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Rough interval-valued intuitionistic fuzzy sets

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Abstract. Theories of fuzzy sets and rough sets are powerful mathematical tools for modeling various types of uncertainty. Dubois and Prade investigated the problem of combining fuzzy sets with rough sets. In this paper we define the notion of rough interval-valued intuitionistic fuzzy sets. The lower and upper approximations of interval-valued intuitionistic fuzzy sets with respect to Pawlak's approximation space are first defined. Properties of interval-valued intuitionistic fuzzy approximation operators are examined.

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Keywords: Fuzzy set, rough set, Rough fuzzy set, Intuitionistic fuzzy set, Rough intuitionistic fuzzy set, Interval-valued intuitionistic fuzzy set, Rough interval-valued intuitionistic fuzzy set.

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1. Introduction

Hybrid models combining fuzzy sets with rough sets have arisen in various guises in different settings. Based on an equivalence relation Dubois and Prade [4] introduced the lower and upper approximations of fuzzy sets in a pawlak's approximation space to obtain an extended notion called rough fuzzy sets in 1990. Interval-valued intuitionistic fuzzy set was introduced by Atanassov and Gargov [2] in 1989. It is characterized by an interval-valued membership degree and an interval-valued non-membership degree. The theory of rough sets, proposed by Pawlak [7] in 1982 is a new mathematical tool for the data reasoning.

Based on a Pawlak's approximation space, the approximation of interval-valued intuitionistic fuzzy set is proposed by us to obtain a hybrid model called rough interval-valued intuitionistic fuzzy sets.

While trying to combine rough set theory and interval-valued intuitionistic fuzzy set theory we find our self at a complicated cross rounds with an abundance of possible ways to proceed. The aim of this paper is to provide the reader with a road map.

In section 3, we introduce the notion of Rough interval-valued intuitionistic fuzzy sets (RIVIFS) along with their properties. In theorem 3.10, it is seen that (3), (4), (5) and (6) are not true in general. Some examples are given in support of the answer.

2. Preliminaries

In this section we recall some basic notions relevant to fuzzy sets, intuitionistic fuzzy sets, interval-valued fuzzy sets, interval-valued intuitionistic fuzzy sets and rough sets.

Definition 2.1 ([9]). Let X be a non empty set. Then a fuzzy set (FS in short) A is a set having the form $A = \{(x, \mu_A(x)) : x \in X\}$, where the function $\mu_A : X \to [0, 1]$ is called the membership function and $\mu_A(x)$ is called the degree of membership of each element $x \in X$.

Definition 2.2 ([7]). Let U be a universe of discourse and R be an equivalence relation on U. The pair (U, R) is called Pawlak approximation space. R will generate a partition $U/R = \{[x]_R : x \in U\}$ on U, where $[x]_R$ is the equivalence class with respect to R containing x. For each $X \subseteq U$, the lower approximation $\underline{R}(X)$ and upper approximation $\overline{R}(X)$ of X with respect to (U, R) are defined as $\underline{R}(X) = \{x \in U : [x]_R \subseteq X\}$ and $\overline{R}(X) = \{x \in U : [x]_R \cap X \neq \phi\}$ is called definable in (U, R) if $\underline{R}(A) = \overline{R}(A)$, otherwise X is called a rough set.

Definition 2.3 ([4]). Let (U, R) be a Pawlak approximation space and $A = \{(x, \mu_A(x) : x \in X\}$ be a fuzzy set. The lower and upper rough approximations of A in (U, R) are denoted by $\underline{R}(A)$ and $\overline{R}(A)$ respectively which are fuzzy subsets of U defined by $\underline{R}(A) = \{(x, \wedge\{\mu_A(y) : y \in [x]_R\}) : x \in U\}$ and $\overline{R}(A) = \{(x, \vee\{\mu_A(y) : y \in [x]_R\}) : x \in U\}$ $\forall x \in U$. The operators $\underline{R}$ and $\overline{R}$ are called the lower and upper rough approximation operators on fuzzy sets. A is called definable in (U, R) if $\underline{R}(A) = \overline{R}(A)$, otherwise A is called a rough fuzzy set.

Definition 2.4 ([5]). An interval-valued fuzzy set A over X is given by a function $\mu_A(x)$ where $\mu_A : X \to Int[0, 1]$, the set of all sub-intervals of the unit interval i.e. for every $x \in X$, $\mu_A(x)$ is an interval within [0,1].

Definition 2.5 ([1]). Let X be a non empty set. An intuitionistic fuzzy set (IFS in short) A in X is an object having the form $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$, where the functions $\mu_A : X \to [0, 1]$ and $\gamma_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\gamma_A(x)$) of each element $x \in X$ to the set A respectively and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$.

Definition 2.6 ([8]). Let (U, R) be a Pawlak approximation space and $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$ be an intuitionistic fuzzy set. The lower and upper rough approximations of A in (U, R) are denoted by $\underline{R}(A)$ and $\overline{R}(A)$ respectively which are intuitionistic fuzzy subsets of U defined by $\underline{R}(A) = \{(x, \wedge\{\mu_A(y) : y \in [x]_R\}, \vee\{\gamma_A(y) : y \in [x]_R\}) : x \in U\}$ and $\overline{R}(A) = \{(x, \vee\{\mu_A(y) : y \in [x]_R\}, \wedge\{\gamma_A(y) : y \in [x]_R\}) : x \in U\}$ $\forall x \in U$. The operators $\underline{R}$ and $\overline{R}$ are called the lower and upper rough approximation operators on intuitionistic fuzzy sets. A is called definable in (U, R) if $\underline{R}(A) = \overline{R}(A)$, otherwise A is called a rough intuitionistic fuzzy set.

Remark 2.7 ([6])**.** If A is an intuitionistic fuzzy set such that $\mu_A(x) + \gamma_A(X) = 1$ for all $x \in U$ then it is easy to observe that $(\underline{R}(A), \overline{R}(A))$ is a rough fuzzy set.

Definition 2.8 ([2])**.** An interval-valued intuitionistic fuzzy set ($IVIFS$ in short) A over a universe set X is defined as the object of the form $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$, where $\mu_A : X \to Int[0,1]$ and $\gamma_A : X \to Int[0,1]$ (where $Int[0,1]$ is the set of all closed intervals of $[0,1]$) are functions such that the condition: $\forall x \in X$ $0 \le sup\mu_A(x) + sup\gamma_A(x) \le 1$ is satisfied.

Definition 2.9 ([2, 3])**.** Let $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$ and $B = \{(x, \mu_B(x), \gamma_B(x)) : x \in X\}$be two interval-valued intuitionistic fuzzy sets then

