On induced fuzzy supra-topological spaces

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Abstract. The concept of induced fuzzy topological space, introduced by Weiss [J. Math. Anal. Appl. 50(1975), 142-150], was defined with the notion of a lower semi-continuous function. The aim of this paper is to introduce and study the concepts of induced fuzzy supra-topological spaces and $s$-lower $\beta$-continuous functions. $s$-Lower $\beta$-continuous functions turn out to be the natural tool for studying the induced fuzzy supra-topological spaces.

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1. Introduction

In 1997 [6], Bhaumik and Mukherjee introduced and studied the concepts of induced fuzzy supra-topological spaces and $s$-Lower semi-continuous functions. In 1998 [10], Mukherjee defined and studied a new class of fuzzy supra topological space under the name of $\alpha$-induced fuzzy supra topological space. Moreover, Mukherjee in 2003 [11], introduced the concept of $\Sigma$—induced $L$—fuzzy supra topological space and Scott $s$-continuity. After the introduction of $\beta$-open subsets by Abd El-Monsef et al. [1], various concepts in topological space were introduced with the help of $\beta$-open subsets instead of open subsets. In Section 2, the concept of $s$-lower $\beta$-continuous function is introduced by using $\beta$-open subsets. Some characterizations and properties of these functions are examined. In Section 3, these functions are used to define a new class of fuzzy supra-topological spaces, called induced fuzzy supra-topological spaces. The fuzzy supra-continuous functions and initial supra-topological spaces are also investigated. The supra-interior and supra-closure of a fuzzy subset $\mu$ are denoted, respectively by $\mu^{*}$ and $\mu^{sc}$ [2].
2. s-Lower $\beta$-continuous functions

**Definition 2.1.** A function $f : (X, \tau) \rightarrow (\mathbb{R}, \sigma)$ from a topological space $(X, \tau)$ to usual topology $(\mathbb{R}, \sigma)$ is said to be s-lower $\beta$-continuous (resp. s-upper $\beta$-continuous) at $x_0 \in X$ if for each $\epsilon > 0$, there exists a $\beta$-open neighbourhood $N(x_0)$ such that $x \in N(x_0)$ implies $f(x) > f(x_0) - \epsilon$ (resp. $f(x) < f(x_0) + \epsilon$)

The following results can easily be proved analogous to the Theorem 2 in [3].

**Result 2.2.** The necessary and sufficient condition for a real-valued function $f$ to be s-lower $\beta$-continuous is that for all $r \in \mathbb{R}$, the set $\{x \in X : f(x) > r\}$ is $\beta$-open (or equivalently $\{x \in X : f(x) \leq r\}$ is $\beta$-closed).

**Result 2.3.** The characteristic function of a $\beta$-open subset is s-lower $\beta$-continuous.

**Result 2.4.** The sum and product of two s-lower $\beta$-continuous functions are not necessarily s-lower $\beta$-continuous functions.

**Result 2.5.** If $\{f_i : j \in J\}$ is an arbitrary family of s-lower $\beta$-continuous functions, then the function $g$, defined by $g(x) = \sup_j f_j(x)$ is s-lower $\beta$-continuous.

**Remark 2.6.** If $f_1, f_2, f_3, ..., f_n$ are s-lower $\beta$-continuous functions, then the function $h$, defined by $h(x) = \inf_{i=1}^n (f_i(x))$, where $i = 1, 2, ..., n$ is not s-lower $\beta$-continuous.

**Result 2.7.** A function $f$ from a topological space $(X, \tau)$ into a space $(\mathbb{R}, \sigma_1)$, where $\sigma_1 = \{\{r, \infty\} : r \in \mathbb{R}\}$ is s-lower $\beta$-continuous iff the inverse image of any open subset of $(\mathbb{R}, \sigma_1)$ is $\beta$-open in $(X, \tau)$.

**Definition 2.8** ([7]). Recall that a function $f : (X, \tau) \rightarrow [0, 1]$ is called Scott continuous (lower $\beta$-continuous) at $a \in X$ if for every $\alpha \in [0, 1)$ with $\alpha < f(a)$ there is a neighborhood $U$ of $a$ such that $\alpha < f(x)$ for every $x \in U$. If it is called Scott continuous (or lower semi continuous) on $X$ if $f$ is Scott continuous (or lower semi continuous) at every point of $X$.

