‘NTV’ metric based entropies of interval-valued intuitionistic fuzzy sets and their applications in decision making

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ABSTRACT. In the present paper, we have proposed new similarity measures for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets based on ‘NTV’ metric along with their weighted form. The proposed similarity measures have been analogously extended to obtain new entropies for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets along with their proofs of validity. A new algorithm for multi-criteria group decision making has been provided using the proposed weighted similarity measure in which the weights have been calculated using the proposed entropies. Further, numerical example for illustrating the proposed methodology has also been provided by taking interval-valued intuitionistic fuzzy sets.

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1. INTRODUCTION

Intuitionistic fuzzy set (IFS), developed by Atanassov [1] is a controlling tool to deal with vagueness and uncertainty. A prominent characteristic of IFS is that it assigns to each element a membership degree and a non-membership degree with certain amount of hesitation degree, and thus, the IFS constitutes an extension of Zadeh’s fuzzy set [40], which only assigns to each element a membership degree. Intuitionistic fuzzy sets can be useful in situations when description of a problem by a (fuzzy) linguistic variable, given in terms of a membership function only, seems insufficient to give best result. Atanassov [2, 3] and many other researchers [22, 8] studied different properties of IFSs in decision making problems, particularly in the case of medical diagnosis, sales analysis, new product marketing, financial services,
etc. Further, Atanassov and Gargov [4] introduced the concept of interval-valued intuitionistic fuzzy sets (IVIFS) and studied various properties. It may be noted that the entropy and similarity measures are two important concepts in the field of fuzzy set theory and are widely investigated by many researchers from different point of view. The similarity measure of IFSs indicates the degree of similarity between two IFSs and plays a significant role in many applications such as pattern recognition, approximate reasoning and decision making.

Vlachos and Sergiadis [25] extended the De Luca and Termini’s [9] non-probabilistic entropy for fuzzy sets in the study of the intuitionistic fuzzy information measure. Burillo and Bustince [6] introduced the notions of entropy of IFSs and interval-valued fuzzy sets (IVFS) to measure the degree of intuitionism of an IFS and IVFS, respectively. Hung and Yang [13] gave their axiomatic definitions and characterization of entropy of IFSs and IVFSs with the help of probability theory. Li and Cheng [10] proposed some similarity measures on IFSs and applied them in pattern recognition problems. Further, Liang and Shi [16] pointed out the drawbacks of Li and Cheng’s methods and to overcome them, they proposed several new similarity measures and also discussed relationships between these measures. Further, Szmidt and Kacprzyk [24] defined a similarity measure using distance measure of IFSs and applied these measures in group decision making problems and medical diagnostic reasoning. Xu [29] defined some similarity measures for IVIFSs and applied these similarity measures in pattern recognitions. Hung and Yang [12] presented a similarity measure of IFSs based on Hausdorff metric and applied it to pattern recognition problems. In the study of fuzzy sets, Wang [26] defined two similarity measures and Pappis and Karacapilidis [19] defined three kinds of similarity measures. Hung and Yang [14] extend these similarity measures from the fuzzy sets to IFSs. Further, Xu [35] generalized some formulas of similarity measures of IFSs to IVIFSs. Zeng and Guo [41] proved that some similarity measures and entropies of IVFSs can be deduced by normalized distances of IVFSs based on their axiomatic definitions. Zeng and Li [42], Zhang et al. [44] showed that similarity measures and entropies of IVFSs can be obtained by the transformation from each other. Zeng et al. [43] put forward some entropy formulas of IFSs according to the relationship between entropies and similarity measures of IFSs. Later on, Cui-Ping Wei et al. [27] proposed the entropy for the IVIFSs and obtained the similarity measure for the IVIFSs on the basis of proposed entropy. Xu and Yager [37] developed some geometric aggregation operators, such as the intuitionistic fuzzy weighted geometric (IFWG) operator, the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator and the intuitionistic fuzzy hybrid geometric (IFHG) operator, and gave an application of the IFHG operator to multi-criteria decision-making problems with intuitionistic fuzzy information. Xu [30] developed some arithmetic aggregation operators, such as the intuitionistic fuzzy weighted averaging (IFWA) operator, the intuitionistic fuzzy ordered weighted averaging (IFOWA) operator and the intuitionistic fuzzy hybrid aggregation (IFHA) operator (for more detail we refer [32]). Xu [31] defined the concept of interval-valued intuitionistic fuzzy number (IVIFN), and gave some basic operational laws of IVIFNs. He put forward an interval-valued intuitionistic fuzzy weighted averaging operator and an interval-valued intuitionistic fuzzy
Let for two IFSs have been defined as using the proposed entropies in section proposed weighted similarity measures in which the weights have been calculated new algorithm for multi-criteria group decision making has been provided using the score function and accuracy function of interval-valued intuitionistic fuzzy numbers. Xu and Chen [34] investigate an interval-valued intuitionistic fuzzy ordered weighted geometric operator and an interval-valued intuitionistic fuzzy hybrid geometric operator. Xu and Yager [38] extended the intuitionistic fuzzy Bonferroni means (IFBMs) to accommodate interval-valued intuitionistic fuzzy environments.

In the present paper, we study some basic definitions related to the intuitionistic fuzzy sets and the interval-valued intuitionistic fuzzy sets in section 2. New similarity measures for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets based on ‘NTV’ metric along with their weighted form have been proposed in section 3. The proposed similarity measures have also been analogously extended to obtain new intuitionistic fuzzy entropies for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets with the proof of their validity in section 4. Further, a new algorithm for multi-criteria group decision making has been provided using the proposed weighted similarity measures in which the weights have been calculated using the proposed entropies in section 5. Numerical example by taking interval-valued intuitionistic fuzzy sets has been illustrated in section 6. Finally, the paper has been concluded in section 7.

2. Preliminaries

In this section, we present some axiomatic definitions of the intuitionistic fuzzy set, interval-valued intuitionistic fuzzy set, similarity measure and entropy measure which are well known in literature.

**Definition 2.1 ([1]).** Let $X$ be the universe of discourse, then an IFS $\tilde{A}$ in $X$ is given by

$$\tilde{A} = \{ \langle x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) \rangle : x \in X \},$$

where $\mu_{\tilde{A}} : X \rightarrow [0, 1]$ and $\nu_{\tilde{A}} : X \rightarrow [0, 1]$ with the condition $0 \leq \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1$, $\forall x \in X$. The numbers $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$ denote the degree of membership and non-membership of an element $x$ in $\tilde{A}$, respectively. For each element $x \in X$, the amount $\pi_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x)$ is called the degree of indeterminacy (hesitation part). It is the degree of uncertainty whether $x$ belongs to $\tilde{A}$ or not. We denote $IFS(X)$ the set of all the IFSs on $X$.

**Definition 2.2 ([1]).** For two IFSs $\tilde{A}$ and $\tilde{B}$ the following relations and operations have been defined as

1. $\tilde{A} \cup \tilde{B} = \{ \langle x, \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \min\{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\} \rangle : x \in X \};$
2. $\tilde{A} \cap \tilde{B} = \{ \langle x, \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, \max\{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\} \rangle : x \in X \};$
3. $\tilde{A}^{c} = \{ \langle x, \nu_{\tilde{B}}(x), \mu_{\tilde{A}}(x) \rangle : x \in X \};$
In many real-world decision problems, the values of the membership function and the non-membership function in an IFS are difficult to express as exact numbers. Instead, the ranges of their values can usually be specified. In order to deal such cases, Atanassov and Gargov [4] generalized the concept of IFS to interval-valued intuitionistic fuzzy set (IVIFS), and define some basic operational laws of IVIFSs.

**Definition 2.3 ([4]).** Let \( X \) be a universe of discourse and \( \text{int}(0,1) \) be the set of all closed subintervals of the interval \([0, 1]\). An interval-valued intuitionistic fuzzy set (IVIFS) \( \hat{A}_* \) in \( X \) is an object having the form:

\[
\hat{A}_* = \left\{ \left( x, \mu_{\hat{A}_*}(x), \nu_{\hat{A}_*}(x) \right) \mid x \in X \right\}
\]

where \( \mu_{\hat{A}_*} : X \to \text{int}(0,1) \), \( \nu_{\hat{A}_*} : X \to \text{int}(0,1) \), with the condition

\[
0 \leq \sup \left( \mu_{\hat{A}_*}(x) \right) + \sup \left( \nu_{\hat{A}_*}(x) \right) \leq 1, \forall x \in X.
\]

Here, the intervals \( \mu_{\hat{A}_*}(x) = [\mu_{\hat{A}_*}^L(x), \mu_{\hat{A}_*}^U(x)] \) and \( \nu_{\hat{A}_*}(x) = [\nu_{\hat{A}_*}^L(x), \nu_{\hat{A}_*}^U(x)] \) denote the degree of the membership and the non-membership of an element \( x \) belonging to IVIFS \( \hat{A}_* \), respectively. For each IVIFS \( \hat{A}_* \) in \( X \), the amount \( \pi_{\hat{A}_*}(x) = \left[ 1 - \mu_{\hat{A}_*}^U(x) - \nu_{\hat{A}_*}^U(x), 1 - \mu_{\hat{A}_*}^L(x) - \nu_{\hat{A}_*}^L(x) \right] \), is called the interval-valued intuitionistic index of \( x \) in \( \hat{A}_* \), which is a hesitancy degree of \( x \) to \( \hat{A}_* \). It is the degree of uncertainty whether an element \( x \) belongs to \( \hat{A}_* \) or not. We denote \( TVIFS(X) \) the set of all the IVIFSs on \( X \).

