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Soft topological group

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Abstract. After making a comparative study of soft topology and usual topology, we introduce the concept of soft topological group and study its various properties. We are interested especially in whether or not familiar results from topological group theory remain true in the context of soft topological group. Finally, based on the observation that soft topological group is a special case of soft uniform space, results motivated from the theory of uniform space are presented.

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1. Introduction

When one wants to understand (or describe) any object from the physical world, one usually build a model of the object and define the notion of the exact solution of the model. Unfortunately, because of the high complexity of the model for instance, it is quite often difficult to find such an exact solution. Also, even if one succeeds in obtaining a solution, it is sometimes dangerous to believe that the obtained solution describes the object perfectly, due to, for example, various uncertainties in the initial data triggered by an (inevitable) error in a measurement.

The importance of developing frameworks which enable us to handle problems with vagueness and uncertainties is then apparent. Indeed, several attempts have already been made: Examples includes the theory of Fuzzy sets by Zadeh [18] and the theory of Rough sets by Pawlak [14]. The starting point of this work - Molodtsov's soft sets [12] - was proposed also in this line of research. Molodtsov's one of the motivations for introducing this mathematical tool is to overcome some of the difficulties the traditional theories have. It also aims at modeling wide range of problems from, e.g, physical science, economics and engineering. Indeed, Molodtsov [12] has already presented applications of his theory to operations research, game theory, probability theory and so on.

Although somewhat different from the initial motivation, the investigation of mathematical aspects of soft sets is of interest as well. Many authors have contributed to this field: Maji et al. [11] defined and studied basic notions of soft set theory. The study of soft topological space was started by Shabir and Naz [15], and independently by Çağman et al. [3]. The notion of soft group was introduced by Aktaş and Çağman [1], and intensive research on this concept has been carried out since then.

This paper introduces the concept of soft topological group and studies its various properties. After fixing basic terminology in section 2, we provide several results on soft topology in section 3, where comparative study of soft topology and usual topology will be made. Soft topological group is defined in section 4 and its various properties as well as comparison to (usual) topological group are presented there. In section 5, we analyze soft topological group from a more general perspective, namely soft uniform space, a concept introduced by Çetkin and Aygün [4]. This paper will conclude with suggestions for future research.

2. Preliminaries

Definition 2.1 ([12])**.** Let U be an initial universe and E be a set of parameters. Then a *soft set* over U is a function $F : E \to \mathcal{P}(U)$.

$F(e)$ can informally be taken as the set of e-*approximate elements* of the soft set.

From now on, we will sometimes identify a soft set $F : E \to \mathcal{P}(U)$ with a subset of $E \times U$. We shall use symbols $F, F', \dots$ for soft sets.

Definition 2.2. Let $\phi : U \to U'$ be a function and F (resp. F') be a soft set over U (resp. U') with a parameter set E. Then $\phi(F)$ (resp. $\phi^{-1}(F')$) is the soft set on U' (resp. U) defined by $(\phi(F))(e) = \phi(F(e))$ (resp. $(\phi^{-1}(F))(e) = \phi^{-1}(F(e))$).

In what follows, we use tilde $\widetilde{(\cdot)}$ to distinguish "soft" objects from usual ones. For example, for a subset X of U, $\tilde{X}$ denotes the soft set satisfying that $\tilde{X}(e) = X$ for all $e \in E$.

Soft versions of basic relations (resp. operations) on sets are obtained by requiring the relations (resp. applying the operations) at each parameter.

Definition 2.3 ([11])**.** Let F and F' be soft sets over U. Then

- (Soft subset) F is a *soft subset* of F', denoted by $F \,\tilde{\subset}\, F'$, if $F(e) \subset F'(e)$ for all $e \in E$.
- (Soft equality) F is *soft equal* to F', denoted by $F \,\tilde{=}\, F'$, if both $F \,\tilde{\subset}\, F'$ and $F' \,\tilde{\subset}\, F$ hold.
- (Soft intersection) The *soft intersection* of F and F', denoted by $F \,\tilde{\cap}\, F'$, is defined by $(F \,\tilde{\cap}\, F')(e) = F(e) \cap F'(e)$ for every $e \in E$.
- (Soft union) The *soft union* of F and F', denoted by $F \,\tilde{\cup}\, F'$, is defined by $(F \,\tilde{\cup}\, F')(e) = F(e) \cup F'(e)$ for every $e \in E$.
- (Soft complement) The *soft complement* of F, denoted by $F^{\tilde{c}}$, is defined by $F^{\tilde{c}}(e) = U \setminus F(e)$ $(e \in E)$.

For properties of these relations and operations, we refer the reader to [3].

Definition 2.4. Let x be an element of U and F be a soft set over U. We say that x is a *soft element* of F, denoted by $x \mathrel{\tilde{\in}} F$, if $x \in F(e)$ for all parameters $e \in E$.

Unless E is singleton, $x \mathrel{\tilde{\notin}} F$ and $\forall e \in E\,(x \notin F(e))$ are different. This simple fact plays a key role throughout this paper.

Note that, even if $\forall x \in U\ (x \mathrel{\tilde{\in}} F \Rightarrow x \mathrel{\tilde{\in}} F')$ holds, it can happen that $F \mathrel{\tilde{\not\subset}} F'$. This means that $(\,\tilde{\in}\,,\,\tilde{=}\,)$ provides a *counter-model* of the axiom of extensionality. In particular, it should be noted that being soft non-empty — $\exists x \in U\,(x \mathrel{\tilde{\notin}} F)$ — is *not* the same as being non-empty — $F \mathrel{\tilde{\neq}} \tilde{\varnothing}$.

3. Soft topology

This section introduces several soft topological concepts and studies them. We assume that the reader has a background in general topology. Information from [5] and [10] is perfectly enough for our purpose.

Definition 3.1 ([3, 15])**.** A family τ of soft sets over U is called a *soft topology* on U if the following three conditions are satisfied:

- $\tilde{\varnothing}$ and $\tilde{U}$ are in τ,
- τ is closed under finite soft intersection,
- τ is closed under (arbitrary) soft union.

We refer to a triplet $\langle U, \tau, E\rangle$ as a *soft topological space*. Each member of τ is called a *soft open* set. Throughout this paper, $\langle U, \tau, E\rangle$ stands for a soft topological space.

Definition 3.2 ([15])**.** For any soft set F over U, the *soft closure* of F, denoted by $\widetilde{\mathrm{Cl}}(F)$, is the soft intersection of all soft closed supersets of F.

Familiar concepts from general topology, such as interior, boundary and limit points, are generalized to the setting of soft sets in a natural way. Interested readers are asked to consult related articles [3, 8, 15]. However, at the referee's request, let us remind the reader of at least the following concepts here:

Definition 3.3. Let x be an element of the universe U. A soft set F is called a *soft neighborhood* of x if there exists a soft open set F' such that $x \mathrel{\tilde{\in}} F' \mathrel{\tilde{\subset}} F$.

The collection of all soft neighborhoods of x is called the *soft neighborhood system* of x.

Definition 3.4. Let $\mathcal{B}$ be a family of soft sets closed under finite soft intersection. Then the *soft topology generated by the base* $\mathcal{B}$ is $\{\tilde{\varnothing}, \tilde{U}\} \cup \{\widetilde{\bigcup}\mathcal{I} \mid \mathcal{I} \subset \mathcal{B}\}$.

Definition 3.5. For any family $\mathcal{S}$ of soft sets, the *soft topology generated by the subbase* $\mathcal{S}$ is the soft topology generated by the base $\{F_1 \mathbin{\tilde{\cap}} \cdots \mathbin{\tilde{\cap}} F_k \mid F_1, \ldots, F_k \in \mathcal{S}\}$.

The following examples may help the reader in understanding the difference between usual topology and soft topology:

Example 3.6. (Soft closure cannot be characterized by soft neighborhood) In general topology, we have the following equivalence for any subset A of a topological space T: a point x is in the closure of $A \iff$ every neighborhood of x intersects A. In soft topology, however, this equivalence is no longer true.

The soft analogue of the left-to-right direction *is* true: Take any $A \subset U$ and $x \in U$ with $x \tilde{\in} \widetilde{\mathrm{Cl}}(\tilde{A})$. Then every soft open neighborhood F' of x satisfies $F' \tilde{\cap} \tilde{A} \tilde{\neq} \tilde{\varnothing}$, because if $F' \tilde{\cap} \tilde{A} \tilde{=} \tilde{\varnothing}$ then $x \tilde{\in} \widetilde{\mathrm{Cl}}(\tilde{A}) \tilde{\subset} F'^{\tilde{c}}$, contradicting the assumption that $x \tilde{\in} F'$.

On the other hand, the soft analogue of the right-to-left direction is *not* true. In order to see this, let us employ the following soft topological space $\langle \mathbb{Z}_2, \tau, E \rangle$, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} \,(= \{\bar{0}, \bar{1}\}), E := \{e_1, e_2\}$ and $\tau := \big\{\tilde{\varnothing}, \{(e_1, \bar{0})\}, E \times \mathbb{Z}_2\big\}$. Then consider the set $A := \{\bar{1}\}$. Since there is exactly one soft open set containing $\bar{0}$, namely $E \times \mathbb{Z}_2$, it holds that $F' \tilde{\cap} \tilde{A} \tilde{\neq} \tilde{\varnothing}$ for all soft neighborhood F' of $\bar{0}$. However, a simple computation shows that $\widetilde{\mathrm{Cl}}(\tilde{A}) \tilde{=} \{(e_1, \bar{0})\}^{\tilde{c}}$, and so $\bar{0} \tilde{\notin} \widetilde{\mathrm{Cl}}(\tilde{A})$.

Example 3.7. (The non-equivalence of two ways to define a soft topology) It is a well-known fact from general topology that giving a neighborhood system for each point is equivalent to giving a topology. Soft topological spaces do not possess this property: The two soft spaces $\langle \mathbb{Z}_2, \tau, E \rangle$ and $\langle \mathbb{Z}_2, \tau', E \rangle$ give an example, where τ and E are as in the above example, and $\tau' := \{\tilde{\varnothing}, E \times \mathbb{Z}_2\}$. While τ and τ' are clearly different as soft topologies, they give rise to the same neighborhood system for $\bar{0}$ and $\bar{1}$: in both spaces, $\bar{0}$ and $\bar{1}$ have the same unique soft neighborhood, namely $E \times \mathbb{Z}_2$.

Let us introduce the notion of continuity in the setting of soft set sets here.

Definition 3.8. Let $\langle U, \tau, E \rangle$ and $\langle U', \tau', E' \rangle$ be soft topological spaces, and $\phi : U \to U'$, $\psi : E \to E'$ be functions. Then a function $\psi \times \phi : E \times U \to E' \times U'$ is called *soft continuous* if the following condition is satisfied:

(SC): For every $x \in U$ and for every soft neighborhood F' of $\psi(x)$, there exists a soft neighborhood F of x such that $\phi(x) \tilde{\in} (\psi \times \phi)(F) \tilde{\subset} F'$.

Remark 3.9. Zorlutuna et al. [19] used slightly different definition in their paper: For every $x \in U$ and for every soft neighborhood F' of $\phi(x)$, there exists a soft neighborhood F of x such that $(\psi \times \phi)(F) \tilde{\subset} F'$. This definition has the disadvantage that $\psi(x)$ may not be a soft element of $(\psi \times \phi)(F)$. Here is a simple example: $\iota \times \mathrm{id} : \{e_1\} \times \{u\} \to \{e_1, e_2\} \times \{u\}$, where $\iota : \{e_1\} \to \{e_1, e_2\}$ is an embedding and both soft spaces are discrete. Clearly, $(\iota \times \mathrm{id})(\{e_1\} \times \{u\})$ is a soft subset of $\{e_!, e_2\} \times \{u\}$, but it does not have $u\,(= \mathrm{id}(u))$ as a soft element. For this reason, we prefer the condition **(SC)** rather than the one given in [19].

The purpose of this paper is to know which results from the topological group theory remain valid in the context of soft topological group. Most properties studied later are of the form "Given subgroups H, K of a group G, consider ...", "Consider the left action $\alpha_L(g) : G \to G; x \mapsto gx$ of $g \in G$. Then ...". These results are on the relationship between subgroups, connected subsets, etc., of a *fixed* group. In such situations, we deal only with the same parameter set, namely the parameter set of the fixed soft topological group. For our purpose, therefore, it suffices to focus on the special case where $E = E'$ and $\psi = \mathrm{id}$ in studying the soft continuous function.

Hence, in this paper, a soft continuous function always refers to the following simpler version of the above definition:

Definition 3.10. $\phi : U \to U'$ is called a *soft continuous function* from $\langle U, \tau, E\rangle$ to $\langle U', \tau', E\rangle$ if the following condition is satisfied:

- **(SC1):** For every $x \in U$ and for every soft neighborhood F' of $\phi(x)$, there exists a soft neighborhood F of x such that $\phi(F) \mathrel{\tilde{\subset}} F'$.