Since every open subset is $\beta$-open, we have the following result.

**Result 2.9.** Every lower $\beta$-continuous function is s-lower $\beta$-continuous.

The converse of the Result 2.9 is not true which can be seen from the following example.

**Example 2.10.** Let $X = \{a, b, c, d\}$ and $Y = \{0, 1, 2\}$. Let

$$\tau = \{X, \emptyset, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}\} \text{ and } \tau_1 = \{Y, \emptyset, \{2\}\}$$

be the topologies on $X$ and $Y$, respectively. A function $g : X \rightarrow Y$ is defined by $g(a) = g(d) = 2, g(b) = 1$ and $g(c) = 0$. Now $g^{-1}(0) = \{c\}, g^{-1}(1) = \{b\}$ and $g^{-1}(2) = \{a, d\}$. We observe that $\{a, d\}$ is $\beta$-open in $(X, \tau)$, since $\{d\} \subseteq \{a, d\} \subseteq \{a, b, d\} = C_1\{d\}$. For all $r \in Y$, by Result 2.2, $g$ is s-lower $\beta$-continuous. Since inverse image of the open subset $\{2\}$ of $Y$ is $\beta$-open, $g$ is not lower $\beta$-continuous.
2.1. **Initial supra-topology.** Finally we shall define an initial supra-topology on $X$.

**Definition 2.11.** Let $(X,\varphi(\tau))$ be an induced fuzzy supra-topological space. The family $\{\sigma_r(\alpha) : \alpha \in \varphi(\tau), r \in I\}$ of all $\beta$-open subsets of $X$ form a supra-topology on $X$, called the initial supra-topology on $X$ and is denoted by $i(\varphi)$. $(X, i(\varphi))$ is called the initial supra-topological space. Thus the relation between the initial supra-topology and the corresponding topology $\tau$ of $\varphi(\tau)$ is $\tau \subseteq i(\varphi)$.

**Example 2.12.** Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{c\}, \{d\}, \{c, d\}, \{a, c, d\}\}$ be a topology on $X$. Besides the members of $\tau$, $\{a, d\}$, $\{b, d\}$ and $\{b, c, d\}$ are also $\beta$-open subsets in $(X, \tau)$.

Now $1_\emptyset$, $1_X$, $1_{\{c\}}$, $1_{\{d\}}$, $1_{\{c,d\}}$, $1_{\{a,d\}}$, $1_{\{b,d\}}$ and $1_{\{b,c,d\}}$ are $s$-lower $\beta$-continuous, since the characteristic function of a $\beta$-open subset is $s$-lower $\beta$-continuous. Thus the collection of all these functions forms an induced fuzzy supra-topology on $X$. Here $i(\varphi) = \{\emptyset, X, \{c\}, \{d\}, \{c, d\}, \{a, d\}, \{b, d\}, \{b, c, d\}, \{a, c, d\}\}$. Thus $\tau \subseteq i(\varphi)$.

**Note.** If we take $\beta O(X)$, the family of all $\beta$-open subsets of $X$, then $\beta O(X) = i(\varphi)$.

### 3. INDUCED FUZZY SUPRA-TOPOLOGICAL SPACE

The notion of supra-topology or pre-topology was due to Garg and Naimpally [8] and that of induced fuzzy topology was due to Weiss [13]. Abd El-Monsef and Ramadan [2] introduced the concept of fuzzy supra-topology as follows: A family $F' \subseteq 1^X$ is called a fuzzy supra-topology on $X$ if $0, 1 \in F'$ and $F'$ is closed under arbitrary union. In this section the notion of induced fuzzy supra-topology is introduced as a generalization of induced fuzzy topology. Its properties and the concepts of fuzzy supra-continuity in induced supra-topological spaces and initial supra-topology are also studied.

3.1. **Induced fuzzy supra-topology and its properties.**

**Theorem 3.1.** Let $(X, \tau)$ be a topological space. The family of all $s$-lower $\beta$-continuous functions from the topological space $(X, \tau)$ to the closed unit interval $I$ forms a fuzzy supra-topology on $X$.

**Proof.** Let $\varphi$ be the collection of all $s$-lower $\beta$-continuous functions from the topological space $(X, \tau)$ to $I$. We will now prove that $\varphi$ is a fuzzy supra-topology on $X$.

