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Intuitionistic fuzzy exterior spaces via rings

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Abstract. In this paper, the concepts of an intuitionistic fuzzy rings, intuitionistic fuzzy structure ring spaces, intuitionistic fuzzy ring exteriors, intuitionistic fuzzy ring exterior B spaces and intuitionistic fuzzy ring exterior V spaces are introduced. Also, the concepts of an intuitionistic fuzzy ring continuous functions, intuitionistic fuzzy ring hardly open functions and somewhat intuitionistic fuzzy ring continuous functions are studied. In this connection, some interesting properties are established and provided necessary examples.

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1. Introduction

The concept of fuzzy sets was introduced by Zadeh [10]. Chang [3] introduced the concepts of fuzzy topological spaces. Atanassov [1] introduced and studied intuitionistic fuzzy sets. On the otherhand, Coker [4, 5] introduced the notions of an intuitionistic fuzzy topological spaces, intuitionistic fuzzy continuity and some other related concepts. G. Balasubramanian [2] introduced the concepts of fuzzy G_δ sets in fuzzy topological spaces. Later R. Narmada Devi [7, 8] was introduced the concepts of intuitionistic fuzzy G_δ sets. Meena and Thomas [6] were introduced the concepts of intuitionistic L-fuzzy subrings. A.A.Salama [9, 10] introuced the concepts of exterior and ideal theory in intuitionistic topological spaces and intuitionistic fuzzy topological spaces respectively. In this paper, the concepts of an intuitionistic fuzzy rings, intuitionistic fuzzy structure ring spaces, intuitionistic fuzzy ring exteriors, intuitionistic fuzzy G_δ rings, intuitionistic fuzzy first category rings, intuitionistic fuzzy ring $G_\delta T_{1/2}$ spaces, intuitionistic fuzzy ring exterior B spaces and intuitionistic fuzzy ring exterior V spaces are introduced. Also, the concepts of an intuitionistic fuzzy ring continuous functions, intuitionistic fuzzy ring hardly open functions and somewhat intuitionistic fuzzy ring continuous functions are studied. In this connection, some interesting properties are established and provided necessary examples.

2. Preliminaries

Definition 2.1 ([1]). Let X be a nonempty fixed set and I is the closed interval [0,1]. An intuitionistic fuzzy set(IFS) A is an object having the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$,where the mapping $\mu_A : X \longrightarrow I$ and $\gamma_A : X \longrightarrow I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) for each element $x \in X$ to the set A respectively and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$. Obviously,every fuzzy set A on a nonempty set X is an IFS of the following form, $A = \{\langle x, \mu_A(x), 1-\mu_A(x)\rangle : x \in X\}$. For the sake of simplicity, we shall use the symbol $A = \langle x, \mu_A, \gamma_A\rangle$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$. For a given nonempty set X, the family of all IFSs in X is denoted by ζ^X.

Definition 2.2 ([1]). Let X be a nonempty set and the IFSs A and B in the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$, $B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle : x \in X\}$. Then

- (i) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$;
- (ii) $\overline{A} = \{\langle x, \gamma_A(x), \mu_A(x)\rangle : x \in X\}$.

Definition 2.3 ([1]). The IFSs $0_\sim$ and $1_\sim$ are defined by $0_\sim$=$\{\langle x, 0, 1\rangle : x \in X\}$ and $1_\sim$=$\{\langle x, 1, 0\rangle : x \in X\}$.

Definition 2.4 ([4, 5]). An intuitionistic fuzzy topology (IFT)on a nonempty set X is a family τ of IFSs in X satisfying the following axioms:

- (i) $0_\sim, 1_\sim \in \tau$;
- (ii) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$;
- (iii) $\cup G_i \in \tau$ for arbitrary family $\{G_i \mid i \in I\} \subseteq \tau$.

In this case the ordered pair (X, τ) or simply by X is called an intuitionistic fuzzy topological space ($IFTS$) on X and each IFS in τ is called an intuitionistic fuzzy open set ($IFOS$). The complement $\overline{A}$ of an $IFOS$ A in X is called an intuitionistic fuzzy closed set ($IFCS$) in X.

Definition 2.5 ([4, 5]). Let A be an IFS in $IFTS$ X. Then

$int(A) = \bigcup\{G \mid G$ is an $IFOS$ in X and $G \subseteq A\}$ is called an intuitionistic fuzzy interior of A;

$clA = \bigcap\{G \mid G$ is an $IFCS$ in X and $G \supseteq A\}$ is called an intuitionistic fuzzy closure of A.

Definition 2.6 ([8]). Let (X, T) be an intuitionistic fuzzy topological space. Let $A = \langle x, \mu_A, \gamma_A\rangle$ be an intuitionistic fuzzy set on an intuitionistic fuzzy topological space (X, T). Then A is said be an intuitionistic fuzzy G_δ set if $A = \bigcap_{i=1}^{\infty} A_i$, where $A_i = \langle x, \mu_{A_i}, \gamma_{A_i}\rangle$ is an intuitionistic fuzzy open set in an intuitionistic fuzzy topological space (X, T). The complement of an intuitionistic fuzzy G_δ set is said to be an intuitionistic fuzzy F_σ set.

Definition 2.7 ([9]). Let $A = \langle A_1, A_2 \rangle$ be an intuitionistic set on a intuitionistic topological space (X, τ). We define an intuitionistic exterior of A as follows: if $A^{IE} = X_I \cap A^C$

3. Properties of intuitionistic fuzzy ring exterior B spaces

In this section, the concepts of an intuitionistic fuzzy rings, intuitionistic fuzzy structure ring spaces, intuitionistic fuzzy ring exterior, intuitionistic fuzzy G_δ rings, intuitionistic fuzzy first category rings, intuitionistic fuzzy ring $G_\delta T_{1/2}$ spaces and intuitionistic fuzzy ring exterior B spaces are introduced. In this connection, some interesting properties are established.

Definition 3.1. Let R be a ring. An intuitionistic fuzzy set $A = \langle x, \mu_A, \gamma_A \rangle$ in R is called an intuitionistic fuzzy ring on R if it satisfies the following conditions:

(i) $\mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y)$,
(ii) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$,
(iii) $\gamma_A(x+y) \leq \gamma_A(x) \vee \gamma_A(y)$,
(iv) $\gamma_A(xy) \leq \mu_A(x) \vee \gamma_A(y)$,

for all $x, y \in R$.

Definition 3.2. Let R be a ring. A family $\mathscr{S}$ of an intuitionistic fuzzy rings in R is said to be intuitionistic fuzzy structure ring on R if it satisfies the following axioms:

(i) $0_\sim, 1_\sim \in \mathscr{S}$.
(ii) $G_1 \cap G_2 \in \mathscr{S}$ for any $G_1, G_2 \in \mathscr{S}$.
(iii) $\cup G_i \in \mathscr{S}$ for arbitrary family $\{G_i \mid i \in I\} \subseteq \mathscr{S}$.

Then the ordered pair $(R, \mathscr{S})$ is called an intuitionistic fuzzy structure ring space. Every member of $\mathscr{S}$ is called an intuitionistic fuzzy open ring in $(R, \mathscr{S})$. The complement $\overline{\text{A}}$ of an intuitionistic fuzzy open ring A in $(R, \mathscr{S})$ is an intuitionistic fuzzy closed ring in $(R, \mathscr{S})$.

Example 3.3. Let $R = \{0, 1\}$ be a set of integers of module 2 with two binary operations as follows:

$+$	0	1
0	0	1
1	1	0

and

$\cdot$	0	1
0	0	0
1	0	1

Then $(R, +, \cdot)$ is a ring. Define intuitionistic fuzzy rings B and C on R as follows:

$$\mu_B(0) = 0.5, \mu_B(1) = 0.7 \text{ and } \gamma_B(0) = 0.3, \gamma_B(1) = 0.2$$

$$\mu_C(0) = 0.3, \mu_C(1) = 0.4 \text{ and } \gamma_C(0) = 0.5, \gamma_C(1) = 0.6$$

Then $\mathscr{S} = \{0_\sim, B, C, 1_\sim\}$ is an intuitionistic fuzzy structure ring on R. Thus the pair $(R, \mathscr{S})$ is an intuitionistic fuzzy structure ring space.

Notation 3.1. Let $(R, \mathscr{S})$ be any intuitionistic fuzzy structure ring space. Then

(i) $O(R)$ denotes the family of all intuitionistic fuzzy open ring of an intuitionistic fuzzy structure ring space $(R, \mathscr{S})$.
(ii) $C(R)$ denotes the family of all intuitionistic fuzzy closed ring of an intuitionistic fuzzy structure ring space $(R, \mathscr{S})$.

Definition 3.4. Let $(R, \mathscr{S})$ be any intuitionistic fuzzy structure ring space. Let $A = \langle x, \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy ring in R. Then

(i) the intuitionistic fuzzy ring interior of A is defined and denoted as

$$IF_R int(A) = \cup\{B = \langle x, \mu_B, \gamma_B \rangle \mid B \in O(R) \text{ and } B \subseteq A\}.$$

(ii) the intuitionistic fuzzy ring closure of A is defined and denoted as

$$IF_R cl(A) = \cap\{B = \langle x, \mu_B, \gamma_B \rangle \mid B \in C(R) \text{ and } A \subseteq B\}.$$

Remark 3.5. Let $(R, \mathscr{S})$ be any intuitionistic fuzzy structure ring space. Let $A = \langle x, \mu_A, \gamma_A \rangle$ be any intuitionistic fuzzy ring in R. Then the following statements hold:

(i) $IF_R cl(A) = A$ if and only if A is an intuitionistic fuzzy closed ring.
(ii) $IF_R int(A) = A$ if and only if A is an intuitionistic open ring.
(iii) $IF_R int(A) \subseteq A \subseteq IF_R cl(A)$.
(iv) $IF_R int(1_\sim) = 1_\sim$ and $IF_R int(0_\sim) = 0_\sim$.
(v) $IF_R cl(1_\sim) = 1_\sim$ and $IF_R cl(0_\sim) = 0_\sim$.
(vi) $IF_R cl(\overline{A}) = \overline{IF_R int(A)}$ and $IF_R int(\overline{A}) = \overline{IF_R cl(A)}$.
(vii) $\cup_{i=1}^{\infty} IF_R cl(A_i) \subseteq IF_R cl(\cup_{i=1}^{\infty} A_i)$.
(viii) $\cap_{i=1}^{n} IF_R cl(A_i) = IF_R cl(\cup_{i=1}^{n} A_i)$.
(ix) $\cap_{i=1}^{\infty} IF_R cl(A_i) \subseteq IF_R cl(\cup_{i=1}^{\infty} A_i)$.
(x) $\cup_{i=1}^{\infty} IF_R int(A_i) \subseteq IF_R int(\cup_{i=1}^{\infty} A_i)$.

