Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 9, No. 1, (January 2015), pp. 1–21
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

'NTV' metric based entropies of interval-valued intuitionistic fuzzy sets and their applications in decision making

Tanuj Kumar, Rakesh Kumar Bajaj

Received 28 October 2013; Revised 10 April 2014; Accepted 30 June 2014

Abstract. In the present paper, we have proposed new similarity measures for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets based on 'NTV' metric along with their weighted form. The proposed similarity measures have been analogously extended to obtain new entropies for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets along with their proofs of validity. A new algorithm for multi-criteria group decision making has been provided using the proposed weighted similarity measure in which the weights have been calculated using the proposed entropies. Further, numerical example for illustrating the proposed methodology has also been provided by taking interval-valued intuitionistic fuzzy sets.

2010 AMS Classification: 04A72, 03E72, 62C86, 90B50, 03F55

Keywords: Intuitionistic fuzzy sets, Interval-valued intuitionistic fuzzy sets, Similarity measures, Entropy measures, Multi-criteria group decision making.

Corresponding Author: Tanuj Kumar (tanujkhutail@gmail.com)

1. Introduction

Intuitionistic fuzzy set (IFS), developed by Atanassov [1] is a controlling tool to deal with vagueness and uncertainty. A prominent characteristic of IFS is that it assigns to each element a membership degree and a non-membership degree with certain amount of hesitation degree, and thus, the IFS constitutes an extension of Zadeh's fuzzy set [40], which only assigns to each element a membership degree. Intuitionistic fuzzy sets can be useful in situations when description of a problem by a (fuzzy) linguistic variable, given in terms of a membership function only, seems insufficient to give best result. Atanassov [2, 3] and many other researchers [22, 8] studied different properties of IFSs in decision making problems, particularly in the case of medical diagnosis, sales analysis, new product marketing, financial services, etc. Further, Atanassov and Gargov [4] introduced the concept of interval-valued intuitionistic fuzzy sets (IVIFS) and studied various properties. It may be noted that the entropy and similarity measures are two important concepts in the field of fuzzy set theory and are widely investigated by many researchers from different point of view. The similarity measure of IFSs indicates the degree of similarity between two IFSs and plays a significant role in many applications such as pattern recognition, approximate reasoning and decision making.

Vlachos and Sergiadis [25] extended the De Luca and Termini's [9] non-probabilistic entropy for fuzzy sets in the study of the intuitionistic fuzzy information measure. Burillo and Bustince [6] introduced the notions of entropy of IFSs and interval-valued fuzzy sets (IVFS) to measure the degree of intuitionism of an IFS and IVFS, respectively. Hung and Yang [13] gave their axiomatic definitions and characterization of entropy of IFSs and IVFSs with the help of probability theory. Li and Cheng [10] proposed some similarity measures on IFSs and applied them in pattern recognition problems. Further, Liang and Shi [16] pointed out the drawbacks of Li and Cheng's methods and to overcome them, they proposed several new similarity measures and also discussed relationships between these measures. Further, Szmidt and Kacprzyk [24] defined a similarity measure using distance measure of IFSs and applied these measures in group decision making problems and medical diagnostic reasoning. Xu [29] defined some similarity measures for IVIFSs and applied these similarity measures in pattern recognitions. Hung and Yang [12] presented a similarity measure of IFSs based on Hausdorff metric and applied it to pattern recognition problems. In the study of fuzzy sets, Wang [26] defined two similarity measures and Pappis and Karacapilidis [19] defined three kinds of similarity measures. Hung and Yang [14] extend these similarity measures from the fuzzy sets to IFSs. Further, Xu [35] generalized some formulas of similarity measures of IFSs to IVIFSs. Zeng and Guo [41] proved that some similarity measures and entropies of IVFSs can be deduced by normalized distances of IVFSs based on their axiomatic definitions. Zeng and Li [42], Zhang *et al.* [44] showed that similarity measures and entropies of IVFSs can be obtained by the transformation from each other. Zeng *et al.* [43] put forward some entropy formulas of IFSs according to the relationship between entropies and similarity measures of IFSs. Later on, Cui-Ping Wei *et al.* [27] proposed the entropy for the IVIFSs and obtained the similarity measure for the IVIFSs on the basis of proposed entropy. Xu and Yager [37] developed some geometric aggregation operators, such as the intuitionistic fuzzy weighted geometric (IFWG) operator, the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator and the intuitionistic fuzzy hybrid geometric (IFHG) operator, and gave an application of the IFHG operator to multi-criteria decision-making problems with intuitionistic fuzzy information. Xu [30] developed some arithmetic aggregation operators, such as the intuitionistic fuzzy weighted averaging (IFWA) operator, the intuitionistic fuzzy ordered weighted averaging (IFOWA) operator and the intuitionistic fuzzy hybrid aggregation (IFHA) operator (for more detail we refer [32]). Xu [31] defined the concept of interval-valued intuitionistic fuzzy number (IVIFN), and gave some basic operational laws of IVIFNs. He put forward an interval-valued intuitionistic fuzzy weighted averaging operator and an interval-valued intuitionistic fuzzy weighted geometric operator, and defines the score function and the accuracy function of IVIFNs. Xu and Chen [33] developed some arithmetic aggregation operators, such as the interval-valued intuitionistic fuzzy weighted averaging (IIFWA) operator, the interval-valued intuitionistic fuzzy ordered weighted averaging (IIFOWA) operator and the interval-valued intuitionistic fuzzy hybrid aggregation (IIFHA) operator, and gave an application of the IIFHA operator to multi-criteria decision-making problems with interval-valued intuitionistic fuzzy information by using the score function and accuracy function of interval-valued intuitionistic fuzzy numbers. Xu and Chen [34] investigate an interval-valued intuitionistic fuzzy ordered weighted geometric operator and an interval-valued intuitionistic fuzzy hybrid geometric operator. Xu and Yager [38] extended the intuitionistic fuzzy Bonferroni means (IFBMs) to accommodate interval-valued intuitionistic fuzzy environments.

In the present paper, we study some basic definitions related to the intuitionistic fuzzy sets and the interval-valued intuitionistic fuzzy sets in section 2. New similarity measures for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets based on 'NTV' metric along with their weighted form have been proposed in section 3. The proposed similarity measures have also been analogously extended to obtain new intuitionistic fuzzy entropies for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets with the proof of their validity in section 4. Further, a new algorithm for multi-criteria group decision making has been provided using the proposed weighted similarity measures in which the weights have been calculated using the proposed entropies in section 5. Numerical example by taking interval-valued intuitionistic fuzzy sets has been illustrated in section 6. Finally, the paper has been concluded in section 7.

2. Preliminaries

In this section, we present some axiomatic definitions of the intuitionistic fuzzy set, interval-valued intuitionistic fuzzy set, similarity measure and entropy measure which are well known in literature.

Definition 2.1 ([1])**.** Let X be the universe of discourse, then an IFS $\tilde{A}$ in X is given by

$$\tilde{A} = \{\langle x,\, \mu_{\tilde{A}}(x),\, \nu_{\tilde{A}}(x)\rangle : x \in X\}, \tag{2.1}$$

where $\mu_{\tilde{A}} : X \to [0,\, 1]$ and $\nu_{\tilde{A}} : X \to [0,\, 1]$ with the condition $0 \le \mu_{\tilde{A}}(x)+\nu_{\tilde{A}}(x) \le 1,\ \forall x \in X$. The numbers $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$ denote the degree of membership and non-membership of an element x in $\tilde{A}$, respectively. For each element $x \in X$, the amount $\pi_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x)$ is called the degree of indeterminacy (hesitation part). It is the degree of uncertainty whether x belongs to $\tilde{A}$ or not. We denote $\mathcal{IFS}(X)$ the set of all the IFSs on X.

Definition 2.2 ([1])**.** For two IFSs $\tilde{A}$ and $\tilde{B}$ the following relations and operations have been defined as

$(P1)$ $\tilde{A} \cup \tilde{B} = \{\langle x,\, \max\{\mu_{\tilde{A}}(x),\, \mu_{\tilde{B}}(x)\},\, \min\{\nu_{\tilde{A}}(x),\, \nu_{\tilde{B}}(x)\}\rangle | x \in X\}$;
$(P2)$ $\tilde{A} \cap \tilde{B} = \{\langle x,\, \min\{\mu_{\tilde{A}}(x),\, \mu_{\tilde{B}}(x)\},\, \max\{\nu_{\tilde{A}}(x),\, \nu_{\tilde{B}}(x)\}\rangle | x \in X\}$;
$(P3)$ $\tilde{A}^c = \{\langle x,\, \nu_{\tilde{B}}(x),\, \mu_{\tilde{A}}(x)\rangle | x \in X\}$;

$(P4)$ $\tilde{A} \subseteq \tilde{B} \Leftrightarrow \mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x),\, \nu_{\tilde{A}}(x) \geq \nu_{\tilde{B}}(x),\, \forall x \in X$;
$(P5)$ $\tilde{A} = \tilde{B} \Leftrightarrow \mu_{\tilde{A}}(x) = \nu_{\tilde{B}}(x),\, \nu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x),\, \forall x \in X$.

In many real-world decision problems, the values of the membership function and the non-membership function in an IFS are difficult to express as exact numbers. Instead, the ranges of their values can usually be specified. In order to deal such cases, Atanassov and Gargov [4] generalized the concept of IFS to interval-valued intuitionistic fuzzy set (IVIFS), and define some basic operational laws of IVIFSs.

Definition 2.3 ([4])**.** Let X be a universe of discourse and $\operatorname{int}(0,\,1)$ be the set of all closed subintervals of the interval $[0,\,1]$. An interval-valued intuitionistic fuzzy set (IVIFS) $\tilde{A}_*$ in X is an object having the form:

$$\tilde{A}_* = \left\{ \left\langle x,\, m^{\mu}_{\tilde{A}_*}(x),\, n^{\nu}_{\tilde{A}_*}(x) \right\rangle | \, x \in X \right\}$$

where $m^{\mu}_{\tilde{A}_*} : X \to \operatorname{int}(0,\,1)$, $n^{\nu}_{\tilde{A}_*} : X \to \operatorname{int}(0,\,1)$, with the condition

$$0 \leq sup\left(m^{\mu}_{\tilde{A}_*}(x)\right) + sup\left(n^{\nu}_{\tilde{A}_*}(x)\right) \leq 1,\ \forall x \in X.$$

Here, the intervals $m^{\mu}_{\tilde{A}_*}(x) = [\mu^L_{\tilde{A}_*}(x),\, \mu^U_{\tilde{A}_*}(x)]$ and $n^{\nu}_{\tilde{A}_*} = [\nu^L_{\tilde{A}_*}(x),\, \nu^U_{\tilde{A}_*}(x)]$ denote the degree of the membership and the non-membership of an element x belonging to IVIFS $\tilde{A}_*$, respectively. For each IVIFS $\tilde{A}_*$ in X, the amount $\pi_{\tilde{A}_*}(x) = \left[1 - \mu^U_{\tilde{A}_*}(x) - \nu^U_{\tilde{A}_*}(x), 1 - \nu^L_{\tilde{A}_*}(x) - \nu^L_{\tilde{A}_*}(x)\right]$, is called the interval-valued intuitionistic index of x in $\tilde{A}_*$, which is a hesitancy degree of x to $\tilde{A}_*$. It is the degree of uncertainty whether an element x belongs to $\tilde{A}_*$ or not. We denote $\mathcal{IVIFS}(X)$ the set of all the IVIFSs on X.

