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Prime bi-ideals and strongly prime fuzzy bi-ideals in near rings

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Abstract. The analysis of prime bi-ideals in near-rings is the primary focus of our research. The concept of strongly prime, semiprime, irreducible and strongly irreducible bi-ideals in near-rings N have been dilated upon by our research team. Now we are taking up the significant concept of fuzzification in prime bi-ideals of near-rings and exploring its behaviour and its operations. Prime, strongly prime and semiprime fuzzy bi-ideals in near-rings have been attempted to be defined systemotically. We have further made a very minute and meticulous study of the process of irreducible and strongly irreducible fuzzy bi-ideals in near-rings.

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1. Introduction

Generalized rings may be nomenclated as near-rings. We could describe them as rings $(N, +, *)$ which needs no additions as abelian and sufficies with only one distributive law. In the historical perspective, a leading role was initiated toward near-rings as an axiomatic research by Dickson in 1905 in [3]. He broughtout that, there are certain fields which exist with only one distributive law called near-fields. After a laps of few years these near-fields re-emerged and proved their utility in co-ordinatizing some significant types of geometric plans.

Zassenhaus [13] was the one who enabled to distinguish and fixate all finite near-fields. Presently near-fields happened to be a very effective tool in discovering and determining bi-transtive, incidence and Ferobenious groups. Whereas it is assumed that two endomorphisms of a non-abelian group $(G, +)$ can not be termed in general usage as endomorphisms. The set $E(G)$ of all finite sums and differences of endomorphisms of G were taken in to consideration. $E(G)$ are near-rings related to the class of the distributely generated near-rings as regards, the addition and composition of these structures.

LA Zadeh in 1965 in [14] launched the theory of fuzzy set as a general abstraction of set theory. The concept of Quasi-ideals in integrative near-rings was poineered by Yakabe [12]. Similarly, the notion of bi-ideals in near-rings was broughtforth by Tamizh Chelvam and N. Ganesan [11]. S. J. Abbassi and Ambreen Zahra Rizvi [2] meditated and research upon the prime ideals in near-rings in 2008. The ideas of fuzzy subnear-rings was initiated by S. Abou-Zaid [1] in 1991 and thoroughly discussed and evaluated fuzzy left(right) ideals of near-rings and discovered some prominent characteristics of fuzzy prime ideals of a near-rings. In this discourse, we have thoroughly thrashed prime bi-ideals, semiprime bi-ideals and not further reducible bi-ideals in near-rings. We have further meditated and research fuzzification of prime bi-ideals, semiprime bi-ideals and irreducible bi-ideals in near-rings inspired from [10].

2. Preliminaries

A near-ring is a set N together with two binary operations addition and multiplication such that $(N,+)$ is a group (not necessarily abilian), $(N,\cdot)$ is a semigroup and for all $n_1,n_2,n_3 \in N$, $(n_{1+}n_2)\cdot n_3 \ = n_1.n_2 + n_2.n_3$ (right distributive law).

In a near-ring only one distributive law holds (left or right). If $(N,+)$ is abelian, we call N an abelian near-ring. If $(N,\cdot)$ is commutative we call N itself a commutative near-ring. The set $N_0 = \{n \in N \mid n0 = 0\}$ is called the zero- symetric part of N. $N_c = \{n \in N \mid n0 = n\} = \{n \in N \mid \text{for all } n' \in N : nn' = n\}$ is called the constant part of N. A near-ring N is regular if for each element a in N there exist an element x in N such that $a = axa$. A near-ring N is strongly regular if for every element a there is an x in N such that $a = xa^2$. A subgroup M of N with $MM \subseteq M$ is called subnear-ring of N and is denoted by $M \leq N$ (see [6]). A subnear-ring M of N is called invariant if $MN \subseteq M$ and $NM \subseteq M$. A normal subgroup I of $(N,+)$ is called ideal of N if

(i) $IN \subseteq I$

(ii) For all $n,n' \in N$ and for all $i \in I$, $n\,(n'+i) - nn' \in I$. This ideal is denoted by $(I \trianglelefteq N)$.

Normal subgroups R of $(N,+)$ with (i) are called right ideals of N denoted by $(R \trianglelefteq_r N)$, while normal subgroups L of $(N,+)$ with (ii) are said to be left ideals of N denoted by $(L \trianglelefteq_l N)$. A non-empty subset I of N is called an invariant N-subgroup of N, if I is a subgroup of $(N,+)$, $IN \subseteq I$ and $NI \subseteq I$. An ideal $P \trianglelefteq N$ is called prime if for all ideals I, $J \trianglelefteq N$ such that $IJ \subseteq P$ implies that either$I \subseteq P$ or $J \subseteq P$. Let $(P_\alpha)_{\alpha\in A}$ be a family of prime ideals totally orderd by inclusion. Then $\underset{\alpha\in A}{\cap} P_\alpha = P$ is also a prime ideal. If $I \trianglelefteq N$ is a direct summand and $P \trianglelefteq N$ is prime then $P \cap I$ is prime ideal in I. An ideal $S \trianglelefteq N$ is semiprime if and only if for all ideals $I \trianglelefteq N$ such that $I^2 \subseteq S$ this implies $I \subseteq S$. Each prime ideal is semiprime. If $(S_\alpha)_{\alpha\in A}$ is a family of semiprime ideals, then $\underset{\alpha\in A}{\cap} S_\alpha$ is again semiprime. Let $I \trianglelefteq N$ be a direct summand and $S \trianglelefteq N$ be semiprime then $S \cap I$ is semiprime in I ([8]). Let N be a near-ring. Given two subsets A and B of N, the product

$AB = \{ab|\ a \in A,\ b \in B\}$. Also we define another operation '$*$' on the class of subsets of N given by $A * B = \{a(a' + b) - aa'|\ a, a' \in A, b \in B\}$. A subgroup B of $(N, +)$ is said to be a bi-ideal of N if $BNB \cap (BN) * B \subseteq B$. A subgroup Q of $(N, +)$ satisfying $QN \cap NQ \cap N * Q \subseteq Q$ is called a quasi-ideal of N ([11]). Note that

Remark 2.1 ([11])**.** Every quasi ideal is a bi-ideal in a near-ring but the converse is not true. For this, let $N = \{0, a, b, c\}$ be the near-ring defined by the caleys tabels

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
c	0	b	0	b

Here $\{0, a\}$ is a bi-ideal but not a quasi-ideal. Note that one sided ideals, N subgroups and invariant subnear-ring are quasi-ideals and so they are also bi-ideals.