(i) Since $X$ is open, it is $\beta$-open and by Result 2.3, $1_x$ is $s$-lower $\beta$-continuous. Thus $1_x \in \varphi$.

(ii) $\emptyset$ is $\beta$-open since it is open in $X$. Thus $1_\emptyset$ is $s$-lower $\beta$-continuous, i.e. $1_\emptyset \in \varphi$.

(iii) Let $\{n_j\}$ be an arbitrary family of $s$-lower $\beta$-continuous functions. By Result 2.3, $\text{Sup}\{n_j\}$ is also $s$-lower $\beta$-continuous. Hence $\bigvee n_j \in \varphi$.

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Thus $\varphi$ satisfies conditions (i) – (iii) of supra-topology. Hence $\varphi$ forms a fuzzy supra-topology.

**Definition 3.2.** The fuzzy supra-topology, obtained as above, is called induced fuzzy supra-topology and the space $(X, \varphi(\tau))$ is called the induced fuzzy supra-topological space. The members of $\varphi(\tau)$ are called fuzzy supra-open subsets.

**Theorem 3.3.** A fuzzy subset $\alpha$ in an induced fuzzy supra-topological space $(X, \varphi(\tau))$ is fuzzy supra-open iff for each $r \in I$, the strong $r$-cut $\sigma_r(\alpha)$ is $\beta$-open in the topological space $(X, \tau)$.

**Proof.** A fuzzy subset $\alpha$ is fuzzy supra-open in $(X, \varphi(\tau))$ if $\alpha \in \varphi(\tau)$ iff $\alpha$ is $s$-lower $\beta$-continuous (by Theorem 3.1). Thus for each $r \in I$, $\{x \in X : \alpha(x) > r\}$ is $\beta$-open in $(X, \tau)$ (by Result 2.2). That is, $\sigma_r(\alpha)$ is $\beta$-open in $(X, \tau)$. \qed

**Corollary 3.4.** A fuzzy subset $\alpha$ in an induced fuzzy supra-topological space $(X, \varphi(\tau))$ is fuzzy supra-closed iff for each $r \in I$, the weak $r$-out $W_r(\alpha)$ is $\beta$-closed in the topological space $(X, \tau)$.

**Theorem 3.5.** If $A$ is $\beta$-open in $(X, \tau)$, then $1_A$ is fuzzy supra $\beta$-open in $(X, \varphi(\tau))$.

**Proof.** Let $A$ be $\beta$-open in $(X, \tau)$, then $A \subseteq cl(int(cl(A)))$, i.e., $1_A \subseteq 1_{cl\,int\,clA} = cl\,int\,clA = cl\,int\,1_A$. Then $1_A$ is fuzzy supra $\beta$-open in $(X, \varphi(\tau))$. \qed

The following theorem follows immediately from Result 2.9.

**Theorem 3.6.** If $\mathcal{F}$ is an induced fuzzy topology and $\varphi$ is an induced supra-topology on $X$, then $\mathcal{F} \subseteq \varphi$.

In [4] the completely induced fuzzy topology was introduced and by Lemma 2.4 of [5] we have the following corollary.

**Corollary 3.7.** $\mathfrak{S} \subseteq \mathcal{F} \subseteq \varphi$, where $\mathfrak{S}$ is a completely induced fuzzy topology.

Let $(X, \mathcal{F})$ be a fuzzy topological space and $\mathcal{F}'$ be a fuzzy supra-topology on $X$. We call $\mathcal{F}'$ a fuzzy supra-topology associated with $\mathcal{F}$ if $\mathcal{F} \subseteq \mathcal{F}'$. The family $\mathcal{F}'\beta O(X)$ of all fuzzy $\beta$-open subsets in $(X, \mathcal{F})$ is fuzzy supra-topology associated with $\mathcal{F}$.

**Theorem 3.8.** Let $(X, \mathcal{F}')$ be a fuzzy supra-topological space where $\mathcal{F}'$ is associated with the fuzzy topology $\mathcal{F}$ on $X$ and $\tau = \mathcal{F} \cap 2^X$. Then the induced fuzzy supra-topology $\varphi(\tau)$ on $X$ is equivalent to the fuzzy supra-topology $\mathcal{F}'$ if for any fuzzy subset $\mu$ and $r \in I$, $W_r^{(\alpha)}(\mu) = \bigcap \{Cl_\tau(W_\ell(\mu)) : t < r\}$ (resp. $\sigma_r^{(\beta)}(\mu) = \bigcup \{Int_\tau(\sigma_\ell(\mu)) : t > r\}$) is $\beta$-closed (resp. $\beta$-open) in $(X, \tau)$.