Definition 2.4 ([4])**.** For all $x \in X$ and $\tilde{A}_*,\, \tilde{B}_* \in \mathcal{IVIFS}(X)$, the following relations and operations have been defined as follows:

$(P1)$ $\tilde{A}_* \cup \tilde{B}_* = \left\{ \left\langle x,\, m^{\mu}_{\tilde{A}_* \cup \tilde{B}_*}(x),\, n^{\nu}_{\tilde{A}_* \cup \tilde{B}_*}(x) | x \in X \right\rangle | x \in X \right\}$, where

$$m^{\mu}_{\tilde{A}_* \cup \tilde{B}_*}(x) = \left[\min\left\{\inf\left(m^{\mu}_{\tilde{A}_*}(x)\right),\, \inf\left(m^{\mu}_{\tilde{B}_*}(x)\right)\right\},\, \max\left\{\sup\left(m^{\mu}_{\tilde{A}_*}(x)\right),\, \sup\left(m^{\mu}_{\tilde{B}_*}(x)\right)\right\}\right],$$

$$n^{\nu}_{\tilde{A}_* \cup \tilde{B}_*}(x) = \left[\max\left\{\inf\left(n^{\nu}_{\tilde{A}_*}(x)\right),\, \inf\left(n^{\nu}_{\tilde{B}_*}(x)\right)\right\},\, \min\left\{\sup\left(n^{\nu}_{\tilde{A}_*}(x)\right),\, \sup\left(n^{\nu}_{\tilde{B}_*}(x)\right)\right\}\right];$$

$(P2)$ $\tilde{A}_* \cap \tilde{B}_* = \left\{ \left\langle x,\, m^{\mu}_{\tilde{A}_* \cap \tilde{B}_*}(x),\, n^{\nu}_{\tilde{A}_* \cap \tilde{B}_*}(x) \right\rangle | x \in X \right\}$, where

$$m^{\mu}_{\tilde{A}_* \cap \tilde{B}_*}(x) = \left[\max\left\{\inf\left(m^{\mu}_{\tilde{A}_*}(x)\right),\, \inf\left(m^{\mu}_{\tilde{B}_*}(x)\right)\right\},\, \min\left\{\sup\left(m^{\mu}_{\tilde{A}_*}(x)\right),\, \sup\left(m^{\mu}_{\tilde{B}_*}(x)\right)\right\}\right],$$

$$n^{\nu}_{\tilde{A}_* \cap \tilde{B}_*}(x) = \left[\min\left\{\inf\left(n^{\nu}_{\tilde{A}_*}(x)\right),\, \inf\left(n^{\nu}_{\tilde{B}_*}(x)\right)\right\},\, \max\left\{\sup\left(n^{\nu}_{\tilde{A}_*}(x)\right),\, \sup\left(n^{\nu}_{\tilde{B}_*}(x)\right)\right\}\right].$$

$(P3)$ $\tilde{A}^c_* = \left\{ \left\langle x,\, [\nu^L_{\tilde{A}_*}(x),\, \nu^R_{\tilde{A}_*}(x)],\, [\mu^L_{\tilde{A}_*}(x),\, \mu^R_{\tilde{A}_*}(x)] \right\rangle \right\}$;
$(P4)$ $\tilde{A}_* \subseteq \tilde{B}_* \Leftrightarrow \mu^L_{\tilde{A}_*}(x) \leq \mu^L_{\tilde{B}_*}(x),\, \mu^U_{\tilde{A}_*}(x) \leq \mu^U_{\tilde{B}_*},\, \nu^L_{\tilde{A}_*}(x) \geq \nu^L_{\tilde{B}_*}(x)$ and $\nu^U_{\tilde{A}_*}(x) \geq \nu^U_{\tilde{B}_*}$;
$(P5)$ $\tilde{A}_* = \tilde{B}_* \Leftrightarrow \mu^L_{\tilde{A}_*}(x) = \mu^L_{\tilde{B}_*}(x),\, \mu^U_{\tilde{A}_*}(x) = \mu^U_{\tilde{B}_*},\, \nu^L_{\tilde{A}_*}(x) = \nu^L_{\tilde{B}_*}(x)$ and $\nu^U_{\tilde{A}_*}(x) = \nu^U_{\tilde{B}_*}$.

Definition 2.5 ([12])**.** A real-valued function $S : \mathcal{IFS}(X) \times \mathcal{IFS}(X) \to [0, 1]$, is called the similarity measure on $\mathcal{IFS}(X)$, if S satisfies the following axiomatic requirements:

($S1$) If $\tilde{A}$ is a crisp set, then $S(\tilde{A}, \tilde{A}^c) = 0$;
($S2$) $S(\tilde{A}, \tilde{B}) = 1 \Leftrightarrow \tilde{A} = \tilde{B}$ i.e., $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$ & $\nu_{\tilde{A}}(x) = \nu_{\tilde{B}}(x)$;
($S3$) $S(\tilde{A}, \tilde{B}) = S(\tilde{B}, \tilde{A})$;
($S4$) If $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$, then $S(\tilde{A}, \tilde{C}) \leq S(\tilde{A}, \tilde{B})$ and $S(\tilde{A}, \tilde{C}) \leq S(\tilde{B}, \tilde{C})$.

Definition 2.6 ([29, 35])**.** A real-valued function $S : \mathcal{IVIFS}(X) \times \mathcal{IVIFS}(X) \to [0, 1]$, is called the similarity measure on $\mathcal{IVIFS}(X)$, if S satisfies the following axiomatic requirements:

($S1$) $0 \leq S(\tilde{A}_*, \tilde{B}_*) \leq 1$;
($S2$) $S(\tilde{A}_*, \tilde{B}_*) = 1 \Leftrightarrow \tilde{A}_* = \tilde{B}_*$;
($S3$) $S(\tilde{A}_*, \tilde{B}_*) = S(\tilde{B}_*, \tilde{A}_*)$;
($S4$) If $\tilde{A}_* \subseteq \tilde{B}_* \subseteq \tilde{C}_*$, then $S(\tilde{A}_*, \tilde{C}_*) \leq S(\tilde{A}_*, \tilde{B}_*)$ and $S(\tilde{A}_*, \tilde{C}_*) \leq S(\tilde{B}_*, \tilde{C}_*)$.

Apart from similarity measures for IFSs, we have the entropies (information measures) for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets. These entropies play an important role in many fields of research such as pattern recognition, approximate reasoning, decision making etc.

Definition 2.7 ([23])**.** A real-valued function $E : \mathcal{IFS}(X) \to [0, 1]$ is called the entropy measure on $\mathcal{IFS}(X)$, if E satisfies the following properties:

($E1$) $E(\tilde{A}) = 0 \Leftrightarrow \tilde{A}$ is crisp set;
($E2$) $E(\tilde{A}) = 1 \Leftrightarrow \mu_{\tilde{A}}(x) = \nu_{\tilde{A}}(x),\ \forall x \in X$;
($E3$) $E(\tilde{A}) \leq E(\tilde{B})$ if $\tilde{A}$ is less fuzzy than $\tilde{B}$, i.e., $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$ and $\nu_{\tilde{A}}(x) \geq \nu_{\tilde{B}}(x)$ for $\mu_{\tilde{B}}(x) \leq \nu_{\tilde{B}}(x)$ or $\mu_{\tilde{A}}(x) \geq \mu_{\tilde{B}}(x)$ and $\nu_{\tilde{A}}(x) \leq \nu_{\tilde{B}}(x)$ for $\mu_{\tilde{B}}(x) \geq \nu_{\tilde{B}}(x)$, $\forall x \in X$;
($E4$) $E(\tilde{A}) = E(\tilde{A}^c)$, where $\tilde{A}^c$ is the complement of $\tilde{A}$.

Definition 2.8 ([17])**.** A real-valued function $E : \mathcal{IVIFS}(X) \to [0, 1]$ is called the entropy measure on $\mathcal{IVIFS}(X)$, if E satisfies the following properties:

($E1$) $E(\tilde{A}_*) = 0 \Leftrightarrow \tilde{A}_*$ is crisp set;
($E2$) $E(\tilde{A}_*) = 1 \Leftrightarrow \mu^L_{\tilde{A}_*}(x) = \mu^U_{\tilde{A}_*}(x)$ and $\nu^L_{\tilde{A}_*}(x) = \nu^U_{\tilde{A}_*}(x)$, $\forall x \in X$;
($E3$) $E(\tilde{A}_*) \leq E(\tilde{B}_*)$ if $\tilde{A}_*$ is less fuzzy than $\tilde{B}_*$, i.e., $\tilde{A}_* \subseteq \tilde{B}_*$, for $\mu^L_{\tilde{B}_*}(x) \leq \nu^L_{\tilde{B}_*}(x)$ and $\mu^U_{\tilde{B}_*}(x) \leq \nu^U_{\tilde{B}_*}(x)$, or $\tilde{B}_* \subseteq \tilde{A}_*$ for $\mu^L_{\tilde{B}_*}(x) \geq \nu^L_{\tilde{B}_*}(x)$ and $\mu^U_{\tilde{B}_*}(x) \geq \nu^U_{\tilde{B}_*}(x), \forall x \in X$;
($E4$) $E(\tilde{A}_*) = E(\tilde{A}^c_*)$, where $\tilde{A}^c_*$ is the complement of $\tilde{A}_*$.

3. Similarity measures for IFSs and IVIFSs

In this section, we propose similarity measures for IFSs and IVIFSs along with their weighted form based on the 'NTV' metric defined by Neito et al. [18] in a natural way on I^n (n-dimensional unit hypercube):

Consider two n-dimensional vectors $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$ in I^n, the distance $d_{NTV}(p, q)$ of p and q is given by

$$d_{NTV}(p, q) = \frac{\sum_{i=1}^{n} |p_i - q_i|}{\sum_{i=1}^{n} \max\{p_i, q_i\}}. \tag{3.1}$$

Let $\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)\rangle\}$ and $\tilde{B} = \{\langle x, \mu_{\tilde{B}}(x), \nu_{\tilde{B}}(x)\rangle\}$ are two single-element IFSs, then we use the distance measure (3.1) to propose a new similarity measure between $\tilde{A}$ and $\tilde{B}$ as follows:

$$S^1_{NTV}(\tilde{A}, \tilde{B}) = 1 - \frac{|\mu_{\tilde{A}}(x) - \mu_{\tilde{B}}(x)| + |\nu_{\tilde{A}}(x) - \nu_{\tilde{B}}(x)| + |\pi_{\tilde{A}}(x) - \pi_{\tilde{B}}(x)|}{\max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} + \max\{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\} + \max\{\pi_{\tilde{A}}(x), \pi_{\tilde{B}}(x)\}}. \tag{3.2}$$

Also, we know that

$$|\mu_{\tilde{A}}(x) - \mu_{\tilde{B}}(x)| = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} - \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\},$$

$$|\nu_{\tilde{A}}(x) - \nu_{\tilde{B}}(x)| = \max\{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\} - \min\{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\},$$

$$|\pi_{\tilde{A}}(x) - \pi_{\tilde{B}}(x)| = \max\{\pi_{\tilde{A}}(x), \pi_{\tilde{B}}(x)\} - \min\{\pi_{\tilde{A}}(x), \pi_{\tilde{B}}(x)\}.$$

Hence, the similarity measure (3.2) reduces to

$$S^1_{NTV}(\tilde{A}, \tilde{B}) = \frac{\min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} + \min\{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\} + \min\{\pi_{\tilde{A}}(x), \pi_{\tilde{B}}(x)\}}{\max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} + \max\{\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)\} + \max\{\pi_{\tilde{A}}(x), \pi_{\tilde{B}}(x)\}}. \tag{3.3}$$

The similarity measure (3.3) is defined for single-element IFS. Further, we define similarity measure of two IFSs $\tilde{A}$ and $\tilde{B}$ under the universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$.