Note that, as the parameter set is identical in this case, it holds that $\phi(x) \mathrel{\tilde{\in}} \phi(F)$ without explicitly assuming so.

A bijection $\phi : U \to U'$ is called a *soft homeomorphism* between $\langle U, \tau, E\rangle$ and $\langle U', \tau', E\rangle$ if both ϕ and ϕ^{-1} are soft continuous.

In order to make explicit the underlying soft topological structure, we sometimes write $\phi : \langle U, \tau, E\rangle \to \langle U', \tau', E\rangle$ instead of just $\phi : U \to U'$.

In general topology, there are several equivalent ways to define the notion of continuity. It is then natural to ask whether this happens even in soft set theory or not. In order to answer this question, let us introduce the following three conditions on a function $\phi : \langle U, \tau, E\rangle \to \langle U', \tau', E\rangle$:

- **(SC2):** For every soft open set $F' \in \tau'$, the inverse image $\phi^{-1}(F')$ is also soft open.
- **(SC3):** For every soft closed set F', the inverse image $\phi^{-1}(F')$ is also soft closed.
- **(SC4):** For every soft set F, we have $\phi(\widetilde{\mathrm{Cl}}(F)) \mathrel{\tilde{\subset}} \widetilde{\mathrm{Cl}}(\phi(F))$.

Theorem 3.11.
(i) The conditions **(SC2)**, **(SC3)** *and* **(SC4)** *are equivalent.*
(ii) The condition **(SC1)** *follows from but not imply* **(SC2)**.

Proof. (i) Firstly, we prove **(SC3)** from **(SC4)**: Pick any soft closed set F'. Then we have $\phi(\phi^{-1}(F')) \mathrel{\tilde{\subset}} F'$. The soft closedness of F', together with the assumption **(SC4)**, implies that $\phi(\widetilde{\mathrm{Cl}}(\phi^{-1}(F'))) \mathrel{\tilde{\subset}} \widetilde{\mathrm{Cl}}(\phi(\phi^{-1}(F'))) \mathrel{\tilde{\subset}} F'$. Therefore, it holds that $\widetilde{\mathrm{Cl}}(\phi^{-1}(F')) \mathrel{\tilde{\subset}} \phi^{-1}(F') \mathrel{\tilde{\subset}} \widetilde{\mathrm{Cl}}(\phi^{-1}(F'))$, which proves that $\phi^{-1}(F')$ is soft closed.

We then deduce **(SC4)** from **(SC3)**: Note that $F \mathrel{\tilde{\subset}} \phi^{-1}(\widetilde{\mathrm{Cl}}(\phi(F)))$ holds for any soft set F. So, by **(SC3)**, we have $\widetilde{\mathrm{Cl}}(F) \mathrel{\tilde{\subset}} \phi^{-1}(\widetilde{\mathrm{Cl}}(\phi(F)))$. Therefore, we get $\phi(\widetilde{\mathrm{Cl}}(F)) \mathrel{\tilde{\subset}} \phi(\phi^{-1}(\widetilde{\mathrm{Cl}}(\phi(F)))) \mathrel{\tilde{=}} \widetilde{\mathrm{Cl}}(\phi(F))$.

The equivalence between **(SC2)** and **(SC3)** can easily be proved, so it is left to the reader.

(ii) Assume **(SC2)**. Then, for every $x \in U$ and a soft open neighborhood F' of $\phi(x)$, $\phi^{-1}(F')$ is a soft open set having x as a soft element. Since $\phi(\phi^{-1}(F')) \mathrel{\tilde{=}} F'$, we have thus deduced **(SC1)** from **(SC2)**.

For the last assertion, let us consider soft topological spaces $S_i := \langle \{u\}, \tau_i, \{e_1, e_2\}\rangle$ for $i = 1, 2$, where

$$\begin{aligned} \tau_1 &= \big\{\tilde{\varnothing}, \{(e_1, u), (e_2, u)\}\big\}, \\ \tau_2 &= \big\{\tilde{\varnothing}, \{(e_2, u)\}, \{(e_1, u), (e_2, u)\}\big\}. \end{aligned}$$

In both soft topologies, $\{e_1, e_2\} \times \{u\}$ is the unique soft neighborhood of the point u. Hence $\mathrm{id} : S_1 \to S_2$ satisfies **(SC1)**. However, $\mathrm{id}^{-1}(\{(e_2, u)\})$ is not soft open in

S_1, showing that the inverse images of soft open sets are, in general, not soft open. Observe that, not only $\mathrm{id} : S_1 \to S_2$ but also $\mathrm{id}^{-1} : S_2 \to S_1$ satisfy **(SC1)**. □

From the above theorem, the reader will be able to see a crucial difference between soft topology and usual topology. However, when concerned is only the soft elementhood relation $\tilde{\in}$, the difference may not be so big — The following proposition says that the inverse image of any soft open set is soft open modulo soft empty set:

Proposition 3.12. *Let $\phi : \langle U, \tau, E\rangle \to \langle U', \tau', E\rangle$ be a soft continuous function. Then for every soft open set $F' \in \tau'$, there exists a soft open set $F \in \tau$ such that $\forall x \in U \left(x \,\tilde{\in}\, F \Leftrightarrow x \,\tilde{\in}\, \phi^{-1}(F')\right)$.*

Proof. For any point $x \in U$ with $\phi(x) \,\tilde{\in}\, F'$, choose a soft open $F_x \in \tau$ such that $x \,\tilde{\in}\, F_x$ and $\phi(F_x) \,\tilde{\subset}\, F'$. Then $F := \widetilde{\bigcup}\{F_x \mid x \in U \text{ satisfies } \phi(x) \,\tilde{\in}\, F'\}$ is the desired soft open set. □

Remark 3.13. One may wonder why, unlike [2], we employed **(SC1)** as the definition of soft continuity. We would like to answer this question as follows: If we use **(SC2)** (or any other equivalent conditions), it means that we are essentially dealing with (not *soft* topological concepts on U but) usual topological concepts on $E \times U$. In order to make concepts more appropriate for the analysis of soft set theory, we prefer **(SC1)** among others.

Zorlutuna et al. [19] investigated the notion of soft continuity given in Definition 3.8, which is more general than the notion employed in this paper. A more general version of **(SC2)** also appear in the same paper (general in the sense that the definition covers the case where the parameter set of the domain is different from that of the range), and is claimed to be equivalent with the notion of soft continuity given in Definition 3.8. However, their proof [19, Theorem 6.3] is not correct; these are actually non-equivalent statements, as we have observed.

Separation axioms can also be generalized. For instance:

Definition 3.14. Let $\langle U, \tau, E\rangle$ be a soft topological space. Then

(1) $\langle U, \tau, E\rangle$ is a *soft T_0 space* if for every elements $x, y \in U$ with $x \neq y$, there exists a soft open set F such that either $x \,\tilde{\in}\, F \wedge \forall e \in E\,(y \notin F(e))$ or $y \,\tilde{\in}\, F \wedge \forall e \in E\,(x \notin F(e))$ holds.
(2) $\langle U, \tau, E\rangle$ is a *soft T_1 space* if for every distinct elements $x, x' \in U$, there exist soft open sets F, F' such that both $x \,\tilde{\in}\, F \wedge \forall e \in E\,(x' \notin F(e))$ and $x' \,\tilde{\in}\, F' \wedge \forall e \in E\,(x \notin F'(e))$ hold.
(3) $\langle U, \tau, E\rangle$ is a *soft Hausdorff (or soft T_2) space* ([15]) if for every pair of distinct points $x, x' \in U$, there exist soft open sets $F, F' \in \tau$ with $x \,\tilde{\in}\, F, x' \,\tilde{\in}\, F'$ and $F \,\tilde{\cap}\, F' = \tilde{\varnothing}$.
(4) $\langle U, \tau, E\rangle$ is a *soft regular space* ([15]) if, for each $x \in U$, every soft neighborhood of x contains a soft closed neighborhood of x.

Remark 3.15. The notion of a soft T_0 space in Shabir and Naz's paper [15] is obtained if we replace $\forall e \in E\,(y \notin F(e))$ (resp. $\forall e \in E\,(x \notin F(e))$) in the above definition by $y \,\tilde{\notin}\, F$ (resp. $x \,\tilde{\notin}\, F$). As has mentioned already, $y \,\tilde{\notin}\, F$ is not equivalent to $\forall e \in E\,(y \notin F(e))$ in general, so our definition here is different from their definition.

From general topology, one knows that a topological space is regular, i.e., every neighborhood of x contains a closed neighborhood for every point x, if and only if any point x and a closed set C not containing x can be separated by open sets. The soft version of this statement is not true, as the following example witnesses:

Example 3.16. (Non-equivalence of the two ways to define soft regularity) If we endow a soft topology on $\mathbb{Z}_2$ by $\tau := \{\tilde{\varnothing}, \{e_2\} \times \mathbb{Z}_2, \widetilde{\mathbb{Z}_2}\}$, where $E := \{e_1, e_2\}$, then this soft space is soft regular, as each point has only one soft clopen neighborhood $\widetilde{\mathbb{Z}_2}$. However, no two soft open sets separate $\bar{0}$ from the soft closed set $\{e_1\} \times \mathbb{Z}_2$, which does not have $\bar{0}$ as a soft element.

Proposition 3.17. *Let $\phi : \langle U, \tau, E\rangle \to \langle U', \tau', E\rangle$ be a soft continuous injection.*
(i) If $\langle U', \tau', E\rangle$ is a soft T_0 space, then so is $\langle U, \tau, E\rangle$.
(ii) If $\langle U', \tau', E\rangle$ is a soft T_1 space, then so is $\langle U, \tau, E\rangle$.
(iii) If $\langle U', \tau', E\rangle$ is a soft Hausdorff space, then so is $\langle U, \tau, E\rangle$.

Proof. We prove only (iii), as the other items can be proved similarly. Take distinct points x and y from U. By assumption, we can separate $\phi(x)$ from $\phi(y)$ by soft open sets, say F'_x, F'_y. By the soft continuity of ϕ, we find soft open neighborhoods F_x, F_y of x, y, respectively, satisfying that $\phi(F_x) \mathrel{\tilde{\subset}} F'_x$ and $\phi(F_y) \mathrel{\tilde{\subset}} F'_y$. It is then clear that this F_x and F_y separate x from y. □

One will naturally ask whether soft regularity is also preserved under soft homeomorphisms or not. This question is answered in the negative:

Example 3.18. (Soft regularity is not always preserved by soft homeomorphisms) The identity function $\mathrm{id} : \langle \mathbb{Z}_3, \tau_1, E\rangle \to \langle \mathbb{Z}_3, \tau_2, E\rangle$ gives a negative answer, where $E := \{e_1, e_2\}$, τ_1 and τ_2 are generated by the following subbases, respectively:

$$\text{Subbase for } \tau_1 : \quad \Big\{\tilde{\varnothing}, E \times \{\bar{0}, \bar{1}\}, E \times \{\bar{2}\}, \{(e_1, \bar{0}), (e_2, \bar{0}), (e_2, \bar{1})\}, \{(e_1, \bar{1}), (e_1, \bar{2}), (e_2, \bar{2})\}, E \times \mathbb{Z}_3\Big\},$$

$$\text{Subbase for } \tau_2 : \quad \Big\{\tilde{\varnothing}, E \times \{\bar{0}, \bar{1}\}, E \times \{\bar{2}\}, \{(e_1, \bar{0}), (e_2, \bar{0}), (e_2, \bar{1})\}, E \times \mathbb{Z}_3\Big\}.$$

The reader will be able to check that the identity function on $\mathbb{Z}_3$ is indeed a soft homeomorphism. It is also clear that the soft topological space $\langle \mathbb{Z}_3, \tau_1, E\rangle$ is soft regular. However, in $\langle \mathbb{Z}_3, \tau_2, E\rangle$, no soft closed set is a soft subset of a soft neighborhood $\{(e_1, \bar{0}), (e_2, \bar{0}), (e_2, \bar{1})\}$ of $\bar{0}$. Hence, $\langle \mathbb{Z}_3, \tau_2, E\rangle$ is not soft regular.

We want to introduce the concept of soft compactness as well. But before doing so, we first need to define what do we mean by soft coverings.

Definition 3.19. A family $\mathcal{C}$ of soft open sets over U is said to be a *soft open covering* of U if for every $x \in U$ there exists an $F \in \mathcal{C}$ such that $x \mathrel{\tilde{\in}} F$.

The reader may have realized that there could be another formulation of soft open covering: $\mathcal{C}$ is a "soft open covering" of U if $\widetilde{\bigcup}_{F\in\mathcal{C}} F \mathrel{\tilde{=}} E \times U$ — Actually, Aygünoğlu and Aygün [2] employed this condition as a definition of soft open covering in their paper. For the reason similar to the one explained in Remark 3.13, we prefer the Definition 3.19.

