Fuzzy n-fold KU-ideals of KU-algebras

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ABSTRACT. In this paper, we introduce the concept of fuzzy n-fold KU-ideal in KU-algebras, which is a generalization of fuzzy KU-ideal of KU-algebras and we obtain a few properties that is similar to the properties of fuzzy KU-ideal in KU-algebras, see [8]. Furthermore, we construct some algorithms for folding theory applied to KU-ideals in KU-algebras.

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1. Introduction

Prabpayak and Leerawat [11, 12] constructed a new algebraic structure which is called KU-algebras and introduced the concept of homomorphisms for such algebras and investigated some related properties. Zadeh [14] introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches, such as groups, functional analysis, probability theory and topology. Mostafa et al [8] introduced the notion of fuzzy KU-ideals of KU-algebras and then they investigated several basic properties which are related to fuzzy KU-ideals. Akram et al and Yaqoob et al [2, 13] introduced the notion of cubic sub-algebras and ideals in KU-algebras. They discussed relationship between a cubic subalgebra and a cubic KU-ideal. Muhiuddin [10] applied the bipolar-valued fuzzy set theory to KU-algebras, and introduced the notions of bipolar fuzzy KU-subalgebras and bipolar fuzzy KU-ideals in KU-algebras. He considered the specifications of a bipolar fuzzy KU-subalgebra, a bipolar fuzzy KU-ideal in KU-algebras and discussed the relations between a bipolar fuzzy KU-subalgebra and a bipolar fuzzy KU-ideal and provided conditions for a bipolar fuzzy KU-subalgebra to be a bipolar fuzzy KU-ideal. Gulistan et al. [4] studied ($\alpha, \beta$)-fuzzy KU-ideals in KU-algebras and discussed some special properties. Jun and Dudek [6] introduced n-fold BCC-ideals and obtained some related results. Jun [5] introduced n-fold fuzzy BCC-ideals and gave a relation...
between an n-fold fuzzy BCC-ideal and a fuzzy BCK-ideal. Mostafa and Kareem [9] introduced n-fold KU-ideals and obtained some related results. Akram et al. [1] introduced the notion of interval-valued ($\tilde{\theta}$, $\tilde{\delta}$)-fuzzy KU-ideals of KU-algebras and obtained some related properties. In this paper, we introduce a generalization of fuzzy KU-ideal of KU-algebras. Therefore, a few properties similar to the properties of fuzzy KU-ideal in KU-algebras can be obtained. Also, a few results of fuzzy n-fold KU-ideals of KU-algebra under homomorphism have been discussed. Moreover, some algorithms for folding theory have been constructed.

2. Preliminaries

In this section, we submit some concepts related to KU-algebra from the literature. Meanwhile, some comments and results are obtained.

**Definition 2.1** ([11, 12]). An algebra $(X, \ast, 0)$ of type $(2, 0)$ is said to be a KU-algebra, if for all $x, y, z \in X$, the following axioms are obtained:

$(ku_1)(x \ast y) \ast [(y \ast z) \ast (x \ast z)] = 0,$

$(ku_2)x \ast 0 = 0,$

$(ku_3)0 \ast x = x,$

$(ku_4)x \ast y = 0$ and $y \ast x = 0$ implies $x = y,$

$(ku_5)x \ast x = 0,$

On a KU-algebra $(x, \ast, 0)$ we can define a binary relation $\leq$ on $X$ by putting:

$x \leq y \iff y \ast x = 0.$

Thus a KU-algebra $X$ satisfies the conditions:

$(ku_{1'}) : (y \ast z) \ast (x \ast z) \leq (x \ast y)$

$(ku_{2'}) : 0 \leq x$

$(ku_{3'}) : x \leq y, y \leq x$ implies $x = y,$

$(ku_{4'}) : y \ast x \leq x.$

**Theorem 2.2** ([8]). *In a KU-algebra $(X, \ast, 0)$, the following axioms are satisfied:*

For all $x, y, z \in X$,

$(1): x \leq y \implies y \ast z \leq x \ast z,$

$(2): x \ast (y \ast z) = y \ast (x \ast z),$

$(3): ((y \ast x) \ast x) \leq y.$

**Definition 2.3** ([8, 11]). A non-empty subset $S$ of a KU-algebra $(X, \ast, 0)$ is called a KU-sub algebra of $X$ if $x \ast y \in S$ whenever $x, y \in S$.