**Definition 2.4 ([4]).** For all \( x \in X \) and \( \hat{A}_*, \hat{B}_* \in TVIFS(X) \), the following relations and operations have been defined as follows:

\((P1)\) \( \hat{A}_* \cup \hat{B}_* = \left\{ \left( x, \min \left( \mu_{\hat{A}_*}(x), \mu_{\hat{B}_*}(x) \right), \max \left( \nu_{\hat{A}_*}(x), \nu_{\hat{B}_*}(x) \right) \right) \mid x \in X \right\} \), where

\[
m_{\hat{A}_* \cup \hat{B}_*}(x) = \left[ \min \left\{ \inf \left( \mu_{\hat{A}_*}(x) \right), \inf \left( \mu_{\hat{B}_*}(x) \right) \right\}, \max \left\{ \sup \left( \mu_{\hat{A}_*}(x) \right), \sup \left( \mu_{\hat{B}_*}(x) \right) \right\} \right],
\]

\[
n_{\hat{A}_* \cup \hat{B}_*}(x) = \left[ \max \left\{ \inf \left( \nu_{\hat{A}_*}(x) \right), \inf \left( \nu_{\hat{B}_*}(x) \right) \right\}, \min \left\{ \sup \left( \nu_{\hat{A}_*}(x) \right), \sup \left( \nu_{\hat{B}_*}(x) \right) \right\} \right];
\]

\((P2)\) \( \hat{A}_* \cap \hat{B}_* = \left\{ \left( x, \min \left( \mu_{\hat{A}_*}(x), \nu_{\hat{B}_*}(x) \right), \max \left( \nu_{\hat{A}_*}(x), \nu_{\hat{B}_*}(x) \right) \right) \mid x \in X \right\} \), where

\[
m_{\hat{A}_* \cap \hat{B}_*}(x) = \left[ \min \left\{ \inf \left( \mu_{\hat{A}_*}(x) \right), \inf \left( \mu_{\hat{B}_*}(x) \right) \right\}, \max \left\{ \sup \left( \mu_{\hat{A}_*}(x) \right), \sup \left( \mu_{\hat{B}_*}(x) \right) \right\} \right],
\]

\[
n_{\hat{A}_* \cap \hat{B}_*}(x) = \left[ \max \left\{ \inf \left( \nu_{\hat{A}_*}(x) \right), \inf \left( \nu_{\hat{B}_*}(x) \right) \right\}, \min \left\{ \sup \left( \nu_{\hat{A}_*}(x) \right), \sup \left( \nu_{\hat{B}_*}(x) \right) \right\} \right];
\]

\((P3)\) \( \hat{A}_*^c = \left\{ \left( x, \max \left( \nu_{\hat{A}_*}(x), \nu_{\hat{B}_*}(x) \right), \min \left( \mu_{\hat{A}_*}(x), \mu_{\hat{B}_*}(x) \right) \right) \right\} \);

\((P4)\) \( \hat{A}_* \subseteq \hat{B}_* \Leftrightarrow \mu_{\hat{A}_*}(x) \leq \mu_{\hat{B}_*}(x), \nu_{\hat{A}_*}(x) \leq \nu_{\hat{B}_*}(x) \) and \( \mu_{\hat{B}_*}^L(x) \geq \nu_{\hat{B}_*}^L(x) \); 

\((P5)\) \( \hat{A}_* = \hat{B}_* \Leftrightarrow \mu_{\hat{A}_*}(x) = \mu_{\hat{B}_*}(x), \nu_{\hat{A}_*}(x) = \nu_{\hat{B}_*}(x) \) and \( \mu_{\hat{B}_*}^L(x) \geq \nu_{\hat{B}_*}^L(x) \).
Definition 2.5 ([12]). A real-valued function \( S : \mathcal{IFS}(X) \times \mathcal{IFS}(X) \rightarrow [0, 1] \), is called the similarity measure on \( \mathcal{IFS}(X) \), if \( S \) satisfies the following axiomatic requirements:

(S1) If \( \tilde{A} \) is a crisp set, then \( S(\tilde{A}, \tilde{A}^c) = 0 \);
(S2) \( S(\tilde{A}, \tilde{B}) = 1 \iff \tilde{A} = \tilde{B} \) i.e., \( \mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x) \) & \( \nu_{\tilde{A}}(x) = \nu_{\tilde{B}}(x) \);
(S3) \( S(\tilde{A}, \tilde{B}) = S(\tilde{B}, \tilde{A}) \);
(S4) If \( \tilde{A} \subseteq \tilde{B} \subseteq \tilde{C} \), then \( S(\tilde{A}, \tilde{C}) \leq S(\tilde{A}, \tilde{B}) \) and \( S(\tilde{A}, \tilde{C}) \leq S(\tilde{B}, \tilde{C}) \).

Definition 2.6 ([29, 35]). A real-valued function \( S : \mathcal{IVIFS}(X) \times \mathcal{IVIFS}(X) \rightarrow [0, 1] \), is called the similarity measure on \( \mathcal{IVIFS}(X) \), if \( S \) satisfies the following axiomatic requirements:

(S1) \( 0 \leq S(\tilde{A}_s, \tilde{B}_s) \leq 1 \);
(S2) \( S(\tilde{A}_s, \tilde{B}_s) = 1 \iff \tilde{A}_s = \tilde{B}_s \);
(S3) \( S(\tilde{A}_s, \tilde{B}_s) = S(\tilde{B}_s, \tilde{A}_s) \);
(S4) If \( \tilde{A}_s \subseteq \tilde{B}_s \subseteq \tilde{C}_s \), then \( S(\tilde{A}_s, \tilde{C}_s) \leq S(\tilde{A}_s, \tilde{B}_s) \) and \( S(\tilde{A}_s, \tilde{C}_s) \leq S(\tilde{B}_s, \tilde{C}_s) \).

Apart from similarity measures for IFSs, we have the entropies (information measures) for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets. These entropies play an important role in many fields of research such as pattern recognition, approximate reasoning, decision making etc.

Definition 2.7 ([23]). A real-valued function \( E : \mathcal{IFS}(X) \rightarrow [0, 1] \) is called the entropy measure on \( \mathcal{IFS}(X) \), if \( E \) satisfies the following properties:

(E1) \( E(\tilde{A}) = 0 \iff \tilde{A} \) is crisp set;
(E2) \( E(\tilde{A}) = 1 \iff \mu_{\tilde{A}}(x) = \nu_{\tilde{A}}(x), \forall x \in X \);
(E3) \( E(\tilde{A}) \leq E(\tilde{B}) \) if \( \tilde{A} \) is less fuzzy than \( \tilde{B} \), i.e., \( \mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x) \) and \( \nu_{\tilde{A}}(x) \geq \nu_{\tilde{B}}(x) \) for \( \mu_{\tilde{B}}(x) \leq \nu_{\tilde{B}}(x) \) or \( \mu_{\tilde{A}}(x) \geq \nu_{\tilde{B}}(x) \) and \( \nu_{\tilde{A}}(x) \leq \nu_{\tilde{B}}(x) \);
(E4) \( E(\tilde{A}) = E(\tilde{A}^c) \), where \( \tilde{A}^c \) is the complement of \( \tilde{A} \).

Definition 2.8 ([17]). A real-valued function \( E : \mathcal{IVIFS}(X) \rightarrow [0, 1] \) is called the entropy measure on \( \mathcal{IVIFS}(X) \), if \( E \) satisfies the following properties:

(E1) \( E(\tilde{A}_s) = 0 \iff \tilde{A}_s \) is crisp set;
(E2) \( E(\tilde{A}_s) = 1 \iff \mu_{\tilde{A}_s}^L(x) = \nu_{\tilde{A}_s}^L(x) \) and \( \nu_{\tilde{A}_s}^U(x) = \nu_{\tilde{A}_s}^U(x) \), \( \forall x \in X \);
(E3) \( E(\tilde{A}_s) \leq E(\tilde{B}_s) \) if \( \tilde{A}_s \) is less fuzzy than \( \tilde{B}_s \), i.e., \( \tilde{A}_s \subseteq \tilde{B}_s \), for \( \mu_{\tilde{B}_s}^L(x) \leq \nu_{\tilde{B}_s}^L(x) \) and \( \mu_{\tilde{B}_s}^U(x) \leq \nu_{\tilde{B}_s}^U(x) \), or \( \tilde{B}_s \subseteq \tilde{A}_s \) for \( \mu_{\tilde{B}_s}^L(x) \geq \nu_{\tilde{B}_s}^L(x) \) and \( \mu_{\tilde{B}_s}^U(x) \geq \nu_{\tilde{B}_s}^U(x) \), \( \forall x \in X \);
(E4) \( E(\tilde{A}_s) = E(\tilde{A}_s^c) \), where \( \tilde{A}_s^c \) is the complement of \( \tilde{A}_s \).