Let $\tilde{A} = \{\langle x_i, \mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\rangle | x_i \in X\}$ and $\tilde{B} = \{\langle x_i, \mu_{\tilde{B}}(x_i), \nu_{\tilde{B}}(x_i)\rangle | x_i \in X\}$ are two IFSs, then similarity measure between $\tilde{A}$ and $\tilde{B}$ is defined as

$$S_{NTV}(\tilde{A}, \tilde{B}) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\min\{\mu_{\tilde{A}}(x_i), \mu_{\tilde{B}}(x_i)\} + \min\{\nu_{\tilde{A}}(x_i), \nu_{\tilde{B}}(x_i)\} + \min\{\pi_{\tilde{A}}(x_i), \pi_{\tilde{B}}(x_i)\}}{\max\{\mu_{\tilde{A}}(x_i), \mu_{\tilde{B}}(x_i)\} + \max\{\nu_{\tilde{A}}(x_i), \nu_{\tilde{B}}(x_i)\} + \max\{\pi_{\tilde{A}}(x_i), \pi_{\tilde{B}}(x_i)\}}\right). \tag{3.4}$$

Theorem 3.1. *The similarity measure between two IFS $\tilde{A}$ and $\tilde{B}$ given by (3.4) is a valid similarity measure.*

Proof. In order to prove that (3.4) is a valid similarity measure, we prove the four properties $(S1)$ to $(S4)$ as listed in definition 2.5.

$(S1)$ By the definition of equality of two IFSs, it is easy to show that $S_{NTV}(\tilde{A}, \tilde{B}) = 1$ if and only if $\tilde{A} = \tilde{B}$.

$(S2)$ If $\tilde{A}$ is a crisp set, then either $\mu_{\tilde{A}}(x_i) = 1, \nu_{\tilde{A}}(x_i) = 0, \pi_{\tilde{A}}(x_i) = 0$ or $\mu_{\tilde{A}}(x_i) = 0, \nu_{\tilde{A}}(x_i) = 1, \pi_{\tilde{A}}(x_i) = 0, \forall x_i \in X$.
Moreover, for $\tilde{A}^c$, either $\mu_{\tilde{A}^c}(x_i) = 0, \nu_{\tilde{A}^c}(x_i) = 1, \pi_{\tilde{A}^c}(x_i) = 0$ or $\mu_{\tilde{A}^c}(x_i) = 1, \nu_{\tilde{A}^c}(x_i) = 0, \pi_{\tilde{A}^c}(x_i) = 0, \forall x_i \in X$;
$\Rightarrow S_{NTV}(\tilde{A}, \tilde{A}^c) = 0$.

$(S3)$ Let $\tilde{A} \subseteq \tilde{B} \subseteq \tilde{C}$, then by the definition 2.2, we have $\mu_{\tilde{A}}(x_i) \le \mu_{\tilde{B}}(x_i) \le \mu_{\tilde{C}}(x_i), \nu_{\tilde{A}}(x_i) \ge \nu_{\tilde{B}}(x_i) \ge \nu_{\tilde{C}}(x_i)$ and

$\pi_{\tilde{A}}(x_i) \leq \pi_{\tilde{B}}(x_i) \leq \pi_{\tilde{C}}(x_i)$, $\forall x_i \in X$ which implies

$$\begin{aligned}
\min\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{B}}(x_i)\} &= \min\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{C}}(x_i)\};\\
\max\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{B}}(x_i)\} &\leq \max\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{C}}(x_i)\};\\
\min\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{B}}(x_i)\} &\geq \min\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{C}}(x_i)\};\\
\max\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{B}}(x_i)\} &= \max\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{C}}(x_i)\};\\
\min\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{B}}(x_i)\} &= \min\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{C}}(x_i)\};\\
\max\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{B}}(x_i)\} &\leq \max\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{C}}(x_i)\},
\end{aligned}$$

which further implies that

$$\begin{aligned}
\frac{\min\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{B}}(x_i)\}}{\max\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{B}}(x_i)\}} &\geq \frac{\min\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{C}}(x_i)\}}{\max\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{C}}(x_i)\}};\\
\frac{\min\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{B}}(x_i)\}}{\max\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{B}}(x_i)\}} &\geq \frac{\min\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{C}}(x_i)\}}{\max\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{C}}(x_i)\}};\\
\frac{\min\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{B}}(x_i)\}}{\max\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{B}}(x_i)\}} &\geq \frac{\min\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{C}}(x_i)\}}{\max\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{C}}(x_i)\}}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\frac{\min\{\mu_{\tilde{A}}(x_i), \mu_{\tilde{B}}(x_i)\}+\min\{\nu_{\tilde{A}}(x_i), \nu_{\tilde{B}}(x_i)\}+\min\{\pi_{\tilde{A}}(x_i), \pi_{\tilde{B}}(x_i)\}}{\max\{\mu_{\tilde{A}}(x_i), \mu_{\tilde{B}}(x_i)\}+\max\{\nu_{\tilde{A}}(x_i), \nu_{\tilde{B}}(x_i)\}+\max\{\pi_{\tilde{A}}(x_i), \pi_{\tilde{B}}(x_i)\}}\\
&\qquad\geq \frac{\min\{\mu_{\tilde{A}}(x_i), \mu_{\tilde{C}}(x_i)\}+\min\{\nu_{\tilde{A}}(x_i), \nu_{\tilde{C}}(x_i)\}+\min\{\pi_{\tilde{A}}(x_i), \pi_{\tilde{C}}(x_i)\}}{\max\{\mu_{\tilde{A}}(x_i), \mu_{\tilde{C}}(x_i)\}+\max\{\nu_{\tilde{A}}(x_i), \nu_{\tilde{C}}(x_i)\}+\max\{\pi_{\tilde{A}}(x_i), \pi_{\tilde{C}}(x_i)\}}.
\end{aligned} \tag{3.5}$$

Similarly,

$$\begin{aligned}
\min\{\mu_{\tilde{B}}(x_i),\, \mu_{\tilde{C}}(x_i)\} &\geq \min\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{C}}(x_i)\};\\
\max\{\mu_{\tilde{B}}(x_i),\, \mu_{\tilde{C}}(x_i)\} &= \max\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{C}}(x_i)\};\\
\min\{\nu_{\tilde{B}}(x_i),\, \nu_{\tilde{C}}(x_i)\} &= \min\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{C}}(x_i)\};\\
\max\{\nu_{\tilde{B}}(x_i),\, \nu_{\tilde{C}}(x_i)\} &\leq \max\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{C}}(x_i)\};\\
\min\{\pi_{\tilde{B}}(x_i),\, \pi_{\tilde{C}}(x_i)\} &\geq \min\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{C}}(x_i)\};\\
\max\{\pi_{\tilde{B}}(x_i),\, \pi_{\tilde{C}}(x_i)\} &= \max\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{C}}(x_i)\},
\end{aligned}$$

whcih implies that

$$\begin{aligned}
\frac{\min\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{B}}(x_i)\}}{\max\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{B}}(x_i)\}} &\geq \frac{\min\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{C}}(x_i)\}}{\max\{\mu_{\tilde{A}}(x_i),\, \mu_{\tilde{C}}(x_i)\}},\\
\frac{\min\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{B}}(x_i)\}}{\max\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{B}}(x_i)\}} &\geq \frac{\min\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{C}}(x_i)\}}{\max\{\nu_{\tilde{A}}(x_i),\, \nu_{\tilde{C}}(x_i)\}},\\
\frac{\min\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{B}}(x_i)\}}{\max\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{B}}(x_i)\}} &\geq \frac{\min\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{C}}(x_i)\}}{\max\{\pi_{\tilde{A}}(x_i),\, \pi_{\tilde{C}}(x_i)\}}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\frac{\min\{\mu_{\tilde{A}}(x_i), \mu_{\tilde{B}}(x_i)\}+\min\{\nu_{\tilde{A}}(x_i), \nu_{\tilde{B}}(x_i)\}+\min\{\pi_{\tilde{A}}(x_i), \pi_{\tilde{B}}(x_i)\}}{\max\{\mu_{\tilde{A}}(x_i), \mu_{\tilde{B}}(x_i)\}+\max\{\nu_{\tilde{A}}(x_i), \nu_{\tilde{B}}(x_i)\}+\max\{\pi_{\tilde{A}}(x_i), \pi_{\tilde{B}}(x_i)\}}\\
&\qquad\geq \frac{\min\{\mu_{\tilde{A}}(x_i), \mu_{\tilde{C}}(x_i)\}+\min\{\nu_{\tilde{A}}(x_i), \nu_{\tilde{C}}(x_i)\}+\min\{\pi_{\tilde{A}}(x_i), \pi_{\tilde{C}}(x_i)\}}{\max\{\mu_{\tilde{A}}(x_i), \mu_{\tilde{C}}(x_i)\}+\max\{\nu_{\tilde{A}}(x_i), \nu_{\tilde{C}}(x_i)\}+\max\{\pi_{\tilde{A}}(x_i), \pi_{\tilde{C}}(x_i)\}}
\end{aligned} \tag{3.6}$$

From equation (3.5) and (3.6),

$S_{NTV}(\tilde{A}, \tilde{B}) \geq S_{NTV}(\tilde{A}, \tilde{C})$ and $S_{NTV}(\tilde{B}, \tilde{C}) \geq S_{NTV}(\tilde{A}, \tilde{C})$.

Therefore, $S_{NTV}(\tilde{A}, \tilde{B})$ is a valid similarity measure between IFSs $\tilde{A}$ and $\tilde{B}$. □

We associate some weights depending upon importance of the elements of the universal set to define the weighted form of the similarity measure (3.4). Let

$w = (w_1, w_2, \ldots, w_n)$ be the weight vector of the elements x_i, $i = 1, 2, \ldots, n$. We propose the following weighted similarity measure:

(3.7)
$$S_{NTV}(\tilde{A}, \tilde{B}) = \sum_{i=1}^{n} w_i \left(\frac{\min\{\mu_{\tilde{A}}(x_i), \mu_{\tilde{B}}(x_i)\} + \min\{\nu_{\tilde{A}}(x_i), \nu_{\tilde{B}}(x_i)\} + \min\{\pi_{\tilde{A}}(x_i), \pi_{\tilde{B}}(x_i)\}}{\max\{\mu_{\tilde{A}}(x_i), \mu_{\tilde{B}}(x_i)\} + \max\{\nu_{\tilde{A}}(x_i), \nu_{\tilde{B}}(x_i)\} + \max\{\pi_{\tilde{A}}(x_i), \pi_{\tilde{B}}(x_i)\}} \right),$$

where $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$.