Proposition 2.2 ([11])**.** *The intersection of all bi-ideals of a near-ring N is a bi-ideal of N.*

Proposition 2.3 ([11])**.** *If B be bi-ideal of a near-ring N and S is a subnear-ring of N, then $B \cap S$ is a bi-ideal of S.*

Proposition 2.4 ([11])**.** *Let N be a zero-symetric near-ring. A subgroup B of N is a bi-ideal if and only if $BNB \subseteq B$.*

Proposition 2.5 ([11])**.** *Let N be a zero-symetric near-ring. If B is a bi-ideal of N then Bn and $n'B$ are bi-ideals of N, where $n, n' \in N$ and n' is distributive element in N.*

Corollary 2.6 ([11])**.** *If B is a bi-ideal of a zero symetric near-ring N and b is a distributive element in N. Then bBc is a bi-ideal of N, where $c \in N$.*

A function f from the non-empty set N to the unit interval $[0, 1]$ of real numbers is called a fuzzy subset of N, that is $f : N \to [0, 1]$. A fuzzy subset $f : N \to [0, 1]$ is non-empty if f is not the constant map which assumes the value 0. For fuzzy subsets f and g of N, $f \leq g$ means that for all $a \in N$, $f(a) \leq g(a)$.

Definition 2.7 ([4])**.** If f and g are fuzzy subsets of a near-ring N. Then $f \cap g$, $f \cup g$, $f + g$, fg and $f * g$ are fuzzy subsets of N defined by

$$(f \cap g)(x) = \min\{f(x), g(x)\}$$

$$(f \cup g)(x) = \max\{f(x), g(x)\}$$

$$(f + g)(x) = \begin{cases} \sup\limits_{x=y+z} \{\min\{f(y), g(y)\}\} & \text{if } x \text{ is expressible as } x = y + z \\ 0 & \text{otherwise} \end{cases}$$

$$(fg) = \begin{cases} \sup\limits_{x=yz} \{\min\{f(y), g(y)\}\} & \text{if } x \text{ is expressible as } x = yz \\ 0 & \text{otherwise} \end{cases}$$

$$(f * g) = \begin{cases} \sup\limits_{x=a(b+c)-ab} \{\min\{f(a), g(c)\}\} & \text{if } x \text{ is expressible as } x = a(b+c) - ab \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.8 ([4]). Let f be a fuzzy subset of N. Then f is called a fuzzy left(right) N-subgroup of N if for all $x, y \in N$

(1) $f(x-y) \geq \min\{f(x), f(y)\}$
(2) $f(xy) \geq f(y)$ $(f(xy) \geq f(x))$
If f is both left and right fuzzy N-subgroup of N, then it is called a fuzzy N-subgroup of N.

Definition 2.9 ([7]). Let N be a near-ring and f be a fuzzy subset of N. We say f a fuzzy subnear-ring of N if

(1) $f(x-y) \geq \min\{f(x), f(y)\}$
(2) $f(xy) \geq \min\{f(x), f(y)\}$ for all $x, y \in N$.

Definition 2.10 ([4]). Let f be a non-empty fuzzy subset of N. f is a fuzzy ideal of N, if for all $x, y, i \in N$ and

(1) $f(x-y) \geq \min\{f(x), f(y)\}$
(2) $f(x) = f(y+x-y)$
(3) $f(xy) \geq f(x)$
(4) $(f(x(y+i) - xy) \geq f(i))$ for any $x, y, i \in N$.
If f satisfies (1), (2) and (3), then it is called a fuzzy right ideal of N. If f satisfies (1), (2) and (4), then it is called a fuzzy left ideal of N , If f is both fuzzy right as well as fuzzy left ideal of N, then f is called a fuzzy ideal of N.

Example 2.11 ([3]). Let $N = \{a, b, c, d\}$ be a set with two binary operatipns as follows,

$+$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	a	b	b

Then $(N, +, \cdot)$ is a (left) near-ring. Define a fuzzy subset $f : N \to [0, 1]$ by $f(c) = f(d) < f(b) < f(a)$. Then f is a fuzzy ideal of N.

Lemma 2.12 ([3]). *If a fuzzy subset of N satisfies the property*

$$f(x-y) \geq \min\{f(x), f(y)\},$$

then
(1) $f(0_N) \geq f(x)$
(2) $f(-x) = f(x)$ *for all* $x, y \in N$.

Definition 2.13 ([5]). Let X be a non-empty fuzzy subset of N, then the characteristic function of X is denoted by f_X and is defined as:

$$f_X(a) = \begin{cases} 1 & \text{if } a \in X \\ 0 & \text{otherwise} \end{cases}$$

Lemma 2.14 ([1])**.** *Let I be a subset of N. Then I is a left(right) ideal of N if and only if f_I is a fuzzy left(right) ideal of N.*

Definition 2.15 ([5])**.** A fuzzy set f of a near-ring N is called a fuzzy bi-ideal of N if

(1) $f(x-y) \geq \min\{f(x), f(y)\}$ for all $x, y, z \in N$.
(2) $f(xyz) \geq \min\{f(x), f(z)\}$ for all $x, y, z \in N$.

Definition 2.16 ([4])**.** A fuzzy subgroup f of N is called a fuzzy quasi-ideal of N if $(f \circ f_N) \cap (f_N \circ f) \cap (f_N * f) \leq f$

A fuzzy subgroup f of N is called a fuzzy bi-ideal of N if $(f \circ f_N \circ f) \cap (f \circ f_N * f) \leq f$

Lemma 2.17 ([4])**.** *For any non-empty subsets X and Y of near-ring N, we have*

(1) $f_X \circ f_Y = f_{XY}$
(2) $f_X \cap f_Y = f_{X \cap Y}$
(3) $f_X * f_Y = f_{X*Y}$.

Lemma 2.18 ([4])**.** *Let Q be a subgroup of $(N, +)$*

(1) Q is a quasi-ideal of N if and only if f_Q is a fuzzy quasi-ideal of N
(2) Q is a bi-ideal of N if and only if f_Q is a fuzzy bi-ideal of N.

Lemma 2.19 ([4])**.** *Let f and g be two fuzzy bi-ideals of a near-ring N. Then $f \wedge g$ is a fuzzy bi-ideal of N.*

3. Prime, strongly prime and semiprime bi-ideals

Definition 3.1. A bi-ideal B of a near-ring N is called a prime bi-ideal of N if $B_1 B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for any bi-ideals B_1, B_2 of N.

Example 3.2. If $N = \{0, 1, 2\}$. Define addition and Multiplication on N as

$+$	0	1	2
0	0	1	2
1	1	0	1
2	2	1	0

$\cdot$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

then N is a commutative near-ring and $\{0\}, \{0, 1\}$ and $\{0, 1, 2\}$ are prime bi-ideals of N.

Definition 3.3. A bi-ideal B of a near-ring N is called a strongly prime bi-ideal of N if $B_1 B_2 \cap B_2 B_1 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for any bi-ideals B_1, B_2 of N.

Definition 3.4. A bi-ideal B of a near-ring N is called a semiprime bi-ideal of N if $B_1^2 \subseteq B$ implies $B_1 \subseteq B$ for any bi-ideal B_1 of N.