3. Similarity measures for IFSs and IVIFSs

In this section, we propose similarity measures for IFSs and IVIFSs along with their weighted form based on the ‘NTV’ metric defined by Neito et al. [18] in a natural way on \( I^n \) (n-dimensional unit hypercube):
Consider two $n$-dimensional vectors $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$ in $I^n$, the distance $d_{NTV}(p, q)$ of $p$ and $q$ is given by

$$d_{NTV}(p, q) = \frac{\sum_{i=1}^{n} |p_i - q_i|}{\sum_{i=1}^{n} \max\{p_i, q_i\}}. \tag{3.1}$$

Let $\tilde{A} = \{(x, \mu_\tilde{A}(x), \nu_\tilde{A}(x))\}$ and $\tilde{B} = \{(x, \mu_\tilde{B}(x), \nu_\tilde{B}(x))\}$ are two single-element IFSs, then we use the distance measure (3.1) to propose a new similarity measure between $\tilde{A}$ and $\tilde{B}$ as follows:

$$S_{NTV}^1(\tilde{A}, \tilde{B}) = 1 - \frac{|\mu_\tilde{A}(x) - \mu_\tilde{B}(x)| + |\nu_\tilde{A}(x) - \nu_\tilde{B}(x)| + |\pi_\tilde{A}(x) - \pi_\tilde{B}(x)|}{\max\{\mu_\tilde{A}(x), \mu_\tilde{B}(x)\} + \max\{\nu_\tilde{A}(x), \nu_\tilde{B}(x)\} + \max\{\pi_\tilde{A}(x), \pi_\tilde{B}(x)\}}. \tag{3.2}$$

Also, we know that

$$|\mu_\tilde{A}(x) - \mu_\tilde{B}(x)| = \max\{\mu_\tilde{A}(x), \mu_\tilde{B}(x)\} - \min\{\mu_\tilde{A}(x), \mu_\tilde{B}(x)\},$$

$$|\nu_\tilde{A}(x) - \nu_\tilde{B}(x)| = \max\{\nu_\tilde{A}(x), \nu_\tilde{B}(x)\} - \min\{\nu_\tilde{A}(x), \nu_\tilde{B}(x)\},$$

$$|\pi_\tilde{A}(x) - \pi_\tilde{B}(x)| = \max\{\pi_\tilde{A}(x), \pi_\tilde{B}(x)\} - \min\{\pi_\tilde{A}(x), \pi_\tilde{B}(x)\}.$$ Hence, the similarity measure (3.2) reduces to

$$S_{NTV}^1(\tilde{A}, \tilde{B}) = \frac{\min\{\mu_\tilde{A}(x), \mu_\tilde{B}(x)\} + \min\{\nu_\tilde{A}(x), \nu_\tilde{B}(x)\} + \min\{\pi_\tilde{A}(x), \pi_\tilde{B}(x)\}}{\max\{\mu_\tilde{A}(x), \mu_\tilde{B}(x)\} + \max\{\nu_\tilde{A}(x), \nu_\tilde{B}(x)\} + \max\{\pi_\tilde{A}(x), \pi_\tilde{B}(x)\}}. \tag{3.3}$$

The similarity measure (3.3) is defined for single-element IFS. Further, we define similarity measure of two IFSs $\tilde{A}$ and $\tilde{B}$ under the universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$.

Let $\tilde{A} = \{(x_i, \mu_\tilde{A}(x_i), \nu_\tilde{A}(x_i))|x_i \in X\}$ and $\tilde{B} = \{(x_i, \mu_\tilde{B}(x_i), \nu_\tilde{B}(x_i))|x_i \in X\}$ are two IFSs, then similarity measure between $\tilde{A}$ and $\tilde{B}$ is defined as

$$S_{NTV}(\tilde{A}, \tilde{B}) = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\min\{\mu_\tilde{A}(x_i), \mu_\tilde{B}(x_i)\} + \min\{\nu_\tilde{A}(x_i), \nu_\tilde{B}(x_i)\} + \min\{\pi_\tilde{A}(x_i), \pi_\tilde{B}(x_i)\}}{\max\{\mu_\tilde{A}(x_i), \mu_\tilde{B}(x_i)\} + \max\{\nu_\tilde{A}(x_i), \nu_\tilde{B}(x_i)\} + \max\{\pi_\tilde{A}(x_i), \pi_\tilde{B}(x_i)\}} \right). \tag{3.4}$$

**Theorem 3.1.** The similarity measure between two IFS $\tilde{A}$ and $\tilde{B}$ given by (3.4) is a valid similarity measure.

**Proof.** In order to prove that (3.4) is a valid similarity measure, we prove the four properties (S1) to (S4) as listed in definition 2.5.

(S1) By the definition of equality of two IFSs, it is easy to show that $S_{NTV}(\tilde{A}, \tilde{B}) = 1$ if and only if $\tilde{A} = \tilde{B}$.

(S2) If $\tilde{A}$ is a crisp set, then either $\mu_\tilde{A}(x_i) = 1, \nu_\tilde{A}(x_i) = 0, \pi_\tilde{A}(x_i) = 0$ or $\mu_\tilde{A}(x_i) = 0, \nu_\tilde{A}(x_i) = 1, \pi_\tilde{A}(x_i) = 0, \forall x_i \in X$. Moreover, for $\tilde{A}^c$, either $\mu_{\tilde{A}^c}(x_i) = 0, \nu_{\tilde{A}^c}(x_i) = 1, \pi_{\tilde{A}^c}(x_i) = 0$ or $\mu_{\tilde{A}^c}(x_i) = 1, \nu_{\tilde{A}^c}(x_i) = 0, \pi_{\tilde{A}^c}(x_i) = 0, \forall x_i \in X$; $\Rightarrow S_{NTV}(\tilde{A}, \tilde{A}^c) = 0$.

(S3) Let $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$, then by the definition 2.2, we have $\mu_\tilde{A}(x_i) \leq \mu_\tilde{B}(x_i) \leq \mu_\tilde{C}(x_i), \nu_\tilde{A}(x_i) \geq \nu_\tilde{B}(x_i) \geq \nu_\tilde{C}(x_i)$ and
\[ \pi_{\tilde{A}}(x_i) \leq \pi_B(x_i) \leq \pi_C(x_i), \forall x_i \in X \] which implies
\[
\begin{align*}
\min \{ \mu_{\tilde{A}}(x_i), \mu_B(x_i) \} &= \min \{ \mu_{\tilde{A}}(x_i), \mu_C(x_i) \}; \\
\max \{ \mu_{\tilde{A}}(x_i), \mu_B(x_i) \} &\leq \max \{ \mu_{\tilde{A}}(x_i), \mu_C(x_i) \}; \\
\min \{ \nu_{\tilde{A}}(x_i), \nu_B(x_i) \} &\geq \min \{ \nu_{\tilde{A}}(x_i), \nu_C(x_i) \}; \\
\max \{ \nu_{\tilde{A}}(x_i), \nu_B(x_i) \} &= \max \{ \nu_{\tilde{A}}(x_i), \nu_C(x_i) \}; \\
\min \{ \pi_{\tilde{A}}(x_i), \pi_B(x_i) \} &= \min \{ \pi_{\tilde{A}}(x_i), \pi_C(x_i) \}; \\
\max \{ \pi_{\tilde{A}}(x_i), \pi_B(x_i) \} &\leq \max \{ \pi_{\tilde{A}}(x_i), \pi_C(x_i) \},
\end{align*}
\]
which further implies that
\[
\begin{align*}
\min \{ \mu_{\tilde{A}}(x_i), \mu_B(x_i) \} &\geq \min \{ \mu_{\tilde{A}}(x_i), \mu_C(x_i) \}; \\
\max \{ \mu_{\tilde{A}}(x_i), \mu_B(x_i) \} &\geq \max \{ \mu_{\tilde{A}}(x_i), \mu_C(x_i) \}; \\
\min \{ \nu_{\tilde{A}}(x_i), \nu_B(x_i) \} &\geq \min \{ \nu_{\tilde{A}}(x_i), \nu_C(x_i) \}; \\
\max \{ \nu_{\tilde{A}}(x_i), \nu_B(x_i) \} &\geq \max \{ \nu_{\tilde{A}}(x_i), \nu_C(x_i) \}; \\
\min \{ \pi_{\tilde{A}}(x_i), \pi_B(x_i) \} &\geq \min \{ \pi_{\tilde{A}}(x_i), \pi_C(x_i) \}; \\
\max \{ \pi_{\tilde{A}}(x_i), \pi_B(x_i) \} &\geq \max \{ \pi_{\tilde{A}}(x_i), \pi_C(x_i) \}.
\end{align*}
\]
Hence, we have
\[
\begin{align*}
\min \{ \mu_{\tilde{A}}(x_i), \mu_B(x_i) \} + \min \{ \nu_{\tilde{A}}(x_i), \nu_B(x_i) \} + \min \{ \pi_{\tilde{A}}(x_i), \pi_B(x_i) \} &\geq \min \{ \mu_{\tilde{A}}(x_i), \mu_C(x_i) \} + \min \{ \nu_{\tilde{A}}(x_i), \nu_C(x_i) \} + \min \{ \pi_{\tilde{A}}(x_i), \pi_C(x_i) \}; \\
\max \{ \mu_{\tilde{A}}(x_i), \mu_B(x_i) \} + \max \{ \nu_{\tilde{A}}(x_i), \nu_B(x_i) \} + \max \{ \pi_{\tilde{A}}(x_i), \pi_B(x_i) \} &\geq \max \{ \mu_{\tilde{A}}(x_i), \mu_C(x_i) \} + \max \{ \nu_{\tilde{A}}(x_i), \nu_C(x_i) \} + \max \{ \pi_{\tilde{A}}(x_i), \pi_C(x_i) \}.
\end{align*}
\]
Similarly,
\[
\begin{align*}
\min \{ \mu_B(x_i), \mu_C(x_i) \} &\geq \min \{ \mu_{\tilde{A}}(x_i), \mu_C(x_i) \}; \\
\max \{ \mu_B(x_i), \mu_C(x_i) \} = \max \{ \mu_{\tilde{A}}(x_i), \mu_C(x_i) \}; \\
\min \{ \nu_B(x_i), \nu_C(x_i) \} = \min \{ \nu_{\tilde{A}}(x_i), \nu_C(x_i) \}; \\
\max \{ \nu_B(x_i), \nu_C(x_i) \} &\leq \max \{ \nu_{\tilde{A}}(x_i), \nu_C(x_i) \}; \\
\min \{ \pi_B(x_i), \pi_C(x_i) \} &\geq \min \{ \pi_{\tilde{A}}(x_i), \pi_C(x_i) \}; \\
\max \{ \pi_B(x_i), \pi_C(x_i) \} = \max \{ \pi_{\tilde{A}}(x_i), \pi_C(x_i) \},
\end{align*}
\]
whih implies that
\[
\begin{align*}
\min \{ \mu_{\tilde{A}}(x_i), \mu_B(x_i) \} &\geq \min \{ \mu_{\tilde{A}}(x_i), \mu_C(x_i) \}; \\
\max \{ \mu_{\tilde{A}}(x_i), \mu_B(x_i) \} &\geq \max \{ \mu_{\tilde{A}}(x_i), \mu_C(x_i) \}; \\
\min \{ \nu_{\tilde{A}}(x_i), \nu_B(x_i) \} &\geq \min \{ \nu_{\tilde{A}}(x_i), \nu_C(x_i) \}; \\
\max \{ \nu_{\tilde{A}}(x_i), \nu_B(x_i) \} &\geq \max \{ \nu_{\tilde{A}}(x_i), \nu_C(x_i) \}; \\
\min \{ \pi_{\tilde{A}}(x_i), \pi_B(x_i) \} &\geq \min \{ \pi_{\tilde{A}}(x_i), \pi_C(x_i) \}; \\
\max \{ \pi_{\tilde{A}}(x_i), \pi_B(x_i) \} &\geq \max \{ \pi_{\tilde{A}}(x_i), \pi_C(x_i) \}.
\end{align*}
\]
Hence, we have
\[
\begin{align*}
\min \{ \mu_{\tilde{A}}(x_i), \mu_B(x_i) \} + \min \{ \nu_{\tilde{A}}(x_i), \nu_B(x_i) \} + \min \{ \pi_{\tilde{A}}(x_i), \pi_B(x_i) \} &\geq \min \{ \mu_{\tilde{A}}(x_i), \mu_C(x_i) \} + \min \{ \nu_{\tilde{A}}(x_i), \nu_C(x_i) \} + \min \{ \pi_{\tilde{A}}(x_i), \pi_C(x_i) \}; \\
\max \{ \mu_{\tilde{A}}(x_i), \mu_B(x_i) \} + \max \{ \nu_{\tilde{A}}(x_i), \nu_B(x_i) \} + \max \{ \pi_{\tilde{A}}(x_i), \pi_B(x_i) \} &\geq \max \{ \mu_{\tilde{A}}(x_i), \mu_C(x_i) \} + \max \{ \nu_{\tilde{A}}(x_i), \nu_C(x_i) \} + \max \{ \pi_{\tilde{A}}(x_i), \pi_C(x_i) \}.
\end{align*}
\]
From equation (3.5) and (3.6),
\[ S_{NTV}(\tilde{A}, B) \geq S_{NTV}(\tilde{A}, C) \] and \[ S_{NTV}(\tilde{B}, \tilde{C}) \geq S_{NTV}(\tilde{A}, \tilde{C}) \].
Therefore, \[ S_{NTV}(\tilde{A}, B) \] is a valid similarity measure between IFSs \( \tilde{A} \) and \( B \). □