- $A \subset B$ iff $inf\mu_A(x) \le inf\mu_B(x)$, $sup\mu_A(x) \le sup\mu_B(x)$ and $inf\gamma_A(x) \ge inf\gamma_B(x)$, $sup\gamma_A(x) \ge sup\gamma_B(x)$.
- The union of A and B is denoted by $A \cup B$ where
$$A \cup B = \{\langle x, [\vee\{inf\mu_A(x), inf\mu_B(x)\}, \vee\{sup\mu_A(x), sup\mu_B(x)\}], [\wedge\{inf\gamma_A(x), inf\gamma_B(x)\}, \wedge\{sup\gamma_A(x), sup\gamma_B(x)\}]\rangle : x \in X\}.$$
- The intersection of A and B is denoted by $A \cap B$ where
$$A \cap B = \{\langle x, [\wedge\{inf\mu_A(x), inf\mu_B(x)\}, \wedge\{sup\mu_A(x), sup\mu_B(x)\}], [\vee\{inf\gamma_A(x), inf\gamma_B(x)\}, \vee\{sup\gamma_A(x), sup\gamma_B(x)\}]\rangle : x \in X\}.$$
- The complement of A is denoted by A^c where $A^c = \{(x, \gamma_A(x), \mu_A(x)) : x \in X\}$.
- $\Box A = \{(x, \mu_A(x), [inf\gamma_A(x), 1 - sup\mu_A(x)]) : x \in X\}$.
- $\Diamond A = \{(x, [inf\mu_A(x), 1 - sup\gamma_A(x)], \gamma_A(x)) : x \in X\}$.
- $A + B = \{(x, [inf\mu_A(x) + inf\mu_B(x) - inf\mu_A(x) \cdot inf\mu_B(x), sup\mu_A(x) + sup\mu_B(x) - sup\mu_A(x) \cdot sup\mu_B(x)], [inf\gamma_A(x) \cdot inf\gamma_B(x), sup\gamma_A(x) \cdot sup\gamma_B(x)]) : x \in X\}$.
- $A \cdot B = \{(x, [inf\mu_A(x) \cdot inf\mu_B(x), sup\mu_A(x) \cdot sup\mu_B(x)], [inf\gamma_A(x) + inf\gamma_B(x) - inf\gamma_A(x) \cdot inf\gamma_B(x), sup\gamma_A(x) + sup\gamma_B(x) - sup\gamma_A(x) \cdot sup\gamma_B(x)]) : x \in X\}$.
- $A@B = \{(x, [\frac{inf\mu_A(x)+inf\mu_B(x)}{2}, \frac{sup\mu_A(x)+sup\mu_B(x)}{2}], [\frac{inf\gamma_A(x)+inf\gamma_B(x)}{2}, \frac{sup\gamma_A(x)+sup\gamma_B(x)}{2}]) : x \in X\}$.
- $A\$B = \{(x, [\sqrt{inf\mu_A(x) \cdot inf\mu_B(x)}, \sqrt{sup\mu_A(x) \cdot sup\mu_B(x)}], [\sqrt{inf\gamma_A(x) \cdot inf\gamma_B(x)}, \sqrt{sup\gamma_A(x) \cdot sup\gamma_B(x)}]) : x \in X\}$.
- $A\#B = \{(x, [\frac{2 \cdot inf\mu_A(x) \cdot inf\mu_B(x)}{inf\mu_A(x)+inf\mu_B(x)}, \frac{2 \cdot sup\mu_A(x) \cdot sup\mu_B(x)}{sup\mu_A(x)+sup\mu_B(x)}], [\frac{2 \cdot inf\gamma_A(x) \cdot inf\gamma_B(x)}{inf\gamma_A(x)+inf\gamma_B(x)}, \frac{2 \cdot sup\gamma_A(x) \cdot sup\gamma_B(x)}{sup\gamma_A(x)+sup\gamma_B(x)}]) : x \in X\}$.

The unit and null interval-valued intuitionistic fuzzy sets are defined by $I = \{(x, [1,1], [0,0]) : x \in X\}$, $\phi = \{(x, [0,0], [1,1]) : x \in X\}$.

3. Rough interval-valued intuitionistic fuzzy set

In this section we introduce the lower and upper rough approximation of an interval-valued intuitionistic fuzzy sets and study their basic properties.

Definition 3.1. Let (U, R) be a Pawlak approximation space and $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in U\}$ be an interval-valued intuitionistic fuzzy set. The lower and upper rough approximations of A in (U, R) are denoted by $\underline{R}(A)$ and $\overline{R}(A)$ respectively which are interval-valued intuitionistic fuzzy subsets of U defined by

$\underline{R}(A) = \{(x, [\wedge\{inf\mu_A(y) : y \in [x]_R\}, \wedge\{sup\mu_A(y) : y \in [x]_R\}], [\vee\{inf\gamma_A(y) : y \in [x]_R\}, \vee\{sup\gamma_A(y) : y \in [x]_R\}]) : x \in U\}$ and $\overline{R}(A) = \{(x, [\vee\{inf\mu_A(y) : y \in [x]_R\}, \vee\{sup\mu_A(y) : y \in [x]_R\}], [\wedge\{inf\gamma_A(y) : y \in [x]_R\}, \wedge\{sup\gamma_A(y) : y \in [x]_R\}]) : x \in U\}$

Now it is easy to show that

$[\wedge\{inf\mu_A(y) : y \in [x]_R\}, \wedge\{sup\mu_A(y) : y \in [x]_R\}] \subseteq [0, 1]$ and $[\vee\{inf\gamma_A(y) : y \in [x]_R\}, \vee\{sup\gamma_A(y) : y \in [x]_R\}] \subseteq [0, 1]$

Also,

$sup\mu_A(y) + sup\gamma_A(y) \leq 1$

$\Rightarrow sup\mu_A(y) \leq 1 - sup\gamma_A(y)$

$\Rightarrow \wedge\{sup\mu_A(y) : y \in [x]_R\} \leq \wedge\{1 - sup\gamma_A(y) : y \in [x]_R\}$

$\Rightarrow \wedge\{sup\mu_A(y) : y \in [x]_R\} \leq 1 - \vee\{sup\gamma_A(y) : y \in [x]_R\}$

$\Rightarrow \wedge\{sup\mu_A(y) : y \in [x]_R\} + \vee\{sup\gamma_A(y) : y \in [x]_R\} \leq 1$

$\therefore \underline{R}(A)$ is an interval-valued intuitionistic fuzzy set. Similarly we can show that $\overline{R}(A)$ is an interval-valued intuitionistic fuzzy set.

If $\underline{R}(A) = \overline{R}(A)$ then A is called definable, otherwise A is called a rough interval-valued intuitionistic fuzzy set ($RIVIFS$ in short). The rough interval-valued intuitionistic fuzzy set $R(A)$ is given by the pair $R(A) = (\underline{R}(A), \overline{R}(A))$ or simply $A = (\underline{A}, \overline{A})$.

Example 3.2. Let $U = \{1, 2, 3, 4, 5\}$ and R be an equivalence relation defined by ${}_xR_y$ iff $(x + y)$ is divisible by 2.

$\therefore R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (1, 3), (3, 1), (1, 5), (5, 1), (3, 5), (5, 3), (2, 4), (4, 2), (2, 6), (6, 2), (4, 6), (6, 4)\}$

$[1]_R = \{1, 3, 5\} = [3]_R = [5]_R$, $[2]_R = \{2, 4, 6\} = [4]_R = [6]_R$.

Let $A = \{(1, [.2, .3], [.3, .4]), (2, [.3, .7], [.2, .3]), (3, [.1, .4], [.2, .5]), (4, [.5, .6], [.1, .3]), (5, [.4, .5], [.2, .4])\}$ then

$\underline{R}(A) = \{(1, [.1, .3], [.3, .5]), (2, [.3, .6], [.2, .3]), (3, [.1, .3], [.3, .5]), (4, [.3, .6], [.2, .3]), (5, [.1, .3], [.3, .5])\}$

$\overline{R}(A) = \{(1, [.4, .5], [.2, .4]), (2, [.5, .7], [.1, .3]), (3, [.4, .5], [.2, .4]), (4, [.5, .7], [.1, .3]), (5, [.4, .5], [.2, .4])\}$

Here, $\underline{R}(A) \neq \overline{R}(A)$, therefore A is a rough interval-valued intuitionistic fuzzy set.

Definition 3.3. Let $R(A) = (\underline{R}(A), \overline{R}(A))$ and $R(B) = (\underline{R}(B), \overline{R}(B))$ be two rough interval-valued intuitionistic fuzzy sets of the interval-valued intuitionistic fuzzy sets A and B respectively. Let us denote $\underline{R}(A)$, $\overline{R}(A)$, $\underline{R}(B)$ and $\overline{R}(B)$ by

$\underline{R}(A) = \{(x, \mu_{\underline{R}(A)}(x), \gamma_{\underline{R}(A)}(x)) : x \in U\}$, $\overline{R}(A) = \{(x, \mu_{\overline{R}(A)}(x), \gamma_{\overline{R}(A)}(x)) : x \in U\}$, $\underline{R}(B) = \{(x, \mu_{\underline{R}(B)}(x), \gamma_{\underline{R}(B)}(x)) : x \in U\}$ and $\overline{R}(B) = \{(x, \mu_{\overline{R}(B)}(x), \gamma_{\overline{R}(B)}(x)) : x \in U\}$.

Then the following relations, operations and operators are valid

1. $\sim R(A) = ((\overline{R}(A))^c, (\underline{R}(A))^c)$

where $(\overline{R}(A))^c$ and $(\underline{R}(A))^c$ are the complements of the interval-valued intuitionistic fuzzy sets $\overline{R}(A)$ and $\underline{R}(A)$ respectively. $\sim R(A)$ is called the rough complement of $R(A)$.

2. $\Box R(A) = (\Box\underline{R}(A), \Box\overline{R}(A))$.
3. $\Diamond R(A) = (\Diamond\underline{R}(A), \Diamond\overline{R}(A))$.