Proposition 3.5. *Every strongly prime bi-ideal of a near-ring N is a prime bi-ideal of N.*

Proof. Let B be a strongly prime bi-ideal of a near-ring N. Now let B_1, B_2 be two bi-ideals of N such that $B_1 B_2 \subseteq B$. Then $B_1 B_2 \cap B_2 B_1 \subseteq B_1 B_2 \subseteq B$ implies $B_1 B_2 \cap B_2 B_1 \subseteq B$. Thus by hypothesis, $B_1 \subseteq B$ or $B_2 \subseteq B$. Hence B is a prime bi-ideal of N. □

Proposition 3.6. *Every prime bi-ideal of a near-ring N is a semiprime bi-ideal of N.*

Proof. Let B be a prime bi-ideal of a near-ring N. Now let B_1 be a bi-ideal of N such that $B_1^2 \subseteq B$. This implies $B_1B_1 \subseteq B$ implies $B_1 \subseteq B$, as B is a prime bi-ideal of a near-ring N. Hence B is a semiprime bi-ideal of N. □

Remark 3.7. A prime bi-ideal of a near-ring N is not necessarily a strongly prime bi-ideal of N.

Example 3.8. Consider $N = \{0, a, b\}$. Define addition and multiplication on N as

+	0	a	b
0	0	a	b
a	a	0	b
b	b	b	0

.	0	a	b
0	0	0	0
a	0	a	a
b	0	b	b

Then $(N, +, .)$ is near-ring and its bi-ideals are $\{0\}, \{0, a\}, \{0, b\}, \{0, a, b\}$. Now $\{0\}$ is a prime bi-ideal of N but not a strongly prime bi-ideal. As $\{0, a\}\{0, b\} \cap \{0, b\}\{0, a\} = \{0, a\} \cap \{0, b\} = \{0\} \subseteq \{0\}$. But neither $\{0, a\}$ nor $\{0, b\}$ contained in $\{0\}$. This example shows that every prime bi-ideal of a near-ring is not a strongly prime bi-ideal.

Lemma 3.9. *Intersection of any family of prime bi-ideals of a near-ring N is a semiprime bi-ideal of N.*

Proof. Let $\{B_i : i \in I\}$ be any family of prime bi-ideals of a near-ring N. We have to show that $\underset{i\in I}{\cap} B_i$ is a semiprime bi-ideal of N. Now $\underset{i\in I}{\cap} B_i$ being the intersection of any family of bi-ideals of a near-ring N is a bi-ideal of N, by Proposition 2.2. Now let B be any bi-ideal of N such that $B^2 \subseteq \underset{i\in I}{\cap} B_i$ implies $BB = B^2 \subseteq B_i$ for all $i \in I$. Thus $B \subseteq B_i$ for all $i \in I$, because each B_i is a prime bi-ideal of N. So $B \subseteq \underset{i\in I}{\cap} B_i$. Hence $\underset{i\in I}{\cap} B_i$ is a semiprime bi-ideal of N. □

Theorem 3.10. *Let X be an arbitrary subset of a near-ring N and B is a bi-ideal of N, then BX is a bi-ideal of N.*

Proof. To show that BX is a bi ideal of N, we have to show that,
(i) BX is a subgroup of $(N, +)$
(ii) $BXNBX \cap BXN * BX \subseteq BX$

Since $0 \in B$ so $0x_1 + 0x_2 + 0x_3 + ... + 0x_n \in BX$. Implies $BX \neq \phi$. Let $\sum\limits_{i=1}^{n} b_i x_i$, $\sum\limits_{j=1}^{m} b_j' x_j' \in BX$, then $\sum\limits_{i=1}^{n} b_i x_i - \sum\limits_{j=1}^{m} b_j' x_j' = \sum\limits_{i=1}^{n} b_i x_i + \sum\limits_{j=1}^{m} (-b_j') x_j' \in BX$ because its again a finite sum. So BX is a subgroup of N.

(ii) Now consider

$$\begin{aligned} BXNBX \cap BXN * BX &= (BXNB \cap BXN * B)X \text{ by right distributive law} \\ &\subseteq (BNNB \cap BNN * B)X \text{ as } X \subseteq N \\ &\subseteq (BNB \cap BN * B)X \text{ as } N^2 \subseteq N \\ &\subseteq BX \text{ as } B \text{ is a bi-ideal of } N \\ BXNBX \cap BXN * BX &\subseteq BX \end{aligned}$$

Hence BX is a bi-ideal of N. □

Corollary 3.11. *Product of two bi-ideals of near-ring N is a bi-ideal of N.*

Proof. Let B_1 and B_2 be two bi-ideals of N. In Theorem 3.10, if we take a bi-ideal B_1 of N and B_2 as an arbitrary subset of N. Then B_1B_2 is a bi-ideal of N. □

4. Irreducible and strongly irreducible bi-ideals

Definition 4.1. A bi-ideal B of a near-ring N is called an irreducible bi-ideal of N if $B_1 \cap B_2 = B$, implies either $B_1 = B$ or $B_2 = B$ for any bi-ideals B_1, B_2 of N.

Definition 4.2. A bi-ideal B of a near-ring N is called a strongly irreducible bi-ideal of N if $B_1 \cap B_2 \subseteq B$ implies either $B_1 \subseteq B$ or $B_2 \subseteq B$ for any bi-ideals B_1, B_2 of N.

Definition 4.3. A bi-ideal B of a near-ring N is called a strongly irreducible semiprime bi-ideal of N if $(B_1 \cap B_2)^2 \subseteq B$ implies either $B_1 \subseteq B$ or $B_2 \subseteq B$ for any bi-ideals B_1, B_2 of N.

Proposition 4.4. *Every strongly irreducible semiprime bi-ideal of N is a strongly prime bi-ideal of N.*

Proof. Let B be a strongly irreducible semiprime bi-ideal of a near-ring N. Let B_1 and B_2 be two bi-ideals of N such that $B_1B_2 \cap B_2B_1 \subseteq B$...**(i)**. Then we have to show that either $B_1 \subseteq B$ or $B_2 \subseteq B$. As $B_1 \cap B_2 \subseteq B_1$ and $B_1 \cap B_2 \subseteq B_2$, implies $(B_1 \cap B_2)^2 \subseteq B_1B_2$ and $(B_1 \cap B_2)^2 \subseteq B_2B_1$. Thus $(B_1 \cap B_2)^2 \subseteq B_1B_2 \cap B_2B_1 \subseteq B$ (using **(i)**). $B_1 \cap B_2$ being the intersection of bi-ideals of N is also a bi-ideal of N, by Proposition 2.2. So $(B_1 \cap B_2)^2 \subseteq B$ implies $(B_1 \cap B_2) \subseteq B$, because B is a semiprime bi-ideal of N. Also $B_1 \subseteq B$ or $B_2 \subseteq B$, because B is a strongly irreducible bi-ideal of N. Hence B is a strongly prime bi-ideal of N. □

Proposition 4.5. *Let B be a bi-ideal of a near-ring N and $a \in N$ such that $a \notin B$, then there exists an irreducible bi-ideal I of N such that $B \subseteq I$ and $a \notin I$.*