We associate some weights depending upon importance of the elements of the universal set to define the weighted form of the similarity measure (3.4). Let
Let \( w = (w_1, w_2, \ldots, w_n) \) be the weight vector of the elements \( x_i, i = 1, 2, \ldots, n. \) We propose the following weighted similarity measure:

\[
S_{NTV}(\tilde{A}, \tilde{B}) = \sum_{i=1}^{n} w_i \left( \frac{\min \{\mu_{\tilde{A}}(x_i), \mu_{\tilde{B}}(x_i)\} + \min \{\nu_{\tilde{A}}(x_i), \nu_{\tilde{B}}(x_i)\} + \min \{\pi_{\tilde{A}}(x_i), \pi_{\tilde{B}}(x_i)\}}{\max \{\mu_{\tilde{A}}(x_i), \mu_{\tilde{B}}(x_i)\} + \max \{\nu_{\tilde{A}}(x_i), \nu_{\tilde{B}}(x_i)\} + \max \{\pi_{\tilde{A}}(x_i), \pi_{\tilde{B}}(x_i)\}} \right),
\]

where \( w_i \geq 0 \) and \( \sum_{i=1}^{n} w_i = 1. \)

**Remark 3.2.** If \( w = (1/n, 1/n, \ldots, 1/n) \), then the weighted similarity measure (3.7) reduces to the similarity measure (3.4).

Let \( \tilde{A} = \{(x, [\mu_{\tilde{A}}^L(x), \mu_{\tilde{A}}^U(x)], [\nu_{\tilde{A}}^L(x), \nu_{\tilde{A}}^U(x)]) | x \in X \} \) and \( \tilde{B} = \{(x, [\mu_{\tilde{B}}^L(x), \mu_{\tilde{B}}^U(x)], [\nu_{\tilde{B}}^L(x), \nu_{\tilde{B}}^U(x)]) | x \in X \} \) are two IVIFSs.

Analogous to the ‘NTV’ similarity measure for IFS in (3.4), we propose the following similarity measure for IVIFSs:

\[
S_{NTV}(\tilde{A}, \tilde{B}) = \sum_{i=1}^{n} \left( \frac{M_L(\mu, \nu) + M_U(\mu, \nu)}{N_L(\mu, \nu) + N_U(\mu, \nu)} \right),
\]

and the weighted form of the above similarity measure is given by

\[
S_{NTV}(\tilde{A}, \tilde{B}) = \sum_{i=1}^{n} w_i \left( \frac{M_L(\mu, \nu) + M_U(\mu, \nu)}{N_L(\mu, \nu) + N_U(\mu, \nu)} \right),
\]

where

\[
M_L(\mu, \nu) = \min \{\mu_{\tilde{A}}^L(x_i), \mu_{\tilde{B}}^L(x_i)\} + \min \{\nu_{\tilde{A}}^L(x_i), \nu_{\tilde{B}}^L(x_i)\} + \min \{\pi_{\tilde{A}}^L(x_i), \pi_{\tilde{B}}^L(x_i)\},
\]

\[
N_L(\mu, \nu) = \max \{\mu_{\tilde{A}}^L(x_i), \mu_{\tilde{B}}^L(x_i)\} + \max \{\nu_{\tilde{A}}^L(x_i), \nu_{\tilde{B}}^L(x_i)\} + \max \{\pi_{\tilde{A}}^L(x_i), \pi_{\tilde{B}}^L(x_i)\},
\]

\[
M_U(\mu, \nu) = \min \{\mu_{\tilde{A}}^U(x_i), \mu_{\tilde{B}}^U(x_i)\} + \min \{\nu_{\tilde{A}}^U(x_i), \nu_{\tilde{B}}^U(x_i)\} + \min \{\pi_{\tilde{A}}^U(x_i), \pi_{\tilde{B}}^U(x_i)\},
\]

\[
N_U(\mu, \nu) = \max \{\mu_{\tilde{A}}^U(x_i), \mu_{\tilde{B}}^U(x_i)\} + \max \{\nu_{\tilde{A}}^U(x_i), \nu_{\tilde{B}}^U(x_i)\} + \max \{\pi_{\tilde{A}}^U(x_i), \pi_{\tilde{B}}^U(x_i)\}.
\]

**Theorem 3.3.** The similarity measure between two IVIFSs \( \tilde{A} \) and \( \tilde{B} \) given by (3.8) is a valid similarity measure.

**Proof.** The proof of the theorem follows on the similar lines as the proof of theorem 3.1. \(\square\)
4. Entropy Measures Based on Proposed Similarity Measures

In this section, we introduce entropy measures based on the proposed similarity measures for IFSs and IVIFSs, respectively. We first recall some entropy formulas for IFSs.

For an IFS $\tilde{A} = \{(x_i, \mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i))|x_i \in X\}$, Szmidt et al. [23] defined two kind of cardinalities of $\tilde{A}$. The least cardinality or min-sigma-count of $\tilde{A}$ given by

$$\min \sum \text{count}(\tilde{A}) = \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i),$$

and the biggest cardinality or max-sigma-count of $\tilde{A}$ given by

$$\max \sum \text{count}(\tilde{A}) = \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i) + \pi_{\tilde{A}}(x_i).$$

Using these two cardinalities, Szmidt et al. [23] proposed an entropy measure for $\tilde{A}$ as

$$E_{SK}(\tilde{A}) = \frac{1}{n} \sum_{i=1}^{n} \max \text{count}(\tilde{A}_i \cap \tilde{A}_i^c)$$

where for each $i$, $\tilde{A}_i$ denote the single-element IFS corresponding to the element $x_i$ in $X$, and described as $\tilde{A}_i = \{(x_i, \mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i))\}$. Also,

$$\tilde{A}_i \cap \tilde{A}_i^c = \{(x_i, \min\{\mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\}, \max\{\mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\})\},$$

$$\tilde{A}_i \cup \tilde{A}_i^c = \{(x_i, \max\{\mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\}, \min\{\mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\})\}.$$  

For an IFS $\tilde{A}$, Wang et al. [26] gave a different entropy formula

$$E_{WL}(\tilde{A}) = \frac{1}{n} \sum_{i=1}^{n} \frac{\min\{\mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\} + \pi_{\tilde{A}}(x_i)}{\max\{\nu_{\tilde{A}}(x_i), \mu_{\tilde{A}}(x_i)\} + \pi_{\tilde{A}}(x_i)}.$$ 

Hung et al. in [11] introduced fuzzy entropy for a vague sets. Using the equivalence of two theories of vague sets and IFSs [7], Ping Wei et al. [27] transform the fuzzy entropy formula for a vague set in [11] to an entropy formula for an IFS $\tilde{A}$ as

$$E_{HL}(\tilde{A}) = \frac{1}{n} \sum_{i=1}^{n} \frac{1 - |\mu_{\tilde{A}}(x_i) - \nu_{\tilde{A}}(x_i)| + \pi_{\tilde{A}}(x_i)}{1 + |\mu_{\tilde{A}}(x_i) - \nu_{\tilde{A}}(x_i)| + \pi_{\tilde{A}}(x_i)}.$$ 

Ping Wei et al. [27] also proved that all these entropies given by (4.3), (4.6) and (4.7) are equivalent. In 1992, Liu [39] find various entropies from the similarity measures for the fuzzy sets by using the following relation:

$$E(\tilde{A}) = S(A, A^c)$$
Similarly, we propose the entropies formula based on the proposed similarity measures (3.4) and (3.8) as follows:

\begin{equation}
E_T(\hat{A}) = S_{NTV}(\hat{A}, \hat{A}^c)
\end{equation}

\begin{equation}
= \frac{1}{n} \sum_{i=1}^{n} \min \left\{ \mu_{\hat{A}}(x_i), \nu_{\hat{A}}(x_i) \right\} + 0.5 \pi_{\hat{A}}(x_i) \left( \frac{\max \left\{ \mu_{\hat{A}}(x_i), \nu_{\hat{A}}(x_i) \right\}}{\max \left\{ \mu_{\hat{A}}(x_i), \nu_{\hat{A}}(x_i) \right\} + 0.5 \pi_{\hat{A}}(x_i)} \right)
\end{equation}

and

\begin{equation}
E_T(\tilde{A}) = \frac{1}{n} \sum_{i=1}^{n} \left( \min \left\{ \mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i) \right\} + \min \left\{ \nu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i) \right\} + 0.5 \left( \mu_{\tilde{A}}(x_i) + \nu_{\tilde{A}}(x_i) \right) \right)
\end{equation}

**Theorem 4.1.** Entropy measure (4.9) is a valid measure for the IFSs.

**Proof.** In order to prove that the entropy (4.9) is a valid measure, we will have to satisfy all the four properties (E1) to (E4) as listed in definition 2.7.