Remark 3.2. If $w = (1/n, 1/n, \ldots, 1/n)$, then the weighted similarity measure (3.7) reduces to the similarity measure (3.4).

Let $\tilde{A}_* = \left\{ \left\langle x, \left[\mu^L_{\tilde{A}_*}(x), \mu^U_{\tilde{A}_*}(x)\right], \left[\nu^L_{\tilde{A}_*}(x), \nu^U_{\tilde{A}_*}(x)\right] \right\rangle | x \in X \right\}$ and $\tilde{B}_* = \left\{ \left\langle x, \left[\mu^L_{\tilde{B}_*}(x), \mu^U_{\tilde{B}_*}(x)\right], \left[\nu^L_{\tilde{B}_*}(x), \nu^U_{\tilde{B}_*}(x)\right] \right\rangle | x \in X \right\}$ are two IVIFSs.

Analogous to the 'NTV' similarity measure for IFS in (3.4), we propose the following similarity measure for IVIFSs:

$$S_{NTV}(\tilde{A}_*, \tilde{B}_*) = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{M_L(\mu, \nu) + M_U(\mu, \nu)}{N_L(\mu, \nu) + N_U(\mu, \nu)} \right), \tag{3.8}$$

and the weighted form of the above similarity measure is given by

$$S_{NTV}(\tilde{A}_*, \tilde{B}_*) = \sum_{i=1}^{n} w_i \left(\frac{M_L(\mu, \nu) + M_U(\mu, \nu)}{N_L(\mu, \nu) + N_U(\mu, \nu)} \right), \tag{3.9}$$

where

$$\begin{aligned} M_L(\mu, \nu) = \min\left\{\mu^L_{\tilde{A}_*}(x_i), \mu^L_{\tilde{B}_*}(x_i)\right\} + \min\left\{\nu^L_{\tilde{A}_*}(x_i), \nu^L_{\tilde{B}_*}(x_i)\right\} \\ + \min\left\{\pi^L_{\tilde{A}_*}(x_i), \pi^L_{\tilde{B}_*}(x_i)\right\}, \end{aligned}$$

$$\begin{aligned} N_L(\mu, \nu) = \max\left\{\mu^L_{\tilde{A}_*}(x_i), \mu^L_{\tilde{B}_*}(x_i)\right\} + \max\left\{\nu^L_{\tilde{A}_*}(x_i), \nu^L_{\tilde{B}_*}(x_i)\right\} \\ + \max\left\{\pi^L_{\tilde{A}_*}(x_i), \pi^L_{\tilde{B}_*}(x_i)\right\}, \end{aligned}$$

$$\begin{aligned} M_U(\mu, \nu) = \min\left\{\mu^U_{\tilde{A}_*}(x_i), \mu^U_{\tilde{B}_*}(x_i)\right\} + \min\left\{\nu^U_{\tilde{A}_*}(x_i), \nu^U_{\tilde{B}_*}(x_i)\right\} \\ + \min\left\{\pi^U_{\tilde{A}_*}(x_i), \pi^U_{\tilde{B}_*}(x_i)\right\}, \end{aligned}$$

$$\begin{aligned} N_U(\mu, \nu) = \max\left\{\mu^U_{\tilde{A}_*}(x_i), \mu^U_{\tilde{B}_*}(x_i)\right\} + \max\left\{\nu^U_{\tilde{A}_*}(x_i), \nu^U_{\tilde{B}_*}(x_i)\right\} \\ + \max\left\{\pi^U_{\tilde{A}_*}(x_i), \pi^U_{\tilde{B}_*}(x_i)\right\}. \end{aligned}$$

Theorem 3.3. *The similarity measure between two IVIFSs $\tilde{A}_*$ and $\tilde{B}_*$ given by (3.8) is a valid similarity measure.*

Proof. The proof of the theorem follows on the similar lines as the proof of theorem 3.1. □

4. Entropy measures based on proposed similarity measures

In this section, we introduce entropy measures based on the proposed similarity measures for IFSs and IVIFSs, respectively. We first recall some entropy formulas for IFSs.

For an IFS $\tilde{A} = \{\langle x_i, \mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\rangle | x_i \in X\}$, Szmidt *et al.* [23] defined two kind of cardinalities of $\tilde{A}$. The least cardinality or min-sigma-count of $\tilde{A}$ given by

$$\min \sum \text{count}(\tilde{A}) = \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i), \tag{4.1}$$

and the biggest cardinality or max-sigma-count of $\tilde{A}$ given by

$$\max \sum \text{count}(\tilde{A}) = \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i) + \pi_{\tilde{A}}(x_i). \tag{4.2}$$

Using these two cardinalities, Szmidt *et al.* [23] proposed an entropy measure for $\tilde{A}$ as

$$E_{SK}(\tilde{A}) = \frac{1}{n} \sum_{i=1}^{n} \frac{\max \text{count}(\tilde{A}_i \cap \tilde{A}_i^c)}{\max \text{count}(\tilde{A}_i \cup \tilde{A}_i^c)}, \tag{4.3}$$

where for each $i, \tilde{A}_i$ denote the single-element IFS corresponding to the element x_i in X, and described as $\tilde{A}_i = \{\langle x_i, \mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\rangle\}$. Also,

$$\tilde{A}_i \cap \tilde{A}_i^c = \{\langle x_i, \min\{\mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\}, \max\{\mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\}\rangle\}, \tag{4.4}$$

$$\tilde{A}_i \cup \tilde{A}_i^c = \{\langle x_i, \max\{\mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\}, \min\{\mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\}\rangle\}. \tag{4.5}$$

For an IFS $\tilde{A}$, Wang *et al.* [26] gave a different entropy formula

$$E_{WL}(\tilde{A}) = \frac{1}{n} \sum_{i=1}^{n} \frac{\min\{\mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\} + \pi_A(x_i)}{\max\{\nu_{\tilde{A}}(x_i), \mu_{\tilde{A}}(x_i)\} + \pi_A(x_i)}. \tag{4.6}$$

Hung *et al.* in [11] introduced fuzzy entropy for a vague sets. Using the equivalence of two theories of vague sets and IFSs [7], Ping Wei *et al.* [27] transform the fuzzy entropy formula for a vague set in [11] to an entropy formula for an IFS $\tilde{A}$ as

$$E_{HL}(\tilde{A}) = \frac{1}{n} \sum_{i=1}^{n} \frac{1 - |\mu_{\tilde{A}}(x_i) - \nu_{\tilde{A}}(x_i)| + \pi_A(x_i)}{1 + |\mu_{\tilde{A}}(x_i) - \nu_{\tilde{A}}(x_i| + \pi_A(x_i)}. \tag{4.7}$$

Ping Wei *et al.* [27] also proved that all these entropies given by (4.3), (4.6) and (4.7) are equivalent. In 1992, Liu [39] find various entropies from the similarity measures for the fuzzy sets by using the following relation:

$$E(\tilde{A}) = S(A, A^c) \tag{4.8}$$

Similarly, we propose the entropies formula based on the proposed similarity measures (3.4) and (3.8) as follows:

$$E_T(\tilde{A}) = S_{NTV}(\tilde{A},\, \tilde{A}^c) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\min\left\{\mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\right\}+0.5\pi_{\tilde{A}}(x_i)}{\max\left\{\mu_{\tilde{A}}(x_i), \nu_{\tilde{A}}(x_i)\right\}+0.5\pi_{\tilde{A}}(x_i)}\right) \tag{4.9}$$

and

(4.10)

$$E_T(\tilde{A}_*) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\min\left\{\mu^L_{\tilde{A}_*}(x_i), \nu^L_{\tilde{A}_*}(x_i)\right\}+\min\left\{\mu^U_{\tilde{A}_*}(x_i), \nu^U_{\tilde{A}_*}(x_i)\right\}+0.5\left(\pi^L_{\tilde{A}_*}(x_i)+\pi^U_{\tilde{A}_*}(x_i)\right)}{\max\left\{\mu^L_{\tilde{A}_*}(x_i), \nu^L_{\tilde{A}_*}(x_i)\right\}+\max\left\{\mu^U_{\tilde{A}_*}(x_i), \nu^L_{\tilde{A}_*}(x_i)\right\}+0.5\left(\pi^L_{\tilde{A}_*}(x_i)+\pi^U_{\tilde{A}_*}(x_i)\right)}\right).$$

Theorem 4.1. *Entropy measure (4.9) is a valid measure for the IFSs.*

Proof. In order to prove that the entropy (4.9) is a valid measure, we will have to satisfy all the four properties $(E1)$ to $(E4)$ as listed in definition 2.7.

$(E1)$ If $\tilde{A}$ is a crisp set, then either $\mu_{\tilde{A}}(x_i) = 1,\ \nu_{\tilde{A}}(x_i) = 0,\ \pi_{\tilde{A}}(x_i) = 0$ or
$\mu_{\tilde{A}}(x_i) = 0,\ \nu_{\tilde{A}}(x_i) = 1, \pi_{\tilde{A}}(x_i) = 0,\ \forall\, x_i \in X$.
From this we have $S(\tilde{A},\, \tilde{A}^c) = 0 \Rightarrow E_T(\tilde{A}) = 0$.
Conversely, if $E_T(\tilde{A}) = 0$, then $\min\{\mu_{\tilde{A}}(x_i),\, \nu_{\tilde{A}}(x_i)\} + 0.5 \times \pi_{\tilde{A}}(x_i) = 0,\ \forall x_i \in X$;
which implies either $\mu_{\tilde{A}}(x_i) = 1, \nu_{\tilde{A}}(x_i) = 0,\ \pi_{\tilde{A}}(x_i) = 0$ or
$\mu_{\tilde{A}}(x_i) = 0,\ \nu_{\tilde{A}}(x_i) = 1,\ \pi_{\tilde{A}}(x_i) = 0,\ \forall\, x_i \in X$;
$\Rightarrow \tilde{A}$ is a crisp set.

$(E2)$ Let $\mu_{\tilde{A}}(x_i) = \nu_{\tilde{A}}(x_i),\ \forall x_i \in X$
$\Leftrightarrow \mu_{\tilde{A}^c}(x_i) = \nu_{\tilde{A}}(x_i) = \mu_{\tilde{A}}(x_i),\ \nu_{\tilde{A}^c}(x_i) = \mu_{\tilde{A}}(x_i) = \nu_{\tilde{A}}(x_i)$,
$\Leftrightarrow \tilde{A}^c = \tilde{A} \Leftrightarrow S_{NTV}(\tilde{A},\, \tilde{A}^c) = 1 \Leftrightarrow E_T(\tilde{A}) = 1$.