Proof. Let X be the collection of all bi-ideals of N which contain B but do not contain a, that is $X = \{Y_i : Y_i \text{ is a bi-ideal of } N,\ B \subseteq Y_i \text{ and } a \notin Y_i\}$. Then X is a non-empty as $B \in X$. The collection X is a partially orderd set under inclusion. If $\{Y_i : i \in I\}$ be any totally orderd subset (chain) of X, Then $\underset{i \in I}{\cup} Y_i = Y$ is also a bi-ideal of N containing B and $a \notin Y$. So Y is an upper bound of $\{Y_i : i \in I\}$. Thus every chain in X has an upper bound in X. Hence by Zorn's lemma, there exists a maximal element I (say) in X. This implies $B \subseteq I$ and $a \notin I$. Now we show that I is an irreducible bi-ideal of N. For this let C, D be two bi-ideals of N such that

$I = C \cap D$. If both C and D properly contain I, then $a \in C$ and $a \in D$. Thus $a \in C \cap D = I$, which is a contradiction to the fact that $a \notin I$. So either $I = C$ or $I = D$. □

Theorem 4.6. *For a near-ring N, the following assertions are equivalent:*

(1) $B^2 = B$ for every bi-ideal B of N.
(2) $B_1 B_2 \cap B_2 B_1 = B_1 \cap B_2$ for any bi-ideals B_1, B_2 of N.
(3) Each bi-ideal of N is semiprime.
(4) Each proper bi-ideal of N is the intersection of irreducible semiprime bi-ideal of N which contain it.

Proof. (1) ⇒ (2): Let B_1 and B_2 be any two bi-ideals of N. Then $B_1 \cap B_2$ is also a bi-ideal of N, by Proposition 2.2. By hypothesis, we have $(B_1 \cap B_2) = (B_1 \cap B_2)^2 = (B_1 \cap B_2)(B_1 \cap B_2) \subseteq B_1 B_2$. Similarly $B_1 \cap B_2 \subseteq B_2 B_1$. So $B_1 \cap B_2 \subseteq B_1 B_2 \cap B_2 B_1$. Now $B_1 B_2$ and $B_2 B_1$, being the product of two bi-ideals of N, are bi-ideals of N by Corollary 3.11. Also $B_1 B_2 \cap B_2 B_1$ is a bi-ideal of N, by Proposition 2.2. Then by hypothesis $B_1 B_2 \cap B_2 B_1 = (B_1 B_2 \cap B_2 B_1)(B_1 B_2 \cap B_2 B_1) \subseteq B_1 B_2 \cdot B_2 B_1 = B_1 B_2^2 B_1 \subseteq B_1 B_2 B_1$
$\subseteq B_1 N B_1 \subseteq B_1$. Similarly, $B_1 B_2 \cap B_2 B_1 \subseteq B_2$. Thus $B_1 B_2 \cap B_2 B_1 \subseteq B_1 \cap B_2$. Hence $B_1 B_2 \cap B_2 B_1 = B_1 \cap B_2$.

(2) ⇒ (3) : Let B be a bi-ideal of N such that $B_1^2 \subseteq B$ for any bi-ideals B_1 of N. Then by hypothesis, we have $B_1 = B_1 \cap B_1 = B_1 B_1 \cap B_1 B_1 = B_1^2 \subseteq B$. Which shows that B is a semiprime bi-ideal of N. Hence every bi-ideal of N is a semiprime bi-ideal of N.

(3) ⇒ (4) : Suppose each bi-ideal of N is semiprime. Now let B be a proper bi-ideal of N. Then by Proposition 4.5, there exists an irreducible bi-ideal of N containing B. If $\underset{\alpha}{\cap} I_\alpha$ be the intersection of all irreducible bi-ideals of N containing B, then $B \subseteq \underset{\alpha}{\cap} I_\alpha$, as $B \subseteq I_\alpha$ for all α. If this inclusion is proper, then there exists $a \in \underset{\alpha}{\cap} I_\alpha$ such that $a \notin B$. This implies $a \in I_\alpha$ for all α. As $a \notin B$, Then by Proposition 4.5, there exists an irreducible bi-ideal I (say) of N such that $B \subseteq I$ and $a \notin I$. Which is the contradiction to the fact that $a \in I_\alpha$ for all α. So $B = \underset{\alpha}{\cap} I_\alpha$. By hypothesis, each bi-ideal of N is semiprime. Thus each proper bi-ideal of N is the intersection of irreducible semiprime bi-ideals of N which contain it.

(4) ⇒ (1): Let each proper bi-ideal of N is the intersection of irreducible semiprime bi-ideals of N which contain it. Now if B is a bi-ideal of N, then $BB = B^2$, being the product of two bi-ideals is also a bi-ideal of N. If $B^2 = N$ (improper bi-ideal), then $N \subseteq B^2$ implies $B \subseteq N \subseteq B^2$. Also $B^2 \subseteq B$, so $B^2 = B$ for each bi-ideal B of N. Now if B^2 is a proper bi-ideal of N, $B^2 \neq N$, then $B^2 = \underset{\alpha}{\cap}\{B_\alpha : B_\alpha$ is an irreducible semiprime bi-ideal of N such that $B^2 \subseteq B_\alpha$ for all $\alpha\}$. Implies $B \subseteq B_\alpha$ for all α, because each B_α is a semiprime bi-ideal of N. Thus $B \subseteq \cap B_\alpha = B^2$. Also $B^2 \subseteq B$ as B is closed under multiplication. Hence $B^2 = B$ for each bi-ideal B of N. □

Theorem 4.7. *If each bi-ideal of a near-ring N is idempotent then the following assertions are equivalent*

(1) B is strongly irreducible.
(2) B is strongly prime.

Proof. (1) $\Rightarrow$ (2) : Let B be a strongly irreducible bi-ideal of N. Then we have to show that B is a strongly prime bi-ideal of N. For this let B_1 and B_2 be any two bi-ideals of N such that $B_1B_2 \cap B_2B_1 \subseteq B$. As each bi-ideal of N is idempotent then by Theorem 4.6, we have $B_1 \cap B_2 = B_1B_2 \cap B_2B_1 \subseteq B$, this implies $B_1 \cap B_2 \subseteq B$. But B is a strongly irreducible bi-ideal of N. Thus we have $B_1 \subseteq B$ or $B_2 \subseteq B$. Hence B is a strongly prime bi-ideal of N.