4. $R(A) \subseteq R(B)$ iff $\underline{R}(A) \subseteq \underline{R}(B)$ and $\overline{R}(A) \subseteq \overline{R}(B)$.
5. $R(A) = R(B)$ iff $R(A) \subseteq R(B)$ and $R(B) \subseteq R(A)$.
6. $R(A) \cup R(B) = (\underline{R}(A) \cup \underline{R}(B), \overline{R}(A) \cup \overline{R}(B))$.
7. $R(A) \cap R(B) = (\underline{R}(A) \cap \underline{R}(B), \overline{R}(A) \cap \overline{R}(B))$.
8. $R(A) + R(B) = (\underline{R}(A) + \underline{R}(B), \overline{R}(A) + \overline{R}(B))$.
9. $R(A) \cdot R(B) = (\underline{R}(A) \cdot \underline{R}(B), \overline{R}(A) \cdot \overline{R}(B))$.
10. $R(A)@R(B) = (\underline{R}(A)@\underline{R}(B), \overline{R}(A)@\overline{R}(B))$.
11. $R(A)\$R(B) = (\underline{R}(A)\S\underline{R}(B), \overline{R}(A)\S\overline{R}(B))$.
12. $R(A)\#R(B) = (\underline{R}(A)\#\underline{R}(B), \overline{R}(A)\#\overline{R}(B))$.

Example 3.4. Let us consider example 3.2. Let us consider another $RIVIFS$ $B = \{(1, [.4, .5], [.1, .3]), (2, [.4, .8], [.1, .2]), (3, [.1, .5], [.2, .3]), (4, [.5, .7], [.1, .2]), (5, [.4, .7], [.1, .3])\}$ then
$\underline{R}(B) = \{(1, [.1, .5], [.2, .3]), (2, [.4, .7], [.1, .2]), (3, [.1, .5], [.2, .3]), (4, [.4, .7], [.1, .2]), (5, [.1, .5], [.2, .3])\}$
$\overline{R}(B) = \{(1, [.4, .7], [.1, .3]), (2, [.5, .8], [.1, .2]), (3, [.4, .7], [.1, .3]), (4, [.5, .8], [.1, .2]), (5, [.4, .7], [.1, .3])\}$.

Let $\sim R(A) = ((\overline{R}(A))^c, (\underline{R}(A))^c)$ then
$(\overline{R}(A))^c = \{(1, [.2, .4], [.4, .5]), (2, [.1, .3], [.5, .7]), (3, [.2, .4], [.4, .5]), (4, [.1, .3], [.5, .7]), (5, [.2, .4], [.4, .5])\}$
$(\underline{R}(A))^c = \{(1, [.3, .5], [.1, .3]), (2, [.2, .3], [.3, .6]), (3, [.3, .5], [.1, .3]), (4, [.2, .3], [.3, .6]), (5, [.3, .5], [.1, .3])\}$.

Let $\Box R(A) = (\Box\underline{R}(A), \Box\overline{R}(A))$ then
$\Box\underline{R}(A) = \{(1, [.1, .3], [.3, .7]), (2, [.3, .6], [.2, .4]), (3, [.1, .3], [.3, .7]), (4, [.3, .6], [.2, .4]), (5, [.1, .3], [.3, .7])\}$
$\Box\overline{R}(A) = \{(1, [.4, .5], [.2, .5]), (2, [.5, .7], [.1, .3]), (3, [.4, .5], [.2, .5]), (4, [.5, .7], [.1, .3]), (5, [.4, .5], [.2, .5])\}$.

Let $\Diamond R(A) = (\Diamond\underline{R}(A), \Diamond\overline{R}(A))$ then
$\Diamond\underline{R}(A) = \{(1, [.1, .5], [.3, .5]), (2, [.3, .7], [.2, .3]), (3, [.1, .5], [.3, .5]), (4, [.3, .7], [.2, .3]), (5, [.1, .5], [.3, .5])\}$
$\Diamond\overline{R}(A) = \{(1, [.4, .6], [.2, .4]), (2, [.5, .7], [.1, .3]), (3, [.4, .6], [.2, .4]), (4, [.5, .7], [.1, .3]), (5, [.4, .6], [.2, .4])\}$.

Let $R(A) \subseteq R(B)$, now we have to show $\underline{R}(A) \subseteq \underline{R}(B)$ and $\overline{R}(A) \subseteq \overline{R}(B)$ which holds in this example.

Let $R(A) \cup R(B) = (\underline{R}(A) \cup \underline{R}(B), \overline{R}(A) \cup \overline{R}(B))$ then
$\underline{R}(A) \cup \underline{R}(B) = \{(1, [.1, .5], [.2, .3]), (2, [.4, .7], [.1, .2]), (3, [.1, .5], [.2, .3]), (4, [.4, .7], [.1, .2]), (5, [.1, .5], [.2, .3])\}$
$\overline{R}(A) \cup \overline{R}(B) = \{(1, [.4, .7], [.1, .3]), (2, [.5, .8], [.1, .2]), (3, [.4, .7], [.1, .3]), (4, [.5, .8], [.1, .2]), (5, [.4, .7], [.1, .3])\}$.

Let $R(A) \cap R(B) = (\underline{R}(A) \cap \underline{R}(B), \overline{R}(A) \cap \overline{R}(B))$ then
$\underline{R}(A) \cap \underline{R}(B) = \{(1, [.1, .3], [.3, .5]), (2, [.3, .6], [.2, .3]), (3, [.1, .3], [.3, .5]), (4, [.3, .6], [.2, .3]), (5, [.1, .3], [.3, .5])\}$
$\overline{R}(A) \cap \overline{R}(B) = \{(1, [.4, .5], [.2, .4]), (2, [.5, .7], [.1, .3]), (3, [.4, .5], [.2, .4]), (4, [.5, .7], [.1, .3]), (5, [.4, .5], [.2, .4])\}$.

Let $R(A) + R(B) = (\underline{R}(A) + \underline{R}(B), \overline{R}(A) + \overline{R}(B))$ then
$\underline{R}(A) + \underline{R}(B) = \{(1, [.19, .65], [.06, .15]), (2, [.58, .88], [.02, .06]), (3, [.19, .65], [.06, .15]),$

$(4,[.58,.88],[.02,.06]),(5,[.19,.65],[.06,.15])\}$
$\overline{R}(A)+\overline{R}(B)=\{(1,[.64,.85],[.02,.12]),(2,[.75,.94],[.01,.06]),(3,[.64,.85],[.02,.12]),(4,[.75,.94],[.01,.06]),(5,[.64,.85],[.02,.12])\}$.

Let $R(A)\cdot R(B)=(\underline{R}(A)\cdot\underline{R}(B),\overline{R}(A)\cdot\overline{R}(B))$ then
$\underline{R}(A)\cdot\underline{R}(B)=\{(1,[.01,.15],[.44,.65]),(2,[.12,.42],[.28,.44]),(3,[.01,.15],[.44,.65]),(4,[.12,.42],[.28,.44]),(5,[.01,.15],[.44,.65])\}$
$\overline{R}(A)\cdot\overline{R}(B)=\{(1,[.16,.35],[.28,.58]),(2,[.25,.56],[.19,.44]),(3,[.16,.35],[.28,.58]),(4,[.25,.56],[.19,.44]),(5,[.16,.35],[.28,.58])\}$.

Let $R(A)@R(B)=(\underline{R}(A)@\underline{R}(B),\overline{R}(A)@\overline{R}(B))$ then
$\underline{R}(A)@\underline{R}(B)=\{(1,[.1,.4],[.25,.4]),(2,[.35,.65],[.15,.25]),(3,[.1,.4],[.25,.4]),(4,[.35,.65],[.15,.25]),(5,[.1,.4],[.25,.4])\}$
$\overline{R}(A)@\overline{R}(B)=\{(1,[.4,.6],[.15,.35]),(2,[.5,.75],[.1,.25]),(3,[.4,.6],[.15,.35]),(4,[.5,.75],[.1,.25]),(5,[.4,.6],[.15,.35])\}$.

Let $R(A)\$R(B)=(\underline{R}(A)\$\underline{R}(B),\overline{R}(A)\$\overline{R}(B))$ then
$\underline{R}(A)\$\underline{R}(B)=\{(1,[.1,.39],[.24,.39]),(2,[.35,.65],[.14,.24]),(3,[.1,.39],[.24,.39]),(4,[.35,.65],[.14,.24]),(5,[.1,.39],[.24,.39])\}$
$\overline{R}(A)\$\overline{R}(B)=\{(1,[.4,.59],[.14,.35]),(2,[.5,.75],[.1,.24]),(3,[.4,.59],[.14,.35]),(4,[.5,.75],[.1,.24]),(5,[.4,.59],[.14,.35])\}$.

