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Zadeh extension principle: A note

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Abstract. In this note, we examine a connection of the Zadeh extension principle with the notion of matrix theories (as used in category theory) in the sense of E.G. Manes. In particular, we employ matrix theories over a complete idempotent semiring, for this purpose.

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1. Introduction

The well- known Zadeh extension principle (ZEP) of fuzzy set theory is regarded as an important tool in fuzzy set theory and its applications (e.g. in fuzzy arithmetic). ZEP is given, e.g., in Zadeh [13], as follows:
For every function $f : X \to Y$ between sets, there exist functions $f^{\to} : I^X \to I^Y$ and $f^{\leftarrow} : I^Y \to I^X$, (where $I = [0,1]$; elements of I^X are called *fuzzy sets* in X) defined as:

$$f^{\to}(\mu)(y) = \vee \left\{\mu(x) : x \in f^{-1}(y)\right\}, \forall \mu \in I^X \text{ and}$$
$$f^{\leftarrow}(\nu) = \nu \circ f, \forall \nu \in I^Y \text{ such that}$$
$$(f^{\to} \circ f^{\leftarrow})(b) \leq b, \forall b \in I^Y \text{ and } (f^{\leftarrow} \circ f^{\to})(a) \geq a, \forall a \in I^X.$$

($f^{\to}$ and $f^{\leftarrow}$ are frequently referred to as the *forward* and the *backward lifting operators*). An introductory account of ZEP has been given by Kerre [5], while a somewhat detailed study of the extension principle for fuzzy sets has been made, e.g., by Gerla and Scarpati [2], Nguyen [9] and Yager [12]. It may be pointed out that the ZEP was initially introduced in the context of fuzzy sets but has since been extended in related areas also (e.g., to define the image of an intuitionistic fuzzy sets under a functions, ZEP is used; see, e.g., Kang et al [4] and Saleh [11]). The ZEP has been also looked into from a category theoretic point of view by a few authors, e.g., Rodabaugh [10], Kotze [6] and Barone [1]. In this note, we point out a connection of the ZEP with the notion of a matrix theory over a complete semiring S, as introduced by Manes [8]. In fact, we show that given such a matrix theory, each function $f : X \to Y$ between sets, naturally gives rise to a function $S^X \to S^Y$ which resembles the forward lifting operators $f^{\to}$ in the ZEP, in case the complete semiring S is additionally assumed to be idempotent.

2. Preliminaries

We shall use the following result from category theory; see, e.g., Herrlich et al [3] and Maclane [7].

Theorem 2.1. *(Freyd's adjoint functor theorem for posets) Let L and M be posets with L closed under arbitrary $\vee$. Let a function $g : L \to M$ preserve arbitrary joins. Then there exists an order-preserving map $f : M \to L$ such that f is right adjoint of g, i.e., $gf(a) \leq a$ and $fg(b) \geq b$ hold $\forall a \in M$ and $\forall b \in L$. Moreover, f is given by*

$$f(a) = \vee \{b \in L : a \geq g(b)\}.$$

Definition 2.2. A semiring $(S, \oplus, \odot)$ is a non empty set S on which two commutative binary operations $\oplus$ and $\odot$ are defined such that the following conditions are satisfied:

(1) $(S, \oplus)$ is a monoid (with identity element 0),
(2) $(S, \odot)$ is a monoid,
(3) Multiplication distributes over addition from either side,
(4) $0 \odot a = 0 = a \odot 0$, $\forall a \in S$,

Further, a semiring $(S, \oplus, \odot)$ is called (i) idempotent if $\oplus$ is idempotent and (ii) complete if for every family $\{a_i \in S : i \in \Omega\}$, there exists an element $\sum\limits_{i\in\Omega} a_i$ of S such that the following conditions hold:
(i) $\sum\limits_{i\in\Omega} a_i = 0$, if $\Omega = \phi$,
(ii) $\sum\limits_{i\in\Omega} a_i = a_1 \oplus a_2 \oplus \ldots \oplus a_n$, if $\Omega = \{1, 2, \ldots, n\}$,
(iii) $b \odot \sum\limits_{i\in\Omega} a_i = \sum\limits_{i\in\Omega} (b \odot a_i)$ and $(\sum\limits_{i\in\Omega} a_i) \odot b = \sum\limits_{i\in\Omega} (a_i \odot b), \forall b \in S$
(iv) $\sum\limits_{i\in\Omega} a_i = \sum\limits_{j\in\pi} (\sum\limits_{k\in\Omega_j} a_k)$ for every partition $\Omega = \bigcup\limits_{j\in\pi} \Omega_j$ of Ω.