(E1) If \( \hat{A} \) is a crisp set, then either \( \mu_{\hat{A}}(x_i) = 1, \nu_{\hat{A}}(x_i) = 0 \) or \( \mu_{\hat{A}}(x_i) = 0, \nu_{\hat{A}}(x_i) = 1, \pi_{\hat{A}}(x_i) = 0, \forall x_i \in X \).

From this we have \( S(\hat{A}, \hat{A}^c) = 0 \Rightarrow E_T(\hat{A}) = 0 \).

Conversely, if \( E_T(\hat{A}) = 0 \), then \( \min \{ \mu_{\hat{A}}(x_i), \nu_{\hat{A}}(x_i) \} + 0.5 \pi_{\hat{A}}(x_i) = 0, \forall x_i \in X \);

which implies either \( \mu_{\hat{A}}(x_i) = 1, \nu_{\hat{A}}(x_i) = 0, \pi_{\hat{A}}(x_i) = 0 \) or \( \mu_{\hat{A}}(x_i) = 0, \nu_{\hat{A}}(x_i) = 1, \pi_{\hat{A}}(x_i) = 0, \forall x_i \in X \);

\( \Rightarrow \hat{A} \) is a crisp set.

(E2) Let \( \mu_{\hat{A}}(x_i) = \nu_{\hat{A}}(x_i), \forall x_i \in X \)

\( \Leftrightarrow \mu_{\hat{A}}(x_i) = \nu_{\hat{A}}(x_i) = \mu_{\hat{A}}(x_i) = \nu_{\hat{A}}(x_i), \pi_{\hat{A}}(x_i) = \pi_{\hat{A}}(x_i), \forall x_i \in X \).

\( \Rightarrow \hat{A}^c = \hat{A} \Leftrightarrow S_{NTV}(\hat{A}, \hat{A}^c) = 1 \Leftrightarrow E_T(\hat{A}) = 1 \).

(E3) As per the definition, \( S_{NTV}(\hat{A}, \hat{A}^c) = S_{NTV}(\hat{A}^c, \hat{A}) \Leftrightarrow E_T(\hat{A}) = E_T(\hat{A}^c) \).

(E4) Suppose that \( \mu_{\hat{B}}(x_i) \leq \nu_{\hat{B}}(x_i) \) for each \( x_i \in X \), then \( \hat{A} \subseteq \hat{B} \), i.e.,

\( \mu_{\hat{A}}(x_i) \leq \mu_{\hat{B}}(x_i), \nu_{\hat{A}}(x_i) \leq \nu_{\hat{B}}(x_i) \);

\( \Rightarrow \mu_{\hat{A}}(x_i) \leq \mu_{\hat{B}}(x_i) \leq \nu_{\hat{B}}(x_i) \leq \nu_{\hat{A}}(x_i) \);

\( \Rightarrow \hat{A} \subseteq \hat{B} \subseteq \hat{B}^c \subseteq \hat{A}^c \).

Therefore, by definition 2.5 of the similarity measure for IFSs, we have \( S_{NTV}(\hat{A}, \hat{A}^c) \leq S_{NTV}(\hat{B}, \hat{A}^c) \leq S_{NTV}(\hat{B}, \hat{B}^c) \).

Similarly, if \( \mu_{\hat{A}}(x_i) \geq \mu_{\hat{B}}(x_i), \nu_{\hat{A}}(x_i) \leq \nu_{\hat{B}}(x_i) \), for \( \mu_{\hat{B}}(x_i) \geq \nu_{\hat{B}}(x_i) \),

then we have \( \nu_{\hat{A}}(x_i) \leq \nu_{\hat{B}}(x_i) \leq \mu_{\hat{B}}(x_i) \leq \mu_{\hat{A}}(x_i) \);

\( \Rightarrow \hat{B}^c \subseteq \hat{A}^c \subseteq \hat{B} \subseteq \hat{A} \).

\( \Rightarrow S_{NTV}(\hat{A}^c, \hat{A}) \leq S_{NTV}(\hat{B}^c, \hat{A}) \leq S_{NTV}(\hat{B}, \hat{B}^c), \)

\( \Rightarrow S_{NTV}(\hat{A}, \hat{A}^c) \leq S_{NTV}(\hat{A}, \hat{B}^c) \leq S_{NTV}(\hat{B}, \hat{B}^c), \)

\( \Rightarrow E_T(\hat{A}) = S_{NTV}(\hat{A}, \hat{A}^c) \leq S_{NTV}(\hat{B}, \hat{B}^c) = E_T(\hat{B}) \).

\( \Rightarrow E_T(\hat{A}) \leq E_T(\hat{B}) \).

Since \( E_T(\hat{A}) \) satisfies all the four properties of an entropy measure, therefore, it is a valid entropy for the IFSs.

**Theorem 4.2.** Entropy measure (4.10) is a valid measure for the IVIFSs.

**Proof.** In order to prove that the entropy (4.10) is a valid measure, we will have to satisfy all the properties (E1) to (E4) as listed in definition 2.8.

(E1) Let \( \hat{A} \), be a crisp set. Then either we have

\[ [\mu_{\tilde{A}}(x_i), \mu_{\tilde{A}}(x_i)] = [1, 1], [\nu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)] = [0, 0] \] 

or

\[ [\mu_{\tilde{A}}(x_i), \mu_{\tilde{A}}(x_i)] = [0, 0], [\nu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)] = [1, 1] \] 

\& \[ [\pi_{\tilde{A}}(x_i), \pi_{\tilde{A}}(x_i)] = [0, 0] \]
If \( t \in X \).

Hence, we have \( S(\mathcal{A}_*, \mathcal{A}^*_\beta) = 0 \Rightarrow E_T(\mathcal{A}_*) = 0 \).

Conversely, suppose that \( E_T(\mathcal{A}_*) = 0 \), then we have

\[
\min \left\{ \mu_{\mathcal{A}^*_\beta}(x), \nu_{\mathcal{A}^*_\beta}(x) \right\} + \min \left\{ \mu_{\mathcal{A}_*}(x), \nu_{\mathcal{A}_*}(x) \right\} + 0.5 \left( \tau_{\mathcal{A}_*}(x) + \tau_{\mathcal{A}^*_\beta}(x) \right) = 0;
\]

Since each term in the above equation is non-negative, therefore,

\[
\min \left\{ \mu_{\mathcal{A}^*_\beta}(x), \nu_{\mathcal{A}^*_\beta}(x) \right\} = 0, \min \left\{ \mu_{\mathcal{A}_*}(x), \nu_{\mathcal{A}_*}(x) \right\} = 0
\]

and \( \tau_{\mathcal{A}_*}(x) + \tau_{\mathcal{A}^*_\beta}(x) = 0 \) for each \( x \in X \);

which further implies that \( \mathcal{A}_* \) is a crisp set.

\[ (E2) \]

If \([\mu_{\mathcal{A}_*}(x), \nu_{\mathcal{A}_*}(x)] = [\mu_{\mathcal{A}^*_\beta}(x), \nu_{\mathcal{A}^*_\beta}(x)] \) for each \( x \in X \), then from equation (4.10) we obtain \( E_T(\mathcal{A}_*) = 1 \).