Let $R(A)\#R(B)=(\underline{R}(A)\#\underline{R}(B),\overline{R}(A)\#\overline{R}(B))$ then
$\underline{R}(A)\#\underline{R}(B)=\{(1,[.1,.38],[.24,.38]),(2,[.34,.65],[.13,.24]),(3,[.1,.38],[.24,.38]),(4,[.34,.65],[.13,.24]),(5,[.1,.38],[.24,.38])\}$
$\overline{R}(A)\#\overline{R}(B)=\{(1,[.4,.58],[.13,.34]),(2,[.5,.75],[.1,.24]),(3,[.4,.58],[.13,.34]),(4,[.5,.75],[.1,.24]),(5,[.4,.58],[.13,.34])\}$.

Theorem 3.5. *Let A, B and C are three interval-valued intuitionistic fuzzy sets in (X,U) then*

1. $\underline{R}(A)\subseteq A\subseteq\overline{R}(A)$.
2. $\overline{R}(A\cup B)=\overline{R}(A)\cup\overline{R}(B)$.
3. $\underline{R}(A\cap B)=\underline{R}(A)\cap\underline{R}(B)$.
4. $\overline{R}(A\cap B)\subseteq\overline{R}(A)\cap\overline{R}(B)$.
5. $\underline{R}(A\cup B)\supseteq\underline{R}(A)\cup\overline{R}(B)$.
6. $\overline{R}(A^c)=(\underline{R}(A))^c$.
7. $\underline{R}(A^c)=(\overline{R}(A))^c$.
8. $\overline{R}\overline{R}(A)=\underline{R}\overline{R}(A)=\overline{R}(A)$.
9. $\underline{R}\underline{R}(A)=\overline{R}\underline{R}(A)=\underline{R}(A)$.
10. $\underline{R}(I)=I=\overline{R}(I)\quad \underline{R}(\phi)=I=\overline{R}(\phi)$.

Proof. Straight forward. 3.5. □

From 6 and 7 of theorem 3.5, we can say rough complement of a rough interval-valued intuitionistic fuzzy set is the rough interval-valued intuitionistic fuzzy set of its complement.

$$\text{i.e. } \sim R(A)=R(A^c).$$

Theorem 3.6. *If $R(A)=(\underline{R}(A),\overline{R}(A))$, $R(B)=(\underline{R}(B),\overline{R}(B))$ and $R(C)=(\underline{R}(C),\overline{R}(C))$ are three rough interval-valued intuitionistic fuzzy sets of interval-valued intuitionistic fuzzy sets A,B and C in (X,U) respectively then*

1. $\sim(\sim R(A))=R(A)$.

2. $R(A) \cup R(B) = R(B) \cup R(A)$.
3. $R(A) \cap R(B) = R(B) \cap R(A)$.
4. $(R(A) \cup R(B)) \cup R(C) = R(A) \cup (R(B) \cup R(C))$.
5. $(R(A) \cap R(B)) \cap R(C) = R(A) \cap (R(B) \cap R(C))$.
6. $(R(A) \cup R(B)) \cap R(C) = (R(A) \cap R(C)) \cup (R(B) \cap R(C))$.
7. $(R(A) \cap R(B)) \cup R(C) = (R(A) \cup R(C)) \cap (R(B) \cup R(C))$.
8. $\sim (R(A) \cup R(B)) = (\sim R(A)) \cap (\sim R(B))$.
9. $\sim (R(A) \cap R(B)) = (\sim R(A)) \cup (\sim R(B))$.

Proof. 1. $\sim (\sim R(A)) =\sim ((\overline{R}(A))^c, (\underline{R}(A))^c)$
$=(((\underline{R}(A))^c)^c, ((\overline{R}(A))^c)^c)$
$=(\underline{R}(A), \overline{R}(A))$
$=R(A)$

2. $R(A) \cup R(B) = (\underline{R}(A) \cup \underline{R}(B), \overline{R}(A) \cup \overline{R}(B))$
$=(\underline{R}(B) \cup \underline{R}(A), \overline{R}(B) \cup \overline{R}(A))$
$=R(B) \cup R(A)$

3. Similar to 2.

4. $(R(A) \cup R(B)) \cup R(C) = (\underline{R}(A) \cup \underline{R}(B), \overline{R}(A) \cup \overline{R}(B)) \cup (\underline{R}(C), \overline{R}(C))$
$=((\underline{R}(A) \cup \underline{R}(B)) \cup \underline{R}(C), (\overline{R}(A) \cup \overline{R}(B)) \cup \overline{R}(C))$
$=(\underline{R}(A) \cup (\underline{R}(B) \cup \underline{R}(C)), \overline{R}(A) \cup (\overline{R}(B) \cup \overline{R}(C)))$
$=(\underline{R}(A) \cup \overline{R}(A)) \cup (\underline{R}(B) \cup \underline{R}(C), \overline{R}(B) \cup \overline{R}(C))$
$=R(A) \cup (R(B) \cup R(C))$

5. Similar to 4.

6. $(R(A) \cup R(B)) \cap R(C) = (\underline{R}(A) \cup \underline{R}(B), \overline{R}(A) \cup \overline{R}(B)) \cap (\underline{R}(C), \overline{R}(C))$
$=((\underline{R}(A) \cup \underline{R}(B)) \cap \underline{R}(C), (\overline{R}(A) \cup \overline{R}(B)) \cap \overline{R}(C))$
$=((\underline{R}(A) \cap \underline{R}(C)) \cup (\underline{R}(B) \cap \underline{R}(C))), ((\overline{R}(A) \cap \overline{R}(C)) \cup (\overline{R}(B) \cap \overline{R}(C)))$
$=(\underline{R}(A)\cap\underline{R}(C), \overline{R}(A)\cap\overline{R}(C))\cup(\underline{R}(B)\cap\underline{R}(C), \overline{R}(B)\cap\overline{R}(C))$
$=(R(A) \cap (R(C))) \cup (R(B) \cap (R(C)))$

7. Similar to 6.

8. $\sim (R(A) \cup R(B)) =\sim (\underline{R}(A) \cup \underline{R}(B), \overline{R}(A) \cup \overline{R}(B))$
$=((\overline{R}(A) \cup \overline{R}(B))^c, (\underline{R}(A) \cup \underline{R}(B))^c)$
$=((\overline{R}(A))^c \cap (\overline{R}(B))^c, (\underline{R}(A))^c \cap (\underline{R}(B))^c)$
$=((\overline{R}(A))^c, (\underline{R}(A))^c) \cap ((\overline{R}(B))^c, (\underline{R}(B))^c)$
$=(\sim R(A)) \cap (\sim R(B))$

9. Similar to 8. 3.6. □

Theorem 3.7. *If A and B be two interval valued-intuitionistic fuzzy sets suct that $A \subseteq B$ then $R(A) \subseteq R(B)$.*

Proof. Let $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$ and $B = \{(x, \mu_B(x), \gamma_B(x)) : x \in X\}$. We have $inf\mu_A(x) \leq inf\mu_B(x)$, $sup\mu_A(x) \leq sup\mu_B(x)$, $inf\gamma_A(x) \geq inf\gamma_B(x)$ and $sup\gamma_A(x) \geq sup\gamma_B(x)$
Let $R(A) = (\underline{R}(A), \overline{R}(A))$, $R(B) = (\underline{R}(B), \overline{R}(B))$
where $\underline{R}(A) = \{(x, [\wedge\{inf\mu_A(y) : y \in [x]_R\}, \wedge\{sup\mu_A(y) : y \in [x]_R\}], [\vee\{inf\gamma_A(y) : y \in [x]_R\}, \vee\{sup\gamma_A(y) : y \in [x]_R\}]) : x \in U\}$
$\underline{R}(B) = \{(x, [\wedge\{inf\mu_B(y) : y \in [x]_R\}, \wedge\{sup\mu_B(y) : y \in [x]_R\}], [\vee\{inf\gamma_B(y) : y \in$