Proof. The proof is simple. □

Definition 3.6. Let $(R, \mathscr{S})$ be any intuitionistic fuzzy structure ring space. Let $A = \langle x, \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy ring in R. Then $IF_R int(\overline{\mathrm{A}})$ is called an intuitionistic fuzzy ring exterior of A and is denoted by $IF_R Ext(A)$.

Proposition 3.7. *Let $(R, \mathscr{S})$ be an intuitionistic fuzzy structure ring space. Let $A = \langle x, \mu_A, \gamma_A \rangle$ and $B = \langle x, \mu_B, \gamma_B \rangle$ be any two intuitionistic fuzzy rings in R. Then the following statements hold:*

(i) $IF_R Ext(A) \subseteq \overline{A}$.
(ii) $IF_R Ext(A) = \overline{IF_R cl(A)}$.
(iii) $IF_R Ext(IF_R Ext(A)) = IF_R int(IF_R cl(A))$.
(iv) *If* $A \subseteq B$ *then* $IF_R Ext(A) \supseteq IF_R Ext(B)$.
(v) $IF_R Ext(1_\sim) = 0_\sim$ *and* $IF_R Ext(0_\sim) = 1_\sim$.
(vi) $IF_R Ext(A \cup B) = IF_R Ext(A) \cap IF_R Ext(B)$.

Proof. The proof is obvious. □

Definition 3.8. Let $(R, \mathscr{S})$ be an intuitionistic fuzzy structure ring space. Let $A = \langle x, \mu_A, \gamma_A \rangle$ be any intuitionistic fuzzy ring in R. Then A is said be to an intuitionistic fuzzy G_δ ring in $(R, \mathscr{S})$ if $A = \bigcap_{i=1}^{\infty} A_i$, where $A_i = \langle x, \mu_{A_i}, \gamma_{A_i} \rangle$ is an intuitionistic fuzzy open ring in $(R, \mathscr{S})$.

The complement of an intuitionistic fuzzy G_δ ring in $(R, \mathscr{S})$ is an intuitionistic fuzzy F_σ ring in $(R, \mathscr{S})$.

Definition 3.9. Let $(R, \mathscr{S})$ be an intuitionistic fuzzy structure ring space. Let $A = \langle x, \mu_A, \gamma_A \rangle$ be any intuitionistic fuzzy ring in R. Then A is said be to an

(i) intuitionistic fuzzy dense ring in $(R, \mathscr{S})$ if there exists no intuitionistic fuzzy closed ring B in $(R, \mathscr{S})$ such that $A \subset B \subset 1_\sim$.
(ii) intuitionistic fuzzy nowhere dense ring in $(R, \mathscr{S})$ if there exists no intuitionistic fuzzy open ring B in $(R, \mathscr{S})$ such that $B \subset IF_Rcl(A)$. That is, $IF_Rint(IF_Rcl(A)) = 0_\sim$.

Definition 3.10. Let $(R, \mathscr{S})$ be an intuitionistic fuzzy structure ring space. Let $A = \langle x, \mu_A, \gamma_A \rangle$ be any intuitionistic fuzzy ring in R. Then A is said be to an intuitionistic fuzzy first category ring in $(R, \mathscr{S})$ if $A = \cup_{i=1}^{\infty} A_i$ where A_i's are intuitionistic fuzzy nowhere dense ring in $(R, \mathscr{S})$.

The complement of an intuitionistic fuzzy first category ring in $(R, \mathscr{S})$ is an intuitionistic fuzzy residual ring in $(R, \mathscr{S})$.

Proposition 3.11. *Let $(R, \mathscr{S})$ be an intuitionistic fuzzy structure ring space. If A is an intuitionistic fuzzy G_δ ring and the intuitionistic fuzzy ring exterior of $\overline{\mathrm{A}}$ is an intuitionistic fuzzy dense ring in $(R, \mathscr{S})$, then $\overline{\mathrm{A}}$ is an intuitionistic fuzzy first category ring in $(R, \mathscr{S})$.*

Proof. Since A is an intuitionistic fuzzy G_δ ring in $(R, \mathscr{S})$, $A = \cap_{i=1}^{\infty} A_i$ where A_i's are intuitionistic fuzzy open rings. Since the intuitionistic fuzzy ring exterior of $\overline{\mathrm{A}}$ is an intuitionistic fuzzy dense ring in $(R, \mathscr{S})$, $IF_Rcl(IF_RExt(\overline{\mathrm{A}})) = 1_\sim$. Since $IF_RExt(\overline{\mathrm{A}}) \subseteq A \subseteq IF_Rcl(A)$, $IF_RExt(\overline{\mathrm{A}}) \subseteq IF_Rcl(A)$.

This implies that $IF_Rcl(IF_RExt(\overline{\mathrm{A}})) \subseteq IF_Rcl(A)$, that is, $1_\sim \subseteq IF_Rcl(A)$. Therefore, $IF_Rcl(A) = 1_\sim$. That is, $IF_Rcl(A) = IF_Rcl(\cap_{i=1}^{\infty} A_i) = 1_\sim$. But $IF_Rcl(\cap_{i=1}^{\infty} A_i) \subseteq \cap_{i=1}^{\infty} IF_Rcl(A_i)$. Hence, $1_\sim \subseteq \cap_{i=1}^{\infty} IF_Rcl(A_i)$.

That is, $\cap_{i=1}^{\infty} IF_Rcl(A_i) = 1_\sim$. This implies that $IF_Rcl(A_i) = 1_\sim$, for each $A_i \in \mathscr{S}$. Hence $IF_Rcl(IF_Rint(A_i)) = 1_\sim$. Now,

$$\begin{aligned} IF_Rint(IF_Rcl(\overline{A_i})) &= IF_Rint(\overline{IF_Rint(A_i)}) \\ &= \overline{IF_Rcl(IF_Rint(A_i))} = 0_\sim. \end{aligned}$$

Therefore, $\overline{A_i}$ is an intuitionistic fuzzy nowhere dense ring in $(R, \mathscr{S})$. Now, $\overline{\mathrm{A}} = \overline{\cap_{i=1}^{\infty} A_i} = \cup_{i=1}^{\infty} \overline{A_i}$. Therefore, $\overline{\mathrm{A}} = \cup_{i=1}^{\infty} \overline{A_i}$ where $\overline{A_i}$'s are intuitionistic fuzzy nowhere dense rings in $(R, \mathscr{S})$. Therefore, $\overline{\mathrm{A}}$ is an intuitionistic fuzzy first category ring in $(R, \mathscr{S})$. □

Proposition 3.12. *If A is an intuitionistic fuzzy first category ring in an intuitionistic fuzzy structure ring space $(R, \mathscr{S})$ such that $B \subseteq \overline{A}$ where B is non-zero intuitionistic fuzzy G_δ ring and the intuitionistic fuzzy ring exterior of $\overline{B}$ is an intuitionistic fuzzy dense ring in $(R, \mathscr{S})$, then A is an intuitionistic fuzzy nowhere dense ring in $(R, \mathscr{S})$.*

Proof. Let A be an intuitionistic fuzzy first category ring in $(R, \mathscr{S})$. Then $A = \cup_{i=1}^{\infty} A_i$ where A_i's are intuitionistic fuzzy nowhere dense rings in $(R, \mathscr{S})$. Now $\overline{IF_Rcl(A_i)}$ is an intuitionistic fuzzy open ring in $(R, \mathscr{S})$. Let $B = \cap_{i=1}^{\infty} \overline{IF_Rcl(A_i)}$. Then B is non-zero intuitionistic fuzzy G_δ ring in $(R, \mathscr{S})$. Now,

$$B = \cap_{i=1}^{\infty} \overline{IF_Rcl(A_i)} = \overline{\cup_{i=1}^{\infty} IF_Rcl(A_i)} \subseteq \overline{\cup_{i=1}^{\infty} A_i} = \overline{\mathrm{A}}.$$

Hence $B \subseteq \overline{\mathrm{A}}$. Then $A \subseteq \overline{B}$. Now,

$$\begin{aligned} IF_Rint(IF_Rcl((A)) &\subseteq IF_Rint(IF_Rcl((\overline{B})) \\ &= IF_Rint(\overline{IF_Rint(B)}) \\ &= \overline{IF_Rcl(IF_Rint(B))} \\ &= \overline{IF_Rcl(IF_RExt(\overline{B}))} \end{aligned}$$

Since $IF_RExt(\overline{B})$ is an intuitionistic fuzzy dense ring in $(R, \mathscr{S})$, $IF_Rcl(Ext(\overline{B}))$ $= 1_\sim$. Therefore, $IF_Rint(IF_Rcl(A)) \subseteq 0_\sim$. Then, $IF_Rint(IF_Rcl(A)) = 0_\sim$. Hence A is an intuitionistic fuzzy nowhere dense ring in $(R, \mathscr{S})$. □

Definition 3.13. Let $(R, \mathscr{S})$ be an intuitionistic fuzzy structure ring space. Let A be any intuitionistic fuzzy ring in R. Then A is said to be an intuitionistic fuzzy regular closed ring in $(R, \mathscr{S})$ if $IF_Rcl(IF_Rint(A)) = A$.

The complement of an intuitionistic fuzzy regular closed ring in $(R.\mathscr{S})$ is an intuitionistic fuzzy regular open ring in $(R.\mathscr{S})$.

Notation 3.2. Every intuitionistic fuzzy regular closed ring is an intuitionistic fuzzy closed ring.

Definition 3.14. Let $(R, \mathscr{S})$ be an intuitionistic fuzzy structure ring space. Then $(R, \mathscr{S})$ is called an intuitionistic fuzzy ring $G_\delta T_{1/2}$ space if every non-zero intuitionistic fuzzy G_δ ring in $(R, \mathscr{S})$ is an intuitionistic fuzzy open ring in $(R, \mathscr{S})$.