**Proof.** Let $\mu$ be $\mathcal{F}'$-closed subset of $X$ and $r \in I$, then by the given condition, $W_r^{(\alpha)}(\mu) = \bigcap \{Cl_\tau(W_\ell(\mu)) : t < r\}$ is $\beta$-closed in $(X, \tau)$. By Theorem 3.3 $\mu$ is $\varphi(\tau)$-closed. Now if $\alpha$ is $\varphi(\tau)$-closed and $r \in I$, then
\[ W_r(\alpha^{sc}) = \bigcap \{ Cl_r(W_t(\alpha)) : t < r \} = \bigcap \{ W_t(\alpha) : t < r \} = W_r(\alpha). \]

Thus \( \alpha^{sc} = \alpha \), i.e. \( \alpha \) is \( \mathcal{F}' \)-closed. This completes the proof. \( \square \)

Analogous to Theorem 3.18 of [12], we have the following theorem.

**Theorem 3.9.** If \( \varphi(\tau) \) is induced fuzzy supra-topology on \( X \), then
\[ \mathcal{F}_{\varphi(\tau)} = \{ \alpha \subseteq X : \mu \in \varphi(\tau) \Rightarrow \alpha \cap \mu \in \varphi(\tau) \} \]
is a fuzzy topology on \( X \) and \( \mathcal{F}_{\varphi(\tau)} \subseteq \varphi(\tau) \).

### 3.2. Fuzzy supra-continuity in induced fuzzy supra-topological spaces.

**Definition 3.10.** Let \( (X, \mathcal{F}) \) and \( (Y, \mathcal{F}') \) be fuzzy topological spaces and \( \mathcal{F}' \) be an associated fuzzy supra-topology with \( \mathcal{F} \). A function \( f : X \to Y \) is a fuzzy \( S \)-continuous if the inverse image of each fuzzy open subset in \( Y \) is \( \mathcal{F}' \)-supra-open in \( X \).

**Theorem 3.11.** Let \( f : (X, \varphi(\tau)) \to (Y, \mathcal{F}) \) be a function from an induced fuzzy supra-topological space \( (X, \varphi(\tau)) \) into a fuzzy topological space \( (Y, \mathcal{F}) \). Then the following statements are equivalent:

1. \( f \) is fuzzy \( S \)-continuous.
2. The inverse image of each fuzzy closed subset in \( Y \) is \( \varphi(\tau) \)-closed.
3. \( (f^{-1}(\gamma))^{Sc} \subseteq f^{-1}(Cl(\gamma)) \) for any fuzzy subset \( \gamma \) in \( Y \).
4. \( f(\alpha^{sc}) \subseteq Cl(f(\alpha)) \) for any fuzzy subset \( \alpha \) in \( X \).
5. For any fuzzy point \( x_p \) in \( X \) and fuzzy open subset \( \gamma \) in \( Y \) containing \( f(x_p) \), there exists \( \alpha \in \varphi(\tau) \) such that \( x_p \in \alpha \) and \( f(\alpha) \subseteq \gamma \).

**Proof.** It is straightforward and hence omitted. \( \square \)

In [2] fuzzy supra-continuity was defined as follows: Let \( (X, \mathcal{F}_1) \) and \( (Y, \mathcal{F}_2) \) be two fuzzy topological spaces, \( (X, \mathcal{F}_1') \) and \( (X, \mathcal{F}_2') \) be two associated fuzzy supra-topological spaces with \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \), respectively. A function \( f : X \to Y \) is a fuzzy supra-continuous if the inverse image of \( \mathcal{F}_2' \)-supra-open subset is \( \mathcal{F}_1' \)-supra-open. Also we know that a function \( f : (X, \tau) \to (Y, \tau_1) \) is \( \beta \)-irresolute if the inverse of \( \beta \)-open subset is \( \beta \)-open.

**Theorem 3.12.** Let \( \varphi(\tau) \) and \( \varphi(\tau_1) \) be two induced fuzzy supra-topological associated with \( \mathcal{F} \) and \( \mathcal{F}_1 \). Then a function \( f : (X, \mathcal{F}) \to (Y, \mathcal{F}_1') \) is fuzzy supra-continuous iff \( f : (X, \tau) \to (Y, \tau_1) \) is \( \beta \)-irresolute function.