Remark 2.3. We can easily verify that if $(S, \oplus, \odot)$ is a complete idempotent semiring and X is any set, then $(S^X, +, .)$ is also a complete idempotent semiring with . and + defined as follows: for all $f, g \in S^X, \{f_i : i \in \Omega\} \subseteq S^X$ and $x \in X$,
$(f.g)(x) = f(x).g(x)$,
$(f + g)(x) = f(x) + g(x), \forall x \in X$, and
$(+\sum\limits_{i\in\Omega} f_i)(x) = \oplus \sum\limits_{i\in\Omega} f_i(x)$.

Definition 2.4 ([8])**.** An algebraic theory in a category $\mathbf{K}$ (in extension form) is a triple $(T, \eta, (-)^{\#})$ where :

(1) T is a function assigning to each $\mathbf{K}$-object X, a $\mathbf{K}$-object $T(X)$,
(2) η is an assignment, assigning to each K-object X, a $\mathbf{K}$-morphism, $X \xrightarrow{\eta_X} TX$,

(3) $(-)^{\#}$ assigns to each **K**-morphism $X \xrightarrow{\alpha} T(Y)$, a **K**-morphism $T(X) \xrightarrow{\alpha^{\#}} T(Y)$ such that for all **K**-objects X and **K**-morphisms $X \xrightarrow{\alpha} T(Y)$, $Y \xrightarrow{\beta} T(Z)$,
$(i)\, \alpha^{\#}.\eta_X = \alpha$, $(ii)\, \eta_X^{\#} = Id_{T(X)}$, $(iii)\, (\beta^{\#}.\alpha)^{\#} = \beta^{\#}.\alpha^{\#}$.

Given a complete semiring $(S, \oplus, \odot)$, an algebraic theory $(T, \eta, (-)^{\#})$ in the category **SET** is called a matrix theory of S if, for every set X

(1) $T(X) = S^X$,
(2) $\eta_X : X \to T(X)$ is defined by $\eta_X(x)(x') = \delta_x^{x'}$,
(3) for sets X, Y and any $X \xrightarrow{\alpha} T(Y)$, $T(X) \xrightarrow{\alpha^{\#}} T(Y)$ is defined by: $(\alpha^{\#}(p))(y) = \sum\limits_{x\in X} (\alpha(x)(y)) \odot p(x), \forall p \in T(X),\ \forall y \in Y$.

$X \xrightarrow{\alpha} T(Y)$, may be thought as a matrix with entries in S with X indexing columns and Y indexing rows.
As is known, T gives rise to a functor $T : \mathbf{SET} \to \mathbf{SET}$, given by $T(X) = S^X$ and for any $f : X \to Y$, $T(f) : S^X \to S^Y$ is defined by $(T(f)(p))(y) = \oplus \sum\limits_{x\in f^{-1}(y)} p(x), \forall p \in S^X$, $\forall y \in Y$.

3. Main observation

Proposition 3.1. *Let $(T, \eta, (-)^{\#})$ be a matrix theory of a complete semiring $(S, \oplus, \odot)$. Then for any function $f : X \to Y$, between sets*

$$T(f)(+\sum_{j\in J} p_j) = +\sum_{j\in J} T(f)(p_j),\ \forall p_j \in S^X, j \in J.$$

Proof. Let $y \in Y$. Then for every $y \in Y$,
$(T(f)(+\sum\limits_{j\in J} p_j))(y) = \oplus\sum\limits_{x\in f^{-1}(y)} (+\sum\limits_{j\in J} p_j)(x)$,
$= \oplus \sum\limits_{x\in f^{-1}(y)} (\oplus \sum\limits_{j\in J} p_j(x))$,
$= \oplus \sum\limits_{j\in J} (\oplus\sum\limits_{x\in f^{-1}(y)} p_j(x))$,
$= \oplus \sum\limits_{j\in J} (T(f)(p_j)(y))$,
$= (+\sum\limits_{j\in J} T(f)(p_j))(y), \forall y \in Y$.
Hence, $(T(f)(+\sum\limits_{j\in J} p_j) = +\sum\limits_{j\in J} T(f)(p_j)$. □

We now confine to a complete semiring $(S, \oplus, \odot)$ which is idempotent also (i.e.,$\oplus$ is idempotent). Then we can define a relation $\leq$ on S as follows: $a \leq b$ iff $a \oplus b = b$. It is easy to see that $\leq$ is a partial order on S and the $\oplus$ operation coincides with the join operation induced by $\leq$.