$(E3)$ As per the definition, $S_{NTV}(\tilde{A},\, \tilde{A}^c) = S_{NTV}(\tilde{A}^c,\, A) \Leftrightarrow E_T(\tilde{A}) = E_T(\tilde{A}^c)$.

$(E4)$ Suppose that $\mu_{\tilde{B}}(x_i) \le \nu_{\tilde{B}}(x_i)$ for each $x_i \in X$, then $\tilde{A} \subseteq \tilde{B}$, i.e.,
$\mu_{\tilde{A}}(x_i) \le \mu_{\tilde{B}}(x_i),\ \nu_{\tilde{A}}(x_i) \ge \nu_{\tilde{B}}(x_i)$;
$\Rightarrow \mu_{\tilde{A}}(x_i) \le \mu_{\tilde{B}}(x_i) \le \nu_{\tilde{B}}(x_i) \le \nu_{\tilde{A}}(x_i)$;
$\Rightarrow \tilde{A} \subseteq \tilde{B} \subseteq \tilde{B}^c \subseteq \tilde{A}^c$.
Therefore, by definition 2.5 of the similarity measure for IFSs, we have
$S_{NTV}(\tilde{A}, \tilde{A}^c) \le S_{NTV}(\tilde{B}, \tilde{A}^c) \le S_{NTV}(\tilde{B}, \tilde{B}^c)$.
Similarly, if $\mu_{\tilde{A}}(x_i) \ge \mu_{\tilde{B}}(x_i), \nu_{\tilde{A}}(x_i) \le \nu_{\tilde{B}}(x_i)$, for $\mu_{\tilde{B}}(x_i) \ge \nu_{\tilde{B}}(x_i)$,
then we have $\nu_{\tilde{A}}(x_i) \le \nu_{\tilde{B}}(x_i) \le \mu_{\tilde{B}}(x_i) \le \mu_{\tilde{A}}(x_i)$,
$\Rightarrow \tilde{A}^c \subseteq \tilde{B}^c \subseteq \tilde{B} \subseteq \tilde{A}$,
$\Rightarrow S_{NTV}(\tilde{A}^c,\, \tilde{A}) \le S_{NTV}(\tilde{B}^c,\, \tilde{A}) \le S_{NTV}(\tilde{B}^c,\, \tilde{B})$,
$\Rightarrow S_{NTV}(\tilde{A},\, \tilde{A}^c) \le S_{NTV}(\tilde{A},\, \tilde{B}^c) \le S_{NTV}(\tilde{B},\, \tilde{B}^c)$,
$\Rightarrow E_T(\tilde{A}) = S_{NTV}(\tilde{A},\, \tilde{A}^c) \le S_{NTV}(\tilde{B},\, \tilde{B}^c) = E_T(\tilde{B})$,
$\Rightarrow E_T(\tilde{A}) \le E_T(\tilde{B})$.

Since $E_T(\tilde{A})$ satisfies all the four properties of an entropy measure, therefore, it is a valid entropy for the IFSs. □

Theorem 4.2. *Entropy measure (4.10) is a valid measure for the IVIFSs.*

Proof. In order to prove that the entropy (4.10) is a valid measure, we will have to satisfy all the properties $(E1)$ to $(E4)$ as listed in definition 2.8.

$(E1)$ Let $\tilde{A}_*$ be a crisp set. Then either we have
$[\mu^L_{\tilde{A}_*}(x_i), \mu^U_{\tilde{A}_*}(x_i)] = [1,\, 1], [\nu^L_{\tilde{A}_*}(x_i),\, \nu^U_{\tilde{A}_*}(x_i)] = [0,\, 0]\ \&\ [\pi^L_{\tilde{A}_*}(x_i),\, \pi^U_{\tilde{A}_*}(x_i)] = [0,\, 0]$
or
$[\mu^L_{\tilde{A}_*}(x_i), \mu^U_{\tilde{A}_*}(x_i)] = [0,\, 0], [\nu^L_{\tilde{A}_*}(x_i), \nu^U_{\tilde{A}_*}(x_i)] = [1,\, 1]\ \&\ [\pi^L_{\tilde{A}_*}(x_i), \pi^U_{\tilde{A}_*}(x_i)] = [0,\, 0]$ for

each $x_i \in X$.
Hence, we have $S(\tilde{A}_*,\, \tilde{A}_*^c) = 0 \Rightarrow E_T(\tilde{A}_*) = 0$.
Conversely, suppose that $E_T(\tilde{A}_*) = 0$, then we have
$\min\left\{\mu^L_{\tilde{A}_*}(x_i), \nu^L_{\tilde{A}_*}(x_i)\right\} + \min\left\{\mu^U_{\tilde{A}_*}(x_i), \nu^U_{\tilde{A}_*}(x_i)\right\} + 0.5\left(\pi^L_{\tilde{A}_*}(x_i) + \pi^U_{\tilde{A}_*}(x_i)\right) = 0;$
Since each term in the above equation is non-negative, therefore,
$\min\left\{\mu^L_{\tilde{A}_*}(x_i), \nu^L_{\tilde{A}_*}(x_i)\right\} = 0, \min\left\{\mu^U_{\tilde{A}_*}(x_i),\, \nu^U_{\tilde{A}_*}(x_i)\right\} = 0$
and $\pi^L_{\tilde{A}_*}(x_i) + \pi^U_{\tilde{A}_*}(x_i) = 0$ for each $x_i \in X$;
which further implies that $\tilde{A}_*$ is a crisp set.

$(E2)$ If $[\mu^L_{\tilde{A}_*}(x),\, \mu^U_{\tilde{A}_*}(x)] = [\nu^L_{\tilde{A}_*}(x),\, \nu^U_{\tilde{A}_*}]$ for each $x_i \in X$, then from equation (4.10) we obtain $E_T(\tilde{A}_*) = 1$.
Conversely, if we suppose that $E_T(\tilde{A}_*) = 1$, then we get

$$\begin{aligned}\min\left\{\mu^L_{\tilde{A}_*}(x_i),\, \nu^L_{\tilde{A}_*}(x_i)\right\} &+ \min\left\{\mu^U_{\tilde{A}_*}(x_i),\, \nu^U_{\tilde{A}_*}(x_i)\right\}\\ &= \max\left\{\mu^L_{\tilde{A}_*}(x_i), \nu^L_{\tilde{A}_*}(x_i)\right\} + \max\left\{\mu^U_{\tilde{A}_*}(x_i),\, \nu^L_{\tilde{A}_*}(x_i)\right\};\end{aligned}$$

which implies that $[\mu^L_{\tilde{A}_*}(x),\, \mu^U_{\tilde{A}_*}(x)] = [\nu^L_{\tilde{A}_*}(x),\, \nu^U_{\tilde{A}_*}],\, \forall\, x_i \in X$.

$(E3)$ As per the definition, $S_{NTV}(\tilde{A}_*,\, \tilde{A}_*^c) = S_{NTV}(\tilde{A}_*^c,\, A_*) \Rightarrow E_T(\tilde{A}_*) = E_T(\tilde{A}_*^c)$.

$(E4)$ Let $\tilde{A}_*$ is less fuzzy than $\tilde{B}_*$, i.e., $\tilde{A}_* \subseteq \tilde{B}_*$
$\Rightarrow \mu^L_{\tilde{A}_*}(x_i) \le \mu^L_{\tilde{B}_*}(x_i), \mu^U_{\tilde{A}_*}(x_i) \le \mu^U_{\tilde{B}_*}(x_i) \,\&\, \nu^L_{\tilde{A}_*}(x_i) \ge \nu^L_{\tilde{B}_*}(x_i), \nu^U_{\tilde{A}_*}(x_i) \ge \nu^U_{\tilde{B}_*}(x_i)$
for $\mu^L_{\tilde{B}_*}(x_i) \le \nu^L_{\tilde{B}_*}(x_i)$ and $\mu^U_{\tilde{B}_*}(x_i) \le \nu^U_{\tilde{B}_*}(x_i), \forall x_i \in X$.
Then it follows that $\mu^L_{\tilde{A}_*}(x_i) \le \mu^L_{\tilde{B}_*}(x_i) \le \nu^L_{\tilde{B}_*}(x_i) \le \nu^L_{\tilde{A}_*}(x_i)$
and $\mu^U_{\tilde{A}_*}(x_i) \le \mu^U_{\tilde{B}_*}(x_i) \le \nu^U_{\tilde{B}_*}(x_i) \le \nu^U_{\tilde{A}_*}(x_i), \forall x_i \in X$;
$\Rightarrow \tilde{A}_* \subseteq \tilde{B}_* \subseteq \tilde{B}_*^c \subseteq \tilde{A}_*^c$.
Therefore, by the definition of similarity measure of IVIFSs, we have
$S_{NTV}(\tilde{A}_*,\, \tilde{A}_*^c) \le S_{NTV}(\tilde{B}_*,\, \tilde{A}_*^c) \le S_{NTV}(\tilde{B}_*,\, \tilde{B}_*^c)$.
Similarly, if $\mu^L_{\tilde{A}_*}(x_i) \ge \mu^L_{\tilde{B}_*}(x_i),\, \mu^U_{\tilde{A}_*}(x_i) \ge \mu^U_{\tilde{B}_*}(x_i)$ and $\nu^L_{\tilde{A}_*}(x_i) \le \nu^L_{\tilde{B}_*}(x_i)$,
$\nu^U_{\tilde{A}_*}(x_i) \le \nu^U_{\tilde{B}_*}(x_i)$ for $\mu^L_{\tilde{B}_*}(x_i) \ge \nu^L_{\tilde{B}_*}(x_i)$ and $\mu^U_{\tilde{B}_*}(x_i) \ge \nu^U_{\tilde{B}_*}(x_i), \forall x_i \in X$;
which follows that $\nu^L_{\tilde{A}_*}(x_i) \le \nu^L_{\tilde{B}_*}(x_i) \le \mu^L_{\tilde{B}_*}(x_i) \le \mu^L_{\tilde{A}_*}(x_i)$
and $\nu^U_{\tilde{A}_*}(x_i) \le \nu^U_{\tilde{B}_*}(x_i) \le \mu^U_{\tilde{B}_*}(x_i) \le \mu^U_{\tilde{A}_*}(x_i), \forall x_i \in X$;
$\Rightarrow \tilde{A}_*^c \subseteq \tilde{B}_*^c \subseteq \tilde{B}_* \subseteq \tilde{A}_*$;
$\Rightarrow S_{NTV}(\tilde{A}_*^c,\, \tilde{A}_*) \le S_{NTV}(\tilde{B}_*^c,\, \tilde{A}_*) \le S_{NTV}(\tilde{B}_*^c,\, \tilde{B}_*)$;
$\Rightarrow S_{NTV}(\tilde{A}_*,\, \tilde{A}_*^c) \le S_{NTV}(\tilde{A}_*,\, \tilde{B}_*^c) \le S_{NTV}(\tilde{B}_*,\, \tilde{B}_*^c)$;
$\Rightarrow E_T(\tilde{A}_*) = S_{NTV}(\tilde{A}_*,\, \tilde{A}_*^c) \le S_{NTV}(\tilde{B}_*,\, \tilde{B}_*^c) = E_T(\tilde{B}_*)$;
$\Rightarrow E_T(\tilde{A}_*) \le E_T(\tilde{B}_*)$.
Since $E_T(\tilde{A}_*)$ satisfies all the four properties of an entropy measure, therefore, it is a valid entropy for the IVIFS.