Definition 3.20. A soft space $\langle U, \tau, E\rangle$ is *soft compact* if, for any soft open covering $\mathcal{C}$ of $\langle U, \tau, E\rangle$, there exist $F_1, \ldots, F_n \in \mathcal{C}$ such that $\{F_1, \ldots, F_n\}$ is a soft open covering.

Likewise, one can define a *a soft open covering* of a subset V of U, and a *soft compact* subset of U. For more information, see [6].

As in the usual topology, soft compactness is preserved by soft continuous functions:

Proposition 3.21. *Let $\phi : \langle U, \tau, E\rangle \to \langle U', \tau', E\rangle$ be a soft continuous function, and $V \subset U$ be a subset. If V is soft compact with respect to τ, then so is $\phi(V)$ with respect to τ'.*

Proof. Let $\mathcal{C}'$ be a soft open covering of $\phi(V)$. For each $v \in V$, there exists an $F'_v \in \mathcal{C}'$ such that $\phi(v) \tilde{\in} F'_v$. Since ϕ is soft continuous, there exists a soft open neighborhood F_v of v such that $\phi(F_v) \tilde{\subset} F'_v$. Then the family $\{F_v \mid v \in V\}$ is a soft covering of V. By the soft compactness of V, there exist $v_1, \ldots, v_n \in V$ such that $\{F_{v_i}\}_{i=1}^n$ is a soft covering of V. Then, since $\phi(F_{v_i}) \tilde{\subset} F'_{v_i}$, $\{F'_{v_i}\}_{i=1}^n$ is a soft covering of $\phi(V)$. □

It would be interesting to examine also the concept of connectedness in the context of soft set theory:

Definition 3.22. A subset X of U is called *soft connected* if, for any soft open covering $\{F_1, F_2\}$ of X subject to the condition that $\nexists x \in X\, (x \tilde{\in} F_1 \wedge x \tilde{\in} F_2)$, either $\forall x \in X\, (x \tilde{\notin} F_1)$ or $\forall x \in X\, (x \tilde{\notin} F_2)$ holds.

It should again be noted that we did not employ the following possible formulation of soft connectedness: X is "soft connected" if for any soft disjoint soft open sets F_1, F_2 with $F_1 \tilde{\cup} F_2 \tilde{=} \tilde{X}$, either $F_1 \tilde{=} \tilde{\varnothing}$ or $F_2 \tilde{=} \tilde{\varnothing}$ holds. The reason for our choice here is similar to that for the choice of soft continuity and soft compactness.

Proposition 3.23. *Let X_1 and X_2 be subsets of U having non-empty intersection. If both X_1 and X_2 are soft connected, then so is $X_1 \cup X_2$.*

Proof. Take any soft open covering $\{F_1, F_2\}$ of $X_1 \cup X_2$. Since $\{F_1, F_2\}$ is a soft open covering also of X_1, we assume $\forall z \in X_1\, (z \tilde{\notin} F_1)$ without loss of generality. In particular, $x \tilde{\notin} F_1$ holds for any $x \in X_1 \cap X_2$. Assume for the contradiction that there were a $z \in X_2$ such that $z \tilde{\in} F_1$. Then, since $\{F_1, F_2\}$ is a soft open covering also of X_2, it would hold that $\forall z \in X_2\, (z \tilde{\notin} F_2)$. In particular, we would have $x \tilde{\notin} F_2$ for every $x \in X_1 \cap X_2$, which gives a contradiction. Thus, $\forall z \in X_2\, (z \tilde{\notin} F_1)$ holds; so we have obtained $\forall z \in X_1 \cup X_2\, (z \tilde{\notin} F_1)$. □

Corollary 3.24. *For each point $x \in U$, the soft connected component of x, i.e., the largest soft connected subset of U containing x, exists.*

Proposition 3.25. *Let $\phi : \langle U, \tau, E\rangle \to \langle U', \tau', E\rangle$ be a soft homeomorphism. Then $X \subset U$ is soft connected if and only if $\phi(X)$ is soft connected.*

Proof. Let X be a soft connected subset of U. Select an arbitrary soft open covering $\{F'_1, F'_2\}$ of $\phi(X)$. By Proposition 3.12, there exist soft open sets $F_i \in \tau$ such that

$\phi(F_i) \subset F'_i$ and $\forall y \in U' \, (y \,\tilde{\in}\, F'_i \Leftrightarrow y \,\tilde{\in}\, \phi(F_i))$ $(i = 1, 2)$. Hence, for each $x \in X$, exactly one of $x \,\tilde{\in}\, F_1$ and $x \,\tilde{\in}\, F_2$ holds. Since X is soft connected, there is no loss of generality in assuming $\forall x \in X \, (x \,\tilde{\notin}\, F_1)$. In this case, we have $\forall y \in \phi(X) \, (y \,\tilde{\notin}\, \phi(F_1))$, and so $\forall y \in \phi(X) \, (y \,\tilde{\notin}\, F'_1)$. Therefore, we have proved that left-to-right direction. The converse direction follows from the same argument applied to ϕ^{-1}. □

Let us say that a property P of soft topological spaces is a *soft topological property* if the following condition holds for any soft space $\langle U, \tau, E\rangle$: A soft space $\langle U, \tau, E\rangle$ has the property $P \iff$ Every soft space which is soft homeomorphic to $\langle U, \tau, E\rangle$ has the property P. Then, the following theorem follows from Proposition 3.21 and 3.25 at once:

Theorem 3.26. *Soft compactness and soft connectedness are soft topological properties.* □

Topological properties such as compactness and connectedness behave well with product. We would like to examine here several properties defined so far in connection with soft product. In order to do so, we first have to make precise what soft product is:

Definition 3.27. For any soft topological spaces $\langle U, \tau, E\rangle$ and $\langle U', \tau', E\rangle$, the set $\{F \times F' \mid F \in \tau, F' \in \tau'\}$ generates a soft topology $\tau_\times$ on $U \times U'$. The soft space $\langle U \times U', \tau_\times, E\rangle$ is called the *soft product* of $\langle U, \tau, E\rangle$ and $\langle U', \tau', E\rangle$.

Here, $F \times F'$ is the soft set over $U \times U'$ defined by $(F \times F')(e) := F(e) \times F'(e)$ for all $e \in E$.

(1) **Soft T_0 spaces, soft T_1 spaces and soft Hausdorff spaces**

Assume $\langle U, \tau, E\rangle$ and $\langle U', \tau', E\rangle$ are soft T_0 spaces. Take distinct pairs $(x, x'), (y, y') \in U \times U'$. Without loss of generality, we assume $x \neq y$. Since $\langle U, \tau, E\rangle$ is a soft T_0 space, there exists a soft open set F such that either $x \,\tilde{\in}\, F \wedge \forall e \in E \big(y \notin F(e)\big)$ or $y \,\tilde{\in}\, F \wedge \forall e \in E \big(x \notin F(e)\big)$ holds. Then we have either of the following:

$$(x, x') \,\tilde{\in}\, F \times \tilde{U} \wedge \forall e \in E \big((y, y') \notin F(e) \times U\big) \text{ or}$$

$$(y, y') \,\tilde{\in}\, F \times \tilde{U} \wedge \forall e \in E \big((x, x') \notin F(e) \times U\big).$$

Therefore, we have shown:

Proposition 3.28. *The soft product of any two soft T_0 spaces is also a soft T_0 space.* □

Similarly, we can prove

Proposition 3.29. *The soft product of any two soft T_1 spaces is also a soft T_1 space.* □

Proposition 3.30 ([17])**.** *The soft product of any two soft Hausdorff spaces is also a soft Hausdorff space.* □

(2) **Soft regular spaces**

Set $U := \mathbb{Z}_2$ and $E := \{e_1, e_2, e_3\}$. Define soft topologies τ_1, τ_2 on U by $\tau_1 := \{\tilde{\varnothing}, \{(e_2, \bar{1})\}, E \times U\}$ and $\tau_2 := \big\{\tilde{\varnothing}, \{(e_1, \bar{0})\}, \{(e_3, \bar{0})\}, E \times U\big\}$. Since, in both soft topological spaces, every point $x \in U$ has only one soft clopen neighborhood $E \times U$, it is obvious that both soft spaces are soft regular. Observe that, no soft closed neighborhood of $(\bar{1}, \bar{0}) \in U \times U$ is a soft subset of the following soft open neighborhood of $(\bar{1}, \bar{0})$:

$$\left\{(e_1, (\bar{0}, \bar{0})), (e_1, (\bar{1}, \bar{0})), (e_2, (\bar{1}, \bar{0})), (e_2, (\bar{1}, \bar{1})), (e_3, (\bar{0}, \bar{0})), (e_3, (\bar{1}, \bar{0}))\right\}$$

Thus, the soft product of two soft regular spaces $\langle U, \tau_1, E\rangle, \langle U, \tau_2, E\rangle$ is not soft regular.

(3) **Soft compact spaces**

For topological spaces, the product of compact spaces is also compact — This fact is proved with the Axiom of Choice for infinite product (known as Tychonoff's theorem), and even without any extra assumption for the case of finite product. This is not always true for soft topological spaces.

Example 3.31. Let us consider the following soft topological spaces $\langle \mathbb{Z}, \tau_1, E\rangle$, $\langle \mathbb{Z}, \tau_2, E\rangle$, where $E := \{e_1, e_2\}$ and

$$\tau_1 := \left\{\tilde{\varnothing}, E \times \mathbb{Z}\right\} \cup \{\{e_1\} \times S \mid S \text{ is a subset of } \mathbb{Z}\},$$

$$\tau_2 := \left\{\tilde{\varnothing}, E \times \mathbb{Z}\right\} \cup \{\{e_2\} \times S \mid S \text{ is a subset of } \mathbb{Z}\}.$$

Note that, in both soft spaces, $E \times \mathbb{Z}$ is the unique soft neighborhood of i for each $i \in \mathbb{Z}$. It is then easy to see that both soft spaces are soft compact (and also soft connected).

Then, in the soft product space, the soft set

$$F_{i_1,i_2} := \{(e_1, (i_1, j)) \mid j \in \mathbb{Z}\} \mathbin{\tilde{\cup}} \{(e_2, (k, i_2)) \mid k \in \mathbb{Z}\}$$

is a soft open neighborhood of (i_1, i_2) for each $(i_1, i_2) \in \mathbb{Z} \times \mathbb{Z}$. It is readily verified that a soft open covering $\{F_{i_1,i_2}\}_{i_1,i_2 \in \mathbb{Z}}$ of $\mathbb{Z} \times \mathbb{Z}$ has no finite subset which soft covers $\mathbb{Z} \times \mathbb{Z}$. This shows that the soft product does not preserve soft compactness.

(4) **Soft connected spaces**

The same soft spaces from Example 3.31 shows that soft connectedness is not preserved by taking soft product. (For example, soft open sets $F_{0,0}$ and $\widetilde{\bigcup}\{F_{i_1,i_2} \mid (i_1, i_2) \neq (0, 0)\}$ witness that the soft product is not soft connected.) A natural question would be "Is the soft product of soft connected spaces also soft connected when the two spaces are identical?" Even for that situation, we have the negative answer.

Example 3.32. Consider the following soft topology τ on $\mathbb{Z}_2$ with a parameter set $E = \{e_1, e_2\}$:

$$\tau := \left\{\tilde{\varnothing}, \{(e_1, \bar{1}), (e_2, \bar{0})\}, \{(e_1, \bar{0}), (e_2, \bar{1})\}, E \times \mathbb{Z}_2\right\}$$

Since both $\bar{0}$ and $\bar{1}$ have only one soft neighborhood $E \times \mathbb{Z}_2$, the soft topological space $\langle \mathbb{Z}_2, \tau, E\rangle$ is soft connected. An easy computation shows that both $E \times \{(\bar{0}, \bar{1}), (\bar{1}, \bar{0})\}$ and $E \times \{(\bar{0}, \bar{0}), (\bar{1}, \bar{1})\}$ are soft open in the soft product space. Hence this soft product is not soft connected.

Remark 3.33. The above soft topological space $\langle \mathbb{Z}_2, \tau, E\rangle$ is an example of soft topological group (see the next section). Therefore, we find that the soft connectedness is not preserved by taking binary soft product even if the soft space is a soft topological group.

What makes it difficult to handle several properties in the context of soft set theory comes from the following fact.

Proposition 3.34. *For any $x_0, x_1 \in U$ and a soft set F on $U \times U$, the following conditions are* not *equivalent:*
(i) F is a soft neighborhood of (x_0, x_1),
(ii) There exist soft open sets F_0, F_1 such that $x_0 \tilde{\in} F_0, x_1 \tilde{\in} F_1$ and $F_0 \times F_1 \tilde{\subset} F$.