**Proof.** Let \( f \) be a fuzzy supra-continuous function and \( A \) be a \( \beta \)-open subset in \( (Y, \tau_1) \). Then \( 1_A \in \varphi(\tau) \) and
\[
\begin{align*}
  f^{-1}(A) &= \{ x \in X : 1_A(f(X)) = 1 \} \\
  &= \{ x \in X : f^{-1}(1_A(x)) > r \text{ and } 0 < r < 1 \} \\
  &= \sigma_r(f^{-1}(1_A)).
\end{align*}
\]
Thus \( f^{-1}(1_A) \) is fuzzy supra-open. Since \( f \) is fuzzy supra-continuous. By Theorem 3.3, \( \sigma_f(f^{-1}(1_A)) \) is \( \beta \)-open in the topological space \((X, \tau)\). Thus \( f \) is \( \beta \)-irresolute function.

Conversely, let \( f : (X, \tau) \to (Y, \tau_1) \) be \( \beta \)-irresolute function and \( \alpha \) is a fuzzy supra-open subset in \((Y, \varphi(\tau_1))\). Now for \( r > 0 \),

\[
\sigma_f(f^{-1}(\alpha)) = \{ x \in X : f^{-1}(\alpha(x)) > r \} = (\alpha f)^{-1}(r, \infty) = f^{-1}(\alpha^{-1}(r, \infty)).
\]

Since \( \alpha \in \varphi(\tau_1) \), \( \alpha \) is \( s \)-lower \( \beta \)-continuous and then \( (\alpha)^{-1}(r, \infty) \) is \( \beta \)-open in \((Y, \tau_1)\). Also by hypothesis, \( f^{-1}(\alpha^{-1}(r, \infty)) \) is \( \beta \)-open in \((X, \tau)\), i.e. \( \sigma_f(f^{-1}(\alpha)) \) is \( \beta \)-open in \((X, \tau)\) which implies \( f^{-1}(\alpha) \in \varphi(\tau) \). Hence the theorem. \( \square \)

Fuzzy supra-open function is defined in [2] as follows:

A function \( f \) from a fuzzy supra-topological space \((X, \mathcal{F}_1')\) into a fuzzy supra-topological space \((Y, \mathcal{F}_2')\) is called fuzzy supra-open if \( f(\alpha) \in \mathcal{F}_2' \) for each \( \alpha \in \mathcal{F}_1' \). We have the following theorem.

**Theorem 3.13.** Let \((X, \varphi(\tau_1))\) and \((Y, \varphi(\tau_2))\) be two induced fuzzy supra-topological spaces. If \( f : (X, \varphi(\tau_1)) \to (Y, \varphi(\tau_2)) \) is an injective fuzzy supra-continuous and fuzzy supra-open, then \( f : (X, \mathcal{F}_1') \to (Y, \mathcal{F}_2') \) is fuzzy continuous.

**Proof.** Let \( f : (X, \varphi(\tau_1)) \to (Y, \varphi(\tau_2)) \) be an injective fuzzy supra-continuous supra-open function. If \( \mu \in \varphi(\tau_1) \) then \( f(\mu) \in \varphi(\tau_2) \) by supra-open function. Now for each \( \alpha \in \mathcal{F}_1' \), \( \alpha \cap f(\mu) \in \varphi(\tau_2) \) by Theorem 3.9. Then \( f^{-1}(\alpha \cap f(\mu)) = f^{-1}(\alpha) \cap \mu \in \varphi(\tau_1) \) by injective supra-continuity of \( f \). Thus \( f^{-1}(\alpha) \in \mathcal{F}_1' \) for each \( \alpha \in \mathcal{F}_1' \) which proves that \( f : (X, \mathcal{F}_1') \to (Y, \mathcal{F}_2') \) is fuzzy continuous. \( \square \)

4. Conclusion

In this paper we studied the concepts of induced fuzzy supra-topological spaces and \( s \)-lower \( \beta \)-continuous functions. We deduced the properties of induced fuzzy supra-topological spaces. Fuzzy supra-continuity in induced fuzzy supra-topological spaces are defined. Finally, we defined the Initial supra-topology.

**References**


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