(2) $\Rightarrow$ (1) : Let B be a strongly prime bi-ideal of N. To show that B is a strongly irreducible bi-ideal of N, let B_1, B_2 be any two bi-ideals of N such that $B_1 \cap B_2 \subseteq B$. By Theorem 4.6, we have $B_1B_2 \cap B_2B_1 = B_1 \cap B_2 \subseteq B$, this implies $B_1B_2 \cap B_2B_1 \subseteq B$. But B is strongly prime bi-ideal of N, thus we have $B_1 \subseteq B$ or $B_2 \subseteq B$. Hence B is a strongly irreducible bi-ideal of N. □

Theorem 4.8. *Each bi-ideal of a near-ring N is strongly prime if and only if each bi-ideal of N is idempotent and the set of bi-ideals of N is totally orderd by inclusion.*

Proof. Suppose that each bi-ideal of N is strongly prime. This implies that each bi-ideal of N is semiprime. Then by Theorem 4.6, each bi-ideal of N is idempotent. Now we show that the set of bi-ideals of N is totally orderd by inclusion. For this let B_1, B_2 be two bi-ideals of N, by Theorem 4.6, we have

$$B_1B_2 \cap B_2B_1 = B_1 \cap B_2 ... \textbf{(i)}$$

By hypothesis, B_1 and B_2 are strongly prime bi-ideals of N, so is $B_1 \cap B_2$. Then **(i)** implies

$$B_1 \subseteq B_1 \cap B_2 \text{ or } B_2 \subseteq B_1 \cap B_2.$$

Thus $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Hence the set of bi-ideals of N is totally orderd by inclusion.

Conversely, assume that each bi-ideal of N is idempotent and the set of bi-ideals of N is totally orderd by inclusion. We have to show that each bi-ideal of N is strongly prime. For this let B be an arbitrary bi-ideal of N and B_1 and B_2 be any two bi-ideals of N such that

$$B_1B_2 \cap B_2B_1 \subseteq B \text{ ... } \textbf{(ii)}$$

By Theorem 4.6, we have $B_1B_2 \cap B_2B_1 = B_1 \cap B_2$. Thus **(ii)** can be written as

$$B_1 \cap B_2 \subseteq B \text{ ... } \textbf{(iii)}$$

Also the set of bi-ideals of N is totally orderd by inclusion, then either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. If $B_1 \subseteq B_2$, then $B_1 \cap B_2 = B_1$. Thus **(iii)** implies $B_1 \subseteq B$. And if $B_2 \subseteq B_1$, then $B_1 \cap B_2 = B_2$. Thus **(iii)** implies $B_2 \subseteq B$. So either $B_1 \subseteq B$ or $B_2 \subseteq B$. Thus B is strongly prime. Hence each bi-ideal of N is strongly prime. □

Theorem 4.9. *If the set of bi-ideals of a near-ring N is totally orderd, then each bi-ideal of N is idempotent if and only if each bi-ideal of N is prime.*

Proof. Let each bi-ideal of N is idempotent, B is an arbitrary bi-ideal of N and B_1, B_2 be any two bi-ideals of N such that $B_1B_2 \subseteq B$. As the set of bi-ideals of N is totally orderd, then either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. If $B_1 \subseteq B_2$, then $B_1B_1 = B_1^2 \subseteq B_1B_2 \subseteq B$, by Theorem 4.6, B is a semiprime bi-ideal of N. Then $B_1^2 \subseteq B$ implies $B_1 \subseteq B$, similarly, if $B_2 \subseteq B_1$, then $B_2B_2 = B_2^2 \subseteq B_1B_2 \subseteq B$. Implies $B_2 \subseteq B$, as B is a semiprime bi-ideal of N. Thus B is a prime bi-ideal of N.

Conversely, suppose that each bi-ideal of N is prime, so is semiprime, by Proposition 3.6. Thus by Theorem 4.6, each bi-ideal of N is idempotent. □

Proposition 4.10. *If the set of bi-ideals of a near-ring N is totally orderd, then the concepts of primeness and strongly primeness coincide.*

Proof. Let B be a prime bi-ideal of N. To show that B is a strongly prime bi-ideal of N, let B_1, B_2 be any two bi-ideals of N such that $B_1B_2 \cap B_2B_1 \subseteq B$. As the set of bi-ideals of near-ring N is totally orderd, then either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. If $B_1 \subseteq B_2$, then

$$B_1B_1 = B_1^2 = B_1^2 \cap B_1^2 \subseteq B_1B_2 \cap B_2B_1 \subseteq B.$$

Implies $B_1 \subseteq B$, as B is a prime bi-ideal of N. Similarly, if $B_2 \subseteq B_1$, then

$$B_2B_2 = B_2^2 = B_2^2 \cap B_2^2 \subseteq B_1B_2 \cap B_2B_1 \subseteq B.$$

Implies $B_2 \subseteq B$, as B is a prime bi-ideal of N. This shows that B is a strongly prime bi-ideal of N. Thus every prime bi-ideal of N is strongly prime. Now let B be a strongly prime bi-ideal of N. To show that B is a prime bi-ideal of N, let B_1, B_2 be any two bi-ideals of N such that $B_1B_2 \subseteq B$. Implies $B_1B_2 \cap B_2B_1 \subseteq B$. Implies either $B_1 \subseteq B$ or $B_2 \subseteq B$, as B is a strongly prime bi-ideal of N. This shows that B is a prime bi-ideal of N. Thus every strongly prime bi-ideal of N is a prime bi-ideal of N. □

Theorem 4.11. *For a near-ring N, the following assertions are equivalent:*

(1) The set of bi-ideals of a near-ring N is totally orderd by inclusion.
(2) Each bi-ideal of N is strongly irreducible .
(3) Each bi-ideal of N is irreducible.

Proof. (1)⇒(2): Let the set of bi-ideals of a near-ring N is totally orderd by set inclusion. To show that each bi-ideal of N is strongly irreducible, let B be an arbitrary bi-ideal of N and B_1 and B_2 be any two bi-ideals of N such that $B_1 \cap B_2 \subseteq B$. By hypothesis, we have either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. If $B_1 \subseteq B_2$, then $B_1 = B_1 \cap B_1 \subseteq B_1 \cap B_2 \subseteq B$. Similarly, if $B_2 \subseteq B_1$, then $B_2 = B_2 \cap B_2 \subseteq B_1 \cap B_2 \subseteq B$. So B is strongly irreducible.

(2)⇒(3): Suppose each bi-ideal of N is strongly irreducible. To show that each bi-ideal of N is irreducible, let B be an arbitrary bi-ideal of N and B_1, B_2 be any two bi-ideals of N such that $B_1 \cap B_2 = B$. This implies $B_1 \cap B_2 \subseteq B$ and $B \subseteq B_1 \cap B_2$. $B_1 \cap B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$, by hypothesis, and $B \subseteq B_1 \cap B_2$ implies $B \subseteq B_1$ and $B \subseteq B_2$. Hence either $B_1 = B$ or $B_2 = B$. Thus B is an irreducible bi-ideal of N. Hence each bi-ideal of N is irreducible.

(3)⇒(1): Let each bi-ideal of N is irreducible. To show that the set of bi-ideals of N is totally orderd by set inclusion, let B_1, B_2 be any two bi-ideals of N. Then $B_1 \cap B_2$, being the intersection of bi-ideals is also a bi-ideal of N, by Proposition 2.2.