□

5. Multi-criteria group decision making with IFS and IVIFS

In this section, we present a new method which is based on the proposed weighted similarity measures, where the objective weights are calculated using the proposed entropies to deal with the multiple criteria group decision making (MCDM) problems under the intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets. Ratings of the alternatives, importance/weights of criteria and importance of decision makers in a group decision committee are the three most significant factors which can affect on the results of decision making problems.

Let $A = \{A_1, A_2, \ldots, A_m\}$ be the set of possible alternatives, $D = \{D_1, D_2, \ldots, D_l\}$ be the set of decision makers and $C = \{C_1, C_2, \ldots, C_n\}$ be the set of criteria with which the performance of alternatives are measured. Assume that the weight information of the criteria and the decision makers are completely unknown. Let $([a_{ij}, b_{ij}], [c_{ij}, d_{ij}])$ be the interval-valued intuitionistic fuzzy number, where $[a_{ij}, b_{ij}]$ indicates the degree that alternative A_i satisfies the criterion C_j, $[c_{ij}, d_{ij}]$ indicates the degree that alternative A_i does not satisfies the criterion C_j and $[a_{ij}, b_{ij}] \subset [0, 1]$, $[c_{ij}, d_{ij}] \subset [0, 1]$ such that $b_{ij} + d_{ij} \leq 1$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Now, we propose the following algorithm to solve the above multi-criteria group decision making problem:

Step 1: Determine the weights of decision makers in the decision group.
Assume that decision group contains l decision makers. The importance/weights of the decision makers in the selection committee may not be equal. The importance/weights of decision makers are considered as linguistic variables expressed by interval-valued intuitionistic fuzzy numbers (IVIFNs). Let $D_k = ([a_k, b_k], [c_k, d_k])$ be an interval-valued intuitionistic fuzzy number for rating of kth decision maker. Then the subjective weight of kth decision maker can be defined as:

$$\lambda_k = \frac{\left(a_k + b_k + (1 - b_k - d_k)\left(\frac{a_k}{a_k + c_k}\right) + (1 - a_k - c_k)\left(\frac{b_k}{b_k + d_k}\right)\right)}{\sum_{k=1}^{l}\left(a_k + b_k + (1 - b_k - d_k)\left(\frac{a_k}{a_k + c_k}\right) + (1 - a_k - c_k)\left(\frac{b_k}{b_k + d_k}\right)\right)} \tag{5.1}$$

and $\sum_{k=1}^{l} \lambda_k = 1$. The linguistic variables for the importance of the decision makers are provided in the Table 1. If the importance of all the decision makers is same namely extremely importance, the rating of the kth decision maker can be expressed as $([1, 1], [0, 0][0, 0])$. Then the weight of each decision maker will be $1/l$.

Step 2: Construct the aggregated interval-valued intuitionistic fuzzy decision matrix by pulling the individual decision opinions into a group opinions.
Let $D^k = \left(r_{ij}^{(k)}\right)_{m \times n}$ is an interval-valued intuitionistic fuzzy decision matrix for kth $(k = 1, 2, \ldots, l)$ decision maker and $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_l$ is the weight vector for decision makers, $\sum_{k=1}^{l} \lambda_k = 1$, $\lambda_k \in [0, 1]$. In group decision-making process, all the individual decision opinions need to be fused into group opinions to construct aggregated interval-valued intuitionistic fuzzy decision matrix. In order to do, we utilize interval-valued intuitionistic fuzzy weighted average (IIFWA) operator due to Xu *et al.* [33] as follows:

$$\begin{aligned} r_{ij} &= IIFWA_\lambda\left(r_{ij}^{(1)}, r_{ij}^{(2)}, \ldots, r_{ij}^{(l)}\right) \\ &= \left(\left[1 - \prod_{k=1}^{l}(1 - a_{ij}^{(k)})^{\lambda_k}, 1 - \prod_{k=1}^{l}(1 - b_{ij}^{(k)})^{\lambda_k}\right], \left[\prod_{k=1}^{l}(c_{ij}^{(k)})^{\lambda_k}, \prod_{k=1}^{l}(d_{ij}^{(k)})^{\lambda_k}\right]\right). \end{aligned} \tag{5.2}$$

The aggregated interval-valued intuitionistic fuzzy decision matrix can be defined as:

$$D = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{pmatrix} \tag{5.3}$$

Step 3: Determine the aggregated interval-valued intuitionistic fuzzy weights of the criteria using IIFWA operator.
All criteria may not be assumed to be of equal importance. Let W represents a set of grades of importance for given criteria's. In order to obtain W, all the individual decision maker opinions for the importance of each of criterion need to be combined. Let $w_j^{(k)} = \left(\left[a_{ij}^{(k)},\, a_{ij}^{(k)}\right],\, \left[c_{ij}^{(k)},\, d_{ij}^{(k)}\right]\right)$ be an IVIFN assigned to criterion C_j by the kth decision maker. Then the aggregated weights of the criteria are calculated using the IIFWA operator due to Xu *et al.* [33] as follows:

$$\begin{aligned} w_j &= IIFWA_\lambda\left(w_j^{(1)},\, w_j^{(2)},\, \ldots,\, w_j^{(l)}\right) \\ &= \left(\left[1-\prod_{k=1}^{l}(1-a_{ij}^{(k)})^{\lambda_k},\, 1-\prod_{k=1}^{l}(1-b_{ij}^{(k)})^{\lambda_k}\right],\, \left[\prod_{k=1}^{l}(c_{ij}^{(k)})^{\lambda_k},\, \prod_{k=1}^{l}(d_{ij}^{(k)})^{\lambda_k}\right]\right). \end{aligned} \tag{5.4}$$

The aggregated weights of the criteria can be defined as:
$W = [w_1,\, w_2,\, \ldots,\, w_n]^T$, here $w_j = ([a_j,\, b_j],\, [a_j,\, b_j])\,,\, j = 1,\, 2,\, \ldots,\, n.$

Step 4: Construct the aggregated weighted interval-valued intuitionistic fuzzy decision matrix.
After the aggregated weights of criteria and the aggregated interval valued intuitionistic fuzzy decision matrix are determined, the aggregated weighted interval-valued intuitionistic fuzzy decision matrix can be defined as follows:

$$D' = D \otimes W = \left(r'_{ij}\right)_{m\times n}, \tag{5.5}$$

where $r'_{ij} = \left(\left[a'_{ij},\, a'_{ij}\right],\, \left[c'_{ij},\, d'_{ij}\right]\right)$ is an element of the aggregated weighted interval-valued intuitionistic fuzzy decision matrix.

Step 5: Determine the objective weights of criteria using the proposed interval-valued intuitionistic fuzzy entropy measure (4.10).
Hwang and Yoon [15] introduced a method based on information entropy to determine the weights of attributes. Rao *et al.* [20, 21] method also suggested the calculation of objective weights using entropy. Xu [28] and Xu *et al.* [36] assigns a small weight to an attribute with similar attribute values across alternatives because such attribute does not help in differentiating alternatives. Furthermore, the method requires all elements in a decision matrix to be normalized to the range [0, 1] so that each column of the decision matrix sums-to-one.

The entropy of the j^{th} criterion C_j, $j = 1,\, 2,\, \ldots,\, n$ for the m available alternatives can be obtained from entropy measure (4.10) as follows:

$$E_j = \frac{1}{m}\sum_{i=1}^{m}\left(\frac{\min\{a_{ij},\, c_{ij}\} + \min\{b_{ij},\, d_{ij}\} + (1-(a_{ij}+b_{ij}+c_{ij}+d_{ij})/2)}{\max\{a_{ij},\, c_{ij}\} + \max\{b_{ij},\, d_{ij}\} + (1-(a_{ij}+b_{ij}+c_{ij}+d_{ij})/2)}\right) \tag{5.6}$$

and the attribute weight w_j for each criterion C_j based on entropy value can be defined as

$$w_j = \frac{1-E_j}{n-\sum\limits_{j=1}^{n}E_j},\, j = 1,\, 2,\, \ldots,\, n. \tag{5.7}$$

Step 6: Obtain the interval-valued intuitionistic fuzzy positive-ideal solution (IVIFPIS) and the interval-valued intuitionistic fuzzy negative-ideal solution (IVIFNIS).
Let J_1 and J_2 be benefit criteria and cost criteria, respectively. The interval-valued intuitionistic fuzzy positive-ideal solution, denoted as A^+, and the interval-valued intuitionistic fuzzy negative-ideal solution, denoted as A^- , are defined as follows:

$$A^{+} = \left(\left(\left[a_1^{+}, b_1^{+}\right], \left[c_1^{+}, d_1^{+}\right] \right), \left(\left[a_2^{+}, b_2^{+}\right], \left[c_2^{+}, d_2^{+}\right] \right), \ldots, \left(\left[a_n^{+}, b_n^{+}\right], \left[c_n^{+}, d_n^{+}\right] \right) \right), \tag{5.8}$$

$$A^{-} = \left(\left(\left[a_1^{-}, b_1^{-}\right], \left[c_1^{-}, d_1^{-}\right] \right), \left(\left[a_2^{-}, b_2^{-}\right], \left[c_2^{-}, d_2^{-}\right] \right), \ldots, \left(\left[a_n^{-}, b_n^{-}\right], \left[c_n^{-}, d_n^{-}\right] \right) \right), \tag{5.9}$$

where for each $j = 1, 2, \ldots, n$,

$$\begin{aligned} \left(\left[a_j^{+}, b_j^{+}\right], \left[c_j^{+}, d_j^{+}\right] \right) = (&\langle [\max a_{ij}, \max b_{ij}], [\min a_{ij}, \min b_{ij}] \,|j \in J_1\rangle, \\ &\langle [\min a_{ij}, \min b_{ij}], [\max a_{ij}, \max b_{ij}] \,|j \in J_2\rangle) \end{aligned} \tag{5.10}$$

$$\begin{aligned} \left(\left[a_j^{-}, b_j^{-}\right], \left[c_j^{-}, d_j^{-}\right] \right) = (&\langle [\min a_{ij}, \min b_{ij}], [\max a_{ij}, \max b_{ij}] \,|j \in J_1\rangle, \\ &\langle [\max a_{ij}, \max b_{ij}], [\min a_{ij}, \min b_{ij}] \,|j \in J_2\rangle). \end{aligned} \tag{5.11}$$