Conversely, if we suppose that \( E_T(\mathcal{A}_*) = 1 \), then we get

\[
\min \left\{ \mu_{\mathcal{A}^*_\beta}(x), \nu_{\mathcal{A}^*_\beta}(x) \right\} + \min \left\{ \mu_{\mathcal{A}_*}(x), \nu_{\mathcal{A}_*}(x) \right\} = \max \left\{ \mu_{\mathcal{A}^*_\beta}(x), \nu_{\mathcal{A}^*_\beta}(x) \right\} + \max \left\{ \mu_{\mathcal{A}_*}(x), \nu_{\mathcal{A}_*}(x) \right\};
\]

which implies that \([\mu_{\mathcal{A}_*}(x), \nu_{\mathcal{A}_*}(x)] = [\mu_{\mathcal{A}^*_\beta}(x), \nu_{\mathcal{A}^*_\beta}(x)] \), \( \forall x \in X \).

\[ (E3) \]

As per the definition, \( S_{\text{NTV}}(\mathcal{A}_*, \mathcal{A}^*_\beta) = S_{\text{NTV}}(\mathcal{A}^*_\beta, \mathcal{A}_*) \Rightarrow E_T(\mathcal{A}_*) = E_T(\mathcal{A}^*_\beta). \)

\[ (E4) \]

Let \( \mathcal{A}_* \) is less fuzzy than \( \mathcal{B}_* \), i.e., \( \mathcal{A}_* \subseteq \mathcal{B}_* \).

\[
\Rightarrow \mu_{\mathcal{A}_*}(x) \leq \mu_{\mathcal{B}_*}(x), \nu_{\mathcal{A}_*}(x) \leq \nu_{\mathcal{B}_*}(x) \Rightarrow \forall x \in X;
\]

which follows that \( \mu_{\mathcal{A}_*}(x) \leq \mu_{\mathcal{B}_*}(x) \), \( \nu_{\mathcal{A}_*}(x) \leq \nu_{\mathcal{B}_*}(x) \), \( \forall x \in X \);

Therefore, by the definition of similarity measure of IVIFSs, we have

\[
S_{\text{NTV}}(\mathcal{A}_*, \mathcal{A}^*_\beta) \leq S_{\text{NTV}}(\mathcal{B}_*, \mathcal{A}^*_\beta) \leq S_{\text{NTV}}(\mathcal{B}_*, \mathcal{B}^*_\beta).
\]

Similarly, if \( \mu_{\mathcal{A}_*}(x) \geq \mu_{\mathcal{B}_*}(x), \nu_{\mathcal{A}_*}(x) \geq \nu_{\mathcal{B}_*}(x) \) and \( \nu_{\mathcal{A}_*}(x) \leq \nu_{\mathcal{B}_*}(x) \), \( \forall x \in X \);

which follows that \( \nu_{\mathcal{A}_*}(x) \leq \nu_{\mathcal{B}_*}(x) \), \( \mu_{\mathcal{A}_*}(x) \leq \mu_{\mathcal{B}_*}(x) \), \( \forall x \in X \);

\[
\Rightarrow \mathcal{A}_* \subseteq \mathcal{B}_* \subseteq \mathcal{B}^*_\beta \subseteq \mathcal{A}^*_\beta;
\]

\[
\Rightarrow S_{\text{NTV}}(\mathcal{A}_*, \mathcal{A}^*_\beta) \leq S_{\text{NTV}}(\mathcal{B}_*, \mathcal{A}^*_\beta) \leq S_{\text{NTV}}(\mathcal{B}_*, \mathcal{B}^*_\beta);
\]

\[
\Rightarrow S_{\text{NTV}}(\mathcal{A}_*, \mathcal{A}^*_\beta) \leq S_{\text{NTV}}(\mathcal{B}_*, \mathcal{B}^*_\beta) = E_T(\mathcal{A}_*) = E_T(\mathcal{B}_*);
\]

\[
\Rightarrow E_T(\mathcal{A}_*) = E_T(\mathcal{B}_*);
\]

Since \( E_T(\mathcal{A}_*) \) satisfies all the four properties of an entropy measure, therefore, it is a valid entropy for the IVIFS.

\[ \square \]

5. Multi-criteria group decision making with IFS and IVIFS

In this section, we present a new method which is based on the proposed weighted similarity measures, where the objective weights are calculated using the proposed entropies to deal with the multiple criteria group decision making (MCDM) problems under the intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets. Ratings of the alternatives, importance/weights of criteria and importance of decision makers in a group decision committee are the three most significant factors which can affect on the results of decision making problems.

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Let $A = \{A_1, A_2, \ldots, A_m\}$ be the set of possible alternatives, $D = \{D_1, D_2, \ldots, D_l\}$ be the set of decision makers and $C = \{C_1, C_2, \ldots, C_n\}$ be the set of criteria with which the performance of alternatives are measured. Assume that the weight information of the criteria and the decision makers are completely unknown. Let $([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])$ be the interval-valued intuitionistic fuzzy number, where $[a_{ij}, b_{ij}]$ indicates the degree that alternative $A_i$ satisfies the criterion $C_j$, $[c_{ij}, d_{ij}]$ indicates the degree that alternative $A_i$ does not satisfy the criterion $C_j$ and $[a_{ij}, b_{ij}] \subseteq [0, 1]$, $[c_{ij}, d_{ij}] \subseteq [0, 1]$ such that $b_{ij} + d_{ij} \leq 1$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Now, we propose the following algorithm to solve the above multi-criteria group decision making problem:

**Step 1:** Determine the weights of decision makers in the decision group.
Assume that decision group contains $l$ decision makers. The importance/weights of the decision makers in the selection committee may not be equal. The importance/weights of decision makers are considered as linguistic variables expressed by interval-valued intuitionistic fuzzy numbers (IVIFNs). Let $D_k = ([a_k, b_k], [c_k, d_k])$ be an interval-valued intuitionistic fuzzy number for rating of $k$th decision maker. Then the subjective weight of $k$th decision maker can be defined as:

$$
\lambda_k = \frac{\left(\frac{a_k + b_k}{a_k + c_k} + (1 - a_k - c_k) \frac{b_k}{b_k + d_k}\right)}{\sum_{k=1}^{l}\left(\frac{a_k + b_k}{a_k + c_k} + (1 - a_k - c_k) \frac{b_k}{b_k + d_k}\right)}
$$

and $\sum_{k=1}^{l} \lambda_k = 1$. The linguistic variables for the importance of the decision makers are provided in the Table 1. If the importance of all the decision makers is same namely extremely important, the rating of the $k$th decision maker can be expressed as $([1, 1], [0, 0], [0, 0])$. Then the weight of each decision maker will be $1/l$.

**Step 2:** Construct the aggregated interval-valued intuitionistic fuzzy decision matrix by pulling the individual decision opinions into a group opinions.
Let $D_k = \left(r_{ij}^{(k)}\right)_{m \times n}$ is an interval-valued intuitionistic fuzzy decision matrix for $k$th ($k = 1, 2, \ldots, l$) decision maker and $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_l$ is the weight vector for decision makers, $\sum_{k=1}^{l} \lambda_k = 1$, $\lambda_k \in [0, 1]$. In group decision-making process, all the individual decision opinions need to be fused into group opinions to construct aggregated interval-valued intuitionistic fuzzy decision matrix. In order to do, we utilize interval-valued intuitionistic fuzzy weighted average (IIFWA) operator due to Xu et al. [33] as follows:

$$
r_{ij} = IIFWA_{\lambda} \left(r_{ij}^{(1)}, r_{ij}^{(2)}, \ldots, r_{ij}^{(l)}\right)
$$

$$
(5.2) = \left[1 - \prod_{k=1}^{l}(1 - a_{ij}^{(k)})^{\lambda_k}, 1 - \prod_{k=1}^{l}(1 - b_{ij}^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(c_{ij}^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(d_{ij}^{(k)})^{\lambda_k}\right].
$$

The aggregated interval-valued intuitionistic fuzzy decision matrix can be defined as:

$$
D = \begin{pmatrix}
    r_{11} & r_{12} & \cdots & r_{1n} \\
    r_{21} & r_{22} & \cdots & r_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    r_{m1} & r_{m2} & \cdots & r_{mn}
\end{pmatrix}
$$

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Step 3: Determine the aggregated interval-valued intuitionistic fuzzy weights of the criteria using IIFWA operator.

All criteria may not be assumed to be of equal importance. Let $W$ represents a set of grades of importance for given criteria’s. In order to obtain $W$, all the individual decision maker opinions for the importance of each of criterion need to be combined. Let $w_j^{(k)} = ([a_{ij}^{(k)}, a_{ij}^{(k)}], [c_{ij}^{(k)}, d_{ij}^{(k)}])$ be an IVIFN assigned to criterion $C_j$ by the $k$th decision maker. Then the aggregated weights of the criteria are calculated using the IIFWA operator due to Xu et al. [33] as follows:

$$w_j = IIFWA \left( w_j^{(1)}, w_j^{(2)}, \ldots, w_j^{(l)} \right)$$

(5.4)

The aggregated weights of the criteria can be defined as:

$$W = [w_1, w_2, \ldots, w_n]^T,$$

where $w_j = ([a_{ij}, b_{ij}], [a_{ij}, b_{ij}]), j = 1, 2, \ldots, n$.

Step 4: Construct the aggregated weighted interval-valued intuitionistic fuzzy decision matrix.

After the aggregated weights of criteria and the aggregated interval valued intuitionistic fuzzy decision matrix are determined, the aggregated weighted interval-valued intuitionistic fuzzy decision matrix can be defined as follows:

$$D' = D \otimes W = (r_{ij}')_{m \times n},$$

where $r_{ij}' = ([a_{ij}', a_{ij}'], [c_{ij}', d_{ij}'])$ is an element of the aggregated weighted interval-valued intuitionistic fuzzy decision matrix.

Step 5: Determine the objective weights of criteria using the proposed interval-valued intuitionistic fuzzy entropy measure (4.10).

Hwang and Yoon [15] introduced a method based on information entropy to determine the weights of attributes. Rao et al. [20, 21] method also suggested the calculation of objective weights using entropy. Xu [28] and Xu et al. [36] assigns a small weight to an attribute with similar attribute values across alternatives because such attribute does not help in differentiating alternatives. Furthermore, the method requires all elements in a decision matrix to be normalized to the range $[0, 1]$ so that each column of the decision matrix sums-to-one.

The entropy of the $j$th criterion $C_j$, $j = 1, 2, \ldots, n$ for the $m$ available alternatives can be obtained from entropy measure (4.10) as follows:

$$E_j = \frac{1}{m} \sum_{i=1}^{m} \left( \frac{\min \{a_{ij}, c_{ij}\} + \min \{b_{ij}, d_{ij}\} + (1 - (a_{ij} + b_{ij} + c_{ij} + d_{ij})/2) \right)$$

and the attribute weight $w_j$ for each criterion $C_j$ based on entropy value can be defined as

$$w_j = \frac{1 - E_j}{n - \sum_{j=1}^{n} E_j}, \quad j = 1, 2, \ldots, n.$$

Step 6: Obtain the interval-valued intuitionistic fuzzy positive-ideal solution (IVIFPIS) and the interval-valued intuitionistic fuzzy negative-ideal solution (IVIFNIS).