Now $B_1 \cap B_2$ is a bi-ideal of N such that $B_1 \cap B_2 = B_1 \cap B_2$. Implies $B_1 = B_1 \cap B_2$ or $B_2 = B_1 \cap B_2$, by hypothesis. Thus $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Hence the set of bi-ideals of N is totally orderd by set inclusion. □

5. Prime, strongly prime and semiprime fuzzy bi-ideals

Definition 5.1. A fuzzy bi-ideal f of a near-ring N is called a prime fuzzy bi-ideal of N if for any fuzzy bi-ideals g, h of N, $g \circ h \leq f$ implies $g \leq f$ or $h \leq f$.

Definition 5.2. A fuzzy bi-ideal f of a near-ring N is called a strongly prime fuzzy bi-ideal of N if for any fuzzy bi-ideals g, h of N, $g \circ h \wedge h \circ g \leq f$ implies $g \leq f$ or $h \leq f$.

Definition 5.3. A fuzzy bi-ideal g of a near-ring N is said to be idempotent if $g = g \circ g = g^2$.

Definition 5.4. A fuzzy bi-ideal f of a near-ring N is said to be a semiprime fuzzy bi-ideal of N if $g \circ g = g^2 \leq f$ implies $g \leq f$ for every fuzzy bi-ideal g of N.

Proposition 5.5. *Every strongly prime fuzzy bi-ideal of a near-ring N is a prime fuzzy bi-ideal of N.*

Proof. Let f be a strongly prime fuzzy bi-ideal of a near-ring N. Now let g, h be two fuzzy bi-ideals of N such that $g \circ h \leq f$. Then $g \circ h \wedge h \circ g \leq f$. Thus by hypothesis, $g \leq f$ or $h \leq f$. Hence f is a prime fuzzy bi-ideal of N. □

Proposition 5.6. *Every prime fuzzy bi-ideal of a near-ring N is a semiprime fuzzy bi-ideal of N.*

Proof. Let f be a prime fuzzy bi-ideal of a near-ring N. Now let g be any fuzzy bi-ideal of N such that $g \circ g \leq f$. Then by hypothesis $g \leq f$. Hence f is a semiprime fuzzy bi- ideal of N. □

Remark 5.7. Every fuzzy bi-ideal of N is semiprime. But every fuzzy bi-ideal of N is not prime.

Proof. Consider the fuzzy bi-ideals f, g and h of N given by

$$\begin{aligned} f(0) &= \cdot 7 \qquad f(1) = \cdot 6 \qquad f(2) = \cdot 4 \\ g(0) &= 1 \qquad g(1) = \cdot 5 \qquad g(2) = \cdot 3 \\ h(0) &= \cdot 7 \qquad h(1) = \cdot 65 \qquad h(2) = \cdot 3 \end{aligned}$$

Then

$$g \circ h(0) = \cdot 7, \quad g \circ h(1) = \cdot 5, \quad g \circ h(2) = \cdot 3$$

Where $g \circ h \leq f$ but neither $g \leq f$ nor $h \leq f$. Hence f is not a prime fuzzy bi-ideal of N. □

Lemma 5.8. *Product of two fuzzy bi-ideals of N is a fuzzy bi-ideal of N.*

Proof. Let f and g be two fuzzy bi-ideals of N. We have to show that $f \circ g$ is a fuzzy bi-ideal of N. For this let

$$\begin{aligned} (f \circ g)(x-y) &= f(g(x-y)) \\ &\geq f\{\min(g(x),\, g(y)\}, \text{ as } g \text{ is a fuzzy bi-ideal of } N \\ &= f\{\min(g(x)\},\, f\{\min(g(y)\} \\ &= \min\{(f \circ g)(x), (f \circ g)(y)\} \\ (f \circ g)(x-y) &\geq \min\{(f \circ g)(x), (f \circ g)(y)\} \end{aligned}$$

$$\begin{aligned} (f \circ g) \circ f_N \circ (f \circ g) \cap (f \circ g) \circ f_N * f \circ g &= [\ (f \circ g) \circ f_N \circ f \cap (f \circ g) \circ f_N * f]\, g \\ &\leq [\ f \circ f_N \circ f_N \circ f \cap (f \circ g) \circ f_N * f]\, g \\ &\leq [\ f \circ f_N \circ f \cap f \circ f_N \circ f_N * f]\, g \\ &\leq f \circ g \end{aligned}$$

Hence $f \circ g$ is a fuzzy bi-ideal of N □

6. Irreducible and strongly irreducible fuzzy bi-ideals

Definition 6.1. A fuzzy bi-ideal f of a near-ring N is said to be an irreducible fuzzy bi-ideal of N if for any fuzzy bi-ideals g and h of N, $g \wedge h = f$ implies $g = f$ or $h = f$.

Definition 6.2. A fuzzy bi-ideal f of a near-ring N is said to be a strongly irreducible fuzzy bi-ideal of N if for any fuzzy bi-ideals g and h of N, $g \wedge h \leq f$ implies $g \leq f$ or $h \leq f$.

Proposition 6.3. *Every strongly irreducible semiprime fuzzy bi-ideal of a near-ring N is a strongly prime fuzzy bi-ideal of N.*

Proof. Let f be a strongly irreducible semiprime fuzzy bi-ideal of a near-ring N. Let g, h be any fuzzy bi-ideals of N such that $g \circ h \wedge h \circ g \leq f$. As $(g \wedge h) \circ (g \wedge h) = (g \wedge h)^2 \leq g \circ h$ and $(g \wedge h) \circ (g \wedge h) = (g \wedge h)^2 \leq h \circ g$, implies $(g \wedge h)^2 \leq (g \circ h) \wedge (h \circ g) \leq f$. So $g \wedge h \leq f$, as f is a semiprime fuzzy bi-ideal of N. Thus either $g \leq f$ or $h \leq f$, because f is a strongly irreducible fuzzy bi-ideal of N. Hence f is a strongly prime fuzzy bi-ideal of N. □

Theorem 6.4. *Let f be a fuzzy bi-ideal of a near-ring N with $f(a) = \alpha$, where $a \in N$ and $\alpha \in [0,1]$. Then there exists an irreducible fuzzy bi-ideal g of N such that $f \leq g$ and $g(a) = \alpha$.*

Proof. Let X=$\{h : h$ is a fuzzy bi-ideal of N, $h(a) = \alpha$ and $f \leq h\}$, then $X \neq \phi$(non-empty), as $f \in X$. The collection X is a partially orderd under inclusion. If $Y = \{h_i : h_i$ is a fuzzy bi-ideal of N, $h_i(a) = \alpha$ and $f \leq h_i$ for all $i \in I\}$ is any totally orderd subset of X, then $\underset{i \in I}{\vee} h_i$ is a fuzzy bi-ideal of N such that $f \leq \underset{i \in I}{\vee} h_i$.