Proof. Firstly, note that (i) follows straightforwardly from (ii). To see that (i) does *not* imply (ii), let us consider this soft space $\langle \mathbb{Z}_2, \tau, E\rangle$, where $E := \{e_1, e_2\}$ and τ is generated by $\{\tilde{\varnothing}, \{(e_1, \bar{0})\}, \{(e_2, \bar{1})\}, E \times \mathbb{Z}_2\}$. An easy computation shows that $\{(e_1,(\bar{0},\bar{0})),(e_1,(\bar{1},\bar{0})),(e_2,(\bar{1},\bar{0})),(e_2,(\bar{1},\bar{1}))\}$ is a soft open neighborhood of $(\bar{1},\bar{0})$. Since both $\bar{0}$ and $\bar{1}$ have only one soft neighborhood $E\times\mathbb{Z}_2$, it is then easy to observe that the condition (ii) does not hold. □

Proposition 3.35 ([8])**.** *Let $\langle U, \tau, E\rangle$ be a soft topological space. Then the family $\tau_e := \{F(e) \mid F \in \tau\}$ induces a topology on U for each parameter $e \in E$.* □

It is also straightforward to see that

Proposition 3.36. *The family $\{F \mid \forall e \in E\,(F(e) \in \tau_e)\}$ is a soft topology.* □

The family given in the above proposition clearly contains the original soft topology τ. One may expect that these are identical; in other words, we can recover the original τ out of τ_e's in this way. The next proposition states, however, that this is not the case

Proposition 3.37. *There exists a soft topological space $\langle U, \tau, E\rangle$ for which τ is a proper subset of $\{F \mid \forall e \in E\,(F(e) \in \tau_e)\}$.*

Proof. Consider $\tau := \{\tilde{\varnothing}, \{(e_1,\bar{0}),(e_2,\bar{1})\}, \{(e_1,\bar{1}),(e_2,\bar{0})\}, \widetilde{\mathbb{Z}_2}\}$ with U and E given by $U := \mathbb{Z}_2, E := \{e_1, e_2\}$. Since both τ_{e_1} and τ_{e_2} are discrete, $F := E\times\{\bar{0}\}$ satisfies $\forall e \in E\,(F(e) \in \tau_e)$ even though it is not soft open. □

The difference between the soft topology τ and the family $\{\tau_e\}_{e\in E}$ of topologies can also be viewed in the light of (soft) closures. To this end, we use the soft space $\langle \mathbb{Z}_3, \tau, E\rangle$, with $E = \{e_1, e_2\}$ and $\tau = \{\tilde{\varnothing}, \{(e_1,\bar{2}),(e_2,\bar{1}),(e_2,\bar{2})\}, E\times\mathbb{Z}_3\}$. Put $H := \{\bar{0}\}$. Then, in the topological space $\langle \mathbb{Z}_3, \tau_{e_1}\rangle$, the closure $\mathrm{Cl}(H)$ of H satisfies $\forall x \in \mathbb{Z}_3\ \big(x \in \mathrm{Cl}(H) \iff x = \bar{0} \text{ or } \bar{1}\big)$. On the other hand, we have $\forall x \in \mathbb{Z}_3\ \big(x \,\tilde{\in}\, \widetilde{\mathrm{Cl}}(\tilde{H}) \iff x = \bar{0}\big)$.

The rest of this section studies the relationship between a soft space $\langle U, \tau, E\rangle$ and the family $\{\langle U, \tau_e\rangle\}_{e\in E}$ of topological spaces in terms of several soft concepts defined so far.

(1) **Soft Hausdorffness**
Shabir and Naz [15] obtained the following result:

Proposition 3.38. ([15]) *If $\langle U, \tau, E\rangle$ is a soft Hausdorff space, then $\langle U, \tau_e\rangle$ is a Hausdorff space for every parameter $e \in E$.* □

One may expect that the converse of the above proposition also holds. This, however, is not the case: The soft space $\langle \mathbb{Z}_2, \tau, E\rangle$ from Example 3.32 witnesses the failure of the converse implication. Clearly, both $\langle U, \tau_{e_1}\rangle$ and $\langle U, \tau_{e_2}\rangle$ are Hausdorff spaces, as they are discrete. However, both $\bar{0}$ and $\bar{1}$ have only one soft clopen neighborhood $E \times \mathbb{Z}_2$; hence this soft space is not a soft Hausdorff space.

(2) **Soft continuity**

Given a function $\phi : \langle U, \tau, E\rangle \to \langle U', \tau', E\rangle$, it would be natural to ask how the soft continuity of the function $\phi : \langle U, \tau, E\rangle \to \langle U', \tau', E\rangle$ is related to the continuity of $\phi : \langle U, \tau_e\rangle \to \langle U', \tau'_e\rangle$ for all $e \in E$. The following two examples shows that these two continuities are independent:

Example 3.39. Let us consider the $+1 \pmod 2$ function on $\langle \mathbb{Z}_2, \tau, E\rangle$, where $E := \{e_1, e_2\}$ and τ is a soft topology on $\mathbb{Z}_2$ generated by:

$$\left\{\tilde{\varnothing}, \{(e_1, \bar{0}), (e_2, \bar{1})\}, \{(e_1, \bar{1}), (e_2, \bar{0})\}, E \times \{\bar{1}\}, E \times \mathbb{Z}_2\right\}.$$

Since both $\langle \mathbb{Z}_2, \tau_{e_1}\rangle$ and $\langle \mathbb{Z}_2, \tau_{e_2}\rangle$ are discrete spaces, the $+1 \pmod 2$ function is clearly continuous in both spaces. However, this function is not soft continuous at $\bar{0}$, as $\bar{0}$ has only one soft neighborhood $E \times \mathbb{Z}_2$ while $\bar{1}$ has $E \times \{\bar{1}\}$ as a soft neighborhood. Thus we find that $\phi : \langle U, \tau, E\rangle \to \langle U', \tau', E\rangle$ is not always a soft continuous function even if $\phi : \langle U, \tau_e\rangle \to \langle U', \tau'_e\rangle$ is continuous at every $e \in E$.

Example 3.40. We then give a soft topology on $\mathbb{Z}$ with a parameter set $E = \{e_1, e_2\}$ by

$$\tau := \left\{\tilde{\varnothing}, E \times \mathbb{Z}\right\} \cup \left\{\{e_1\} \times S \mid S \text{ is a non-empty subset of } \mathbb{Z} \setminus \{5\}\right\}.$$

Then, the function $- : \mathbb{Z} \to \mathbb{Z}; x \mapsto -x$ is not continuous at 5 with respect to the topology τ_{e_1}. However, it *is* soft continuous as a function on $\langle \mathbb{Z}, \tau, E\rangle$, as every point $i \in \mathbb{Z}$ has only one soft neighborhood $E \times \mathbb{Z}$. This example shows that the continuity of $\phi : \langle U, \tau_e\rangle \to \langle U', \tau'_e\rangle$ for every $e \in E$ does not follow from the soft continuity of ϕ.

(3) **Soft compactness**

Firstly, let us consider the soft space $\langle \mathbb{Z}, \tau, E\rangle$, where $E := \{e_1, e_2\}$ and $\tau := \{\tilde{\varnothing}, E \times \mathbb{Z}\} \cup \{\{e_1\} \times A \mid A \subset \mathbb{Z} \text{ is non-empty}\}$. Observe that any soft open covering of $\langle \mathbb{Z}, \tau, E\rangle$ has $E \times \mathbb{Z}$ as an element. Hence $\langle \mathbb{Z}, \tau, E\rangle$ is soft compact. On the other hand, the topological space $\langle \mathbb{Z}, \tau_{e_1}\rangle$ is discrete and is not compact. This shows that, even if E is finite, the soft compactness of $\langle U, \tau, E\rangle$ does not imply the compactness of $\langle U, \tau_e\rangle$ for all $e \in E$.

Our next example uses an ordinal number $\omega + 1\,(= \{0, 1, 2, \ldots, \omega\})$ and $E := \{e_1, e_2\}$. Let τ be the soft topology on $\omega + 1$ generated by

$$\{\tilde{\varnothing}, F', E \times (\omega + 1)\} \cup \{F_n \mid n \in \omega\}, \text{ where}$$

$F' := \{(e_1, \alpha), (e_2, \omega) \mid 5 \le \alpha \le \omega\}$ and $F_n := \{(e_1, n), (e_2, \alpha) \mid \alpha \in \omega + 1\}$. It is not hard to see that both $\langle \omega + 1, \tau_{e_1}\rangle$ and $\langle \omega + 1, \tau_{e_2}\rangle$ are compact.

However, $\{F', F_0, F_1 \dots\}$ is a soft open covering of which no finite subset soft covers $\omega+1$. From this example, we see that even if $\langle U, \tau_e\rangle$ is compact for every $e \in E$, the soft space $\langle U, \tau, E\rangle$ is not necessarily soft compact.

(4) **Soft non-connectedness**

Consider the soft space $\langle \mathbb{Z}_2, \tau, E\rangle$, where $E := \{e_1, e_2\}$ and τ is the soft topology on $\mathbb{Z}_2$ generated by $\{\tilde{\varnothing}, \{(e_1, \bar{1})\}^{\tilde{c}}, \{(e_2, \bar{0})\}^{\tilde{c}}, E \times \mathbb{Z}_2\}$. Then $\{\{(e_1, \bar{1})\}^{\tilde{c}}, \{(e_2, \bar{0})\}^{\tilde{c}}\}$ soft covers $\mathbb{Z}_2$, witnessing that $\mathbb{Z}_2$ is not soft connected. However, we see at once that both $\langle \mathbb{Z}_2, \tau_{e_1}\rangle$ and $\langle \mathbb{Z}_2, \tau_{e_2}\rangle$ are connected. This shows that the soft non-connectedness of $\langle U, \tau, E\rangle$ does not entail the connectedness of $\langle U, \tau_e\rangle$ for every $e \in E$.

For the converse direction, we employ another soft topological space $\langle \mathbb{Z}_3, \tau, E\rangle$, where $E := \{e_1, e_2\}$ and

$$\tau := \{\tilde{\varnothing}, \{(e_1, \bar{0}), (e_2, \bar{0}), (e_2, \bar{1})\}, \{(e_1, \bar{1}), (e_1, \bar{2}), (e_2, \bar{2})\}, E \times \mathbb{Z}_3\}.$$

One can check straightforwardly that $\langle \mathbb{Z}_3, \tau_{e_i}\rangle$ is non-connected for each $i = 1, 2$, while $\langle \mathbb{Z}_3, \tau, E\rangle$ is soft-connected.

Remark 3.41. Needless to say, all negative results around soft continuous function presented in this section are negative with the more general definition of soft continuous function (Definition 3.8). Note, on the other hand, that the proofs of the most positive results related to soft continuity given in this section do not rest on the fact that the parameter set of domain is identical with that of the range. Therefore we obtain proofs of the same assertion with more general interpretation of soft continuous function once we replace the words "soft continuous function" in those proofs by "soft continuous function (in the sense of Definition 3.8)". Those results are Propositions 3.12, 3.17, 3.21, 3.25 and Theorem 3.26.

4. Soft topological group

In this section, we introduce the concept of soft topological group, and investigate its properties. Those who do not have a background in the theory of topological group can consult, e.g, [7]. Before proceeding any further, let us see one example.

Example 4.1. Let $\langle \mathbb{R}, \tau, E\rangle$ be a soft topological space, where $E := \{e_1, e_2\}$ and τ is a soft topology on $\mathbb{R}$ generated by the following subbase:

$$\begin{aligned}\big\{\tilde{\varnothing}, E \times \mathbb{R}\big\} \cup \{E \times (-\varepsilon, \varepsilon) \mid \varepsilon > 0\} \cup \big\{\{(e_1, r)\} \mid r \in \mathbb{R}\big\} \\ \cup \big\{\{(e_2, x) \mid r - \varepsilon < x < r + \varepsilon\} \mid r \in \mathbb{R}, \varepsilon > 0, 0 \notin (r - \varepsilon, r + \varepsilon)\big\}\end{aligned}$$

If we view $\mathbb{R}$ as an additive group $(\mathbb{R}, +)$, then $^{-1} : \mathbb{R} \to \mathbb{R}$ is soft continuous. This can be seen from the fact that if a soft set F is in the above subbase of τ, then F^{-1} is also in the subbase.