Step 7: Calculate the similarity measures of alternatives with the IVIFPIS and IVIFNIS based on proposed weighted similarity measure (3.9), respectively as follows:
The similarity between alternatives can be found based on the proposed weighted similarity measure (3.9) as follows:

$$S(A_i, A^{+}) = \sum_{j=1}^{n} w_j \left(\frac{p+q}{s+t} \right), \tag{5.12}$$

and

$$S(A_i, A^{-}) = \sum_{j=1}^{n} w_j \left(\frac{p'+q'}{s'+t'} \right), \tag{5.13}$$

where

$$\begin{aligned} p &= \min\left\{a_{ij}, a_j^{+}\right\} + \min\left\{c_{ij}, c_j^{+}\right\} + \min\left\{1 - b_{ij} - d_{ij}, 1 - b_j^{+} - d_j^{+}\right\}, \\ q &= \min\left\{b_{ij}, b_j^{+}\right\} + \min\left\{d_{ij}, d_j^{+}\right\} + \min\left\{1 - a_{ij} - c_{ij}, 1 - a_j^{+} - c_j^{+}\right\}, \\ s &= \max\left\{a_{ij}, a_j^{+}\right\} + \max\left\{c_{ij}, c_j^{+}\right\} + \max\left\{1 - b_{ij} - d_{ij}, 1 - b_j^{+} - d_j^{+}\right\}, \\ t &= \max\left\{b_{ij}, b_j^{+}\right\} + \max\left\{d_{ij}, d_j^{+}\right\} + \max\left\{1 - a_{ij} - c_{ij}, 1 - a_j^{+} - c_j^{+}\right\}, \\ p' &= \min\left\{a_{ij}, a_j^{-}\right\} + \min\left\{c_{ij}, c_j^{-}\right\} + \min\left\{1 - b_{ij} - d_{ij}, 1 - b_j^{-} - d_j^{-}\right\}, \\ q' &= \min\left\{b_{ij}, b_j^{-}\right\} + \min\left\{d_{ij}, d_j^{-}\right\} + \min\left\{1 - a_{ij} - c_{ij}, 1 - a_j^{-} - c_j^{-}\right\}, \\ s' &= \max\left\{a_{ij}, a_j^{-}\right\} + \max\left\{c_{ij}, c_j^{-}\right\} + \max\left\{1 - b_{ij} - d_{ij}, 1 - b_j^{-} - d_j^{-}\right\}, \\ t' &= \max\left\{b_{ij}, b_j^{-}\right\} + \max\left\{d_{ij}, d_j^{-}\right\} + \max\left\{1 - a_{ij} - c_{ij}, 1 - a_j^{-} - c_j^{-}\right\}, \end{aligned}$$

Step 8: Calculate the relative closeness coefficient to the interval-valued intuitionistic fuzzy ideal solution.
The relative closeness coefficient of an alternative A_i with respect A^{+} and A^{-} is defined as follows:

$$C_{i^*} = \frac{S(A_i, A^{+})}{S(A_i, A^{+}) + S(A_i, A^{-})}, \; i = 1, 2, \ldots, m. \tag{5.14}$$

Step 9: Rank all the alternatives.
After the relative closeness coefficient of each alternative is determined, alternatives are ranked according to descending order of C_{i^*}'s and select one that has largest rank, denoted by C_{k^*} among the values C_{i^*}, $i = 1, 2, \ldots, m$. Hence, C_{i^*} is the best choice.

Remark 5.1. Since the intuitionistic fuzzy set is a particular case of interval-valued intuitionistic fuzzy set, therefore above proposed algorithm for IVIFSs may similarly be outline for IFSs. For this, we will have to make the following changes:

- In step 1, the subjective weight given by the equation (5.1) will be replaced by the weight as suggested in Boran [5].
- In step 2 and 3, the interval-valued intuitionistic fuzzy weighted average (IIFWA) operator [33] will be replaced by the intuitionistic fuzzy weighted average (IFWA) operator [30].
- In step 5, the entropy measure given by the equation (4.10) will be replaced by the entropy measure given by the equation (4.9).
- In step 5, the weighted similarity measure given by the equation (3.9) will be replaced by the weighted similarity measure given by the equation (3.7).

Table 1. The importance of decision makers and their weights.

	DM_1	DM_2	DM_3
linguistic terms	Very Important	Medium	Important
Weight	0.393	0.236	0.372

Table 2. Linguistic terms for rating the importance of criteria and the decision makers

Linguistic terms	IFNs	IVIFNs
Very Important (VI)	(0.90, 0.10)	$([0.90, 0.95], [0.00, 0.05])$
Important (I)	(0.85, 0.10)	$([0.85, 0.90], [0.05, 0.10])$
Medium (M)	(0.50, 0.40)	$([0.50, 0.55], [0.35, 0.40])$
Unimportant (U)	(0.20, 0.70)	$([0.20, 0.25], [0.65, 0.70])$
Very Unimportant (VU)	(0.05, 0.90)	$([0.05, 0.10], [0.85, 0.90])$

Table 3. Linguistic terms for rating the alternatives

Linguistic terms	IFNs	IVIFNs
Extremely Good (EG)/Extremely High (EH)	(0.95, 0.05)	$([0.90, 95.00], [0.00, 0.05])$
Very Very Good (VVG)/Very Very High (VVH)	(0.85, 0.10)	$([0.85, 0.90], [0.05, 0.10])$
Very good (VG)/Very High (VH)	(0.80, 0.15)	$([0.80, 0.85], [0.10, 0.15])$
Good (G)/High (H)	(0.75, 0.20)	$([0.75, 0.80], [0.15, 0.20])$
Medium Good (MG)/Medium High (MH)	(0.60, 0.25)	$([0.60, 0.65], [0.20, 0.25])$
Fair (F)/Medium (M)	(0.50, 0.35)	$([0.50, 0.55], [0.30, 0.35])$
Medium Poor (MP)/Medium Low (ML)	(0.40, 0.55)	$([0.40, 0.45], [0.50, 0.55])$
Poor (P)/Low (L)	(0.30, 0.65)	$([0.30, 0.35], [0.60, 0.65])$
Very Poor (VP)/Very Low (VL)	(0.20, 0.75)	$([0.20, 0.25], [0.70, 0.75])$
Very Very Poor (VVP)/Very Very Low (VVL)	(0.10, 0.85)	$([0.10, 0.15], [0.80, 0.85])$

6. Numerical examples

Example 6.1. An automobile company desires to select the most appropriate supplier for one of the key elements in its manufacturing process. After pre-evaluation, five suppliers (A_1, A_2, A_3, A_4, A_5) have remained as alternatives for further evaluation. In order to evaluate alternative suppliers, a committee of three decision makers DM_1, DM_2 and DM_3 has been formed. Four criteria are considered as:

- X_1: Product quality.
- X_2: Relationship closeness.
- X_3: Delivery performance.
- X_4: Price.

The proposed method is currently applied to solve this problem and the computational procedure is as follows:

Importance degree of the decision makers on group decision is shown in Table 1. Linguistic terms used for the ratings of the decision makers and criteria are given in Table 2. In order to obtain the weights of the decision makers, Equation 5.1 is utilized:

$$\lambda_{DM_1} = 0.393,\ \lambda_{DM_2} = 0.372,\ \lambda_{DM_2} = 0.236.$$

Now the aggregated interval-valued intuitionistic fuzzy decision matrix based on the opinions of decision makers is constructed using IIFWA operator. The linguistic terms shown in Table 3 are used to rate each alternative supplier with respect to each criterion by three decision makers. The ratings given by the decision makers to five alternatives is shown in Table 4.

TABLE 4. The rating the alternatives

Criteria	Suppliers	Decisions makers			Criteria	Suppliers	Decisions makers		
		DM_1	DM_2	DM_3			DM_1	DM_2	DM_3
X_1	A_1	G	G	G	X_3	A_1	VG	G	VG
	A_2	MG	G	F		A_2	G	MG	MG
	A_3	VVG	VG	VG		A_3	VG	VG	G
	A_4	MG	G	G		A_4	VG	G	G
	A_5	F	MG	MG		A_5	G	G	MG
X_2	A_1	MG	G	MG	X_4	A_1	H	H	H
	A_2	F	MG	G		A_2	MH	M	MH
	A_3	VG	G	VG		A_3	VH	VH	H
	A_4	F	F	MG		A_4	H	MH	MH
	A_5	MP	F	F		A_5	M	MH	M

TABLE 5. The importance weight of the criteria

Criteria	DM_1	DM_2	DM_3
X_1	VI	VI	I
X_2	I	I	I
X_3	I	I	M
X_4	M	I	M

The aggregated interval-valued intuitionistic fuzzy decision matrix based on aggregation of decision makers opinions is constructed shown in Matrix representation A.

The importance weights of the criteria provided by decision makers can be linguistic terms. These linguistic terms is represented as interval-valued intuitionistic fuzzy numbers in Table 5 and opinions of decision makers on criteria are aggregated using Equation 5.4 to determine the aggregated weights of criteria. The interval-valued intuitionistic fuzzy weights of criteria after aggregation of opinions of decision makers is:

$$W = \begin{bmatrix} ([0.884,\ 0.936],\ [0.000,\ 0.065]) \\ ([0.850,\ 0.900],\ [0.050,\ 0.100]) \\ ([0.766,\ 0.825],\ [0.103,\ 0.167]) \\ ([0.624,\ 0.685],\ [0.221,\ 0.288]) \end{bmatrix}$$

Matrix representation A

$$D = \begin{bmatrix} ([0.750, 0.800], [0.150, 0.120]) & ([0.642, 0.694], [0.187, 0.237]) & ([0.790, 0.840], [0.110, 0.160]) & ([0.750, 0.800], [0.150, 0.200]) \\ ([0.611, 0.664], [0.217, 0.268]) & ([0.634, 0.687], [0.210, 0.262]) & ([0.668, 0.719], [0.178, 0.229]) & ([0.579, 0.629], [0.220, 0.270]) \\ ([0.822, 0.872], [0.076, 0.128]) & ([0.790, 0.840], [0.110, 0.160]) & ([0.783, 0.833], [0.116, 0.167]) & ([0.783, 0.833], [0.116, 0.167]) \\ ([0.700, 0.751], [0.168, 0.218]) & ([0.540, 0.591], [0.258, 0.309]) & ([0.771, 0.822], [0.128, 0.178]) & ([0.668, 0.719], [0.178, 0.229]) \\ ([0.564, 0.614], [0.234, 0.285]) & ([0.463, 0.514], [0.366, 0.418]) & ([0.703, 0.754], [0.167, 0.217]) & ([0.526, 0.576], [0.272, 0.323]) \end{bmatrix}$$

Matrix representation B

$$D' = \begin{bmatrix} ([0.663, 0.749], [0.150, 0.251]) & ([0.546, 0.624], [0.227, 0.313]) & ([0.605, 0.693], [0.201, 0.301]) & ([0.468, 0.548], [0.338, 0.430]) \\ ([0.540, 0.621], [0.217, 0.316]) & ([0.539, 0.618], [0.250, 0.336]) & ([0.511, 0.594], [0.263, 0.358]) & ([0.361, 0.431], [0.392, 0.481]) \\ ([0.726, 0.816], [0.076, 0.184]) & ([0.671, 0.756], [0.154, 0.244]) & ([0.600, 0.688], [0.207, 0.306]) & ([0.489, 0.571], [0.311, 0.407]) \\ ([0.619, 0.703], [0.168, 0.268]) & ([0.459, 0.532], [0.295, 0.378]) & ([0.591, 0.678], [0.217, 0.316]) & ([0.417, 0.493], [0.360, 0.451]) \\ ([0.498, 0.575], [0.234, 0.331]) & ([0.394, 0.462], [0.398, 0.476]) & ([0.538, 0.623], [0.252, 0.348]) & ([0.328, 0.395], [0.433, 0.518]) \end{bmatrix}$$

After the weights of the criteria and the rating of the alternatives has been determined, the aggregated weighted interval-valued intuitionistic fuzzy decision matrix is constructed utilizing Equation 5.5 shown in Matrix representation B.