Indeed, if $a, b, x \in N$, then

$$\begin{aligned}
\left(\underset{i\in I}{\vee} h_i\right)(a-b) &= \underset{i\in I}{\vee}\left(h_i\left(a-b\right)\right) \\
&\geq \underset{i\in I}{\vee}\left(h_i\left(a\right)\wedge h_i\left(b\right)\right) \text{ as each } h_i \text{ is a fuzzy bi-ideal of } N. \\
&= \left(\underset{i\in I}{\vee} h_i\left(a\right)\right)\wedge\left(\underset{i\in I}{\vee} h_i\left(b\right)\right) \\
&= \left(\underset{i\in I}{\vee} h_i\right)(a)\wedge\left(\underset{i\in I}{\vee} h_i\right)(b)
\end{aligned}$$

and

$$\begin{aligned}
\left(\underset{i\in I}{\vee} h_i\right)(ab) &= \underset{i\in I}{\vee}\left(h_i\left(ab\right)\right) \\
&\geq \underset{i\in I}{\vee}\left(h_i\left(a\right)\wedge h_i\left(b\right)\right) \text{ as each } h_i \text{ is a fuzzy bi-ideal of } N. \\
&= \left(\underset{i\in I}{\vee} h_i\left(a\right)\right)\wedge\left(\underset{i\in I}{\vee} h_i\left(b\right)\right) \\
&= \left(\underset{i\in I}{\vee} h_i\right)(a)\wedge\left(\underset{i\in I}{\vee} h_i\right)(b)
\end{aligned}$$

Now

$$\begin{aligned}
\left(\underset{i\in I}{\vee} h_i\right)(axb) &= \underset{i\in I}{\vee}\left(h_i\left(axb\right)\right) \\
&= \underset{i\in I}{\vee}\left(h_i\left(a\right)\wedge h_i\left(b\right)\right) \quad \text{as each } \mathrm{h}_i \text{ is a fuzzy bi-ideal of } N. \\
&= \left(\underset{i\in I}{\vee} h_i\left(a\right)\right)\wedge\left(\underset{i\in I}{\vee} h_i\left(b\right)\right) \\
&= \left(\underset{i\in I}{\vee} h_i\right)(a)\wedge\left(\underset{i\in I}{\vee} h_i\right)(b)
\end{aligned}$$

Hence $\underset{i\in I}{\vee} h_i$ is a fuzzy bi-ideal of N. As $f \leq h_i$ for all $i \in I$. This implies $f \leq \underset{i\in I}{\vee} h_i$. Also

$$\left(\underset{i\in I}{\vee} h_i\right)(a) = \underset{i\in I}{\vee} h_i\left(a\right) = \alpha.$$

Thus $\underset{i\in I}{\vee} h_i \in X$ and $\underset{i\in I}{\vee} h_i$ is an upper bound of Y. Hence by Zorn's lemma, there exists a fuzzy bi-ideal g of N which is maximal with the property $f \leq g$ and $g\left(a\right) = \alpha$. Now we show that g is an irreducible fuzzy bi-ideal of N. For this, suppose that for any fuzzy bi-ideals g_1, g_2 of N, we have $g_1 \wedge g_2 = g$. This implies $g \leq g_1$ and $g \leq g_2$. We claim that $g = g_1$ or $g = g_2$. On contrary, suppose that $g \neq g_1$ and $g \neq g_2$. This implies $g \lessdot g_1$ and $g \lessdot g_2$. So $g_1\left(a\right) \neq \alpha$ and $g_2\left(a\right) \neq \alpha$, as $g\left(a\right) = \alpha$. Hence $\left(g_1 \wedge g_2\right)\left(a\right) = g_1\left(a\right) \wedge g_2\left(a\right) \neq \alpha$. Which is a contradiction to the fact that $g_1\left(a\right) \wedge g_2\left(a\right) = g\left(a\right) = \alpha$. Hence either $g = g_1$ or $g = g_2$. Thus g is an irreducible fuzzy bi-ideal of N. □

Theorem 6.5. *For a near-ring N, the following assertions are equivalent:*

(1) $f \circ f = f$ for every fuzzy bi-ideal of N.
(2) $g \wedge h = g \circ h \wedge h \circ g$ for all fuzzy bi-ideals g and h of N.
(3) Each fuzzy bi-ideal of N is fuzzy semiprime.
(4) Each proper fuzzy bi-ideal of N is the intersection of irreducible semiprime fuzzy bi-ideals of N which contain it.

Proof. (1) $\Rightarrow$(2) : Let g and h be two fuzzy bi-ideals of N. Then by Lemma 2.19, $g \wedge h$ is also a fuzzy bi-ideal of N. Thus by hypothesis, we have $g \wedge h = (g \wedge h) \circ (g \wedge h) \leq g \circ h$. Similarly $g \wedge h \leq h \circ g$. Implies $g \wedge h \leq g \circ h \wedge h \circ g$. Now $g \circ h$ and $h \circ g$, being the products of two fuzzy bi-ideals of N, are fuzzy bi-ideals of N. Also $g \circ h \wedge h \circ g$ is a fuzzy bi-ideal of N, by Lemma 2.19. Thus by hypothesis, we have

$$\begin{aligned} g \circ h \wedge h \circ g &= (g \circ h \wedge h \circ g) \circ (g \circ h \wedge h \circ g) \\ &\leq (g \circ h) \circ (h \circ g) \\ &= g \circ h \circ g \qquad \text{as } h \circ h = h \text{ (by hypothesis)} \\ &\leq g \circ f_N \circ g \qquad \text{as } h \leq f_N \\ &\leq g \qquad \text{as } g \text{ is a fuzzy bi-ideal of } N. \end{aligned}$$

Similarly $g \circ h \wedge h \circ g \leq h$. Thus $g \circ h \wedge h \circ g \leq g \wedge h$. Hence $g \circ h \wedge h \circ g = g \wedge h$.

(2) $\Rightarrow$ (3) : Let g be a fuzzy bi-ideal of N such that $f^2 \leq g$ for any bi-ideal f of N. Then by hypothsis,

$$f = f \wedge f = f \circ f \wedge f \circ f = f \circ f = f^2 \leq g.$$

This implies $f \leq g$. Thus f is a semiprime fuzzy bi-ideal of N. Hence every fuzzy bi-ideal of N is semiprime.

(3) $\Rightarrow$ (4) : Let f be a proper fuzzy bi-ideal of N and $\{ f_i : i \in I\}$ be the collection of all irreducible fuzzy bi-ideals of N such that $f \leq f_i$ for all $i \in I$. This implies $f \leq \underset{i\in I}{\wedge} f_i$. Let $a \in N$ then by Theorem 6.4, there exists an irreducible fuzzy bi-ideal f_α of N such that $f \leq f_\alpha$ and $f(a) = f_\alpha(a)$. This implies $f_\alpha \in \{ f_i : i \in I\}$. Thus $\underset{i\in I}{\wedge} f_i \leq f_\alpha$. So $\underset{i\in I}{\wedge} f_i(a) \leq f_\alpha(a) = f(a)$ for all $a \in N$. This implies $\underset{i\in I}{\wedge} f_i \leq f$. Hence $\underset{i\in I}{\wedge} f_i = f$. By hypothesis, each fuzzy bi-ideal of N is semiprime. Thus each fuzzy bi-ideal of N is the intersection of all irreducible semiprime fuzzy bi-ideals of N which contain it.