Proposition 3.15. *If the intuitionistic fuzzy structure ring space $(R, \mathscr{S})$ is an intuitionistic fuzzy ring $G_\delta T_{1/2}$ space and if A is an intuitionistic fuzzy first category ring in $(R, \mathscr{S})$, then A is not an intuitionistic fuzzy dense ring in $(R, \mathscr{S})$.*

Proof. Assume the contrary, suppose that A is an intuitionistic fuzzy first category ring in $(R, \mathscr{S})$ such that A is an intuitionistic fuzzy dense ring in $(R, \mathscr{S})$, that is, $IF_Rcl(A) = 1_\sim$. Then, $\underline{A = \cup_{i=1}^{\infty} A_i}$ where A_i's are intuitionistic fuzzy nowhere dense rings in $(R, \mathscr{S})$. Now, $\overline{IF_Rcl(A_i)}$ is an intuitionistic fuzzy open ring in $(R, \mathscr{S})$. Let $B = \cap_{i=1}^{\infty} \overline{IF_Rcl(A_i)}$. Then, $\underline{B}$ is non-zero intuitionistic fuzzy G_δ ring in $(R, \mathscr{S})$. Now, $B = \cap_{i=1}^{\infty} \overline{IF_Rcl(A_i)} = \overline{\cup_{i=1}^{\infty} IF_Rcl(A_i)} \subseteq \overline{\cup_{i=1}^{\infty} A_i} = \overline{\mathrm{A}}$. Hence $B \subseteq \overline{\mathrm{A}}$. Then, $IF_Rint(B) \subseteq IF_Rint(\overline{\mathrm{A}}) \subseteq \overline{IF_Rcl(A)} = 0_\sim$. That is, $IF_Rint(B) = 0_\sim$.

Since $(R, \mathscr{S})$ is an intuitionistic fuzzy ring $G_\delta T_{1/2}$ space, $B = IF_Rint(B)$, which implies that $B = 0_\sim$. This is a contradiction. Hence A is not an intuitionistic fuzzy dense ring in $(R, \mathscr{S})$. □

Proposition 3.16. *If $(R, \mathscr{S})$ is an intuitionistic fuzzy ring $G_\delta T_{1/2}$ space, then* $IF_RExt(\cup_{i=1}^{\infty} \overline{A_i}) = \cap_{i=1}^{\infty} A_i$.

Proof. Let $(R, \mathscr{S})$ is an intuitionistic fuzzy ring $G_\delta T_{1/2}$ space. Assume that A_i's are intuitionistic fuzzy regular closed rings in $(R, \mathscr{S})$. Then, A_i's are intuitionistic fuzzy closed rings in $(R, \mathscr{S})$, which implies that $\overline{A_i}$'s are intuitionistic fuzzy open rings in $(R, \mathscr{S})$.

Let $B = \cap_{i=1}^{\infty} A_i$. Then B is non-zero intuitionistic fuzzy G_δ ring in $(R, \mathscr{S})$. Since $(R, \mathscr{S})$ is an intuitionistic fuzzy ring $G_\delta T_{1/2}$ space, $B = IF_Rint(B)$ is an intuitionistic fuzzy open ring, which implies that $IF_Rint(\cap_{i=1}^{\infty} A_i) = \cap_{i=1}^{\infty} A_i$. Now,

$IF_RExt(\cup_{i=1}^{\infty}\overline{A_i}) = IF_Rint(\overline{\cup_{i=1}^{\infty}\overline{A_i}}) = IF_Rint(\cap_{i=1}^{\infty}A_i) = \cap_{i=1}^{\infty}A_i$. Hence the proof. □

Definition 3.17. Let $(R,\mathscr{S})$ be an intuitionistic fuzzy structure ring space. Then $(R,\mathscr{S})$ is called an intuitionistic fuzzy ring exterior B (in short, $ExtB$) space if $IF_RExt(\cap_{i=1}^{\infty}\overline{A_i}) = 0_\sim$ where A_i's are intuitionistic fuzzy nowhere dense rings in $(R,\mathscr{S})$.

Example 3.18. Let $R = \{0,1\}$ be a set of integers of module 2 with two binary operations as follows:

+	0	1
0	0	1
1	1	0

and

·	0	1
0	0	0
1	0	1

Then $(R,+,\cdot)$ is a ring. Define intuitionistic fuzzy rings A, B, C, D, E, F and G on R as follows:

$$\mu_A(0) = 0.5, \mu_A(1) = 0.7 \text{ and } \gamma_A(0) = 0.3, \gamma_A(1) = 0.3$$
$$\mu_B(0) = 0.5, \mu_B(1) = 0.7 \text{ and } \gamma_B(0) = 0.3, \gamma_B(1) = 0.2$$
$$\mu_C(0) = 0.3, \mu_C(1) = 0.4 \text{ and } \gamma_C(0) = 0.5, \gamma_C(1) = 0.6$$
$$\mu_D(0) = 0.4, \mu_D(1) = 0.5 \text{ and } \gamma_D(0) = 0.3, \gamma_D(1) = 0.5$$
$$\mu_E(0) = 0.3, \mu_E(1) = 0.2 \text{ and } \gamma_E(0) = 0.5, \gamma_E(1) = 0.7$$
$$\mu_F(0) = 0.3, \mu_B(1) = 0.2 \text{ and } \gamma_F(0) = 0.5, \gamma_F(1) = 0.8$$
$$\mu_G(0) = 0.3, \mu_B(1) = 0.2 \text{ and } \gamma_G(0) = 0.6, \gamma_G(1) = 0.8$$
$$\mu_H(0) = 0.3, \mu_B(1) = 0.2 \text{ and } \gamma_H(0) = 0.6, \gamma_H(1) = 0.8$$

Then $\mathscr{S} = \{0_\sim, A, B, C, D, 1_\sim\}$ is an intuitionistic fuzzy structure ring on R. Thus the pair $(R,\mathscr{S})$ is an intuitionistic fuzzy structure ring space. Let $\{E, F, G, H\}$ be intuitionistic fuzzy nowhere dense rings in $(R,\mathscr{S})$. Then

$$IF_RExt(\cap\{\overline{E},\overline{F},\overline{G},\overline{H}\}) = IF_RExt(\overline{E}) = IF_Rint(E) = 0_\sim.$$

Therefore, $(R,\mathscr{S})$ is an intuitionistic fuzzy ring $ExtB$ space.

Proposition 3.19. *Let $(R,\mathscr{S})$ be an intuitionistic fuzzy structure ring space. Then the following statements are equivalent:*

(i) *$(R,\mathscr{S})$ is an intuitionistic fuzzy ring $ExtB$ space.*
(ii) *$IF_Rint(A) = 0_\sim$, for every intuitionistic fuzzy first category ring A in $(R,\mathscr{S})$.*
(iii) *$IF_Rcl(A) = 1_\sim$, for every intuitionistic fuzzy residual ring A in $(R,\mathscr{S})$.*

Proof. **(i)⇒(ii)**

Let A be any intuitionistic fuzzy first category ring in $(R,\mathscr{S})$. Then $A = \cup_{i=1}^{\infty}A_i$ where A_i's are intuitionistic fuzzy nowhere dense rings in $(R,\mathscr{S})$. Now, $IF_Rint(A) = IF_Rint(\cup_{i=1}^{\infty}A_i) = IF_Rint(\overline{\cap_{i=1}^{\infty}\overline{A_i}}) = IF_RExt(\cap_{i=1}^{\infty}\overline{A_i})$.

Since $(R,\mathscr{S})$ is an intuitionistic fuzzy ring $ExtB$ space, $IF_RExt(\cap_{i=1}^{\infty}\overline{A_i}) = 0_\sim$. Therefore, $IF_Rint(A) = 0_\sim$. Hence (i) ⇒ (ii).

(ii)⇒(iii)

Let A be any intuitionistic fuzzy residual ring in $(R,\mathscr{S})$. Then $\overline{A}$ is an intuitionistic fuzzy first category ring in $(R,\mathscr{S})$. By (ii), $IF_Rint(\overline{A}) = 0_\sim$. That is,

$IF_Rint(\overline{\mathrm{A}}) = 0_\sim = \overline{IF_Rcl(A)}$. Therefore, $IF_Rcl(A) = 1_\sim$. Hence (ii) $\Rightarrow$ (iii).

(iii)⇒(i)

Let A be any intuitionistic fuzzy first category ring in $(R, \mathscr{S})$. Then $A = \cup_{i=1}^{\infty} A_i$ where A_i's are intuitionistic fuzzy nowhere dense rings in $(R, \mathscr{S})$. Since A is an intuitionistic fuzzy first category ring in $(R, \mathscr{S})$, $\overline{\mathrm{A}}$ is an intuitionistic fuzzy residual ring in $(R, \mathscr{S})$. Then by (iii), $IF_Rcl(\overline{A}) = 1_\sim$. Now,

$$\begin{aligned} IF_RExt(\cap_{i=1}^{\infty}\overline{A_i}) &= IF_Rint(\overline{\cap_{i=1}^{\infty}\overline{A_i}}) \\ &= IF_Rint(\cup_{i=1}^{\infty}A_i) \\ &= IF_Rint(A) \\ &= \overline{IF_Rcl(\overline{\mathrm{A}})} = 0_\sim. \end{aligned}$$

Hence, $IF_RExt(\cap_{i=1}^{\infty}\overline{A_i}) = 0_\sim$ where A_i's are intuitionistic fuzzy nowhere dense rings in $(R, \mathscr{S})$. Therefore, $(R, \mathscr{S})$ is an intuitionistic fuzzy ring $ExtB$ space. □

Proposition 3.20. *If A is an intuitionistic fuzzy first category ring in an intuitionistic fuzzy structure ring space $(R, \mathscr{S})$ such that $B \subseteq \overline{A}$ where B is non-zero intuitionistic fuzzy G_δ ring and the intuitionistic fuzzy ring exterior of $\overline{B}$ is an intuitionistic fuzzy dense ring in $(R, \mathscr{S})$, then $(R, \mathscr{S})$ is an intuitionistic fuzzy ring $ExtB$ space.*

Proof. Let A be an intuitionistic fuzzy first category ring in $(R, \mathscr{S})$ such that $B \subseteq \overline{\mathrm{A}}$ where B is non-zero intuitionistic fuzzy G_δ ring and the intuitionistic fuzzy ring exterior of $\overline{B}$ is an intuitionistic fuzzy dense ring in $(R, \mathscr{S})$. Then by Proposition 3.3., A is an intuitionistic fuzzy nowhere dense ring $(R, \mathscr{S})$, that is, $IF_Rint(IF_Rcl(A)) = 0_\sim$. Then, $IF_Rint(A) \subseteq IF_Rint(IF_Rcl(A))$ implies that $IF_Rint(A) = 0_\sim$. By Proposition 3.6., $(R, \mathscr{S})$ is an intuitionistic fuzzy ring $ExtB$ space. □