$[x]_R\}, \vee\{sup\gamma_B(y) : y \in [x]_R\}]) : x \in U\}$
Now $inf\mu_A(x) \leq inf\mu_B(x)$
$\Rightarrow \wedge\{inf\mu_A(y) : y \in [x]_R\} \leq \wedge\{inf\mu_B(y) : y \in [x]_R\}$
and $sup\mu_A(x) \leq sup\mu_B(x)$
$\Rightarrow \wedge\{sup\mu_A(y) : y \in [x]_R\} \leq \wedge\{sup\mu_B(y) : y \in [x]_R\}$
Similarly, $\vee\{inf\gamma_A(y) : y \in [x]_R\} \geq \vee\{inf\gamma_B(y) : y \in [x]_R\}$
and $\vee\{sup\gamma_A(y) : y \in [x]_R\} \leq \vee\{sup\gamma_B(y) : y \in [x]_R\}$
$\therefore \underline{R}(A) \subseteq \underline{R}(B)$
Similarly it can be shown that $\overline{R}(A) \subseteq \overline{R}(A)$
$\therefore R(A) \subseteq R(B)$ 3.7. □

Theorem 3.8. *For every rough interval-valued intuitionistic fuzzy set* $R(A) = (\underline{R}(A), \overline{R}(A))$

1. $\sim(\Box(\sim R(A))) = \Diamond R(A)$
2. $\sim(\Diamond(\sim R(A))) = \Box R(A)$
3. $\Box\Box R(A) = \Box R(A)$
4. $\Diamond\Diamond R(A) = \Diamond R(A)$
5. $\Box\Diamond R(A) = \Diamond R(A)$
6. $\Diamond\Box R(A) = \Box R(A)$

Proof. 1. $\sim(\Box(\sim R(A))) =\sim(\Box((\overline{R}(A))^c, (\underline{R}(A))^c))$
$=\sim(\Box(\overline{R}(A))^c, \Box(\underline{R}(A))^c)$
$=((\Box(\underline{R}(A))^c)^c, (\Box(\overline{R}(A))^c)^c)$
$=(\Diamond(\underline{R}(A)), \Diamond(\overline{R}(A)))$
$=\Diamond R(A)$

2. Similar to 1.
3. $\Box\Box R(A) = \Box(\Box\underline{R}(A), \Box\overline{R}(A))$
$=(\Box\Box\underline{R}(A), \Box\Box\overline{R}(A))$
$=(\Box\underline{R}(A), \Box\overline{R}(A))$
$=\Box R(A)$
4. Similar to 3.
5. $\Box\Diamond R(A) = \Box(\Diamond\underline{R}(A), \Diamond\overline{R}(A))$
$=(\Box\Diamond\underline{R}(A), \Box\Diamond\overline{R}(A))$
$=(\Diamond\underline{R}(A), \Diamond\overline{R}(A))$
$=\Diamond R(A)$
6. Similar to 5. 3.8. □

Theorem 3.9. *For any two rough interval-valued intuitionistic fuzzy sets* $R(A) = (\underline{R}(A), \overline{R}(A))$ *and* $R(B) = (\underline{R}(B), \overline{R}(B))$

1. $\Box(R(A) \cup R(B)) = \Box R(A) \cup \Box R(B)$
2. $\Box(R(A) \cap R(B)) = \Box R(A) \cap \Box R(B)$
3. $\Diamond(R(A) \cup R(B)) = \Diamond R(A) \cup \Diamond R(B)$
4. $\Diamond(R(A) \cap R(B)) = \Diamond R(A) \cap \Diamond R(B)$.

Proof. 1. $\Box(R(A) \cup R(B)) = \Box(\underline{R}(A) \cup \underline{R}(B), \overline{R}(A) \cup \overline{R}(B))$
$=(\Box(\underline{R}(A) \cup \underline{R}(B)), \Box(\overline{R}(A) \cup \overline{R}(B)))$
$=(\Box\underline{R}(A) \cup \Box\underline{R}(B), \Box\overline{R}(A) \cup \Box\overline{R}(B))$
$=(\Box\underline{R}(A), \Box\overline{R}(A)) \cup (\Box\underline{R}(B), \Box\overline{R}(B))$

$=\square R(A)\cup\square R(B)$

2-4. Similar to 1. 3.9. $\square$

Theorem 3.10. *For any two rough interval-valued intuitionistic fuzzy sets* $R(A)=(\underline{R}(A),\overline{R}(A))$ *and* $R(B)=(\underline{R}(B),\overline{R}(B))$

1. $\square(R(A)@R(B))=\square R(A)@\square R(B)$
2. $\lozenge(R(A)@R(B))=\lozenge R(A)@\lozenge R(B)$
3. $\square(R(A)\$R(B))=\square R(A)\$\square R(B)$ *provided* $sup\mu_{\underline{R}(A)}(x)=sup\mu_{\underline{R}(B)}(x)$ *and* $sup\mu_{\overline{R}(A)}(x)=sup\mu_{\overline{R}(B)}(x)$
4. $\lozenge(R(A)\$R(B))=\lozenge R(A)\$\lozenge R(B)$ *provided* $sup\mu_{\underline{R}(A)}(x)=sup\mu_{\underline{R}(B)}(x)$ *and* $sup\mu_{\overline{R}(A)}(x)=sup\mu_{\overline{R}(B)}(x)$
5. $\square(R(A)\#R(B))=\square R(A)\#\square R(B)$ *provided* $sup\mu_{\underline{R}(A)}(x)=sup\mu_{\underline{R}(B)}(x)$ *and* $sup\mu_{\overline{R}(A)}(x)=sup\mu_{\overline{R}(B)}(x)$
6. $\lozenge(R(A)\#R(B))=\lozenge R(A)\#\lozenge R(B)$ *provided* $sup\mu_{\underline{R}(A)}(x)=sup\mu_{\underline{R}(B)}(x)$ *and* $sup\mu_{\overline{R}(A)}(x)=sup\mu_{\overline{R}(B)}(x)$

Proof. 1. $\square(R(A)@R(B))=\square(\underline{R}(A)@\underline{R}(B),\overline{R}(A)@\overline{R}(B))$

$=(\square(\underline{R}(A)@\underline{R}(B)),\square(\overline{R}(A)@\overline{R}(B)))$

$$\square(\underline{R}(A)@\underline{R}(B))=\square\{(x,[\frac{inf\mu_{\underline{R}(A)}(x)+inf\mu_{\underline{R}(B)}(x)}{2},\frac{sup\mu_{\underline{R}(A)}(x)+sup\mu_{\underline{R}(B)}(x)}{2}],[\frac{inf\gamma_{\underline{R}(A)}(x)+inf\gamma_{\underline{R}(B)}(x)}{2},\frac{sup\gamma_{\underline{R}(A)}(x)+sup\gamma_{\underline{R}(B)}(x)}{2}]):x\in X\}$$

$$=\{(x,[\frac{inf\mu_{\underline{R}(A)}(x)+inf\mu_{\underline{R}(B)}(x)}{2},\frac{sup\mu_{\underline{R}(A)}(x)+sup\mu_{\underline{R}(B)}(x)}{2}],[\frac{inf\gamma_{\underline{R}(A)}(x)+inf\gamma_{\underline{R}(B)}(x)}{2},1-\frac{sup\mu_{\underline{R}(A)}(x)+sup\mu_{\underline{R}(B)}(x)}{2}]):x\in X\}$$

$\square R(A)@\square R(B)=(\square\underline{R}(A),\square\overline{R}(A))@(\square\underline{R}(B),\square\overline{R}(B))$

$=(\square\underline{R}(A)@\square\underline{R}(A)),(\square\overline{R}(B)@\square\overline{R}(B))$

Now $\square\underline{R}(A)=\{(x,\mu_{\underline{R}(A)}(x),[inf\gamma_{\underline{R}(A)}(x),1-sup\mu_{\underline{R}(A)}(x)]):x\in X\}$

$\square\underline{R}(B)=\{(x,\mu_{\underline{R}(B)}(x),[inf\gamma_{\underline{R}(B)}(x),1-sup\mu_{\underline{R}(B)}(x)]):x\in X\}$

$$\square\underline{R}(A)@\square\underline{R}(B)=\{(x,[\frac{inf\mu_{\underline{R}(A)}(x)+inf\mu_{\underline{R}(B)}(x)}{2},\frac{sup\mu_{\underline{R}(A)}(x)+sup\mu_{\underline{R}(B)}(x)}{2}],[\frac{inf\gamma_{\underline{R}(A)}(x)+inf\gamma_{\underline{R}(B)}(x)}{2},\frac{1-sup\mu_{\underline{R}(A)}(x)+1-sup\mu_{\underline{R}(B)}(x)}{2}]):x\in X\}$$

$$=\{(x,[\frac{inf\mu_{\underline{R}(A)}(x)+inf\mu_{\underline{R}(B)}(x)}{2},\frac{sup\mu_{\underline{R}(A)}(x)+sup\mu_{\underline{R}(B)}(x)}{2}],[\frac{inf\gamma_{\underline{R}(A)}(x)+inf\gamma_{\underline{R}(B)}(x)}{2},1-\frac{sup\mu_{\underline{R}(A)}(x)+sup\mu_{\underline{R}(B)}(x)}{2}]):x\in X\}$$