We then claim that $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is soft continuous. The soft continuity at (a, b) with $a, b \neq 0$ is easy, and so left to the reader. Suppose $a = 0$ and $b \neq 0$. For any soft neighborhood F of b, $\{(e_1, b), (e_2, x) \mid b - \varepsilon < x < b + \varepsilon\} \mathrel{\tilde{\subset}} F$ holds for some $\varepsilon > 0$. Then, for $\varepsilon' > 0$ small enough, it holds that $(0, b) \mathrel{\tilde{\in}} F'$ and that the image of F' by $+$ is a soft subset of $\{(e_1, b), (e_2, x) \mid b - \varepsilon < x < b + \varepsilon\} (\mathrel{\tilde{\subset}} F)$, where F' is given by

$$\Big(\{(e_1,0)\}\times\{(e_1,b)\}\Big)\,\tilde{\cup}\,\Big(\big(E\times(-\varepsilon',\varepsilon')\big)\times\{(e_2,x)\mid b-\varepsilon'<x<b+\varepsilon'\}\Big).$$

This shows that $+$ is soft continuous at $(0,b)$. The reader will be able to prove the soft continuity at $(a,0)$ with $a\neq 0$ and also at $(0,0)$.

However, this soft space $\langle\mathbb{R},\tau,E\rangle$ does *not* satisfy the following condition:

- For every $(r_1,r_2)\in\mathbb{R}\times\mathbb{R}$ and a soft neighborhood F of $r_1+r_2\in\mathbb{R}$, there exist soft neighborhoods F_1 and F_2 of r_1 and r_2, respectively, such that $F_1+F_2\,\tilde{\subset}\,F$

To see this, take a soft neighborhood $\{(e_1,5),(e_2,x)\mid 4<x<6\}$ of 5. Observe that, for every soft neighborhood F of 0, there exists an $\varepsilon_F>0$ such that $(E\times(-\varepsilon_F,\varepsilon_F))\,\tilde{\subset}\,F$. Then one finds that no soft neighborhood F' of 5 satisfies $F+F'\,\tilde{\subset}\,\{(e_1,5),(e_2,x)\mid 4<x<6\}$, by noting that $(F+F')(e_1)$ contains $(5-\varepsilon_F,5+\varepsilon_F)$.

For our purposes, it is convenient to have, for every $(g,h)\in G\times G$ and a soft neighborhood F of $g\cdot h\in G$, soft neighborhoods F_g and F_h of g and h, respectively, such that $F_g\cdot F_h\,\tilde{\subset}\,F$. The above example shows that this property does *not* follow from the soft continuity of the group operation $\cdot: G\times G\to G$ with the soft product topology on $G\times G$. (The reader should remind Proposition 3.34 here. In soft product topology, not all soft neighborhood F of $(x_0,x_1)\in U\times U$ contains a soft set of the form $F_0\times F_1$ with $x_0\,\tilde{\in}\,F_0$ and $x_1\,\tilde{\in}\,F_1$.) Hence we have to explicitly require this property in the definition of soft topological group. Here is the definition:

Definition 4.2. Let $G=(G,\cdot,1_G)$ be a group, where 1_G is the neutral element of G. We say that $\langle G,\tau,E\rangle$ is a *soft topological group* if τ is a soft topology on G with a parameter set E and the following conditions are satisfied:

- For every $(g,h)\in G\times G$ and a soft neighborhood F of $g\cdot h\in G$, there exist soft neighborhoods F_g and F_h of g and h, respectively, such that $F_g\cdot F_h\,\tilde{\subset}\,F$.
- The inversion function ${}^{-1}: G\to G$ is soft continuous.

In what follows, $\langle G,\tau,E\rangle$ stands for a soft topological group. We use the symbol e only for parameters and not for the neutral element of the group G; The neutral element will be denoted by 1_G, or just 1 when no confusion can arise. Also, we will write gh for $g\cdot h$, $(g,h\in G)$.

Remark 4.3. In the literature, e.g., [1], the terminology *soft group* is used to refer to a soft set F such that $F(e)$ is a subgroup of G for every $e\in E$. Here, we do *not* require for each soft open set to be a soft group — Our concept is literally a soft topological structure on a group. We chose this definition for the following reason: The notion of soft topological group should be a natural extension of the usual notion of topological group. In particular, when the parameter set E is singleton, two definitions should coincide. If, however, we require for each soft open set to be a soft group, two notions are different even when E is singleton. (Consider the discrete topological space $(\mathbb{Z},+)$, in which not all open sets are subgroup.) The reader, therefore, should keep in mind that our definition of soft topological group is different from [13, 16].

Proposition 4.4. *$\langle G, \tau, E\rangle$ is a soft topological group $\iff$ For every $x, y \in G$ and for every soft open set F with $xy^{-1} \tilde{\in} F$, there exist soft open sets F_1, F_2 such that $x \tilde{\in} F_1, y \tilde{\in} F_2$ and $F_1 F_2^{-1} \tilde{\subset} F$.*

Proof. Left-to-right: By the definition of the soft topological group, there exist soft open sets F_1', F_2' such that $x \tilde{\in} F_1', y^{-1} \tilde{\in} F_2'$ and $F_1' F_2' \tilde{\subset} F$. By the soft continuity of the inversion, there exists a soft open set F_2'' satisfying $y \tilde{\in} F_2''$ and $(F_2'')^{-1} \tilde{\subset} F_2'$. Hence we have $x \tilde{\in} F_1', y \tilde{\in} F_2''$ and $F_1'(F_2'')^{-1} \tilde{\subset} F_1' F_2' \tilde{\subset} F$.
Right-to-left: Let F be a soft open set such that $x^{-1} \tilde{\in} F$. Since $x^{-1} = \mathtt{1} \cdot x^{-1}$, there exist soft open sets F_1, F_2 such that $\mathtt{1} \tilde{\in} F_1, x \tilde{\in} F_2$ and $F_1 F_2^{-1} \tilde{\subset} F$. In particular, we have $F_2^{-1} \tilde{\subset} F$, which shows that the inversion is soft continuous.

Let F be a soft open set satisfying $xy \tilde{\in} F$. By noting $xy = x(y^{-1})^{-1}$, we find soft open sets F_1, F_2 such that $x \tilde{\in} F_1, y^{-1} \tilde{\in} F_2$ and $F_1 F_2^{-1} \tilde{\subset} F$. Since we have already shown that $^{-1} : G \to G$ is soft continuous, we find a soft open set F_2' such that $y \tilde{\in} F_2'$ and $(F_2')^{-1} \tilde{\subset} F_2$. Thus, we have $F_1 F_2' \tilde{=} F_1((F_2')^{-1})^{-1} \tilde{\subset} F_1 F_2^{-1} \tilde{\subset} F$. This completes the proof of the soft continuity of $\cdot : G \times G \to G$. □

Observe that it does *not* follow from the soft continuity of the two operations $\cdot : G \times G \to G$, $^{-1} : G \to G$ that $\alpha_L(g) : G \to G; x \mapsto gx$ is soft continuous for every $g \in G$. Indeed, the soft topological space $\langle \mathbb{R}, \tau, E\rangle$ given in Example 4.1 is a witness: Although both $\cdot : G \times G \to G$ and $^{-1} : G \to G$ are soft continuous in the soft space, the function $\alpha_L(r)$ is not soft continuous at 0 for every $r \in \mathbb{R} \setminus \{0\}$. However, we have

Proposition 4.5. *Let g be an arbitrary element of a group G. Then*
(i) $\alpha_L(g) : G \to G; x \mapsto gx$ (resp. $\alpha_R(g) : G \to G; x \mapsto xg$) is a soft homeomorphism.
(ii) $\beta(g) : G \to G; x \mapsto gxg^{-1}$ is a soft homeomorphism. □

Proof. For any $h, x \in G$ and a soft neighborhood F of hx, by the definition of soft topological group, there exist soft neighborhood F_h and F_x of h and x respectively such that $F_h \cdot F_x \tilde{\subset} F$. Hence we have $\alpha_L(h)(F_x) \tilde{=} h \cdot F_x \tilde{\subset} F_h \cdot F_x \tilde{\subset} F$, showing that $\alpha_L(h)$ is soft continuous. Since $\alpha_L(h)$ is soft continuous for every $h \in G$, in particular for both g and g^{-1}, the first claim follows at once by noting that $\alpha_L(g)^{-1} = \alpha_L(g^{-1})$.

The second claim can be proved similarly. □

As noted earlier in Theorem 3.11, soft open sets are not always preserved by soft homeomorphisms. This is the case even when the underlying soft topological space is a soft topological group.

Example 4.6. (Soft open sets are not always preserved by soft homeomorphisms) Let $E := \{e_1, e_2\}$ and consider the following soft topology τ on an additive group $\mathbb{Z}_2$: $\tau = \{\tilde{\varnothing}, \{(e_1, \bar{1})\}, E \times \mathbb{Z}_2\}$. The reader will be able to check easily that $\langle \mathbb{Z}_2, \tau, E\rangle$ is indeed a soft topological group. In this example, even though $\{(e_1, \bar{1})\}$ is soft open, $\alpha_L(\bar{1})^{-1}(\{(e_1, \bar{1})\}) = \{(e_1, \bar{0})\}$ is not soft open.

Here, we examine the relationship between $\langle G, \tau, E\rangle$ and $\langle G, \tau_e\rangle$:

Example 4.7. Let us consider the soft space $\langle \mathbb{Z}_2, \tau, E\rangle$, with $E = \{e_1, e_2\}$ and τ is generated by $\{\tilde{\varnothing}, \{(e_1, \bar{1})\}, \{(e_2, \bar{0})\}, E \times \mathbb{Z}_2\}$. Note that both $\bar{0}$ and $\bar{1}$ have exactly one soft neighborhood, namely $E \times \mathbb{Z}_2$. It is then evident that, with the additive structure on $\mathbb{Z}_2$, the soft space $\langle \mathbb{Z}_2, \tau, E\rangle$ is indeed a soft topological group. However, a trivial verification shows that neither $\langle \mathbb{Z}_2, \tau_{e_1}\rangle = \langle \mathbb{Z}_2, \{\tilde{\varnothing}, \{\bar{1}\}, \mathbb{Z}_2\}\rangle$ nor $\langle \mathbb{Z}_2, \tau_{e_2}\rangle = \langle \mathbb{Z}_2, \{\tilde{\varnothing}, \{\bar{0}\}, \mathbb{Z}_2\}\rangle$ is a topological group.

Recall the soft space $\langle \mathbb{Z}_2, \tau, E\rangle$ from Example 3.39. Here, we view $\mathbb{Z}_2$ as an additive group. Then, both $\langle \mathbb{Z}_2, \tau_{e_1}\rangle$ and $\langle \mathbb{Z}_2, \tau_{e_2}\rangle$ are topological group, as τ_{e_1} and τ_{e_2} are discrete. However, because $\alpha_L(\bar{1}) : \mathbb{Z}_2 \to \mathbb{Z}_2$ is not soft continuous at $\bar{0}$, the soft space $\langle \mathbb{Z}_2, \tau, E\rangle$ is not a soft topological group.

From the theory of topological group, one knows the following equivalence: A topological group G is a T_0 space $\iff$ G is a T_1 space $\iff$ G is a Hausdorff space. The following theorem is a soft analogue of this result:

Theorem 4.8. *For any soft topological group $\langle G, \tau, E\rangle$, the following are equivalent:*
(i) $\langle G, \tau, E\rangle$ is a soft T_0 space;
(ii) $\langle G, \tau, E\rangle$ is a soft T_1 space;
(iii) $\langle G, \tau, E\rangle$ is a soft Hausdorff space.

Proof. (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are easy, and left to the reader.
(i) $\Rightarrow$ (ii): We first show that $\widetilde{\{1\}}$ is soft closed. For this, it suffices to prove that every $x (\neq 1)$ can be separated from 1 by a soft open set. Take an $x \in G \setminus \{1\}$ arbitrarily. By the assumption, there exists a soft open set F such that either $x \,\tilde{\in}\, F \wedge \forall e \in E\,(1 \notin F(e))$ or $1 \,\tilde{\in}\, F \wedge \forall e \in E\,(x \notin F(e))$ holds. If the first case happens, we are done. In the second case, the soft continuity of $\alpha_L(x) : G \to G$ and the inversion ${}^{-1} : G \to G$ guarantees the existence of a soft set F' satisfying that $x \,\tilde{\in}\, F'$ and $xF'^{-1} \,\tilde{\subset}\, F$. Hence, we have $F' \,\tilde{\subset}\, F^{-1}x$. If 1 were in $F'(e)$ for some $e \in E$, then we would have $1 = u^{-1}x$ for some $u \in F(e)$. But then, x is equal to $u\,(\in F(e))$, contradicting the assumption that $\forall e \in E\,(x \notin F(e))$. Therefore, 1 is not in $F'(e)$ for any $e \in E$, and hence $\widetilde{\{1\}} \,\tilde{\cap}\, F' \,\tilde{=}\, \tilde{\varnothing}$ holds for this soft neighborhood F' of x.