Proposition 3.21. *If $(R, \mathscr{S})$ is an intuitionistic fuzzy ring $ExtB$ space and if $\cup_{i=1}^{\infty} A_i = 1_\sim$ where A_i's are intuitionistic fuzzy regular closed rings in $(R, \mathscr{S})$, then $IF_Rcl(\cup_{i=1}^{\infty} IF_RExt(\overline{A_i})) = 1_\sim$.*

Proof. Let $(R, \mathscr{S})$ be an intuitionistic fuzzy ring $ExtB$ space. Assume that A_i's are intuitionistic fuzzy regular closed rings in $(R, \mathscr{S})$. Suppose that $IF_Rint(A_i) = 0_\sim$, for each $i \in J$. Since A_i is an intuitionistic fuzzy regular closed ring in $(R, \mathscr{S})$, A_i is an intuitionistic fuzzy closed ring in $(R, \mathscr{S})$. Also, $IF_Rint(A_i) = 0_\sim$ implies that $IF_Rint(IF_Rcl(A_i)) = 0_\sim$. Therefore, A_i's are intuitionistic fuzzy nowhere dense rings in $(R, \mathscr{S})$. Since $\cup_{i=1}^{\infty} A_i = 1_\sim$,

$$\begin{aligned} IF_RExt(\cap_{i=1}^{\infty}\overline{A_i}) &= IF_RExt(\overline{\cup_{i=1}^{\infty}A_i}) \\ &= IF_Rint(\cup_{i=1}^{\infty}A_i) \\ &= IF_Rint(1_\sim) = 1_\sim. \end{aligned}$$

Hence, $IF_RExt(\cap_{i=1}^{\infty}\overline{A_i}) = 1_\sim$. Since $(R, \mathscr{S})$ is an intuitionistic fuzzy ring $ExtB$ space, $IF_RExt(\cap_{i=1}^{\infty}\overline{A_i}) = 0_\sim$, which is a contradiction.

Hence $IF_Rint(A_i) \neq 0_\sim$, for atleast one $i \in J$. Therefore, $\cup_{i=1}^{\infty} IF_Rint(A_i) \neq 0_\sim$. Since A_i is an intuitionistic fuzzy regular closed rings in $(R, \mathscr{S})$ and $\cup_{i=1}^{\infty} IF_Rcl(A_i) \subseteq$

$IF_Rcl(\cup_{i=1}^{\infty} A_i)$,

$$\Rightarrow \cup_{i=1}^{\infty} IF_Rcl(IF_Rint(A_i)) \subseteq IF_Rcl(\cup_{i=1}^{\infty} IF_Rint(A_i))$$
$$\Rightarrow \cup_{i=1}^{\infty} A_i \subseteq IF_Rcl(\cup_{i=1}^{\infty} IF_Rint(A_i))$$
$$\Rightarrow \cup_{i=1}^{\infty} A_i \subseteq IF_Rcl(\cup_{i=1}^{\infty} IF_RExt(\overline{A_i}))$$
$$\Rightarrow 1_{\sim} \subseteq IF_Rcl(\cup_{i=1}^{\infty} IF_RExt(\overline{A_i})).$$

But $1_{\sim} \supseteq IF_Rcl(\cup_{i=1}^{\infty} IF_RExt(\overline{A_i}))$. Therefore, $IF_Rcl(\cup_{i=1}^{\infty} IF_RExt(\overline{A_i})) = 1_{\sim}$. □

4. On intuitionistic fuzzy structure ring exterior V spaces

In this section, the concepts of an intuitionistic fuzzy ring exterior V spaces, intuitionistic fuzzy ring continuous functions, intuitionistic fuzzy ring open functions, intuitionistic fuzzy ring hardly open functions and somewhat intuitionistic fuzzy ring continuous functions are introduced. In this connection, some interesting properties among these functions are discussed. Necessary examples are provided.

Definition 4.1. Let $(R, \mathscr{S})$ be an intuitionistic fuzzy structure ring space. Then $(R, \mathscr{S})$ is called an intuitionistic fuzzy ring exterior V (in short, $ExtV$)space if $IF_Rcl(\cap_{i=1}^{n} A_i) = 1_{\sim}$ where A_i's are intuitionistic fuzzy G_δ rings and the intuitionistic fuzzy ring exterior of $\overline{A_i}$'s are intuitionistic fuzzy dense rings in $(R, \mathscr{S})$.

Example 4.2. Let $R = \{0, 1, 2\}$ be a set of integers of module 3 with two binary operations as follows:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

and

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Then $(R, +, \cdot)$ is a ring. Define intuitionistic fuzzy rings A, B and C on R as follows:

$$\mu_A(0) = 1, \mu_A(1) = 0.2, \mu_A(2) = 0.9 \text{ and } \gamma_A(0) = 0, \gamma_A(1) = 0.8, \gamma_A(2) = 0.1$$

$$\mu_B(0) = 0.3, \mu_B(1) = 1, \mu_B(2) = 0.2 \text{ and } \gamma_B(0) = 0.7, \gamma_B(1) = 0, \gamma_B(2) = 0.8$$

$$\mu_C(0) = 0.7, \mu_C(1) = 0.4, \mu_C(2) = 1 \text{ and } \gamma_C(0) = 0.3, \gamma_C(1) = 0.6, \gamma_C(2) = 0$$

Then $\mathscr{S} = \{0_{\sim}, A, B, C, A \cap B, A \cup B, A \cap C, A \cup C, B \cap C, B \cup C, C \cap (A \cup B), A \cup (B \cap C), B \cup (A \cap C), 1_{\sim}\}$ is an intuitionistic fuzzy structure ring on R. Thus the pair $(R, \mathscr{S})$ is an intuitionistic fuzzy structure ring space.

Now, $A \cap C = \cap\{B \cup (A \cap C), C \cap (A \cup B), C, A\}$ and $C \cap (A \cup B) = \cap\{A \cup B, C \cap (A \cup B), A \cup C\}$ are intuitionistic fuzzy G_δ rings in $(R, \mathscr{S})$. Also, the intuitionistic fuzzy ring exterior of $\overline{A \cap C}$ and $\overline{C \cap (A \cup B)}$ are intuitionistic fuzzy dense rings in $(R, \mathscr{S})$. Now,

$$IF_Rcl(\cap\{A \cap C, C \cap (A \cup B)\}) = IF_Rcl(A \cap C) = 1_{\sim}.$$

Therefore, $(R, \mathscr{S})$ is an intuitionistic fuzzy ring $ExtV$ space.

Proposition 4.3. *Let $(R, \mathscr{S})$ be an intuitionistic fuzzy structure ring space. Then $(R, \mathscr{S})$ is an intuitionistic fuzzy ring ExtV space if and only if $IF_Rint(\cup_{i=1}^n \overline{A_i})$ $= 0_\sim$ where A_i's are intuitionistic fuzzy G_δ rings and the intuitionistic fuzzy ring exterior of $\overline{A_i}$'s are intuitionistic fuzzy dense rings in $(R, \mathscr{S})$.*

Proof. Let $(R, \mathscr{S})$ be an intuitionistic fuzzy ring $ExtV$ space. Assume that A_i's are intuitionistic fuzzy G_δ rings and the intuitionistic fuzzy ring exterior of $\overline{A_i}$'s are intuitionistic fuzzy dense rings in $(R, \mathscr{S})$.

Since $(R, \mathscr{S})$ is an intuitionistic fuzzy ring $ExtV$ space, $IF_Rcl(\cap_{i=1}^n A_i) = 1_\sim$. Now,

$$IF_Rint(\cup_{i=1}^n \overline{A_i}) = IF_Rint(\overline{\cap_{i=1}^n A_i}) = \overline{IF_Rcl(\cap_{i=1}^n A_i)} = 0_\sim.$$

Therefore, $IF_Rint(\cup_{i=1}^n \overline{A_i}) = 0_\sim$ where A_i's are intuitionistic fuzzy G_δ rings and the intuitionistic fuzzy ring exterior of $\overline{A_i}$'s are intuitionistic fuzzy dense rings in $(R, \mathscr{S})$.

Conversely, let $IF_Rint(\cup_{i=1}^n \overline{A_i}) = 0_\sim$ where A_i's are intuitionistic fuzzy G_δ rings and the intuitionistic fuzzy ring exterior of $\overline{A_i}$'s are intuitionistic fuzzy dense rings in $(R, \mathscr{S})$. Now,

$$IF_Rcl(\cap_{i=1}^n A_i) = IF_Rcl(\overline{\cup_{i=1}^n A_i}) = \overline{IF_Rint(\cup_{i=1}^n A_i)} = 1_\sim.$$

Therefore, $(R, \mathscr{S})$ is an intuitionistic fuzzy ring $ExtV$ space. □

Proposition 4.4. *Let $(R, \mathscr{S})$ be an intuitionistic fuzzy structure ring space. If every intuitionistic fuzzy first category ring in $(R, \mathscr{S})$ is formed from the intuitionistic fuzzy G_δ rings and the intuitionistic fuzzy ring exterior of its complements are intuitionistic fuzzy dense rings in an intuitionistic fuzzy ring ExtV space $(R, \mathscr{S})$, then $(R, \mathscr{S})$ is an intuitionistic fuzzy ring ExtB space.*

Proof. Assume that A_i's are intuitionistic fuzzy G_δ rings in $(R, \mathscr{S})$ and the intuitionistic fuzzy ring exterior of $\overline{A_i}$'s are intuitionistic fuzzy dense rings in $(R, \mathscr{S})$, for $i = 1, ..., n$. Since $(R, \mathscr{S})$ is an intuitionistic fuzzy ring $ExtV$ space and by Proposition 4.1., $IF_Rint(\cup_{i=1}^n \overline{A_i}) = 0_\sim$. But $\cup_{i=1}^n IF_Rint(\overline{A_i}) \subseteq IF_Rint(\cup_{i=1}^n \overline{A_i})$, which implies that $\cup_{i=1}^n IF_Rint(\overline{A_i}) = 0_\sim$. Then $IF_Rint(\overline{A_i}) = 0_\sim$. Since A_i's are intuitionistic fuzzy G_δ rings in $(R, \mathscr{S})$ and the intuitionistic fuzzy ring exterior of $\overline{A_i}$'s are intuitionistic fuzzy dense rings in $(R, \mathscr{S})$, for $i = 1, ..., n$. By Proposition 4.2., $\overline{A_i}$'s are intuitionistic fuzzy first category rings in $(R, \mathscr{S})$, for $i = 1, ..., n$. Therefore, $IF_Rint(\overline{A_i}) = 0_\sim$, for every $\overline{A_i}$ is an intuitionistic fuzzy first category rings in $(R, \mathscr{S})$. By Proposition 3.6., $(R, \mathscr{S})$ is an intuitionistic fuzzy ring $ExtB$ space. □

Definition 4.5. Let $(R_1, \mathscr{S}_1)$ and $(R_2, \mathscr{S}_2)$ be any two intuitionistic fuzzy structure ring spaces. Let $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ be any function. Then f is said to be an

(i) intuitionistic fuzzy ring continuous function if $f^{-1}(A)$ is an intuitionistic fuzzy open ring in $(R_1, \mathscr{S}_1)$, for every intuitionistic fuzzy open ring A in $(R_2, \mathscr{S}_2)$.