$\therefore\square(\underline{R}(A)@\underline{R}(B))=\square\underline{R}(A)@\square\underline{R}(B)$

Similarly we can prove that $\square(\overline{R}(A)@\overline{R}(B))=\square\overline{R}(A)@\square\overline{R}(B)$

Consequently $\square(R(A)@R(B))=\square R(A)@\square R(B)$

2. Similar to 1.

3. Let $sup\mu_{\underline{R}(A)}(x)=sup\mu_{\underline{R}(B)}(x)$ and $sup\mu_{\overline{R}(A)}(x)=sup\mu_{\overline{R}(B)}(x)$

$\square(R(A)\$R(B))=\square(\underline{R}(A)\$\underline{R}(B),\overline{R}(A)\$\overline{R}(B))$

$=(\square(\underline{R}(A)\$\underline{R}(B)),\square(\overline{R}(A)\$\overline{R}(B)))$

$$\square(\underline{R}(A)\$\underline{R}(B))=\square\{(x,[\sqrt{inf\mu_{\underline{R}(A)}(x)\cdot inf\mu_{\underline{R}(B)}(x)},\sqrt{sup\mu_{\underline{R}(A)}(x)\cdot sup\mu_{\underline{R}(B)}(x)}],[\sqrt{inf\gamma_{\underline{R}(A)}(x)\cdot inf\gamma_{\underline{R}(B)}(x)},\sqrt{sup\gamma_{\underline{R}(A)}(x)\cdot sup\gamma_{\underline{R}(B)}(x)}]):x\in X\}$$

$$=\{(x,[\sqrt{inf\mu_{\underline{R}(A)}(x)\cdot inf\mu_{\underline{R}(B)}(x)},\sqrt{sup\mu_{\underline{R}(A)}(x)\cdot sup\mu_{\underline{R}(B)}(x)}],[\sqrt{inf\gamma_{\underline{R}(A)}(x)\cdot inf\gamma_{\underline{R}(B)}(x)},1-\sqrt{sup\mu_{\underline{R}(A)}(x)\cdot sup\mu_{\underline{R}(B)}(x)}]):x\in X\}$$

$$=\{(x,[\sqrt{inf\mu_{\underline{R}(A)}(x)\cdot inf\mu_{\underline{R}(B)}(x)},\sqrt{sup\mu_{\underline{R}(A)}(x)\cdot sup\mu_{\underline{R}(B)}(x)}],$$

$[\sqrt{inf\gamma_{\underline{R}(A)}(x)\cdot inf\gamma_{\underline{R}(B)}(x)}, 1-sup\mu_{\underline{R}(A)}(x)]) : x \in X\}$

$\Box R(A)\$\Box R(B) = (\Box\underline{R}(A), \Box\overline{R}(A))\$(\Box\underline{R}(B), \Box\overline{R}(B))$
$=(\Box\underline{R}(A)\$\Box\underline{R}(A), \Box\overline{R}(B)\$\Box\overline{R}(B))$

Now $\Box\underline{R}(A) = \{(x, \mu_{\underline{R}(A)}(x), [inf\gamma_{\underline{R}(A)}(x), 1-sup\mu_{\underline{R}(A)}(x)]) : x \in X\}$
$\Box\underline{R}(B) = \{(x, \mu_{\underline{R}(B)}(x), [inf\gamma_{\underline{R}(B)}(x), 1-sup\mu_{\underline{R}(B)}(x)]) : x \in X\}$

$\Box\underline{R}(A)\$\Box\underline{R}(B) = \{(x, [\sqrt{inf\mu_{\underline{R}(A)}(x)\cdot inf\mu_{\underline{R}(B)}(x)}, \sqrt{sup\mu_{\underline{R}(A)}(x)\cdot sup\mu_{\underline{R}(B)}(x)}],$
$[\sqrt{inf\gamma_{\underline{R}(A)}(x)\cdot inf\gamma_{\underline{R}(B)}(x)}, \sqrt{(1-sup\mu_{\underline{R}(A)}(x))\cdot(1-sup\mu_{\underline{R}(B)}(x))}]) : x \in X\}$

$=\{(x, [\sqrt{inf\mu_{\underline{R}(A)}(x)\cdot inf\mu_{\underline{R}(B)}(x)}, \sqrt{sup\mu_{\underline{R}(A)}(x)\cdot sup\mu_{\underline{R}(B)}(x)}],$
$[\sqrt{inf\gamma_{\underline{R}(A)}(x)\cdot inf\gamma_{\underline{R}(B)}(x)}, 1-sup\mu_{\underline{R}(A)}(x)]) : x \in X\}$

$\therefore \Box(\underline{R}(A)\$\underline{R}(B)) = \Box\underline{R}(A)\$\Box\underline{R}(B)$

Similarly we can prove that $\Box(\overline{R}(A)\$\overline{R}(B)) = \Box\overline{R}(A)\$\Box\overline{R}(B)$

Consequently $\Box(R(A)\$R(B)) = \Box R(A)\$\Box R(B)$

4. Similar to 3.
5. Let $sup\mu_{\underline{R}(A)}(x) = sup\mu_{\underline{R}(B)}(x)$ and $sup\mu_{\overline{R}(A)}(x) = sup\mu_{\overline{R}(B)}(x)$

$\Box(R(A)\#R(B)) = \Box(\underline{R}(A)\#\underline{R}(B), \overline{R}(A)\#\overline{R}(B))$
$=(\Box(\underline{R}(A)\#\underline{R}(B)), \Box(\overline{R}(A)\#\overline{R}(B)))$

$\Box(\underline{R}(A)\#\underline{R}(B)) = \Box\{(x, [\frac{2\cdot inf\mu_{\underline{R}(A)}(x)\cdot inf\mu_{\underline{R}(B)}(x)}{inf\mu_{\underline{R}(A)}(x)+inf\mu_{\underline{R}(B)}(x)}, \frac{2\cdot sup\mu_{\underline{R}(A)}(x)\cdot sup\mu_{\underline{R}(B)}(x)}{sup\mu_{\underline{R}(A)}(x)+sup\mu_{\underline{R}(B)}(x)}],$
$[\frac{2\cdot inf\gamma_{\underline{R}(A)}(x)\cdot inf\gamma_{\underline{R}(B)}(x)}{inf\gamma_{\underline{R}(A)}(x)+inf\gamma_{\underline{R}(B)}(x)}, \frac{2\cdot sup\gamma_{\underline{R}(A)}(x)\cdot sup\gamma_{\underline{R}(B)}(x)}{sup\gamma_{\underline{R}(A)}(x)+sup\gamma_{\underline{R}(B)}(x)}]) : x \in X\}$

$=\{(x, [\frac{2\cdot inf\mu_{\underline{R}(A)}(x)\cdot inf\mu_{\underline{R}(B)}(x)}{inf\mu_{\underline{R}(A)}(x)+inf\mu_{\underline{R}(B)}(x)}, \frac{2\cdot sup\mu_{\underline{R}(A)}(x)\cdot sup\mu_{\underline{R}(B)}(x)}{sup\mu_{\underline{R}(A)}(x)+sup\mu_{\underline{R}(B)}(x)}],$
$[\frac{2\cdot inf\gamma_{\underline{R}(A)}(x)\cdot inf\gamma_{\underline{R}(B)}(x)}{inf\gamma_{\underline{R}(A)}(x)+inf\gamma_{\underline{R}(B)}(x)}, 1-\frac{2\cdot sup\mu_{\underline{R}(A)}(x)\cdot sup\mu_{\underline{R}(B)}(x)}{sup\mu_{\underline{R}(A)}(x)+sup\mu_{\underline{R}(B)}(x)}]) : x \in X\}$