Let $J_1$ and $J_2$ be benefit criteria and cost criteria, respectively. The interval-valued intuitionistic fuzzy positive-ideal solution, denoted as $A^+$, and the interval-valued intuitionistic fuzzy negative-ideal solution, denoted as $A^-$, are defined as follows:
Calculate the similarity measures of alternatives with the IVIFPIS and IVIFNIS.

\[
A^+ = \left(\left(\left[a_1^+, b_1^-\right], \left[c_1^+, d_1^-\right]\right), \left(\left[a_2^+, b_2^-\right], \left[c_2^+, d_2^-\right]\right)\right), \ldots, \left(\left(a_n^+, b_n^-\right], \left(c_n^+, d_n^-\right)\right)\right),
\]

\[
A^- = \left(\left(\left[a_1^-, b_1^-\right], \left[c_1^-, d_1^-\right]\right), \left(\left[a_2^-, b_2^-\right], \left[c_2^-, d_2^-\right]\right)\right), \ldots, \left(\left(a_n^-, b_n^-\right], \left(c_n^-, d_n^-\right)\right)\right),
\]

where for each \( j = 1, 2, \ldots, n \),

\[
\left(a_j^+, b_j^+\right], \left[c_j^+, d_j^+\right) = \left(\left(max a_{ij}, max b_{ij}\right], \left[min a_{ij}, min b_{ij}\right]\right) j \in J_1,
\]

\[
\left(a_j^-, b_j^-\right], \left[c_j^-, d_j^-\right) = \left(\left[min a_{ij}, max b_{ij}\right], \left[max a_{ij}, max b_{ij}\right]\right) j \in J_1,
\]

\[
\left(\left[min a_{ij}, max b_{ij}\right], \left[min a_{ij}, max b_{ij}\right]\right) j \in J_2.
\]

**Step 7:** Calculate the similarity measures of alternatives with the IVIFPIS and IVIFNIS based on proposed weighted similarity measure (3.9), respectively as follows:

The similarity between alternatives can be found based on the proposed weighted similarity measure (3.9) as follows:

\[
S(A_i, A^+) = \sum_{j=1}^{n} w_j \left(\frac{p + q}{s + t}\right),
\]

and

\[
S(A_i, A^-) = \sum_{j=1}^{n} w_j \left(\frac{p' + q'}{s' + t'}\right),
\]

where

\[
p = \min \left\{ a_{ij}, a_{ij}^+ \right\} + \min \left\{ c_{ij}, c_{ij}^+ \right\} + \min \left\{ 1 - b_{ij} - d_{ij}, 1 - b_{ij}^+ - d_{ij}^+ \right\},
\]

\[
q = \min \left\{ b_{ij}, b_{ij}^+ \right\} + \min \left\{ d_{ij}, d_{ij}^+ \right\} + \min \left\{ 1 - a_{ij} - c_{ij}, 1 - a_{ij}^+ - c_{ij}^+ \right\},
\]

\[
s = \max \left\{ a_{ij}, a_{ij}^+ \right\} + \max \left\{ c_{ij}, c_{ij}^+ \right\} + \max \left\{ 1 - b_{ij} - d_{ij}, 1 - b_{ij}^+ - d_{ij}^+ \right\},
\]

\[
t = \max \left\{ b_{ij}, b_{ij}^+ \right\} + \max \left\{ d_{ij}, d_{ij}^+ \right\} + \max \left\{ 1 - a_{ij} - c_{ij}, 1 - a_{ij}^+ - c_{ij}^+ \right\},
\]

\[
p' = \min \left\{ a_{ij}, a_{ij}^- \right\} + \min \left\{ c_{ij}, c_{ij}^- \right\} + \min \left\{ 1 - b_{ij} - d_{ij}, 1 - b_{ij}^- - d_{ij}^- \right\},
\]

\[
q' = \min \left\{ b_{ij}, b_{ij}^- \right\} + \min \left\{ d_{ij}, d_{ij}^- \right\} + \min \left\{ 1 - a_{ij} - c_{ij}, 1 - a_{ij}^- - c_{ij}^- \right\},
\]

\[
s' = \max \left\{ a_{ij}, a_{ij}^- \right\} + \max \left\{ c_{ij}, c_{ij}^- \right\} + \max \left\{ 1 - b_{ij} - d_{ij}, 1 - b_{ij}^- - d_{ij}^- \right\},
\]

\[
t' = \max \left\{ b_{ij}, b_{ij}^- \right\} + \max \left\{ d_{ij}, d_{ij}^- \right\} + \max \left\{ 1 - a_{ij} - c_{ij}, 1 - a_{ij}^- - c_{ij}^- \right\}.
\]

**Step 8:** Calculate the relative closeness coefficient to the interval-valued intuitionistic fuzzy ideal solution.

The relative closeness coefficient of an alternative \( A_i \) with respect \( A^+ \) and \( A^- \) is defined as follows:

\[
C_i^* = \frac{S(A_i, A^+)}{S(A_i, A^+) + S(A_i, A^-)}, \quad i = 1, 2, \ldots, m.
\]

**Step 9:** Rank all the alternatives.

After the relative closeness coefficient of each alternative is determined, alternatives are ranked according to descending order of \( C_i^* \)'s and select one that has largest rank, denoted by \( C_k^* \) among the values \( C_i^* \), \( i = 1, 2, \ldots, m \). Hence, \( C_k^* \) is the best choice.

**Remark 5.1.** Since the intuitionistic fuzzy set is a particular case of interval-valued intuitionistic fuzzy set, therefore above proposed algorithm for IVIFSs may similarly be outline for IFSs. For this, we will have to make the following changes:
In step 1, the subjective weight given by the equation (5.1) will be replaced by the weight as suggested in Boran [5].

In step 2 and 3, the interval-valued intuitionistic fuzzy weighted average (IIFWA) operator [33] will be replaced by the intuitionistic fuzzy weighted average (IFWA) operator [30].

In step 5, the entropy measure given by the equation (4.10) will be replaced by the entropy measure given by the equation (4.9).

In step 5, the weighted similarity measure given by the equation (3.9) will be replaced by the weighted similarity measure given by the equation (3.7).

### Table 1. The importance of decision makers and their weights.

<table>
<thead>
<tr>
<th>Linguistic terms</th>
<th>$DM_1$</th>
<th>$DM_2$</th>
<th>$DM_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight</td>
<td>0.393</td>
<td>0.236</td>
<td>0.372</td>
</tr>
</tbody>
</table>

### Table 2. Linguistic terms for rating the importance of criteria and the decision makers

<table>
<thead>
<tr>
<th>Linguistic terms</th>
<th>IFNs</th>
<th>IVIFNs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Very Important (VI)</td>
<td>$(0.90, 0.10)$</td>
<td>$(0.90, 0.95), [0.00, 0.05])$</td>
</tr>
<tr>
<td>Important (I)</td>
<td>$(0.85, 0.10)$</td>
<td>$(0.85, 0.90), [0.05, 0.10])$</td>
</tr>
<tr>
<td>Medium (M)</td>
<td>$(0.50, 0.40)$</td>
<td>$(0.50, 0.55), [0.35, 0.40])$</td>
</tr>
<tr>
<td>Unimportant (U)</td>
<td>$(0.20, 0.70)$</td>
<td>$(0.20, 0.25), [0.65, 0.70])$</td>
</tr>
<tr>
<td>Very Unimportant (VU)</td>
<td>$(0.05, 0.90)$</td>
<td>$(0.05, 0.10), [0.85, 0.90])$</td>
</tr>
</tbody>
</table>

### Table 3. Linguistic terms for rating the alternatives

<table>
<thead>
<tr>
<th>Linguistic terms</th>
<th>IFNs</th>
<th>IVIFNs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Extremely Good (EG)/Extremely High (EH)</td>
<td>$(0.95, 0.05)$</td>
<td>$(0.90, 0.95), [0.00, 0.05])$</td>
</tr>
<tr>
<td>Very Very Good (VVG)/Very Very High (VVH)</td>
<td>$(0.85, 0.10)$</td>
<td>$(0.85, 0.90), [0.05, 0.10])$</td>
</tr>
<tr>
<td>Very good (VG)/Very High (VH)</td>
<td>$(0.80, 0.15)$</td>
<td>$(0.80, 0.85), [0.10, 0.15])$</td>
</tr>
<tr>
<td>Good (G)/High (H)</td>
<td>$(0.75, 0.20)$</td>
<td>$(0.75, 0.80), [0.15, 0.20])$</td>
</tr>
<tr>
<td>Medium Good (MG)/Medium High (MH)</td>
<td>$(0.60, 0.25)$</td>
<td>$(0.60, 0.65), [0.20, 0.25])$</td>
</tr>
<tr>
<td>Fair (F)/Medium (M)</td>
<td>$(0.50, 0.35)$</td>
<td>$(0.50, 0.55), [0.30, 0.35])$</td>
</tr>
<tr>
<td>Medium Poor (MP)/Medium Low (ML)</td>
<td>$(0.40, 0.55)$</td>
<td>$(0.40, 0.45), [0.50, 0.55])$</td>
</tr>
<tr>
<td>Poor (P)/Low (L)</td>
<td>$(0.30, 0.65)$</td>
<td>$(0.30, 0.35), [0.60, 0.65])$</td>
</tr>
<tr>
<td>Very Poor (VP)/Very Low (VL)</td>
<td>$(0.20, 0.75)$</td>
<td>$(0.20, 0.25), [0.70, 0.75])$</td>
</tr>
<tr>
<td>Very Very Poor (VVP)/Very Very Low (VVL)</td>
<td>$(0.10, 0.85)$</td>
<td>$(0.10, 0.15), [0.80, 0.85])$</td>
</tr>
</tbody>
</table>

### 6. Numerical examples

**Example 6.1.** An automobile company desires to select the most appropriate supplier for one of the key elements in its manufacturing process. After pre-evaluation, five suppliers ($A_1, A_2, A_3, A_4, A_5$) have remained as alternatives for further evaluation. In order to evaluate alternative suppliers, a committee of three decision makers $DM_1, DM_2$ and $DM_3$ has been formed. Four criteria are considered as:
The proposed method is currently applied to solve this problem and the computational procedure is as follows:

Importance degree of the decision makers on group decision is shown in Table 1. Linguistic terms used for the ratings of the decision makers and criteria are given in Table 2. In order to obtain the weights of the decision makers, Equation 5.1 is utilized:

$\lambda_{DM_1} = 0.393$, $\lambda_{DM_2} = 0.372$, $\lambda_{DM_3} = 0.236$.