(ii) somewhat intuitionistic fuzzy ring continuous function if $A \in \mathscr{S}_2$ and $f^{-1}(A)$ $\neq 0_\sim$ implies that there exists an intuitionistic fuzzy open ring B in $(R_1, \mathscr{S}_1)$ such that $B \neq 0_\sim$ and $B \subseteq f^{-1}(A)$.

(iii) intuitionistic fuzzy ring hardly open function if for each intuitionistic fuzzy dense ring A in $(R_2, \mathscr{S}_2)$ such that $A \subseteq B \subset 1_\sim$ for some intuitionistic fuzzy open ring B in $(R_2, \mathscr{S}_2)$, $f^{-1}(A)$ is an intuitionistic fuzzy dense ring in $(R_1, \mathscr{S}_1)$.

(iv) intuitionistic fuzzy ring open function if $f(A)$ is an intuitionistic fuzzy open ring in $(R_2, \mathscr{S}_2)$, for every intuitionistic fuzzy open ring A in $(R_1, \mathscr{S}_1)$.

Proposition 4.6. *Let $(R_1, \mathscr{S}_1)$ and $(R_2, \mathscr{S}_2)$ be any two intuitionistic fuzzy structure ring spaces. Let $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ be any function. Then the following statements are equivalent:*

(i) *f is an intuitionistic fuzzy ring continuous function.*

(ii) *$f^{-1}(B)$ is an intuitionistic fuzzy closed ring in $(R_1, \mathscr{S}_1)$, for every intuitionistic fuzzy closed ring B in $(R_2, \mathscr{S}_2)$.*

(iii) *$IF_Rcl(f^{-1}(A)) \subseteq f^{-1}(IF_Rcl(A))$, for each intuitionistic fuzzy ring A in $(R_2, \mathscr{S}_2)$.*

(iv) *$f^{-1}(IF_Rint(A)) \subseteq IF_Rint(f^{-1}(A))$, for each intuitionistic fuzzy ring A in $(R_2, \mathscr{S}_2)$.*

Proof. The proof is simple. □

Remark 4.7. Let $(R_1, \mathscr{S}_1)$ and $(R_2, \mathscr{S}_2)$ be any two intuitionistic fuzzy structure ring spaces. If $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ is an intuitionistic fuzzy ring continuous function, then $f^{-1}(IF_RExt(\overline{\mathrm{A}})) \subseteq IF_RExt(\overline{f^{-1}(A)})$, for each intuitionistic fuzzy ring A in $(R_2, \mathscr{S}_2)$.

Proof. The proof is follows from the Definition 3.4. and Proposition 4.3.. □

Proposition 4.8. *If a function $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ from an intuitionistic fuzzy structure ring space $(R_1, \mathscr{S}_1)$ into another intuitionistic fuzzy structure ring space $(R_2, \mathscr{S}_2)$ is intuitionistic fuzzy ring continuous, 1-1 and if A is an intuitionistic fuzzy dense ring in $(R_1, \mathscr{S}_1)$, then $f(A)$ is an intuitionistic fuzzy dense ring in $(R_2, \mathscr{S}_2)$.*

Proof. Suppose that $f(A)$ is not an intuitionistic fuzzy dense ring in $(R_2, \mathscr{S}_2)$. Then there exists an intuitionistic fuzzy closed ring in $(R_2, \mathscr{S}_2)$ such that $f(A) \subset C \subset 1_\sim$. Then, $f^{-1}(f(A)) \subset f^{-1}(C) \subset f^{-1}(1_\sim)$. Since f is 1-1, $f^{-1}(f(A)) = A$. Hence $A \subset f^{-1}(C) \subset 1_\sim$.

Since f is an intuitionistic fuzzy ring continuous function and C is an intuitionistic fuzzy closed ring in $(R_2, \mathscr{S}_2)$, $f^{-1}(C)$ is an intuitionistic fuzzy closed ring in $(R_1, \mathscr{S}_1)$. Then $IF_Rcl(A) \neq 1_\sim$, which is a contradiction. Therefore $f(A)$ is an intuitionistic fuzzy dense ring in $(R_2, \mathscr{S}_2)$. □

Remark 4.9. Let $(R_1, \mathscr{S}_1)$ and $(R_2, \mathscr{S}_2)$ be any two intuitionistic fuzzy structure ring spaces. Then

(i) the intuitionistic fuzzy ring continuous image of an intuitionistic fuzzy ring $ExtV$ space $(R_1, \mathscr{S}_1)$ may fail to be an intuitionistic fuzzy ring $ExtV$ space $(R_2, \mathscr{S}_2)$.

(ii) the intuitionistic fuzzy ring open image of an intuitionistic fuzzy ring $ExtV$ space $(R_1, \mathscr{S}_1)$ may fail to be an intuitionistic fuzzy ring $ExtV$ space $(R_2, \mathscr{S}_2)$.

Proof. It is clearly from the following Example 4.2. and Example 4.3. □

Example 4.10. Let $R = \{0, 1, 2\}$ be a set of integers of module 3 with two binary operations as follows:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

and

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Then $(R, +, \cdot)$ is a ring. Define intuitionistic fuzzy rings A, B, C, D, E, and F on R as follows:

$$\mu_A(0) = 1, \mu_A(1) = 0.2, \mu_A(2) = 0.9 \text{ and } \gamma_A(0) = 0, \gamma_A(1) = 0.8, \gamma_A(2) = 0.1$$

$$\mu_B(0) = 0.3, \mu_B(1) = 1, \mu_B(2) = 0.2 \text{ and } \gamma_B(0) = 0.7, \gamma_B(1) = 0, \gamma_B(2) = 0.8$$

$$\mu_C(0) = 0.7, \mu_C(1) = 0.4, \mu_C(2) = 1 \text{ and } \gamma_C(0) = 0.3, \gamma_C(1) = 0.6, \gamma_C(2) = 0$$

$$\mu_D(0) = 0.9, \mu_D(1) = 1, \mu_D(2) = 0.2 \text{ and } \gamma_D(0) = 0.1, \gamma_D(1) = 0, \gamma_D(2) = 0.8$$

$$\mu_E(0) = 0.2, \mu_E(1) = 0.2, \mu_E(2) = 1 \text{ and } \gamma_E(0) = 0.8, \gamma_E(1) = 0.8, \gamma_E(2) = 0$$

$$\mu_F(0) = 1, \mu_F(1) = 0.7, \mu_F(2) = 0.4 \text{ and } \gamma_F(0) = 0, \gamma_F(1) = 0.3, \gamma_F(2) = 0.6.$$

Then $\mathscr{S}_1 = \{0_\sim, A, B, C, A \cap B, A \cup B, A \cap C, A \cup C, B \cap C, B \cup C, C \cap (A \cup B), A \cup (B \cap C), B \cup (A \cap C), 1_\sim\}$ and $\mathscr{S}_2 = \{0_\sim, D, E, F, D \cap E, D \cup E, D \cap F, D \cup F, E \cap F, E \cup F, F \cap (D \cup E), D \cup (E \cap F), E \cup (D \cap F), 1_\sim\}$ are two intuitionistic fuzzy structure rings on R. Thus the pair $(R, \mathscr{S}_1)$ and $(R, \mathscr{S}_2)$ are intuitionistic fuzzy structure ring spaces.

Now, $A \cap C = \cap\{B \cup (A \cap C), C \cap (A \cup B), C, A\}$ and $C \cap (A \cup B) = \cap\{A \cup B, C \cap (A \cup B), A \cup C\}$ are intuitionistic fuzzy G_δ rings in $(R, \mathscr{S}_1)$. Also, the intuitionistic fuzzy ring exterior of $\overline{A \cap C}$ and $\overline{C \cap (A \cup B)}$ are intuitionistic fuzzy dense rings in $(R, \mathscr{S}_1)$. Now, $IF_Rcl(\cap\{A \cap C, C \cap (A \cup B)\}) = IF_Rcl(A \cap C) = 1_\sim$. Therefore, $(R, \mathscr{S}_1)$ is an intuitionistic fuzzy ring $ExtV$ space.

Define a function $f : (R, \mathscr{S}_1) \to (R, \mathscr{S}_2)$ by $f(0) = 1, f(1) = 2$ and $f(2) = 0$. Clearly, f is an intuitionistic fuzzy ring continuous function. Also, $f(A) = D, f(B) = E$ and $f(C) = F$.

Now, $D = \cap\{D, D \cup E, D \cup (E \cap F)\}$, $D \cap F = \cap\{F, D \cup F, D \cap F, F \cap (D \cup E)\}$ and $E = \cap\{E, E \cup F, E \cup (D \cap F)\}$ are intuitionistic fuzzy G_δ rings in $(R, \mathscr{S}_2)$.

Also, the intuitionistic fuzzy ring exterior of $\overline{D}$, $\overline{E}$ and $\overline{D \cap F}$ are intuitionistic fuzzy G_δ rings in $(R, \mathscr{S}_2)$. But, $IF_Rcl(\cap\{D, E, D \cap F\}) = \overline{E \cap F} \neq 1_\sim$. Therefore, $(R, \mathscr{S}_2)$ is not an intuitionistic fuzzy ring $ExtV$ space.

Therefore the intuitionistic fuzzy ring continuous image of an intuitionistic fuzzy ring $ExtV$ space $(R_1, \mathscr{S}_1)$ may fail to be an intuitionistic fuzzy ring $ExtV$ space $(R_2, \mathscr{S}_2)$.