Theorem 3.2. *Let $(S, \oplus, \odot)$ be a complete idempotent semiring, $f : X \to Y$ a map between sets X and Y, and $T(f)$ be denoted by $f^{\to}$. Then there exists a function $f^{\leftarrow} : S^Y \to S^X$ such that*

$$(f^{\to} \circ f^{\leftarrow})(q) \leq q,\ \forall q \in S^Y, \text{ and}$$
$$(f^{\leftarrow} \circ f^{\to})(p) \geq p,\ \forall p \in S^X.$$

Moreover, $f^{\leftarrow}$ is given by,

$$f^{\leftarrow}(q) = q \circ f, \forall q \in S^Y.$$

Proof. For a proof, we view $f^{\rightarrow} : S^X \rightarrow S^Y$, as a functor from the poset S^X to the poset S^Y, considered as categories. Then from Proposition 3.1 and Theorem 2.1, $\exists$ an order-preserving map $f^{\leftarrow} : S^Y \rightarrow S^X$ such that $f^{\leftarrow}$, considered as functor, is right adjoint of $(f^{\rightarrow})$, i.e.,

$$(f^{\rightarrow} \circ f^{\leftarrow})(q) \leq q,\ \forall q \in S^Y, \text{ and}$$
$$(f^{\leftarrow} \circ f^{\rightarrow})(p) \geq p,\ \forall p \in S^X.$$

We now show that $f^{\leftarrow}(q) = q \circ f$. By Theorem 2.1,

$$\forall q \in S^Y,\ f^{\leftarrow}(q) = +\Sigma \left\{p \in S^X : q \geq (f^{\rightarrow})(p)\right\} = +\sum_{p \in \Omega_q} p,$$

where $\Omega_q = \left\{p \in S^X : q \geq (f^{\rightarrow})(p)\right\}$. First, we show that $f^{\leftarrow}(q) \geq q \circ f$.
Now, $\forall y \in Y$,
$(f^{\rightarrow}(q \circ f))(y) = \oplus \sum_{x \in f^{-1}(y)} (q \circ f)(x) = \oplus \sum_{x \in f^{-1}(y)} q(f(x)) = \oplus \Sigma q(y) = q(y)$.
Hence $q \circ f \in \Omega_q$, whereby $f^{\leftarrow}(q) = +\sum_{p \in \Omega_q} p \geq q \circ f$. (1)
Now we have to show that
$q \circ f \geq f^{\leftarrow}(q)$, i.e., $(q \circ f)(x) \geq f^{\leftarrow}(q)(x)$, $\forall x \in X$
We note that,

$$p \in \Omega_q \Rightarrow q(y) \geq f^{\rightarrow}(p)(y), \forall y \in Y \Rightarrow q(y) \geq \oplus \sum_{z \in f^{-1}(y)} p(z), \forall y \in Y.$$

In particular, for any given $x \in X$ and $p \in \Omega_q$,

$$q(f(x)) \geq \underset{z \in f^{-1}(f(x))}{\oplus\Sigma} p(z), \forall p \in \Omega_q.$$

Taking summation over $p \in \Omega_q$, we get

$$q(f(x)) \geq \oplus \sum_{p \in \Omega_q} \Big(\underset{z \in f^{-1}(f(x))}{\oplus\Sigma} p(z) \Big).$$

But as $\underset{z \in f^{-1}(f(x))}{\oplus\Sigma} p(z) \geq p(x)$,
$\oplus \sum_{p \in \Omega_q} \Big(\underset{z \in f^{-1}(f(x))}{\oplus\Sigma} p(z) \Big) \geq \oplus \sum_{p \in \Omega_q} p(x)$, whereby $q(f(x)) \geq \oplus \sum_{p \in \Omega_q} p(x)$,
i.e., $(q \circ f)(x) \geq (+\sum_{p \in \Omega_q} p)(x) = (f^{\leftarrow}(q))(x), \forall x \in X$.
Hence $(q \circ f) \geq f^{\leftarrow}(q)$. (2)
From (1) and(2), we find that $f^{\leftarrow}(q) = q \circ f$. □

Remark 3.3. In view of Theorem 3.2 and in particular due to the properties $(f^{\rightarrow} \circ f^{\leftarrow})(q) \leq q$, $\forall q \in S^Y$ and $(f^{\leftarrow} \circ f^{\rightarrow})(p) \geq p$, $\forall p \in S^X$, it appears appropriate to refer to the functions $f^{\rightarrow}$ and $f^{\leftarrow}$ as the *forward* and *backward* operators, as in the case of the ZEP.

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