Take any distinct x, y from G. Then, since $x^{-1}y$ is a soft element of a soft open set $\widetilde{\{1\}}^{\tilde{c}}$, the soft continuity of $\alpha_L(x^{-1})$ implies the existence of a soft open set F such that $y \,\tilde{\in}\, F$ and $x^{-1}F \,\tilde{\subset}\, \widetilde{\{1\}}^{\tilde{c}}$. Thus, this soft open set F satisfies $\forall e \in E\,(x \notin F(e))$.
(ii) $\Rightarrow$ (iii): Take $x \neq y$ from G. Since $1 \neq x^{-1}y$, the assumption implies that $\widetilde{\{x^{-1}y\}}^{\tilde{c}}$ is soft open. Select a soft neighborhood F of 1 such that $FF^{-1} \,\tilde{\subset}\, \widetilde{\{x^{-1}y\}}^{\tilde{c}}$. Assume for the contradiction that, for some $e \in E$, the soft sets $xF(e)$ and $yF(e)$ had a common element, say g. Put $g = xh = yk$ for $h, k \in F(e)$. But then we would have $x^{-1}y = hk^{-1} \in F(e)F(e)^{-1} \subset \{x^{-1}y\}^c$, a contradiction. Hence $xF \,\tilde{\cap}\, yF \,\tilde{=}\, \tilde{\varnothing}$.

In view of the soft continuity of $\alpha_L(x^{-1})$ (resp. $\alpha_L(y^{-1})$), we find a soft open F_x (resp. F_y) such that $x \,\tilde{\in}\, F_x \,\tilde{\subset}\, xF$ (resp. $y \,\tilde{\in}\, F_y \,\tilde{\subset}\, yF$). Clearly, F_x and F_y are soft disjoint, as $F_x \,\tilde{\cap}\, F_y \,\tilde{\subset}\, xF \,\tilde{\cap}\, yF \,\tilde{=}\, \tilde{\varnothing}$. □

We then investigate the notion of soft-connectedness in the context of soft topological groups. We present several results familiar from topological group theory which remain valid in the context of soft topological groups:

Definition 4.9. The *soft connected component* of G is the soft connected component of $1 \in G$.

Proposition 4.10. *Let C be the soft connected component of G. Then, for each $g \in G$, the connected component of g is gC.*

Proof. Since $\alpha_L(g) : G \to G$ is soft homeomorphic from Proposition 4.5, and soft-connectedness is a soft topological property by Proposition 3.25, gC is a soft connected set containing $g\,(= g\mathtt{1})$. Hence gC is a subset of the soft connected component of g. By repeating the same argument for g^{-1}, it is immediate that gC is actually the soft connected component of g. □

Proposition 4.11. *The soft connected component N of G is a normal subgroup.*

Proof. Let $a, b \in N$. Since both ${}^{-1} : G \to G$ and $\alpha_L(a) : G \to G$ are soft homeomorphic, aN^{-1} is also soft connected. Since aN^{-1} contains $\mathtt{1} = aa^{-1}$, we have $aN^{-1} \subset N$. Clearly, ab^{-1} is in aN^{-1}, so we have $ab^{-1} \in aN^{-1} \subset N$. This shows that N is a subgroup of G.

Note that both $a^{-1}Na$ and aNa^{-1} are also soft connected, and contain $\mathtt{1}$. By the fact that N is the largest soft connected subset containing $\mathtt{1}$, we see that both $a^{-1}Na$ and aNa^{-1} are a subset of N, from which the normality of N follows. □

The following is a basic result from the theory of topological group:

Proposition 4.12. *If both H and K are connected subsets of a topological group G, then $HK \subset G$ is also connected.*

Proof. Note that $H \times K$ is a connected subset of $G \times G$. From the continuity of $\cdot : G \times G \to G$ and the fact that the image of a connected set under a continuous function is also connected, the connectedness of HK follows. □

The above easiest proof does not work for soft topological space, as soft connectedness is not preserved by taking soft product (see Example 3.32 and Remark 3.33). The assertion itself is, however, still true for soft topological groups:

Proposition 4.13. *If both H and K are soft connected subsets of a soft topological group G, then $HK \subset G$ is also soft connected.*

Proof. Let $\{F_1, F_2\}$ be a soft open covering of HK such that no $g \in HK$ satisfies both $g \,\tilde{\in}\, F_1$ and $g \,\tilde{\in}\, F_2$. In view of Proposition 3.25, $hK = (\alpha_L(h))(K)$ is soft connected for every $h \in H$. Note that $\{F_1, F_2\}$ is a soft covering of hK for every $h \in H$. Take an $h \in H$ arbitrarily. We assume $\forall g \in hK\,(g \,\tilde{\notin}\, F_1)$ without loss of generality. Assume for the contradiction that $\exists g' \in h'K\,(g' \,\tilde{\in}\, F_1)$ holds for some $h' \in H$. Pick such a g' from $h'K$, and put $g' = h'k'$ $(k' \in K)$. Then both $hk' \,\tilde{\notin}\, F_1$ and $h'k' = g' \,\tilde{\in}\, F_1$ are true, contradicting the soft connectedness of Hk'. Therefore, $\forall g \in hK\,(g \,\tilde{\notin}\, F_1)$ holds for every $h \in H$. In other words, $\forall g \in HK\,(g \,\tilde{\notin}\, F_1)$. Thus, HK is soft connected. □

From the theory of topological group, we know that every open subgroup, i.e., a subgroup which is open with respect to the given topology, is closed. Here is its soft version:

Proposition 4.14. *Let $H \subset G$ be a subgroup of G such that $\tilde{H}$ is soft open. Then $\tilde{H}$ is soft closed.*

Proof. Let $G = H \sqcup \bigsqcup_{\lambda\in\Lambda} Hg_\lambda$ be a right coset decomposition. We first show that $\widetilde{Hg_\lambda}$ is soft open for each $\lambda \in \Lambda$. For every $h \in H$, from the soft continuity of $\alpha_R(g_\lambda^{-1}) : G \to G$, we can pick a soft neighborhood F_h of hg_λ such that $F_h g_\lambda^{-1} \tilde{\subset} \tilde{H}$. Then, for all $h \in H$, we have $h \tilde{\in} \widetilde{\bigcup}_{h\in H} F_h g_\lambda^{-1} \tilde{\subset} \tilde{H}$. Hence $\tilde{H} \tilde{=} \widetilde{\bigcup}_{h\in H} F_h g_\lambda^{-1}$, and so $\widetilde{Hg_\lambda}$ is soft equal to $\widetilde{\bigcup}_{h\in H} F_h$. As a soft union of soft open sets F_h's, $\widetilde{Hg_\lambda}$ is also soft open.

Then $\widetilde{\bigsqcup_{\lambda\in\Lambda} Hg_\lambda}$ is soft open, as it is the soft union of soft open sets. Hence $\tilde{H} = \tilde{G} \setminus \widetilde{\bigsqcup_{\lambda\in\Lambda} Hg_\lambda}$ is soft closed. □

Another result from the topological group theory which has a soft analogue is: For every subgroup H, H is open $\iff$ H has a non-empty interior. Here is the soft version:

Proposition 4.15. *Let $H \subset G$ be a subgroup. $\tilde{H}$ is soft open $\iff$ There exist an $h \in H$ and a soft neighborhood F of h such that $F \tilde{\subset} \tilde{H}$.*

Proof. We prove only the right-to-left direction: Take h and F as above. Then, for any $h' \in H$, there exists a soft neighborhood $F'_{h'}$ of h' such that $hh'^{-1} F'_{h'} \tilde{\subset} F$, as $\alpha_L(hh'^{-1}) : G \to G$ is soft continuous. Since $F'_{h'} \tilde{\subset} h'h^{-1}F$ and H is a subgroup, we have $F'_{h'} \tilde{\subset} h'h^{-1}F \tilde{\subset} \tilde{H}$. Therefore, $\tilde{H} \tilde{=} \widetilde{\bigcup}_{h\in H} F'_h$ is soft open. □

Not all results from the theory of topological group carry over to the soft setting. For instance, the closure of any subgroup of a topological group is again a subgroup. But, we have:

Example 4.16. (The closure of a subgroup is not necessarily a subgroup) Recall the soft space from Example 3.40. If we view $\mathbb{Z}$ as the additive group $(\mathbb{Z}, +)$, then this provides an example of a soft topological group. Clearly, $\{0\}$ is a subgroup of $\mathbb{Z}$. Observe that 5 is a soft element of any soft closed set F subject to the condition that $0 \tilde{\in} F$. In fact, $\{i \in \mathbb{Z} \mid i \tilde{\in} \widetilde{\mathrm{Cl}}(\widetilde{\{0\}})\}$ is equal to $\{0, 5\}$. Hence, we conclude that, in soft topological groups, the set $\{g \in G \mid g \tilde{\in} \widetilde{\mathrm{Cl}}(H)\}$ is not necessarily a subgroup of G even if H is.

Also, even though $\mathrm{Cl}(A) \cdot \mathrm{Cl}(B) \subset \mathrm{Cl}(A \cdot B)$ holds for any subsets A and B of a topological group, we have the following:

Example 4.17. ($\widetilde{\mathrm{Cl}}(\tilde{A}) \cdot \widetilde{\mathrm{Cl}}(\tilde{B}) \tilde{\subset} \widetilde{\mathrm{Cl}}(\tilde{A} \cdot \tilde{B})$ does not hold in general) The same soft topological group from Example 3.40 is a witness. Indeed, if we set $A := \{2\}$ and $B := \{3\}$, then

$$\widetilde{\mathrm{Cl}}(\tilde{A}) \tilde{=} \{(e_1, 2), (e_1, 5)\} \cup (\{e_2\} \times \mathbb{Z}),$$

$$\widetilde{\mathrm{Cl}}(\tilde{B}) \tilde{=} \{(e_1, 3), (e_1, 5)\} \cup (\{e_2\} \times \mathbb{Z}).$$

However, $\widetilde{\mathrm{Cl}}(\tilde{A} \cdot \tilde{B})$ is $\{(e_1, 5)\} \cup (\{e_2\} \times \mathbb{Z})$, which clearly does not contain $\widetilde{\mathrm{Cl}}(\tilde{A}) \cdot \widetilde{\mathrm{Cl}}(\tilde{B})$ as a soft subset.

Every topological group (more generaly, every uniform space) is regular [9]. On the other hand, we have:

Example 4.18. (A soft topological group which is not soft regular) Take the following soft topological group $\langle \mathbb{R}, \tau, E\rangle$, where $E := \{e_1, e_2\}$ and τ is the soft topology generated by the following subbase:

$$\left\{\tilde{\varnothing}, E \times \mathbb{R}\right\} \cup \{\{(e_1, r), (e_2, x) \mid r - \varepsilon < x < r + \varepsilon\} \mid r \in \mathbb{R}, \varepsilon > 0\}.$$

A trivial verification shows that, seen as an additive group, $\langle \mathbb{R}, \tau, E\rangle$ is indeed a soft Hausdorff topological group. Then, for any positive $\varepsilon > 0$, take a soft open neighborhood $\{(e_1, 0), (e_2, x) \mid -\varepsilon < x < \varepsilon\}$ of 0. Let F be a soft closed neighborhood of 0 such that $F \tilde{\subset} \{(e_1, 0), (e_2, x) \mid -\varepsilon < x < \varepsilon\}$.

We claim that no soft open neighborhood of 0 is a soft subset of this F. To see this, we first note that for any $r \in \mathbb{R}$, and a soft neighborhood F_r of r, there exists a $\varepsilon_r > 0$ with the following property:

$$\{(e_1, r), (e_2, x) \mid r - \varepsilon_r < x < r + \varepsilon_r\} \tilde{\subset} F_r.$$

Since $F^{\tilde{c}}$ is soft open, for each $r \in \mathbb{R} \setminus \{0\}$, there exists an $\varepsilon_r > 0$ satisfying that

$$\{(e_1, r), (e_2, x) \mid r - \varepsilon_r < x < r + \varepsilon_r\} \tilde{\subset} F^{\tilde{c}}.$$

It is clear that $0 \notin (r - \varepsilon_r, r + \varepsilon_r)$. It is also not hard to see the following soft equality:

$$\widetilde{\bigcup}_{r \neq 0} \left\{(e_1, r), (e_2, x) \,\middle|\, r - \varepsilon_r < x < r + \varepsilon_r\right\} \tilde{=} E \times (\mathbb{R} \setminus \{0\}).$$

This implies $F^{\tilde{c}} \tilde{=} E \times (\mathbb{R} \setminus \{0\})$, and so F is soft equal to $E \times \{0\}$. Since no soft open neighborhood of 0 is a soft subset of $E \times \{0\}$, this shows that $\langle \mathbb{R}, \tau, E\rangle$ is not soft regular.

Recall the following equivalences from the theory of topological group: a topological group G is a T_0 space $\Longleftrightarrow$ G is a T_1 space $\Longleftrightarrow$ G is a Hausdorff space $\Longleftrightarrow$ G is a Hausdorff space and is regular. From the above example, we see that the last equivalence is not valid for soft topological groups. Hence Theorem 4.8 is the best we can say in full generality.