Example 4.11. Let $R = \{0, 1, 2\}$ be a set of integers of module 3 with two binary operations as follows:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

and

.	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Then $(R, +, \cdot)$ is a ring. Define intuitionistic fuzzy rings A, B, C and D on R as follows:

$$\mu_A(0) = 1, \mu_A(1) = 0.2, \mu_A(2) = 0.9 \text{ and } \gamma_A(0) = 0, \gamma_A(1) = 0.8, \gamma_A(2) = 0.1$$

$$\mu_B(0) = 0.3, \mu_B(1) = 1, \mu_B(2) = 0.2 \text{ and } \gamma_B(0) = 0.7, \gamma_B(1) = 0, \gamma_B(2) = 0.8$$

$$\mu_C(0) = 0.7, \mu_C(1) = 0.4, \mu_C(2) = 1 \text{ and } \gamma_C(0) = 0.3, \gamma_C(1) = 0.6, \gamma_C(2) = 0$$

$$\mu_D(0) = 0.5, \mu_D(1) = 0.6, \mu_D(2) = 0.4 \text{ and } \gamma_D(0) = 0.5, \gamma_D(1) = 0.4, \gamma_D(2) = 0.6$$

Then $\mathscr{S}_1 = \{0_\sim, A, B, C, A\cap B, A\cup B, A\cap C, A\cup C, B\cap C, B\cup C, C\cap(A\cup B), A\cup(B\cap C), B\cup(A\cap C), 1_\sim\}$ and $\mathscr{S}_2 = \{0_\sim, A, B, C, D, A\cup B, A\cup C, A\cup D, B\cup C, B\cup D, C\cup D, A\cap B, A\cap C, A\cap D, B\cap C, B\cap D, C\cap D, D\cup(A\cap C), C\cap(A\cup B), A\cup(B\cap C), B\cup(A\cap C), 1_\sim\}$ are two intuitionistic fuzzy structure rings on R. Thus the pair $(R, \mathscr{S}_1)$ and $(R, \mathscr{S}_2)$ are intuitionistic fuzzy structure ring spaces.

Now, $A\cap C = \cap\{B\cup(A\cap C), C\cap(A\cup B), C, A\}$ and $C\cap(A\cup B) = \cap\{A\cup B, C\cap(A\cup B), A\cup C\}$ are intuitionistic fuzzy G_δ rings in $(R, \mathscr{S}_1)$. Also, the intuitionistic fuzzy ring exterior of $\overline{A\cap C}$ and $\overline{C\cap(A\cup B)}$ are intuitionistic fuzzy dense rings in $(R, \mathscr{S}_1)$. Now, $IF_Rcl(\cap\{A\cap C, C\cap(A\cup B)\}) = IF_Rcl(A\cap C) = 1_\sim$. Therefore, $(R, \mathscr{S}_1)$ is an intuitionistic fuzzy ring $ExtV$ space.

Define a function $f : (R, \mathscr{S}_1) \to (R, \mathscr{S}_2)$ by $f(0) = 0, f(1) = 1$ and $f(2) = 2$. Clearly, f is an intuitionistic fuzzy ring open function. Also, $f(A) = A, f(B) = B, f(C) = C$ and $f(D) = D$.

Now, $A = \cap\{A, A\cup B, A\cup C, A\cup(B\cap C)\}$, $D\cup(A\cap C) = \cap\{C, C\cup D, A\cap C, D\cup(A\cap C), C\cap(A\cup B)\}$ and $B = \cap\{B, B\cup C, B\cup D, B\cup(A\cap C)\}$ are intuitionistic fuzzy G_δ rings in $(R, \mathscr{S}_2)$. Also, the intuitionistic fuzzy ring exterior of $\overline{\text{A}}$, $\overline{B}$ and $\overline{D\cup(A\cap C)}$ are intuitionistic fuzzy G_δ rings in $(R, \mathscr{S}_2)$.

But, $IF_Rcl(\cap\{A, B, D\cup(A\cap C)\}) = \overline{B\cap C} \neq 1_\sim$. Therefore, $(R, \mathscr{S}_2)$ is not an intuitionistic fuzzy ring $ExtV$ space.

Therefore the intuitionistic fuzzy ring open image of an intuitionistic fuzzy ring $ExtV$ space $(R_1, \mathscr{S}_1)$ may fail to be an intuitionistic fuzzy ring $ExtV$ space $(R_2, \mathscr{S}_2)$.

Proposition 4.12. *Let $(R_1, \mathscr{S}_1)$ and $(R_2, \mathscr{S}_2)$ be any two intuitionistic fuzzy structure ring spaces. If $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ is onto function, then the following statements are equivalent:*

(i) *f is an intuitionistic fuzzy ring hardly open function.*

(ii) *$IF_Rint(f(A)) \neq 0_\sim$, for all intuitionistic fuzzy ring A in $(R_1, \mathscr{S}_1)$ with $IF_Rint(A) \neq 0_\sim$ and there exists an intuitionistic fuzzy closed ring $B \neq 0_\sim$ in $(R_2, \mathscr{S}_2)$ such that $B \subseteq f(A)$.*

(iii) *$IF_Rint(f(A)) \neq 0_\sim$, for all intuitionistic fuzzy ring A in $(R_1, \mathscr{S}_1)$ with $IF_Rint(A) \neq 0_\sim$ and there exists an intuitionistic fuzzy closed ring $B \neq 0_\sim$ in $(R_2, \mathscr{S}_2)$ such that $f^{-1}(B) \subseteq A$.*

Proof. **(i)⇒(ii)**

Assume that (i) is true. Let A be intuitionistic fuzzy ring A in $(R_1, \mathscr{S}_1)$ with $IF_Rint(A) \neq 0_\sim$ and $B \neq 0_\sim$ be an intuitionistic fuzzy closed ring in $(R_2, \mathscr{S}_2)$ such that $B \subseteq f(A)$. Suppose that $IF_Rint(A) = 0_\sim$. This implies that $IF_Rcl(\overline{f(A)}) = 1_\sim$. Thus, $\overline{f(A)}$ is an intuitionistic fuzzy dense ring in $(R_2, \mathscr{S}_2)$ and $\overline{f(A)} \subseteq \overline{B}$. By assumption, $f^{-1}(\overline{f(A)})$ is an intuitionistic fuzzy dense ring in $(R_1, \mathscr{S}_1)$. That is, $IF_Rcl(f^{-1}(\overline{f(A)})) = 1_\sim$. Now,

$$IF_Rint(A) = IF_Rint(f^{-1}(f(A))) = \overline{IF_Rcl(\overline{f^{-1}(f(A))})} = \overline{IF_Rcl(f^{-1}(\overline{f(A)}))} = 0_\sim.$$

This is a contradiction. Hence (i)⇒(ii).

(ii)⇒(iii)

Assume that (ii) is true. Since f is onto function and by assumption, $B \subseteq f(A)$. This implies that $f^{-1}(B) \subseteq f^{-1}(f(A))$, that is, $f^{-1}(B) \subseteq A$. Hence (ii)⇒(iii).

(iii)⇒(i)

Let $C \subseteq \overline{D}$ where C is an intuitionistic fuzzy dense ring and D is non-zero intuitionistic fuzzy open ring in $(R_2, \mathscr{S}_2)$. Let $A = f^{-1}(\overline{C})$ and $B = \overline{D}$. Now, $f^{-1}(B) = f^{-1}(\overline{D}) \subseteq f^{-1}(\overline{C}) = A$. Consider,

$$IF_Rint(f(A)) = IF_Rint(f(f^{-1}(\overline{C})) = IF_Rint(\overline{C}) = \overline{IF_Rint(C)} = 0_\sim.$$

Therefore, $IF_Rint(A) = 0_\sim$, which implies that

$$IF_Rint(f^{-1}(\overline{C})) = IF_Rint(\overline{f^{-1}(C)}) = 0_\sim.$$

Therefore, $\overline{IF_Rcl(f^{-1}(C))} = 0_\sim$. Thus, $IF_Rcl(f^{-1}(C)) = 1_\sim$. Therefore, $f^{-1}(C)$ is an intuitionistic fuzzy dense ring in $(R_1, \mathscr{S}_1)$. This implies that f is an intuitionistic fuzzy ring hardly open function. Hence (iii)⇒(i). Hence the proof. □

Proposition 4.13. *If a function $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ from an intuitionistic fuzzy structure ring space $(R_1, \mathscr{S}_1)$ onto another intuitionistic fuzzy structure ring space $(R_2, \mathscr{S}_2)$ is intuitionistic fuzzy ring continuous, 1-1 and intuitionistic fuzzy ring hardly open function and if $(R_1, \mathscr{S}_1)$ is an intuitionistic fuzzy ring $ExtV$ space, then $(R_2, \mathscr{S}_2)$ is an intuitionistic fuzzy ring $ExtV$ space.*

Proof. Let $(R_1, \mathscr{S}_1)$ be an intuitionistic fuzzy ring $ExtV$ space. Assume that A_i's $(i = 1, ..., n)$ are intuitionistic fuzzy G_δ rings in $(R_2, \mathscr{S}_2)$ and the intuitionistic fuzzy ring exterior of $\overline{A_i}$'s are intuitionistic fuzzy dense ring in $(R_2, \mathscr{S}_2)$.

Then $IF_Rcl(IF_RExt(\overline{A_i})) = 1_\sim$ and $A_i = \cap_{j=1}^{\infty} B_{ij}$ where B_{ij}'s are intuitionistic fuzzy open rings in $(R_2, \mathscr{S}_2)$. Hence

$$f^{-1}(A_i) = f^{-1}(\cap_{j=1}^{\infty} B_{ij}) = \cap_{j=1}^{\infty} f^{-1}(B_{ij}) \tag{4.1}$$

Since f is an intuitionistic fuzzy ring continuous function and B_{ij}'s are intuitionistic fuzzy open rings in $(R_2, \mathscr{S}_2)$, $f^{-1}(B_{ij})$'s are intuitionistic fuzzy open rings in $(R_1, \mathscr{S}_1)$. Hence $f^{-1}(A_i) = \cap_{j=1}^{\infty} f^{-1}(B_{ij})$ is an intuitionistic fuzzy G_δ rings in $(R_1, \mathscr{S}_1)$.