(4) $\Rightarrow$ (1) : Let f be a fuzzy bi-ideal of N. Then by the definition of fuzzy bi-ideal we have, $f^2 = f \circ f \leq f$. Also $f^2 = f \circ f$, being the product of two fuzzy bi-ideals of N is a fuzzy bi-ideal of N. Then by hypothesis, $f^2 = \underset{i\in I}{\wedge} f_i$, where each f_i is an irreducible semiprime fuzzy bi-ideal of N such that $f^2 \leq f_i$ for all $i \in I$. This implies $f \leq f_i$ for all $i \in I$, because each f_i is a semiprime fuzzy bi-ideal of N. Thus $f \leq \underset{i\in I}{\wedge} f_i = f^2$. Hence $f^2 = f$. □

Proposition 6.6. *Let each fuzzy bi-ideal of a near-ring N is idempotent. Then the following assertions for a fuzzy bi-ideal of N are equivalent:*

(1) f is strongly irreducible.
(2) f is strongly prime.

Proof. (1) $\Rightarrow$ (2) : Let each fuzzy bi-ideal of a near-ring N is idempotent and f be a strongly irreducible fuzzy bi-ideal of N. Suppose that g and h be two fuzzy bi-ideals of N such that $g \circ h \wedge h \circ g \leq f$. By Theorem 6.5, $g \wedge h = g \circ h \wedge h \circ g \leq f$. Implies either $g \leq f$ or $h \leq f$, as f is strongly irreducible. So f is strongly prime fuzzy bi-ideal of N.

(2) $\Rightarrow$ (1) : Suppose f is a strongly prime fuzzy bi-ideal of N. Let g and h be any fuzzy bi-ideals of N such that $g \wedge h \leq f$. By Theorem 6.5, $g \circ h \wedge h \circ g = g \wedge h \leq f$, so $g \circ h \wedge h \circ g \leq f$. Implies either $g \leq f$ or $h \leq f$, as f is a strongly prime fuzzy bi-ideal of N. Thus f is strongly irreducible. □

Theorem 6.7. *Each fuzzy bi-ideal of a near-ring N is strongly prime if and only if each fuzzy bi-ideal of N is idempotent and the set of fuzzy bi-ideals of N is totally orderd by inclusion.*

Proof. Suppose that each fuzzy bi-ideal of a near-ring N is strongly prime, then each fuzzy bi-ideal of N is semiprime. Thus by Theorem 6.5, each fuzzy bi-ideal of N is idempotent. Now we show that the set of fuzzy bi-ideals of N is totally orderd by inclusion. For this let g and h be any two fuzzy bi-ideals of N. Then by Theorem 6.5, $g \circ h \wedge h \circ g = g \wedge h$, implies $g \circ h \wedge h \circ g \leq g \wedge h$. As each fuzzy bi-ideal of N is strongly prime, so is $g \wedge h$. Thus either $g \leq g \wedge h$ or $h \leq g \wedge h$. If $g \leq g \wedge h$, implies $g \leq h$ and if $h \leq g \wedge h$, implies $h \leq g$. So the set of fuzzy bi-ideals of N is totally orderd by inclusion.

Conversely, assume that each fuzzy bi-ideal of N is idempotent and the set of fuzzy bi-ideals of N is totally orderd by inclusion. Let f be an arbitrary fuzzy bi-ideal of N and g, h be any fuzzy bi-ideals of N such that $g \circ h \wedge h \circ g \leq f$. By Theorem 6.5 $g \wedge h = g \circ h \wedge h \circ g \leq f$ implies

$$g \wedge h \leq f \quad ... \qquad \textbf{(i)}$$

Since the set of fuzzy bi-ideals of N is totally orderd by inclusion. So either $g \leq h$ or $h \leq g$, implies either $g \wedge h = g$ or $g \wedge h = h$. Then **(i)** implies either $g \leq f$ or $h \leq f$. □

Theorem 6.8. *If the set of fuzzy bi-ideals of a near-ring N is totally orderd by inclusion, then each fuzzy bi-ideal of N is idempotent if and only if each fuzzy bi-ideal of N is prime.*

Proof. Suppose each fuzzy bi-ideal of N is idempotent. Let f be an arbitrary fuzzy bi-ideal and g, h be fuzzy bi-ideals of N such that $g \circ h \leq f$. Since the set of fuzzy bi-ideals of N is totally orderd by inclusion, so either $g \leq h$ or $h \leq g$. If $g \leq h$ then $g \circ g \leq g \circ h \leq f$, implies $g \leq f$ as f is semiprime by Theorem 6.5. If $h \leq g$ then $h \circ h \leq g \circ h \leq f$, implies $h \leq f$ as f is semiprime by Theorem 6.5. So each fuzzy bi-ideal of N is prime.

Conversely, suppose that every fuzzy bi-ideal of N is prime. Since every prime fuzzy bi-ideal of N is semiprime. So by Theorem 6.5, each fuzzy bi-ideal of N is idempotent. □

Theorem 6.9. *For a near-ring N the following assertions are equivalent:*

(1) Set of fuzzy bi-ideals of a near-ring N is totally orderd by inclusion.
(2) Each fuzzy bi-ideal of N is strongly irreducible.
(3) Each fuzzy bi-ideal of N is irreducible.

Proof. (1) $\Rightarrow$ (2) : Let f be an arbitrary fuzzy bi-ideal of N and g, h be fuzzy bi-ideals of N such that $g \wedge h \leq f$. Since the set of fuzzy bi-ideals of N is totally orderd by inclusion, so either $g \leq h$ or $h \leq g$. Thus either $g \wedge h = g$ or $g \wedge h = h$, implies either $g \leq f$ or $h \leq f$.

(2) $\Rightarrow$ (3) : Let f be an arbitrary fuzzy bi-ideal of N and g, h be fuzzy bi-ideals of N such that

$$g \wedge h = f \; ...(i)$$

Thus $g \geq f$ and $h \geq f$. (i) implies $g \wedge h \leq f$. So $g \leq f$ or $h \leq f$, as f is strongly irreducible. Hence either $g = f$ or $h = f$.

(3) $\Rightarrow$ (1) : Let g and h be two fuzzy bi-ideals of N. Then by Lemma 2.19, $g \wedge h$ is also a fuzzy bi-ideal of N. Also $g \wedge h = g \wedge h$, implies either $g = g \wedge h$ or $h = g \wedge h$. Thus $g \leq h$ or $h \leq g$. □

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