$=\{(x, [\frac{2\cdot inf\mu_{\underline{R}(A)}(x)\cdot inf\mu_{\underline{R}(B)}(x)}{inf\mu_{\underline{R}(A)}(x)+inf\mu_{\underline{R}(B)}(x)}, \frac{2\cdot sup\mu_{\underline{R}(A)}(x)\cdot sup\mu_{\underline{R}(B)}(x)}{sup\mu_{\underline{R}(A)}(x)+sup\mu_{\underline{R}(B)}(x)}],$
$[\frac{2\cdot inf\gamma_{\underline{R}(A)}(x)\cdot inf\gamma_{\underline{R}(B)}(x)}{inf\gamma_{\underline{R}(A)}(x)+inf\gamma_{\underline{R}(B)}(x)}, 1-sup\mu_{\underline{R}(A)}(x)]) : x \in X\}$

$\Box R(A)\#\Box R(B) = (\Box\underline{R}(A), \Box\overline{R}(A))\#(\Box\underline{R}(B), \Box\overline{R}(B))$
$=(\Box\underline{R}(A)\#\Box\underline{R}(A), \Box\overline{R}(B)\#\Box\overline{R}(B))$

Now $\Box\underline{R}(A) = \{(x, \mu_{\underline{R}(A)}(x), [inf\gamma_{\underline{R}(A)}(x), 1-sup\mu_{\underline{R}(A)}(x)]) : x \in X\}$
$\Box\underline{R}(B) = \{(x, \mu_{\underline{R}(B)}(x), [inf\gamma_{\underline{R}(B)}(x), 1-sup\mu_{\underline{R}(B)}(x)]) : x \in X\}$

$\Box\underline{R}(A)\#\Box\underline{R}(B) = \{(x, [\frac{2\cdot inf\mu_{\underline{R}(A)}(x)\cdot inf\mu_{\underline{R}(B)}(x)}{inf\mu_{\underline{R}(A)}(x)+inf\mu_{\underline{R}(B)}(x)}, \frac{2\cdot sup\mu_{\underline{R}(A)}(x)\cdot sup\mu_{\underline{R}(B)}(x)}{sup\mu_{\underline{R}(A)}(x)+sup\mu_{\underline{R}(B)}(x)}],$
$[\frac{2\cdot inf\gamma_{\underline{R}(A)}(x)\cdot inf\gamma_{\underline{R}(B)}(x)}{inf\gamma_{\underline{R}(A)}(x)+inf\gamma_{\underline{R}(B)}(x)}, \frac{2\cdot(1-sup\mu_{\underline{R}(A)}(x))\cdot(1-sup\mu_{\underline{R}(B)}(x))}{(1-sup\mu_{\underline{R}(A)}(x))+(1-sup\mu_{\underline{R}(B)}(x))}]) : x \in X\}$

$= \{(x, [\frac{2\cdot inf\mu_{\underline{R}(A)}(x)\cdot inf\mu_{\underline{R}(B)}(x)}{inf\mu_{\underline{R}(A)}(x)+inf\mu_{\underline{R}(B)}(x)}, \frac{2\cdot sup\mu_{\underline{R}(A)}(x)\cdot sup\mu_{\underline{R}(B)}(x)}{sup\mu_{\underline{R}(A)}(x)+sup\mu_{\underline{R}(B)}(x)}],$
$[\frac{2\cdot inf\gamma_{\underline{R}(A)}(x)\cdot inf\gamma_{\underline{R}(B)}(x)}{inf\gamma_{\underline{R}(A)}(x)+inf\gamma_{\underline{R}(B)}(x)}, 1-sup\mu_{\underline{R}(A)}(x)]) : x \in X\}$

$\therefore \Box(\underline{R}(A)\#\underline{R}(B)) = \Box\underline{R}(A)\#\Box\underline{R}(B)$

Similarly we can prove that $\Box(\overline{R}(A)\#\overline{R}(B)) = \Box\overline{R}(A)\#\Box\overline{R}(B)$

Consequently $\Box(R(A)\#R(B)) = \Box R(A)\#\Box R(B)$

6. Similar to 5. 3.10. □

Now let us show by an example that in theorem 3.10 -(3) (4), (5) and (6) are not true in general. For this let us take two RIVIFS A and B as stated in examples 3.2 and 3.4, i.e.

$A = \{(1, [.2, .3], [.3, .4]), (2, [.3, .7], [.2, .3]), (3, [.1, .4], [.2, .5]), (4, [.5, .6], [.1, .3]),$

$(5,[.4,.5],[.2,.4])\}$
$\underline{R}(A)=\{(1,[.1,.3],[.3,.5]),(2,[.3,.6],[.2,.3]),(3,[.1,.3],[.3,.5]),(4,[.3,.6],[.2,.3]),$
$(5,[.1,.3],[.3,.5])\}$
$\overline{R}(A)=\{(1,[.4,.5],[.2,.4]),(2,[.5,.7],[.1,.3]),(3,[.4,.5],[.2,.4]),(4,[.5,.7],[.1,.3]),$
$(5,[.4,.5],[.2,.4])\}$
$B=\{(1,[.4,.5],[.1,.3]),(2,[.4,.8],[.1,.2]),(3,[.1,.5],[.2,.3]),(4,[.5,.7],[.1,.2]),$
$(5,[.4,.7],[.1,.3])\}$
$\underline{R}(B)=\{(1,[.1,.5],[.2,.3]),(2,[.4,.7],[.1,.2]),(3,[.1,.5],[.2,.3]),(4,[.4,.7],[.1,.2]),$
$(5,[.1,.5],[.2,.3])\}$
$\overline{R}(B)=\{(1,[.4,.7],[.1,.3]),(2,[.5,.8],[.1,.2]),(3,[.4,.7],[.1,.3]),(4,[.5,.8],[.1,.2]),$
$(5,[.4,.7],[.1,.3])\}.$
$\Box\underline{R}(A)=\{(1,[.1,.3],[.3,.7]),(2,[.3,.6],[.2,.4]),(3,[.1,.3],[.3,.7]),(4,[.3,.6],[.2,.4]),$
$(5,[.1,.3],[.3,.7])\}$
$\Box\overline{R}(A)=\{(1,[.4,.5],[.2,.5]),(2,[.5,.7],[.1,.3]),(3,[.4,.5],[.2,.5]),(4,[.5,.7],[.1,.3]),$
$(5,[.4,.5],[.2,.5])\}.$
$\Diamond\underline{R}(A)=\{(1,[.1,.5],[.3,.5]),(2,[.3,.7],[.2,.3]),(3,[.1,.5],[.3,.5]),(4,[.3,.7],[.2,.3]),$
$(5,[.1,.5],[.3,.5])\}$
$\Diamond\overline{R}(A)=\{(1,[.4,.6],[.2,.4]),(2,[.5,.7],[.1,.3]),(3,[.4,.6],[.2,.4]),(4,[.5,.7],[.1,.3]),$
$(5,[.4,.6],[.2,.4])\}.$
$\Box\underline{R}(B)=\{(1,[.1,.5],[.2,.5]),(2,[.4,.7],[.1,.3]),(3,[.1,.5],[.2,.5]),(4,[.4,.7],[.1,.3]),$
$(5,[.1,.5],[.2,.5])\}$
$\Box\overline{R}(B)=\{(1,[[.4,.7],[.1,.3]),(2,[.5,.8],[.1,.2]),(3,[.4,.7],[.1,.3]),(4,[.5,.8],[.1,.2]),$
$(5,[.4,.7],[.1,.3])\}.$
$\Diamond\underline{R}(B)=\{(1,[.1,.7],[.2,.3]),(2,[.4,.8],[.1,.2]),(3,[.1,.7],[.2,.3]),(4,[.4,.8],[.1,.2]),$
$(5,[.1,.7],[.2,.3])\}$
$\Diamond\overline{R}(B)=\{(1,[.4,.7],[.1,.3]),(2,[.5,.8],[.1,.2]),(3,[.4,.7],[.1,.3]),(4,[.5,.8],[.1,.2]),$
$(5,[.4,.7],[.1,.3])\}.$
For theorem 3
$\underline{R}(A)\$\underline{R}(B)=\{(1,[.1,.39],[.24,.39]),(2,[.35,.65],[.14,.24]),(3,[.1,.39],[.24,.39]),$
$(4,[.35,.65],[.14,.24]),(5,[.1,.39],[.24,.39])\}$
$\Box(\underline{R}(A)\$\underline{R}(B))=\{(1,[.1,.39],[.24,.61]),(2,[.35,.65],[.14,.35]),(3,[.1,.39],[.24,.61]),$
$(4,[.35,.65],[.14,.35]),(5,[.1,.39],[.24,.61])\}$
$\Box(\underline{R}(A)\$\Box\underline{R}(B))=\{(1,[.1,.39],[.24,.59]),(2,[.35,.65],[.14,.35]),(3,[.1,.39],[.24,.59]),$
$(4,[.35,.65],[.14,.35]),(5,[.1,.39],[.24,.59])\}$
$\overline{R}(A)\$\overline{R}(B)=\{(1,[.4,.59],[.14,.35]),(2,[.5,.75],[.1,.24]),(3,[.4,.59],[.14,.35]),$
$(4,[.5,.75],[.1,.24]),(5,[.4,.59],[.14,.35])\}$
$\Box(\overline{R}(A)\$\overline{R}(B))=\{(1,[.4,.59],[.14,.41]),(2,[.5,.75],[.1,.25]),(3,[.4,.59],[.14,.41]),$
$(4,[.5,.75],[.1,.25]),(5,[.4,.59],[.14,.41])\}$
$\Box(\overline{R}(A)\$\Box\overline{R}(B))=\{(1,[.4,.59],[.14,.39]),(2,[.5,.75],[.1,.24]),(3,[.4,.59],[.14,.39]),$
$(4,[.5,.75],[.1,.24]),(5,[.4,.59],[.14,.39])\}$