Since f is an intuitionistic fuzzy ring hardly open function and $IF_RExt(\overline{A_i})$ is an intuitionistic fuzzy dense ring in $(R_2, \mathscr{S}_2)$, $f^{-1}(IF_RExt(\overline{A_i}))$ is an intuitionistic fuzzy dense ring in $(R_1, \mathscr{S}_1)$. Now,

$$\begin{aligned} f^{-1}(IF_RExt(\overline{A_i})) &= f^{-1}(IF_Rint(A_i)) \\ &\subseteq IF_Rint(f^{-1}(A_i)) \\ &= IF_RExt(\overline{f^{-1}(A_i)}). \end{aligned}$$

Therefore $1_\sim = IF_Rcl(f^{-1}(IF_RExt(\overline{A_i}))) \subseteq IF_Rcl(IF_RExt(\overline{f^{-1}(A_i)}))$, which implies that $1_\sim = IF_Rcl(IF_RExt(\overline{f^{-1}(A_i)}))$. Hence $IF_RExt(\overline{f^{-1}(A_i)})$ is an intuitionistic fuzzy dense ring in $(R_1, \mathscr{S}_1)$. Since $(R_1, \mathscr{S}_1)$ is an intuitionistic fuzzy ring $ExtV$ space, $IF_Rcl(\cap_{i=1}^n f^{-1}(A_i)) = 1_\sim$ where $f^{-1}(A_i)$'s are intuitionistic fuzzy G_δ rings in $(R_1, \mathscr{S}_1)$ and the intuitionistic fuzzy ring exterior of $\overline{f^{-1}(A_i)}$'s are intuitionistic fuzzy dense ring in $(R_1, \mathscr{S}_1)$. Thus, $IF_Rcl(\cap_{i=1}^n f^{-1}(A_i)) = 1_\sim = IF_Rcl(f^{-1}(\cap_{i=1}^n A_i))$. Therefore, $f^{-1}(\cap_{i=1}^n A_i)$ is an intuitionistic fuzzy dense ring in $(R_1, \mathscr{S}_1)$. Since f is an intuitionistic fuzzy ring continuous, 1-1 and by Proposition 3.4., $f(f^{-1}(\cap_{i=1}^n A_i))$ is an intuitionistic fuzzy dense ring in $(R_2, \mathscr{S}_2)$. Hence $IF_Rcl(f(f^{-1}(\cap_{i=1}^n A_i))) = 1_\sim$. Since f is 1-1, $f(f^{-1}(\cap_{i=1}^n A_i)) = \cap_{i=1}^n A_i$. Then, $IF_Rcl(\cap_{i=1}^n A_i) = 1_\sim$. Therefore, $(R_2, \mathscr{S}_2)$ is an intuitionistic fuzzy ring $ExtV$ space.

Conversely, let $(R_2, \mathscr{S}_2)$ be an intuitionistic fuzzy ring $ExtV$ space. Assume that A_i's $(i = 1, ..., n)$ are intuitionistic fuzzy G_δ rings in $(R_2, \mathscr{S}_2)$ and the intuitionistic fuzzy ring exterior of $\overline{A_i}$'s are intuitionistic fuzzy dense ring in $(R_2, \mathscr{S}_2)$.

Then $IF_Rcl(IF_RExt(\overline{A_i})) = 1_\sim$ and $A_i = \cap_{j=1}^\infty B_{ij}$ where B_{ij}'s are intuitionistic fuzzy open rings in $(R_2, \mathscr{S}_2)$. Hence

$$f^{-1}(A_i) = f^{-1}(\cap_{j=1}^\infty B_{ij}) = \cap_{j=1}^\infty f^{-1}(B_{ij}) \tag{4.2}$$

Since f is an intuitionistic fuzzy ring continuous function and B_{ij}'s are intuitionistic fuzzy open rings in $(R_2, \mathscr{S}_2)$, $f^{-1}(B_{ij})$'s are intuitionistic fuzzy open rings in $(R_1, \mathscr{S}_1)$. Hence $f^{-1}(A_i) = \cap_{j=1}^\infty f^{-1}(B_{ij})$ is an intuitionistic fuzzy G_δ rings in $(R_1, \mathscr{S}_1)$.

Since f is an intuitionistic fuzzy ring hardly open function and $IF_RExt(\overline{A_i})$ is an intuitionistic fuzzy dense ring in $(R_2, \mathscr{S}_2)$, $f^{-1}(IF_RExt(\overline{A_i}))$ is an intuitionistic fuzzy dense ring in $(R_1, \mathscr{S}_1)$. By Remark 4.2., $f^{-1}(IF_RExt(\overline{A_i})) \subseteq IF_RExt(\overline{f^{-1}(A_i)})$. Thus,

$$IF_Rcl(f^{-1}(IF_RExt(\overline{A_i}))) = 1_\sim \subseteq IF_Rcl(IF_RExt(\overline{f^{-1}(A_i)})).$$

Hence, $IF_RExt(\overline{f^{-1}(A_i)})$ is an intuitionistic fuzzy dense ring in $(R_1, \mathscr{S}_1)$.

Suppose that $IF_Rcl(\cap_{i=1}^n f^{-1}(A_i)) \neq 1_\sim$. This implies that

$$\begin{gathered}\overline{IF_Rcl(\cap_{i=1}^n f^{-1}(A_i))} \neq 0_\sim \\ \Rightarrow IF_Rint(\cup_{i=1}^n \overline{f^{-1}(A_i)}) \neq 0_\sim \\ \Rightarrow IF_Rint(\cup_{i=1}^n f^{-1}(\overline{A_i})) \neq 0_\sim.\end{gathered}$$

Then, there is an non-zero intuitionistic fuzzy open ring C_i in $(R_1, \mathscr{S}_1)$ such that $C_i \subseteq \cup_{i=1}^n f^{-1}(\overline{A_i})$. Now,

$$\begin{aligned} f(C_i) &\subseteq f(\cup_{i=1}^n f^{-1}(\overline{A_i})) \\ &\subseteq \cup_{i=1}^n f(f^{-1}(\overline{A_i})) \\ &\subseteq \cup_{i=1}^n \overline{A_i} \\ &= \overline{\cap_{i=1}^n A_i}. \end{aligned}$$

$$\text{Then, } IF_Rint(f(C_i)) \subseteq IF_Rint(\overline{\cap_{i=1}^n A_i}) = \overline{IF_Rcl(\cap_{i=1}^n A_i)}. \tag{4.3}$$

Since $(R_2, \mathscr{S}_2)$ is an intuitionistic fuzzy ring $ExtV$ space, $IF_Rcl(\cap_{i=1}^n A_i) = 1_\sim$. Hence from (4.3), $IF_Rint(f(C_i)) \subseteq 0_\sim$, which implies that $IF_Rint(f(C_i)) = 0_\sim$, which is a contradiction. Hence $IF_Rcl(\cap_{i=1}^n f^{-1}(A_i)) = 1_\sim$. Therefore, $(R_1, \mathscr{S}_1)$ is an intuitionistic fuzzy ring $ExtV$ space. □

Proposition 4.14. *Let $(R_1, \mathscr{S}_1)$ and $(R_2, \mathscr{S}_2)$ be any two intuitionistic fuzzy structure ring spaces. Let $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ be any bijective function. Then the following statements are equivalent:*

(i) *f is somewhat intuitionistic fuzzy ring continuous function.*
(ii) *If A is an intuitionistic fuzzy closed ring in $(R_2, \mathscr{S}_2)$ such that $f^{-1}(A) \neq 1_\sim$, then there exists an intuitionistic fuzzy closed ring $0_\sim \neq C \neq 1_\sim$ in $(R_1, \mathscr{S}_1)$ such that $f^{-1}(A) \subset C$.*
(iii) *If A is an intuitionistic fuzzy dense ring in $(R_1, \mathscr{S}_1)$, then $f(A)$ is an intuitionistic fuzzy dense ring in $(R_2, \mathscr{S}_2)$.*

Proof. **(i)⇒(ii)**
Assume that (i) is true. Let A be an intuitionistic fuzzy closed ring in $(R_2, \mathscr{S}_2)$ such that $f^{-1}(A) \neq 1_\sim$. Then $\overline{\mathrm{A}}$ is an intuitionistic fuzzy open ring in $(R_2, \mathscr{S}_2)$ such that $\overline{f^{-1}(A)} = f^{-1}(\overline{\mathrm{A}}) \neq 0_\sim$. Since f is somewhat intuitionistic fuzzy ring continuous, there exists an intuitionistic fuzzy open ring C in $(R_1, \mathscr{S}_1)$ such that $C \subseteq f^{-1}(\overline{\mathrm{A}})$. Then there exists an intuitionistic fuzzy closed ring $\overline{C} \neq 0_\sim$ in $(R_1, \mathscr{S}_1)$ such that $\overline{C} \subset f^{-1}(A)$. Hence (i)⇒(ii).
(ii)⇒(iii)
Assume that (ii) is true. Let A be an intuitionistic fuzzy dense ring in $(R_1, \mathscr{S}_1)$ such that $f(A)$ is an intuitionistic fuzzy dense ring in $(R_2, \mathscr{S}_2)$. Then, there exists an intuitionistic fuzzy closed ring C in $(R_2, \mathscr{S}_2)$ such that

$$f(A) \subset C \subset 1_\sim.$$

This implies that $f^{-1}(C) \neq 1_\sim$. Then by (ii), there exists an intuitionistic fuzzy closed ring $0_\sim \neq D \neq 1_\sim$ such that $A \subset f^{-1}(C) \subset D \subset 1_\sim$. This is a contradiction. Hence (ii)⇒(iii).
(iii)⇒(ii)
Assume that (iii) is true. Suppose (ii) is not true. Then there exists an intuitionistic fuzzy closed ring A in $(R_2, \mathscr{S}_2)$ such that $f^{-1}(A) \neq 1_\sim$. But there is no intuitionistic fuzzy closed ring $0_\sim \neq C \neq 1_\sim$ in $(R_1, \mathscr{S}_1)$ such that $f^{-1}(A) \subseteq C$. This implies that $f^{-1}(A)$ is an intuitionistic fuzzy dense ring in $(R_1, \mathscr{S}_1)$. But from hypothesis $f(f^{-1}(A)) = A$ must be intuitionistic fuzzy dense ring in $(R_2, \mathscr{S}_2)$, which is a contradiction. Hence (iii)⇒(ii).
(ii)⇒(i)
Let A be an intuitionistic fuzzy open ring in $(R_2, \mathscr{S}_2)$ and $f^{-1}(A) \neq 0_\sim$. Then, $f^{-1}(\overline{\mathrm{A}}) = \overline{f^{-1}(A)} = 0_\sim$. Then by (ii), there exists an intuitionistic fuzzy closed ring $0_\sim \neq B \neq 1_\sim$ such that $f^{-1}(\overline{\mathrm{A}}) \subset B$.