5. Soft topological group as soft uniform space

This last section treats soft topological groups from more general viewpoint, i.e., soft uniform space, a notion defined by Çetkin and Aygün [4]. Those who are not so familiar with uniform spaces can consult some textbook, e.g, [9].

Definition 5.1 ([4])**.** Let U be an initial universe and E be a set of parameters. Then a *soft uniformity (or soft uniform structure)* over U is a family $\mathcal{D}$ of soft sets on $U \times U$ satisfying the following five conditions:

- $\widetilde{\Delta_U} := \left\{(e, \{(x, x) \mid x \in U\}) \mid e \in E\right\}$ is a soft subset of D for each $D \in \mathcal{D}$,
- $D_1 \tilde{\cap} D_2 \in \mathcal{D}$ for each $D_1, D_2 \in \mathcal{D}$,
- If $D \in \mathcal{D}$ and $D \tilde{\subset} D'$ then $D' \in \mathcal{D}$,
- For each $D \in \mathcal{D}$, there exists a $D' \in \mathcal{D}$ such that $D'^{-1} \tilde{\subset} D$,
- For each $D \in \mathcal{D}$, there exists a $D' \in \mathcal{D}$ such that $D' \circ D' \tilde{\subset} D$.

In the above definition, we used the following usual notation: D'^{-1} is the *inverse* of D', given by $D'^{-1}(e) := \{(y,x) \mid (x,y) \in D'(e)\}$. The *composition* $D' \circ D'$ is defined by $(D' \circ D')(e) := \{(u,v) \in U^2 \mid \exists z \in U\, ((u,z) \in D'(e) \text{ and } (z,v) \in D'(e))\}$. Each member of $\mathcal{D}$ is called a *soft entourage*.

The word *soft uniform space* refers to the triplet $\langle U, \mathcal{D}, E\rangle$. In what follows, unless otherwise stated, $\langle U, \mathcal{D}, E\rangle$ denotes a soft uniform space.

Definition 5.2 ([4])**.** Let $\mathcal{B}$ be a family of soft sets over $U \times U$. We say that $\mathcal{B}$ is a *base* of a soft uniformity on U if $\mathcal{B}$ satisfies:

- $\widetilde{\Delta_U}$ is a soft subset of D for each $D \in \mathcal{B}$,
- For each $D_1, D_2 \in \mathcal{B}$, there exists a $D_3 \in \mathcal{B}$ such that $D_3 \tilde{\subset} D_1 \tilde{\cap} D_2$,
- For each $D \in \mathcal{B}$, there exists a $D' \in \mathcal{B}$ such that $D'^{-1} \tilde{\subset} D$,
- For each $D \in \mathcal{B}$, there exists a $D' \in \mathcal{B}$ such that $D' \circ D' \tilde{\subset} D$.

A base $\mathcal{B}$ of a soft uniformity generates a soft uniformity on U in the obvious way (see [4]). Several basic properties of soft uniformity have been established in [4], so we do not go into detail here.

It is well-known that topological groups are special cases of uniform spaces [9]. Our next task is to extend this paradigm to the context of soft sets:

Definition 5.3. Let F be a soft set over a group G. Then
(i) The soft set $L_F : E \to G \times G$ is defined by $L_F(e) := \{(x,y) \mid x^{-1}y \in F(e)\}$.
(ii) The soft set $R_F : E \to G \times G$ is defined by $R_F(e) := \{(x,y) \mid yx^{-1} \in F(e)\}$.

Clearly, L_F and R_F coincide when G is an abelian group.

Proposition 5.4. *The family $\{L_F \mid F$ is a soft neighborhood of $\mathtt{1}\}$ is a base of a soft uniformity on G.*

Proof. The only non-trivial condition would be the fourth one: For any soft neighborhood F of $\mathtt{1}$, there exists a soft neighborhood F' of $\mathtt{1}$ with the property that $L_{F'} \circ L_{F'} \tilde{\subset} L_F$. But this is just an another way of saying $F' \cdot F' \tilde{\subset} F$, which follows at once from the definition of soft topological group. □

Likewise, the family $\{R_F \mid F$ is a soft neighborhood of $\mathtt{1}\}$ is a base of a soft uniformity on G. The resulting soft uniformity on the group G is called the *soft left (resp. right) uniformity*, and is denoted by $\mathcal{D}^L_{G,\tau}$ (resp. $\mathcal{D}^R_{G,\tau}$).

Remark 5.5. What have been discussed above are soft versions of the left uniformity and the right uniformity. The reader should recall the fact that these are not the only ways to view topological groups as uniform spaces — A uniformity generated by $\{L_N \cap R_N \mid N$ is a neighborhood of $\mathtt{1}\}$, the so-called *two-sided uniformity*, can be used. The family $\{L_N \cup R_N \mid N$ is a neighborhood of $\mathtt{1}\}$ also generates a uniformity. The soft versions of these two uniformities also allow us one to view soft topological groups as soft uniform spaces.

The notion of uniform continuity is important in the study of uniform spaces. Here, we provide its soft version.

Definition 5.6. $\phi : U \to U'$ is called a *soft uniformly continuous function* from $\langle U, \mathcal{D}, E\rangle$ to $\langle U', \mathcal{D}', E\rangle$ if $(\phi \times \phi)^{-1}(D')$ is in $\mathcal{D}$ for all $D' \in \mathcal{D}'$.

As before, we may use the notation $\phi : \langle U, \mathcal{D}, E\rangle \to \langle U', \mathcal{D}', E\rangle$ to explicitly mention the underlying soft uniform structures.

Theorem 5.7. *For every group homomorphism $\phi : G \to G'$, the following are equivalent:*
(i) $\phi : \langle G, \tau, E\rangle \to \langle G', \tau', E\rangle$ is soft continuous;
(ii) $\phi : \langle G, \mathcal{D}^L_{G,\tau}, E\rangle \to \langle G', \mathcal{D}^L_{G',\tau'}, E\rangle$ is soft uniformly continuous.

Proof. (i) $\Rightarrow$ (ii): We have to show that $(\phi\times\phi)^{-1}L_{F'} \in \mathcal{D}^L_{G,\tau}$ holds for every soft neighborhood F' of $1_{G'}$. From the soft continuity of ϕ, we can pick a soft neighborhood F of 1_G with the property that $\phi(F) \tilde{\subset} F'$. Take an arbitrary parameter $e \in E$. If $(g, h) \in (L_F)(e)$, then $\phi(g)^{-1}\phi(h) = \phi(g^{-1}h) \in \phi(F(e)) \subset F'(e)$. So, $(\phi(g), \phi(h))$ is in $(L_{F'})(e)$. This means that $L_F \tilde{\subset} (\phi\times\phi)^{-1}L_{F'}$. Since soft uniformities are upward closed (with respect to the soft inclusion), we see that $(\phi\times\phi)^{-1}L_{F'}$ is in $\mathcal{D}^L_{G,\tau}$.
(i) $\Leftarrow$ (ii): Choose a $g \in G$ and a soft neighborhood F' of $\phi(g)$ arbitrarily. Since $\alpha_L(\phi(g)) : G' \to G'$ is soft continuous, we know that there exists a soft neighborhood F'_1 of $1_{G'}$ such that $\phi(g)F'_1 \tilde{\subset} F'$. Since $1_{G'} \tilde{\in} F'_1 \tilde{\subset} \phi(g^{-1})F'$, the soft continuity of ϕ implies that $(\phi\times\phi)^{-1}(L_{\phi(g^{-1})F'})$ is a soft entourage on G. By the definition of the soft left uniformity, $L_{F_1} \tilde{\subset} (\phi\times\phi)^{-1}(L_{\phi(g^{-1})F'})$ holds for some soft neighborhood F_1 of 1_G. This means, $x^{-1}y \in F_1(e)$ implies $\phi(x^{-1}y) \in \phi(g^{-1})F'(e)$ for every $e \in E$. Thus $\phi(F_1(e)) \subset \phi(g^{-1})F'(e)$ for all $e \in E$, so $\phi(gF_1) \tilde{\subset} F'$. The soft continuity of $\alpha_L(g^{-1}) : G \to G$ yields the existence of a soft neighborhood F of g such that $g^{-1}F \tilde{\subset} F_1$. Therefore, $g \tilde{\in} F \tilde{\subset} gF_1$. Since $\phi(F) \tilde{\subset} \phi(gF_1) \tilde{\subset} F'$, this completes the proof. □

It would be interesting to investigate the notion of totally boundedness in the context of soft uniform spaces. In the next definition, $D[s]$ denotes the soft set over U given by $D[s](e) = \{x \in U \mid (s, x) \in D(e)\}$.

Definition 5.8. A soft uniform space $\langle U, \mathcal{D}, E\rangle$ is called *soft totally bounded* if for each soft entourage $D \in \mathcal{D}$, there exists a finite set $S \subset U$ with the property that $\forall x \in U\, \exists s \in S\, (x \,\tilde{\in}\, D[s])$.

Suppose a uniformity is derived from some soft topological group as its soft left uniformity. In that situation, by observing the equivalence $g \in sN \iff g \in L_N[s]$, we can rephrase the above condition using the terminology from soft topological group theory as follows:

Proposition 5.9. *A soft left uniform space $\langle G, \mathcal{D}^L_{G,\tau}, E\rangle$ is soft totally bounded $\iff$ For every soft neighborhood N of 1, there exists a finite subset S of G such that $\forall g \in G\, \exists s \in S\, (g \,\tilde{\in}\, sN)$.* □

The reader will be able to formulate and prove the right version of the above proposition.

Theorem 5.10. *Let $\langle G, \tau, E\rangle$ be a soft topological group. If the soft left uniform space $\langle G, \mathcal{D}^L_{G,\tau}, E\rangle$ is soft totally bounded, then so is the soft right uniform space $\langle G, \mathcal{D}^R_{G,\tau}, E\rangle$.*

Proof. Take an arbitrary soft topological group $\langle G, \tau, E\rangle$ with $\langle G, \mathcal{D}^L_{G,\tau}, E\rangle$ soft totally bounded. We first prepare the following lemma:

Lemma 5.11. *For any soft neighborhood F of $\mathtt{1}$, there exists a soft neighborhood F' of $\mathtt{1}$ such that $F' \tilde{\subset} F$ and $F' \tilde{=} gF'g^{-1}$ for every $g \in G$.*

Proof. Select a soft neighborhood F_1 of $\mathtt{1}$ so that $F_1^{-1}F_1F_1 \tilde{\subset} F$. Since the soft left uniform space $\langle G, \mathcal{D}^L_{G,\tau}, E\rangle$ is soft totally bounded by assumption, Proposition 5.9 assures the existence of finitely many elements $g_1, \ldots, g_n \in G$ with the property that $\forall g \in G\, \exists i \leq n\, (g \tilde{\in} g_iF_1)$. Since the function $\beta(g_i^{-1}) : x \mapsto g_i^{-1}xg_i$ is soft homeomorphic for each i, we have a soft neighborhood $F^{(i)}$ of $\mathtt{1}$ such that $g_i^{-1}F^{(i)}g_i \tilde{\subset} F_1$. Pick up an element g from G arbitrarily; then take an $i \leq n$ so that $g \tilde{\in} g_iF_1$. Define a soft neighborhood F'' of $\mathtt{1}$ by $F'' := F^{(1)} \tilde{\cap} \cdots \tilde{\cap} F^{(n)}$. Then we have

$$\begin{array}{rcl} g^{-1}F''g & \tilde{\subset} & F_1^{-1}g_i^{-1}F''g_iF_1 \\ & \tilde{\subset} & F_1^{-1}F_1F_1 \\ & \tilde{\subset} & F \end{array}$$

It is straightforward that $F' := \widetilde{\bigcup}_{g\in G}\, g^{-1}F''g$ has the desired property. □

Let F be a soft neighborhood of $\mathtt{1}$, and take F' as in the above lemma. In view of Proposition 5.9, one can pick up finitely many elements $h_1, \ldots, h_m \in G$ such that $\forall g \in G\, \exists i \leq m\, (g \tilde{\in} h_iF')$. Since F' satisfies $F' \tilde{=} h_iF'h_i^{-1}$, we have $h_iF' \tilde{=} F'h_i \tilde{\subset} Fh_i$. Therefore, $\forall g \in G\, \exists i \leq m\, (g \tilde{\in} Fh_i)$ holds., which finishes the proof. □

Needless to say, the converse of the above theorem is also true. Therefore, when dealing with soft totally boundedness, it is not important to specify which soft uniformity — left or right — is used.

