6] N. Kuroki, On fuzzy ideals and fuzzy bi-ideals in semigroups, Fuzzy Sets and Systems 5(1) (198) 203–215.

[7] N. Kuroki, Fuzzy semiprime ideals in semigroups, Fuzzy Sets and Systems 8 (1982) 71–79.

[8] N. Kuroki, On fuzzy semigroups, Inform. Sci. 53 (1991) 203–236.

[9] N. Kuroki, Fuzzy generalized bi-ideals in semigroups, Inform. Sci. 66 (1992) 235–243.

[10] N. Kuroki, On fuzzy semiprime quasi-ideals in semigroups, Inform. Sci. 75 (1993) 201–211.

[11] N. Kuroki, Fuzzy bi-ideals in Semigroups, Comment. Math. Univ. St. Paul. 28(1) (1980) 17–21.

[12] N. Kuroki, On fuzzy ideals and fuzzy bi-ideals in semigroups, Fuzzy Sets and Systems 5 (1981) 203–215.

[13] J. N. Mordeson, K. R. Bhutani and A. Rosenfeld, Fuzzy Group Theory, Studies in Fuzziness and Soft Computing, vol. 182, Springer-Verlag, Berlin, Heidelberg, 2005.

[14] J. N. Mordeson, D. S. Malik and N. Kuroki, Fuzzy semigroups, Studies in Fuziness and Soft Computing, vol. 131, Springer-Verlag, Berlin, Heidelberg, 2003.

[15] J. N. Mordeson and D. S. Malik, Fuzzy automata and languages, theory and applications, Computational Mathematics Series, Chapman and Hall/CRC, Boca Raton, 2002.

[16] V. Murali, Fuzzy points of equivalent fuzzy subsets, Inform. Sci. 158 (2004) 277–288.

[17] P. M. Pu and M. liu, Fuzzy topology I: nieghbourhood structure of a fuzzy point and moore-smith convergence, J. Math. Anal. Appl. 76 (1980) 571–599.

[18] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512–517.

[19] A. B. Saeid and Y. B. Jun, Redefined fuzzy subalgebras of BCK/BCI-algebras, Iran. J. Fuzzy Syst. 5(2) (2008) 63–70.

[20] A. B. Saeid, D. P. Williams and M. K. Rafsanjani, A new generalization of fuzzy BCK/BCI-algebras, Neural Comput. & Applic. 21 (2012) 813–819.

[21] M. Shabir and Y. Nawaz, Semigroups characterized by the properties of their anti fuzzy ideals. J. Adv. Res. Pure Math. 1(3) (2009) 42–59.

[22] A. K. Suschkewitsch, Ũber die endlichen Gruppen ohne Gesetz der eindeutigen Umkehrbarkeit, Mathematical Annalysis 99 (1928) 30–50.

[23] L. A. Zadeh, Fuzzy sets, Information and Control 8(1965) 338–353.

S. Abdullah (saleemabdullah81@yahoo.com, saleem@math.qau.edu.pk)
Department of Mathematics, Quaid-i-Azam University 45320, Islamabad 44000, Pakistan

M. Aslam (draslamqau@yahoo.com)
Department of Mathematics, King Khalid University, Abha, Saudi Arabia

B. Davvaz (bdavvaz@yahoo.com)
Department of Mathematics, Yazd University, P.O.Box 89195-741, Yazd, Iran