This implies that $\overline{B} \subset f^{-1}(A)$ and $\overline{B} \neq 0_\sim$ is an intuitionistic fuzzy open ring in $(R_1, \mathscr{S}_1)$. Hence (ii)⇒(i). Hence the proof. □

Proposition 4.15. *If a function $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ from an intuitionistic fuzzy structure ring space $(R_1, \mathscr{S}_1)$ onto another intuitionistic fuzzy structure ring* *space* $(R_2, \mathscr{S}_2)$ *is somewhat intuitionistic fuzzy ring continuous, 1-1 and intuitionistic fuzzy ring open function and if* $(R_1, \mathscr{S}_1)$ *is an intuitionistic fuzzy ring* $ExtV$ *space, then* $(R_2, \mathscr{S}_2)$*is an intuitionistic fuzzy ring* $ExtV$ *space.*

Proof. Let $(R_1, \mathscr{S}_1)$ be an intuitionistic fuzzy ring $ExtV$ space. Assume that A_i's $(i = 1, ..., n)$ are intuitionistic fuzzy G_δ rings in $(R_1, \mathscr{S}_1)$ and the intuitionistic fuzzy ring exterior of $\overline{A_i}$'s are intuitionistic fuzzy dense rings in $(R_1, \mathscr{S}_1)$.

Then, $IF_Rcl(IF_RExt(\overline{A_i})) = 1_\sim$ and $A_i = \cap_{j=1}^{\infty} B_{ij}$ where B_{ij}'s are intuitionistic fuzzy open rings in $(R_1, \mathscr{S}_1)$. Since f is an intuitionistic fuzzy ring open function, $f(B_{ij})$'s are intuitionistic fuzzy open rings in $(R_2, \mathscr{S}_2)$. Now, $\cap_{j=1}^{\infty} f(B_{ij})$ is an intuitionistic fuzzy G_δ rings in $(R_2, \mathscr{S}_2)$. Since f is 1-1,

$$f^{-1}(\cap_{j=1}^{\infty} f(B_{ij})) = \cap_{j=1}^{\infty} f^{-}(f(B_{ij})) = \cap_{j=1}^{\infty} B_{ij} = A_i \tag{4.4}$$

$$\text{Since } f \text{ is onto, } f(A_i) = f(f^{-1}(\cap_{j=1}^{\infty} f(B_{ij}))) = \cap_{j=1}^{\infty} f(B_{ij}) \tag{4.5}$$

Therefore, $f(A_i)$ is an intuitionistic fuzzy G_δ rings in $(R_2, \mathscr{S}_2)$. Since f is somewhat intuitionistic fuzzy ring continuous function, $IF_RExt(\overline{A_i})$ is an intuitionistic fuzzy dense ring in $(R_1, \mathscr{S}_1)$ and by Proposition 4.7., $f(IF_RExt(\overline{A_i}))$ is an intuitionistic fuzzy dense ring in $(R_2, \mathscr{S}_2)$, which implies that $IF_RExt(\overline{f(A_i)})$. Now we claim that $IF_Rcl(\cap_{i=1}^{\infty} f(A_i)) = 1_\sim$. Suppose that $IF_Rcl(\cap_{i=1}^{n} f(A_i)) \neq 1_\sim$. This implies that

$$\begin{gathered} \overline{IF_Rcl(\cap_{i=1}^{n} f(A_i))} \neq 0_\sim \\ \Rightarrow IF_Rint(\cup_{i=1}^{n} \overline{f(A_i)}) \neq 0_\sim \\ \Rightarrow IF_Rint(\cup_{i=1}^{n} f(\overline{A_i})) \neq 0_\sim. \end{gathered}$$

Therefore there is an non-zero intuitionistic fuzzy open ring C_i in $(R_2, \mathscr{S}_2)$ such that $C_i \subseteq \cup_{i=1}^{n} f(\overline{A_i})$. Then $f^{-1}(C_i) \subseteq f^{-1}(\cup_{i=1}^{n} f(\overline{A_i}))$. Since f is somewhat intuitionistic fuzzy ring continuous function and $C_i \in \mathscr{S}_2$, $IF_Rint(f^{-1}(C_i)) \neq 0_\sim$ implies that $IF_Rint(f^{-1}(\cup_{i=1}^{n} f(\overline{A_i}))) \neq 0_\sim$. Then $IF_Rint(\cup_{i=1}^{n} f^{-1}(f(\overline{A_i}))) \neq 0_\sim$. Since f is a bijective function, $IF_Rint(\cap_{i=1}^{n} \overline{A_i}) \neq 0_\sim$, which implies that $\overline{IF_Rcl(\cap_{i=1}^{n} A_i)} \neq 0_\sim$. That is, $IF_Rcl(\cap_{i=1}^{n} A_i) \neq 1_\sim$. This is a contradiction. Hence $(R_2, \mathscr{S}_2)$ is an intuitionistic fuzzy ring $ExtV$ space.

Conversely, let $(R_2, \mathscr{S}_2)$ be an intuitionistic fuzzy ring $ExtV$ space. Assume that A_i's $(i = 1, ..., n)$ are intuitionistic fuzzy G_δ rings in $(R_1, \mathscr{S}_1)$ and the intuitionistic fuzzy ring exterior of $\overline{A_i}$'s are intuitionistic fuzzy dense ring in $(R_1, \mathscr{S}_1)$.

Then $IF_Rcl(IF_RExt(\overline{A_i})) = 1_\sim$ and $A_i = \cap_{j=1}^{\infty} B_{ij}$ where B_{ij}'s are intuitionistic fuzzy open rings in $(R_1, \mathscr{S}_1)$. Since f is somewhat intuitionistic fuzzy ring continuous function, $IF_RExt(\overline{A_i})$'s are intuitionistic fuzzy dense rings in $(R_1, \mathscr{S}_1)$ and By Proposition 4.7., $f(IF_RExt(\overline{A_i}))$ is an intuitionistic fuzzy dense ring in $(R_2, \mathscr{S}_2)$. That is, $IF_Rcl(IF_RExt(\overline{A_i})) = 1_\sim$. Since f is an intuitionistic fuzzy ring open function and B_{ij}'s are intuitionistic fuzzy open rings in $(R_1, \mathscr{S}_1)$, $f(B_{ij})$'s are intuitionistic fuzzy open rings in $(R_2, \mathscr{S}_2)$. Hence $\cap_{j=1}^{\infty} f(B_{ij})$ is an intuitionistic fuzzy G_δ ring in $(R_2, \mathscr{S}_2)$. Since f is 1-1,

$$f^{-1}(\cap_{i=1}^{n} f(B_{ij})) = \cap_{i=1}^{n} (f^{-1}(f(B_{ij})) = \cap_{i=1}^{n} B_{ij}. \tag{4.6}$$

Since f is onto,

$$f(A_i) = f(f^{-1}(\cap_{j=1}^{\infty} f(B_{ij}))) = \cap_{j=1}^{\infty} f(B_{ij}). \tag{4.7}$$

Hence $f(A_i)$ is an intuitionistic fuzzy G_δ ring in $(R_2, \mathscr{S}_2)$. Now,

$$\begin{aligned} IF_Rcl(IF_RExt(\overline{f(A_i)}) &= IF_Rcl(IF_RExt(f(\overline{A_i})) \\ &= IF_Rcl(IF_Rint(f(A_i)) \\ &\supseteq IF_Rcl(f(IF_Rint(A_i)) \\ &\supseteq f(IF_Rcl(IF_Rint(A_i))) \\ &= f(1_\sim) = 1_\sim. \end{aligned}$$

This implies that $IF_RExt(\overline{f(A_i)}$ is an intuitionistic fuzzy dense ring in $(R_2, \mathscr{S}_2)$. Hence the intuitionistic fuzzy ring exterior of $\overline{f(A_i)}$ is an intuitionistic fuzzy dense ring in $(R_2, \mathscr{S}_2)$. Since $(R_2, \mathscr{S}_2)$ is an intuitionistic fuzzy ring $ExtV$ space,

$$IF_Rcl(\cap_{i=1}^{n} f(A_i)) = 1_\sim.$$

Now we claim that $IF_Rcl(\cap_{i=1}^{n} f(A_i)) = 1_\sim$ where A_i's $(i = 1, ..., n)$ are intuitionistic fuzzy G_δ rings in $(R_1, \mathscr{S}_1)$ and the intuitionistic fuzzy ring exterior of $\overline{A_i}$'s are intuitionistic fuzzy dense rings in $(R_1, \mathscr{S}_1)$.

Suppose that $IF_Rcl(\cap_{i=1}^{n} A_i) \neq 1_\sim$. This implies that

$$\begin{aligned} &\overline{IF_Rcl(\cap_{i=1}^{n} A_i)} \neq 0_\sim \\ &\Rightarrow IF_Rint(\overline{\cap_{i=1}^{n} A_i}) \neq 0_\sim \\ &\Rightarrow IF_Rint(\cup_{i=1}^{n} \overline{A_i}) \neq 0_\sim. \end{aligned}$$

Then there is a non-zero intuitionistic fuzzy open ring C_i in $(R_1, \mathscr{S}_1)$ such that $C_i \subseteq \cup_{i=1}^{n} \overline{A_i}$. Now,

$$\begin{aligned} f(C_i) &\subseteq f(\cup_{i=1}^{n} \overline{A_i}) \\ &\subseteq \cup_{i=1}^{n} f(\overline{A_i}) \\ &\subseteq \cup_{i=1}^{n} \overline{f(A_i)} \\ &= \overline{\cap_{i=1}^{n} f(A_i)}. \end{aligned}$$

$$\text{Then, } IF_Rint(f(C_i)) \subseteq IF_Rint(\overline{\cap_{i=1}^{n} f(A_i)}) \subseteq \overline{IF_Rcl(\cap_{i=1}^{n} f(A_i))} \tag{4.8}$$

Since $(R_2, \mathscr{S}_2)$ is an intuitionistic fuzzy ring $ExtV$ space, $IF_Rcl(\cap_{i=1}^{n} f(A_i)) = 1_\sim$. Hence from (4.8), $IF_Rint(f(C_i)) \subseteq 0_\sim$, which implies that $IF_Rint(f(C_i)) = 0_\sim$, which is a contradiction. Hence $IF_Rcl(\cap_{i=1}^{n} A_i) = 1_\sim$. Therefore $(R_1, \mathscr{S}_1)$ is an intuitionistic fuzzy ring $ExtV$ space. □

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