$$\therefore \Box(R(A)\$R(B))\neq\Box R(A)\$\Box R(B)$$

For theorem 4
$\Diamond(\underline{R}(A)\$\underline{R}(B))=\{(1,[.1,.61],[.24,.39]),(2,[.35,.76],[.14,.24]),(3,[.1,.61],[.24,.39]),$
$(4,[.35,.76],[.14,.24]),(5,[.1,.61],[.24,.39])\}$
$\Diamond(\underline{R}(A)\$\Box\underline{R}(B))=\{(1,[.1,.59],[.24,.39]),(2,[.35,.75],[.14,.24]),(3,[.1,.59],[.24,.39]),$
$(4,[.35,.65],[.14,.35]),(5,[.1,.59],[.24,.39])\}$

$\Diamond(\overline{R}(A)\$\overline{R}(B)) = \{(1,[.4,.65],[.14,.35]),(2,[.5,.76],[.1,.24]),(3,[.4,.65],[.14,.35]),$
$(4,[.5,.76],[.1,.24]),(5,[.4,.65],[.14,.35])\}$
$\Diamond(\overline{R}(A)\$\Box\overline{R}(B)) = \{(1,[.4,.65],[.14,.35]),(2,[.5,.75],[.1,.24]),(3,[.4,.65],[.14,.35]),$
$(4,[.5,.75],[.1,.24]),(5,[.4,.65],[.14,.35])\}$

$$\therefore \Diamond(R(A)\$R(B)) \neq \Diamond R(A)\$\Diamond R(B)$$

For theorem 5

$\underline{R}(A)\#\underline{R}(B) = \{(1,[.1,.38],[.24,.38]),(2,[.34,.65],[.13,.24]),(3,[.1,.38],[.24,.38]),$
$(4,[.34,.65],[.13,.24]),(5,[.1,.38],[.24,.38])\}$
$\Box(\underline{R}(A)\#\underline{R}(B)) = \{(1,[.1,.38],[.24,.62]),(2,[.34,.65],[.13,.35]),(3,[.1,.38],[.24,.62]),$
$(4,[.34,.65],[.13,.35]),(5,[.1,.38],[.24,.62])\}$
$\Box\underline{R}(A)\#\Box\underline{R}(B) = \{(1,[.1,.38],[.24,.58]),(2,[.34,.65],[.13,.34]),(3,[.1,.38],[.24,.58]),$
$(4,[.34,.65],[.13,.34]),(5,[.1,.38],[.24,.58])\}$
$\overline{R}(A)\#\overline{R}(B) = \{(1,[.4,.58],[.13,.34]),(2,[.5,.75],[.1,.24]),(3,[.4,.58],[.13,.34]),$
$(4,[.5,.75],[.1,.24]),(5,[.4,.58],[.13,.34])\}$
$\Box(\overline{R}(A)\#\overline{R}(B)) = \{(1,[.4,.58],[.13,.42]),(2,[.5,.75],[.1,.25]),(3,[.4,.58],[.13,.42]),$
$(4,[.5,.75],[.1,.25]),(5,[.4,.58],[.13,.42])\}$
$\Box\overline{R}(A)\#\Box\overline{R}(B) = \{(1,[.4,.58],[.13,.38]),(2,[.5,.75],[.1,.24]),(3,[.4,.58],[.13,.38]),$
$(4,[.5,.75],[.1,.24]),(5,[.4,.58],[.13,.38])\}$

$$\therefore \Box(R(A)\#R(B)) \neq \Box R(A)\#\Box R(B)$$

For theorem 6

$\Diamond(\underline{R}(A)\#\underline{R}(B)) = \{(1,[.1,.62],[.24,.38]),(2,[.34,.76],[.13,.24]),(3,[.1,.62],[.24,.38]),$
$(4,[.34,.76],[.13,.24]),(5,[.1,.62],[.24,.38])\}$
$\Diamond\underline{R}(A)\#\Box\underline{R}(B) = \{(1,[.1,.58],[.24,.38]),(2,[.34,.75],[.13,.24]),(3,[.1,.58],[.24,.38]),$
$(4,[.34,.75],[.13,.24]),(5,[.1,.58],[.24,.38])\}$
$\Diamond(\overline{R}(A)\#\overline{R}(B)) = \{(1,[.4,.66],[.13,.34]),(2,[.5,.76],[.1,.24]),(3,[.4,.66],[.13,.34]),$
$(4,[.5,.76],[.1,.24]),(5,[.4,.66],[.13,.34])\}$
$\Diamond\overline{R}(A)\#\Box\overline{R}(B) = \{(1,[.4,.65],[.13,.34]),(2,[.5,.75],[.1,.24]),(3,[.4,.65],[.13,.34]),$
$(4,[.5,.75],[.1,.24]),(5,[.4,.65],[.13,.34])\}$

$$\therefore \Diamond(R(A)\#R(B)) \neq \Diamond R(A)\#\Diamond R(B)$$

Example 3.11. Les us give an example where interval-valued intutionistic fuzzy concept and rough concept appear together. Let us consider the universal set of all faculties in colleges in a state. We define ${}_xR_y$ if and only if x and y belong to the same college. Then obviously this is an equivalence relation and decomposes the set of faculties into equivalence classes, which are colleges in the state.

Let us now define two interval-valued intuitionistic fuzzy sets G and Y over the universe as the set of "good faculty" and "young faculty" respectively. These two concepts can be defined through interval-valued intuitionistic fuzzy sets. For example a faculty x can be considered as a member of G as $(x,[.4,.6],[.2,.4])$ and the same faculty can be defined as a member of Y as $(x,[.2,.5],[.1,.4])$. Then the following four cases arise.

Case I- If $\underline{R}(G) = \underline{R}(Y)$ then the colleges for which all the faculties are good also have all the faculties young.

Case II- Let $\underline{R}(G)$ and $\underline{R}(Y)$ are ϕ or not ϕ together then either there is no college which contain all good faculty or young faculty or there are some colleges which contain all good faculties or all young faculties. Unlike case-I, here the set of colleges

may not be same.
Case III- If $\overline{R}(G) = \overline{R}(Y)$ then the set of colleges which have at least one good faculty is same as the set of colleges which have at least one young faculty.
Case IV- If $\underline{R}(G)$ and $\underline{R}(Y)$ are U or not U together. If both are U then all the colleges have at least one good faculty as well as at least one young faculty. If both are not U it implies there are some colleges which do not have any young faculty and there some colleges which do not have any good faculty. Unlike case- III, here the same colleges may not have this feature.

4. Conclusions

In this paper we combine two different theories rough set theory and interval-valued intuitionistic fuzzy set theory and define the notion of rough interval-valued intuitionistic fuzzy sets. Just like rough set theory, interval-valued intuitionistic fuzzy set theory addresses the topic of dealing with imperfect knowledge. Recent investigations have shown how both theories can be combined into a more flexible, more impressive framework for modeling and processing incomplete information in information systems.

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