Proposition 5.12. *If both H and K are soft totally bounded subsets of G, then so is $H \cdot K$.*

Proof. For any soft neighborhood F' of $\mathtt{1}$, $\forall h \in H\, \exists s \in S_1\, (h \tilde{\in} F's)$ holds for some finite set $S_1 \subset H$. Since S_1K is a soft union of finitely many soft totally bounded sets sK's, it is also soft totally bounded. Hence, there exists a finite set $S_2 \subset S_1K$ satisfying $\forall g \in S_1K\, \exists s \in S_2\, (g \tilde{\in} F's)$. Fix an $h \in H$ and a $k \in K$. Then, we find an $s \in S_1$ such that $h \tilde{\in} F's$. For this $s \in S_1$, there exists an $s' \in S_2$ such that $sk \tilde{\in} F's'$. In total, we have $hk \tilde{\in} F'F's'$.

For any soft neighborhood F of $\mathtt{1}$, there exists a soft neighborhood F' of $\mathtt{1}$ such that $F' \cdot F' \tilde{\subset} F$. The above argument applied to this F' shows that $H \cdot K$ is soft totally bounded. □

We introduce another concept here:

Definition 5.13. Let $\langle G, \tau, E\rangle$ be a soft topological group and $H \subset G$ be a subgroup. We say that H is *soft neutral* in G if for each soft neighborhood F of $\mathtt{1}$, there exists a soft neighborhood F' of $\mathtt{1}$ such that $F' \cdot H \tilde{\subset} H \cdot F$.

Proposition 5.14. *Every soft totally bounded subgroup H of a soft topological group G is soft neutral in G.*

Proof. Fix a soft neighborhood F of $\mathtt{1}$ arbitrarily. Then take a soft neighborhood F'' of $\mathtt{1}$ with the property that $F'' \cdot F'' \mathrel{\tilde{\subset}} F$. Since H is soft totally bounded, there exist $h_1, \ldots, h_n \in H$ such that $\forall h \in H\, \exists i \leq n\, (h \mathrel{\tilde{\in}} h_i F'')$. In particular, $H \mathrel{\tilde{\subset}} \{h_1, \ldots, h_n\} \cdot F''$. Since, for each $i \leq n$, the function $\beta(h_i^{-1}) : G \to G$ is soft homeomorphic, we have a soft neighborhood F'_i of $\mathtt{1}$ such that $h_i^{-1} F'_i h_i \mathrel{\tilde{\subset}} F''$, in other words $F'_i h_i \mathrel{\tilde{\subset}} h_i F''$. If we put $F' = F'_1 \mathrel{\tilde{\cap}} \cdots \mathrel{\tilde{\cap}} F'_n$, then $F'\{h_1, \ldots, h_n\} \mathrel{\tilde{\subset}} \{h_1, \ldots, h_n\} F''$ holds. Thus

$$\begin{array}{rcl} F' \cdot H & \tilde{\subset} & F' \cdot \{h_1, \ldots, h_n\} \cdot F'' \\ & \tilde{\subset} & \{h_1, \ldots, h_n\} \cdot F'' \cdot F'' \\ & \tilde{\subset} & \{h_1, \ldots, h_n\} \cdot F \\ & \tilde{\subset} & H \cdot F, \end{array}$$

which completes the proof. □

We conclude this section with a few results around the soft version of uniform topology.

Definition 5.15 ([4]). For any soft uniform space $\langle U, \mathcal{D}, E\rangle$, define the operator $\widetilde{\mathrm{Cl}}^u$ on the soft sets by putting $\widetilde{\mathrm{Cl}}^u(F) := \widetilde{\bigcap}\{D[F] \mid D \in \mathcal{D}\}$, where $D[F]$ is a soft set given by $D[F](e) := \{x \in U \mid (x, y) \in D(e) \text{ for some } y \in F(e)\}$. Then the *soft uniform topology* on U is the soft topology on U specified by taking $\widetilde{\mathrm{Cl}}^u$ as the closure operator.

Lemma 5.16. *Assume that the parameter set E is a finite set. Then, for each $x \in U$, the soft neighborhood filter of x (in the soft uniform topology) is generated by the family $\{D[x] \mid D \in \mathcal{D}\}$.*

Proof. For any soft entourage D, pick a symmetric soft entourage D' so that $D' \mathrel{\tilde{\subset}} D$ holds. If we put $F := (D'[x])^{\tilde{c}}$, then the symmetricity of D' implies that $x \notin D'[F](e)$ for every parameter $e \in E$. Since we have $F \mathrel{\tilde{\subset}} \widetilde{\mathrm{Cl}}^u(F) \mathrel{\tilde{\subset}} D'[F]$, it follows that $x \mathrel{\tilde{\in}} (\widetilde{\mathrm{Cl}}^u(F))^{\tilde{c}}$. The soft set $(\widetilde{\mathrm{Cl}}^u(F))^{\tilde{c}}$ is clearly soft open (with respect to the soft uniform topology) and is a soft subset of $F^{\tilde{c}} (\mathrel{\tilde{\doteq}} D'[x])$.

Conversely, for any soft open neighborhood F of x, $F^{\tilde{c}}$ is soft closed and, for every parameter e, x is not in $F^{\tilde{c}}(e) = (\widetilde{\mathrm{Cl}}^u(F^{\tilde{c}}))(e)$. Therefore, for each $e \in E$, we can pick a symmetric soft entourage D_e such that $x \notin D_e[F^{\tilde{c}}](e)$. Since E is finite by assumption, we find a symmetric soft entourage D' satisfying $x \notin D'[F^{\tilde{c}}](e)$ for all $e \in E$. Then $F^{\tilde{c}}$ does not soft intersect $D'[x]$; and so $D'[x] \mathrel{\tilde{\subset}} F$, as required. □

Proposition 5.17. *Let $\langle G, \tau, E\rangle$ be a soft topological group with E finite. Then for every $x \in G$, the soft neighborhood system at x with respect to τ is equal to the soft neighborhood system at x with respect to the soft uniform topology given by the soft left uniformity $\mathcal{D}^L_{G,\tau}$.*

Proof. Take a soft neighborhood F of $\mathtt{1}$ arbitrarily. Then, we have $L_F[x] \mathrel{\tilde{\doteq}} xF$, since $L_F[x](e) = \{g \in G \mid x^{-1}g \in F(e)\} = xF(e)$ hold for all $e \in E$. By the soft continuity of $\alpha_L(x^{-1}) : G \to G$, there exists a soft open neighborhood (with respect to τ) F' of x such that $x^{-1}F' \mathrel{\tilde{\subset}} F$. Hence, $F' \mathrel{\tilde{\subset}} xF \mathrel{\tilde{\doteq}} L_F[x]$. From Lemma 5.16, we know that the soft neighborhood filter of x (with respect to the soft uniform topology) is generated by $L_F[x]$'s. Therefore, we conclude that any soft neighborhood of x in the soft uniform topology is also a soft neighborhood of x with respect to τ.

For the converse inclusion, take any soft neighborhood F of x with respect to τ. Pick a soft neighborhood F' of $\mathtt{1}$ such that $xF' \tilde{\subset} F$. Then we have $L'_F[x] \tilde{=} xF' \tilde{\subset} F$. □

As pointed out already in Example 3.7, even if the soft neighborhood systems in two different soft topologies are identical at every point, two soft topologies can differ. Indeed, the above proposition is the best we can say about the relationship between two soft topologies.

Example 5.18. (Two soft topologies can be different) Consider the soft topological group $\langle \mathbb{Z}_2, \tau, E\rangle$, with $E := \{e_1, e_2\}$ and $\tau := \{\tilde{\varnothing}, \{(e_2, \bar{0})\}^{\tilde{c}}, \{(e_2, \bar{1})\}^{\tilde{c}}, E \times \mathbb{Z}_2\}$. In the soft uniform topology, we have

$$\begin{aligned}\widetilde{\mathrm{Cl}}^u(\{(e_1,\bar{0})\}) &\tilde{=} L_{\{(e_2,\bar{1})\}^{\tilde{c}}}[\{(e_1,\bar{0})\}] \,\tilde{\cap}\, L_{E\times\mathbb{Z}_2}[\{(e_1,\bar{0})\}] \\ &\tilde{=} L_{\{(e_2,\bar{1})\}^{\tilde{c}}}[\{(e_1,\bar{0})\}] \\ &\tilde{=} \{e_1\} \times \mathbb{Z}_2,\end{aligned}$$

which is not soft closed in the original soft topology τ.

Next, take the following soft topological group $\langle \mathbb{Z}_2, \tau \cup \{(e_1, \bar{0})\}, E\rangle$, where τ and E are as above. The soft set $\{(e_1, \bar{0})\}^{\tilde{c}}$ is soft closed in the initial soft topology τ. On the other hand, a simple computation shows that

$$\begin{aligned}\widetilde{\mathrm{Cl}}^u(\{(e_1,\bar{0})\}^{\tilde{c}}) &\tilde{=} L_{\{(e_2,\bar{1})\}^{\tilde{c}}}[\{(e_1,\bar{0})\}^{\tilde{c}}] \,\tilde{\cap}\, L_{E\times\mathbb{Z}_2}[\{(e_1,\bar{0})\}^{\tilde{c}}] \\ &\tilde{=} L_{\{(e_2,\bar{1})\}^{\tilde{c}}}[\{(e_1,\bar{0})\}^{\tilde{c}}] \\ &\tilde{=} E \times \mathbb{Z}_2 \\ &\tilde{\neq} \{(e_1,\bar{0})\}^{\tilde{c}},\end{aligned}$$

which shows that $\{(e_1, \bar{0})\}^{\tilde{c}}$ is not soft closed with respect to the soft uniform topology.

6. Future work

In this final section, we present several future work related to soft topological group:

(1) It is well-known in the theory of topological group that $H \cdot K \subset G$ is compact when both H and K are compact subsets of a topological group G. The proof is folklore: The (binary) product of compact spaces is also compact, and the image of compact set under any continuous function, in particular $\cdot : G \times G \to G$, is again compact. This standard proof does *not* generalize to soft topological group, as the soft product of soft compact sets may not be soft compact (See section 3). It is natural to ask if the soft version is true or not, i.e., if both H and K are soft compact subset of a soft topological group G, is $H \cdot K \subset G$ also soft compact?

(2) We used two different soft topological groups to give an example of the fact that the soft product of soft compact topological groups is not soft compact in general. However, we do not know if it is also the case when we take

soft product of the *same* soft compact topological group. More precisely, we would like to know the answer to the following question: If $\langle G, \tau, E\rangle$ is a soft compact topological group, is the soft product $G \times G$ also soft compact?

(3) Our initial motivation is of the following form: Fix a group G. What happen if we replace a topological structure on G by a *soft* topological structure? Which results remain valid and which does not? In order to compare properties of G with soft topological structure on it with properties of G with topological structure on it, we used not soft points but crisp points. It would be interesting to develop the theory of soft topological group with soft points.

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References

[1] H. Aktaş and N. Çağman, Soft sets and soft groups, Inform. Sci. 77(13) (2007) 2726–2735.
[2] A. Aygünoğlu and A. Aygün, Some notes on soft topological spaces, Neural Comput. & Appl. 22(1) (2012) 113–119.
[3] N. Çağman, S. Karataş and S. Enginoglu, Soft topology, Comput. Math. Appl. 62(1) (2011) 351–358.
[4] V. Çetkin and H. Aygün, Uniformity structure in the context of soft set, Ann. Fuzzy Math. Inform. 6(1) (2013) 69–76.
[5] K. P. Hart, J. -I. Nagata and J. E. Vaughan (eds.), Encyclopedia of General Topology, Elsevier, 2004.
[6] T. Hida, A comparison of two formulations of soft compactness, Ann. Fuzzy Math. Inform. 8(4) (2014) 511–525.
[7] P. J. Higgins, Introduction to topological groups, Cambridge University Press, 1974.
[8] S. Hussain and B. Ahmad, Some properties of soft topological spaces, Comput. Math. Appl. 62(11) (2011) 4058–4067.
[9] I. M. James, Introduction to Uniform Spaces, Cambridge University Press, 1990.
[10] J. L. Kelley, General topology, Springer Verlag, 1975.
[11] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, Comput. Math. Appl. 45(4-5) (2003) 555–562.
[12] D. A. Molodtsov, Soft set theory-First results, Comput. Math. Appl. 37(4-5) (1999) 19–31.
[13] S. Nazmul and S. K. Samanta, Soft topological soft groups, Math. Sci. (Springer) 6 (2012), Art. 66, 10 pp.
[14] Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci. 11(5) (1982) 341–356.
[15] M. Shabir and M. Naz, On soft topological spaces, Comput. Math. Appl. 61(7) (2011) 1786–1799.
[16] T. Shah and S. Shaheen, Soft topological groups and rings, Ann. Fuzzy Math. Inform. 7(5) (2014) 725–743.
[17] B. P. Varol and H. Aygün, On soft Hausdorff spaces, Ann. Fuzzy Math. Inform. 5(1) (2013) 15–24.
[18] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[19] İ. Zorlutuna, M. Akdag, W. K. Min and S. Atmaca, Remarks on soft topological spaces, Ann. Fuzzy Math. Inform. 3(2) (2012) 171–185.

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