The entropy of the j^{th} criterion X_j, $j = 1, 2, \ldots, 4$ for the available alternatives can be obtained from entropy measure (4.10). The objectives weights of criteria based on entropy are $w_1 = 0.359$, $w_2 = 0.230$, $w_3 = 0.303$, $w_4 = 0.108$.

Product quality, relationship closeness and delivery performance are benefit criteria $J_1 = \{X_1, X_2, X_3\}$ and price is cost criteria $J_2 = \{X_4\}$. Then interval-valued intuitionistic fuzzy positive-ideal solution and interval-valued intuitionistic fuzzy negative ideal solution are

$$\begin{aligned} A^+ &= \{([0.726, 0.816], [0.076, 0.184]), ([0.671, 0.756], [0.154, 0.244]), \\ &\quad ([0.605, 0.693], [0.201, 0.301]), ([0.328, 0.395], [0.433, 0.518])\} \end{aligned}$$

and

$$\begin{aligned} A^- &= \{([0.498, 0.575], [0.234, 0.331]), ([0.394, 0.462], [0.398, 0.476]), \\ &\quad ([0.511, 0.594], [0.263, 0.358]), ([0.489, 0.571], [0.311, 0.407])\}. \end{aligned}$$

Similarity of each alternative with the IVIFPIS and IVIFPIN based on proposed weighted similarity measure (3.9) is calculated in Table 6.

TABLE 6. Similarities with the IVIFPIS and IVIFPIN

Alternatives	S^+	S^-
A_1	0.873	0.772
A_2	0.769	0.883
A_3	0.966	0.711
A_4	0.818	0.818
A_5	0.722	0.953

Finally, using Equation 5.14, the value of relative closeness of each alternative for the final ranking is shown in Table 6.

TABLE 7. Relative closeness coefficient

Alternatives	C_{i^*}
A_1	0.531
A_2	0.465
A_3	0.576
A_4	0.500
A_5	0.431

Thus, the preference order of alternatives is A_1, A_2, A_3, A_4 and A_5 according to decreasing order of C_{i^*} is

$$A_3 > A_1 > A_4 > A_2 > A_5.$$

7. Conclusions

The proposed new similarity measures for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets based on 'NTV' metric along with their weighted form are valid similarity measures. The new intuitionistic fuzzy entropies for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets analogously obtained through the proposed similarity measures are also valid information measures. Further, a new algorithm for MCDM using the proposed weighted similarity measures in which the weights have been calculated using the proposed entropies, has been illustrated through a numerical example.

Acknowledgements. The authors would like to thank the Editor-in-chief and anonymous reviewers for their critical comments and valuable suggestions. The general and technical remarks of the editor and reviewers were of great help for improving the quality and presentation of the manuscript.

References

[1] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20(1) (1986) 87–96.
[2] K. T. Atanassov, More on intuitionistic fuzzy sets, Fuzzy sets and Systems 33 (1989) 37–46.
[3] K. T. Atanassov, Intuitionistic fuzzy sets: theory and applications, Physica-Verlag, Heidelberg (1999).
[4] K. T. Atanassov and G. Gargov, Interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 31(3) (1989) 343–349.
[5] F. M. Boran, S. Genç, M. Kurt and D. Akay, A multi-criteria intuitionistic fuzzy group decision making for supplier selection with TOPSIS method, Expert Systems with Applications 36(8) (2009) 11363–11368.
[6] P. Burillo and H. Bustince, Entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets, Fuzzy Sets and Systems 78(3) (1996) 305–316.
[7] H. Bustince and P. Burillo, Vague sets are intuitionistic fuzzy sets, Fuzzy Sets and Systems 79(3) (1996) 403–405.
[8] S. K. De, R. Biswas and A. R. Roy, An application of intuitionistic fuzzy sets in medical diagnosis, Fuzzy Sets and Systems 117 (2001) 209–213.
[9] A. Deluca and S. Termini, A definition of a non-probabilistic entropy in the setting of fuzzy sets theory, Information and Control 20(4) (1972) 301–312.
[10] L. Dengfeng and C. Chuntain, New similarity measures of intuitionistic fuzzy sets and application to pattern recognition, Pattern Recognition Letters 23(1–3) (2002) 221–225.
[11] G. S. Hung and Y. S. Liu, The fuzzy entropy of vague sets based on non-fuzzy sets, Computer Engineering and Applications 22 (2005) 16–17.
[12] W. L. Hung and M. S. Yang, Similarity measures of intuitionistic fuzzy sets based on Hausdorff distance, Pattern Recognition Letters 25(14) (2004) 1603–1611.
[13] W. L. Hung and M. S. Yang, Fuzzy entropy on intuitionistic fuzzy sets, International Journal of Intelligent Systems 21(4) (2006) 443–451.
[14] W. L. Hung and M. S. Yang, On similarity measures between intuitionistic fuzzy sets, International Journal of Intelligent Systems 23(3) (2008) 364–383.
[15] C. L. Hwang and K. Yoon, Multiple Attribute Decision Making: Methods and Applications, Springer, Heidelberg, (1981).
[16] Z. Liang and P. Shi, Similarity measures on intuitionistic fuzzy sets, Pattern Recognition Letters 24(15) (2003) 2687–2693.
[17] X. Liu, S. Zheng and F. Xiong, Entropy and subsethood for general interval-valued intuitionistic fuzzy sets, Fuzzy Systems and Knowledge Discovery 3613 (2005) 42–52.
[18] J. J. Nieto, A. Torres and M. M. Vzquez-Trasande, A metric space to study differences between polynucleotides, Appl. Math. Lett. 16(8) (2003) 1289–1294.
[19] C. P. Pappis and N. I. Karacapilidis, A comparative assessment of measures of similarity of fuzzy values, Fuzzy Sets and Systems 56(2) (1993) 171–174.

[20] R. V. Rao, Decision Making in the Manufacturing Environment using Graph Theory and Fuzzy Multiple Attribute Decision making methods, Springer-Verlag, London, (2007).
[21] R. V. Rao and D. Singh, Weighted Euclidean distance based approach as a multiple attribute decision making method for plant or facility layout design selection, International Journal of Industrial Engineering Computations 3(3) (2012) 365–382
[22] E. Szmidt and J. Kacprzyk, Group decision making under intuitionistic fuzzy preference relations, Proc. of 7th IPMU Conf. Paris (1998) 172–178.
[23] E. Szmidt and J. Kacprzyk, Entropy for intuitionistic fuzzy sets, Fuzzy Sets and Systems 118 (2001) 467–477.
[24] E. Szmidt and J. Kacprzyk, A new concept of a similarity measure for intuitionistic fuzzy sets and its use in group decision making, Modeling Decisions for Artificial Intelligence 3558 (2005) 272–282.
[25] I. K. Vlachos and G. D. Sergiadis, Intuitionistic fuzzy information applications to pattern recognition, Pattern Recognition Letters 28(2) (2007) 197–206.
[26] W. J. Wang, New similarity measures on fuzzy sets and on elements, Fuzzy Sets and Systems 85(3) (1997) 305–309.
[27] C. P. Wei, P. Wang and Y. Z. Zhang, Entropy, similarity measure of interval-valued intuitionistic fuzzy sets and their applications, Inform. Sci. 181(19) (2011) 4273–4286.
[28] X. Xu, A note on the subjective and objective integrated approach to determine attribute weights, European J. Oper. Res. 156(2) (2004) 530–532.
[29] Z. S. Xu, On similarity measures of interval-valued intuitionistic fuzzy sets and their application to pattern recognitions, J. Southeast Univ. 23 (2007) 139–143.
[30] Z. S. Xu, Intuitionistic fuzzy aggregation opterators, IEEE Transaction of Fuzzy Systems 15(6) (2007) 1179–1187.
[31] Z. S. Xu, Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision making, Control Decis. 22 (2007) 215–219.
[32] Z. S. Xu and X. Cai, Intuitionistic Fuzzy Information Aggregation: Theory and Applications, Springer, Jointly published with Science Press, (2013).
[33] Z. S. Xu and J. Chen, Approach to group decision making based on interval-valued intuitionistic judgment matrices, System Engineering Theory & Practice 27(4) (2007) 126–133.
[34] Z. S. Xu and J. Chen, On geometric aggregation over interval-valued intuitionistic fuzzy information. The 4th International Conference on Fuzzy Systems and Knowledge Discovery (FSKD07), Haikou, China: (2007) 466–471.
[35] Z. S. Xu and J. Chen, An overview of distance and similarity measures of intuitionistic sets, Internat. J. Uncertain. Fuzziness Knowledge-Based Systems 16(04) (2008) 529–555.
[36] Z. S. Xu and H. Hui, Entropy-based procedures for intuitionistic fuzzy multiple attribute decision making, Journal of Systems Engineering and Electronics 20(5) (2009) 1001–1011.
[37] Z. S. Xu and R. R. Yager, Some geometric aggregation operators based on intuitionistic fuzzy sets, Int. J. Gen. Syst. 35(4) (2006) 417–433.
[38] Z. S. Xu and R. R. Yager, Intuitionistic fuzzy Bonferroni means, IEEE Transactions on Systems, Man, and Cybernetics-Part B 41(2) (2011) 568–578.
[39] L. Xuecheng, Entropy, distance measure and similarity measure of fuzzy sets and their relations, Fuzzy Sets and Systems 52(3) (1992) 305–318.
[40] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–356.
[41] W. Zeng and P. Guo, Normalized distance, similarity measure, inclusion measure and entropy of interval-valued fuzzy sets and their relationship, Inform. Sci. 178(5) (2008) 1334–1342.
[42] W. Zeng and H. Li, Relationship between similarity measure and entropy of interval-valued fuzzy sets, Fuzzy Sets and Systems 157(11) (2006) 1477–1484.
[43] W. Zeng, W. Yu, F. Yu, X. Chen, H. Wu and S. Wu, Entropy of intuitionistic fuzzy set based on similarity measure, International Journal of Innovative Computing, Information and Control 5 12(**A**) (2009) 4737–4744.
[44] H. Zhang, W. Zhang and C. Mei, Entropy of interval-valued fuzzy sets based on distance and its relationship with similarity measure, Knowledge-Based Systems 22(6) (2009) 449–454.

TANUJ KUMAR (tanujkhutail@gmail.com)
Department of Mathematics, Jaypee University of Information Technology, Waknaghat, Solan, Himachal Pradesh, Postal Code– 173234.

RAKESH KUMAR BAJAJ (rakesh.bajaj@gmail.com)
Department of Mathematics, Jaypee University of Information Technology, Waknaghat, Solan, Himachal Pradesh, Postal Code– 173234.