Now the aggregated interval-valued intuitionistic fuzzy decision matrix based on the opinions of decision makers is constructed using IIFWA operator. The linguistic terms shown in Table 3 are used to rate each alternative supplier with respect to each criterion by three decision makers. The ratings given by the decision makers to five alternatives is shown in Table 4.

Table 4. The rating the alternatives

<table>
<thead>
<tr>
<th>Criteria</th>
<th>Suppliers</th>
<th>Decisions makers</th>
<th>Criteria</th>
<th>Suppliers</th>
<th>Decisions makers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>$A_1$</td>
<td>G</td>
<td>$X_3$</td>
<td>$A_1$</td>
<td>VG</td>
</tr>
<tr>
<td></td>
<td>$A_2$</td>
<td>MG</td>
<td></td>
<td>$A_2$</td>
<td>G</td>
</tr>
<tr>
<td></td>
<td>$A_3$</td>
<td>VG</td>
<td></td>
<td>$A_3$</td>
<td>VG</td>
</tr>
<tr>
<td></td>
<td>$A_4$</td>
<td>G</td>
<td></td>
<td>$A_4$</td>
<td>VG</td>
</tr>
<tr>
<td></td>
<td>$A_5$</td>
<td>F</td>
<td></td>
<td>$A_5$</td>
<td>G</td>
</tr>
<tr>
<td>$X_2$</td>
<td>$A_1$</td>
<td>MG</td>
<td>$X_4$</td>
<td>$A_1$</td>
<td>H</td>
</tr>
<tr>
<td></td>
<td>$A_2$</td>
<td>F</td>
<td></td>
<td>$A_2$</td>
<td>MH</td>
</tr>
<tr>
<td></td>
<td>$A_3$</td>
<td>VG</td>
<td></td>
<td>$A_3$</td>
<td>VH</td>
</tr>
<tr>
<td></td>
<td>$A_4$</td>
<td>F</td>
<td></td>
<td>$A_4$</td>
<td>H</td>
</tr>
<tr>
<td></td>
<td>$A_5$</td>
<td>MP</td>
<td></td>
<td>$A_5$</td>
<td>M</td>
</tr>
</tbody>
</table>

Table 5. The importance weight of the criteria

<table>
<thead>
<tr>
<th>Criteria</th>
<th>$DM_1$</th>
<th>$DM_2$</th>
<th>$DM_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>VI</td>
<td>VI</td>
<td>I</td>
</tr>
<tr>
<td>$X_2$</td>
<td>I</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>$X_3$</td>
<td>I</td>
<td>I</td>
<td>M</td>
</tr>
<tr>
<td>$X_4$</td>
<td>M</td>
<td>I</td>
<td>M</td>
</tr>
</tbody>
</table>

The aggregated interval-valued intuitionistic fuzzy decision matrix based on aggregation of decision makers opinions is constructed shown in Matrix representation A.

The importance weights of the criteria provided by decision makers can be linguistic terms. These linguistic terms is represented as interval-valued intuitionistic fuzzy numbers in Table 5 and opinions of decision makers on criteria are aggregated using Equation 5.4 to determine the aggregated weights of criteria. The interval-valued intuitionistic fuzzy weights of criteria after aggregation of opinions of decision makers is:

$W = \begin{bmatrix}
(0.884, 0.936], [0.000, 0.065]) \\
(0.850, 0.900], [0.050, 0.100]) \\
(0.766, 0.825], [0.103, 0.167]) \\
(0.624, 0.685], [0.221, 0.288])
\end{bmatrix}$
Matrix representation A

\[ D = \begin{bmatrix}
  ([0.750, 0.800], [0.150, 0.120]) & ([0.642, 0.694], [0.187, 0.237]) & ([0.790, 0.840], [0.110, 0.160]) & ([0.750, 0.800], [0.150, 0.200]) \\
  ([0.611, 0.664], [0.217, 0.268]) & ([0.634, 0.687], [0.210, 0.262]) & ([0.668, 0.719], [0.178, 0.229]) & ([0.579, 0.629], [0.220, 0.270]) \\
  ([0.822, 0.872], [0.076, 0.128]) & ([0.790, 0.840], [0.110, 0.160]) & ([0.783, 0.833], [0.116, 0.167]) & ([0.783, 0.833], [0.116, 0.167]) \\
  ([0.700, 0.751], [0.168, 0.218]) & ([0.540, 0.591], [0.258, 0.309]) & ([0.771, 0.822], [0.128, 0.178]) & ([0.668, 0.719], [0.178, 0.229]) \\
  ([0.564, 0.614], [0.234, 0.285]) & ([0.463, 0.514], [0.366, 0.418]) & ([0.703, 0.754], [0.167, 0.217]) & ([0.526, 0.576], [0.272, 0.323])
\end{bmatrix} \]

Matrix representation B

\[ D' = \begin{bmatrix}
  ([0.663, 0.749], [0.150, 0.251]) & ([0.546, 0.624], [0.227, 0.313]) & ([0.605, 0.693], [0.201, 0.301]) & ([0.468, 0.548], [0.338, 0.430]) \\
  ([0.540, 0.621], [0.217, 0.316]) & ([0.539, 0.618], [0.250, 0.336]) & ([0.511, 0.594], [0.263, 0.358]) & ([0.361, 0.431], [0.392, 0.481]) \\
  ([0.726, 0.816], [0.076, 0.184]) & ([0.671, 0.756], [0.154, 0.244]) & ([0.600, 0.688], [0.207, 0.306]) & ([0.489, 0.571], [0.311, 0.407]) \\
  ([0.619, 0.703], [0.168, 0.268]) & ([0.459, 0.532], [0.295, 0.378]) & ([0.591, 0.678], [0.217, 0.316]) & ([0.417, 0.493], [0.360, 0.451]) \\
  ([0.498, 0.575], [0.234, 0.331]) & ([0.394, 0.462], [0.398, 0.476]) & ([0.538, 0.623], [0.252, 0.348]) & ([0.328, 0.395], [0.433, 0.518])
\end{bmatrix} \]
After the weights of the criteria and the rating of the alternatives has been determined, the aggregated weighted interval-valued intuitionistic fuzzy decision matrix is constructed utilizing Equation 5.5 shown in Matrix representation B.

The entropy of the $j^{th}$ criterion $X_j, j = 1, 2, \ldots, 4$ for the available alternatives can be obtained from entropy measure (4.10). The objectives weights of criteria based on entropy are $w_1 = 0.359, w_2 = 0.230, w_3 = 0.303, w_4 = 0.108$. 

Product quality, relationship closeness and delivery performance are benefit criteria $J_1 = \{X_1, X_2, X_3\}$ and price is cost criteria $J_2 = \{X_4\}$. Then interval-valued intuitionistic fuzzy positive-ideal solution and interval-valued intuitionistic fuzzy negative ideal solution are

$$A^+ = \{(0.726, 0.816), (0.076, 0.184), (0.671, 0.756), (0.154, 0.244), (0.605, 0.693), (0.201, 0.301), (0.328, 0.395), (0.433, 0.518)\}$$

and

$$A^- = \{(0.498, 0.575), (0.234, 0.331), (0.394, 0.462), (0.398, 0.476), (0.511, 0.594), (0.263, 0.358), (0.489, 0.571), (0.311, 0.407)\}.$$

Similarity of each alternative with the IVIFPIS and IVIFPIN based on proposed weighted similarity measure (3.9) is calculated in Table 6.

**Table 6. Similarities with the IVIFPIS and IVIFPIN**

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>$S^+$</th>
<th>$S^-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>0.873</td>
<td>0.772</td>
</tr>
<tr>
<td>$A_2$</td>
<td>0.769</td>
<td>0.883</td>
</tr>
<tr>
<td>$A_3$</td>
<td>0.966</td>
<td>0.711</td>
</tr>
<tr>
<td>$A_4$</td>
<td>0.818</td>
<td>0.818</td>
</tr>
<tr>
<td>$A_5$</td>
<td>0.722</td>
<td>0.953</td>
</tr>
</tbody>
</table>

Finally, using Equation 5.14, the value of relative closeness of each alternative for the final ranking is shown in Table 6.

**Table 7. Relative closeness coefficient**

<table>
<thead>
<tr>
<th>Alternatives</th>
<th>$C_i^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>0.531</td>
</tr>
<tr>
<td>$A_2$</td>
<td>0.465</td>
</tr>
<tr>
<td>$A_3$</td>
<td>0.576</td>
</tr>
<tr>
<td>$A_4$</td>
<td>0.500</td>
</tr>
<tr>
<td>$A_5$</td>
<td>0.431</td>
</tr>
</tbody>
</table>

Thus, the preference order of alternatives is $A_1, A_2, A_3, A_4$ and $A_5$ according to decreasing order of $C_i^*$ is

$$A_3 > A_1 > A_4 > A_2 > A_5.$$
7. Conclusions

The proposed new similarity measures for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets based on ‘NTV’ metric along with their weighted form are valid similarity measures. The new intuitionistic fuzzy entropies for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets analogously obtained through the proposed similarity measures are also valid information measures. Further, a new algorithm for MCDM using the proposed weighted similarity measures in which the weights have been calculated using the proposed entropies, has been illustrated through a numerical example.

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References


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