**Definition 2.4.** [11] A non-empty subset $I$ of a KU-algebra $(X, \ast, 0)$ is called an ideal of $X$ if for any $x, y \in X$,

(i) $0 \in I,$

(ii) $x \ast y, x \in I$ imply $y \in I.$

We will refer to $X$ is a KU-algebra unless otherwise indicated.

**Lemma 2.5.** *In a KU-algebra $X$, any ideal is a KU-sub algebra.*

*Proof.* Let $I$ be an ideal. Then $0 \in I$ and $y \ast (x \ast y) = 0$ for all $x, y \in X$. Thus for $x, y \in I$ we have $y \ast (x \ast y) \in I$, which implies $x \ast y \in I$. \hfill $\Box$
**Definition 2.6 ([8, 11]).** Let $I$ be a non empty subset of a KU-algebra $X$. Then $I$ is said to be a KU-ideal of $X$, if

1. $(I_1) 0 \in I$
2. $(I_2) \forall x, y, z \in X,$ if $x \ast (y \ast z) \in I$ and $y \in I,$ imply $x \ast z \in I.$

**Theorem 2.7.** In a KU-algebra $X$, any KU-ideal is an ideal.

**Proof.** Indeed, by putting $x = 0$ in Definition 2.6 $(I_2)$, we obtain the result. □

Combining Lemma 2.5 and Theorem 2.7, we have the following corollary.

**Corollary 2.8.** Any KU-ideal of a KU-algebra $X$ is a KU-sub algebra.

Now, we review some fuzzy logic concepts.

**Definition 2.9 ([14]).** Let $X$ be a set, a fuzzy set $\mu$ in $X$ is a function $\mu : X \to [0, 1].$

For a fuzzy set $\mu$ in $X$ and $t \in [0, 1]$ define $U(\mu, t)$ to be the set $U(\mu, t) = \{x \in X : \mu(x) \geq t\},$ which is called a level set of $\mu.$

**Definition 2.10 ([8]).** A fuzzy set $\mu$ in a KU-algebra $X$ is called a fuzzy sub-algebra of $X$ if $\mu(x \ast y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X.$

**Definition 2.11.** Let $X$ be a KU-algebra, a fuzzy set $\mu$ in $X$ is called a fuzzy ideal of $X$ if it satisfies the following conditions:

1. $(F_1) \mu(0) \geq \mu(x)$ for all $x \in X.$
2. $(F_2) \forall x, y \in X, \mu(y) \geq \min\{\mu(x \ast y), \mu(x)\}.$

**Definition 2.12 ([8]).** Let $X$ be a KU-algebra, a fuzzy set $\mu$ in $X$ is called a fuzzy KU-ideal of $X$ if it satisfies the following conditions:

1. $(FI_1) \mu(0) \geq \mu(x)$ for all $x \in X.$
2. $(FI_2) \forall x, y, z \in X, \mu(x \ast z) \geq \min\{\mu(x \ast (y \ast z)), \mu(y)\}.$

**Example 2.13.** Let $X = \{0, 1, 2, 3, 4\}$ with $\ast$ is defined by the following table

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
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</table>

Using the algorithms in Appendix A, we can prove that $(X, \ast, 0)$ is KU-algebra. We define $\mu : X \to [0, 1]$ in $X$ by

$\mu(0) = t_0, \mu(1) = \mu(2) = t_1, \mu(3) = \mu(4) = t_2,$ where $t_0, t_1, t_2 \in [0, 1]$ with $t_0 > t_1 > t_2.$ By routine calculations, we know that $\mu$ is a fuzzy KU-ideal of KU-algebra $X.$

**Lemma 2.14 ([8]).** If $\mu$ is a fuzzy ideal of KU-algebra $X$ and if $x \leq y,$ then $\mu(x) \geq \mu(y).$

**Lemma 2.15 ([8]).** Let $\mu$ be a fuzzy KU-ideal of KU-algebra $X,$ if the inequality $x \ast y \leq z$ hold in $X.$ Then $\mu(y) \geq \min\{\mu(x), \mu(z)\}.$

**Lemma 2.16.** Any fuzzy KU-ideal of KU-algebras $X$ is a fuzzy ideal.
Lemma 2.17. In a KU-algebra $X$ any fuzzy KU-ideal is a fuzzy sub-algebra.

Proof. Let $\mu$ be a fuzzy KU-ideal of a KU-algebra $X$, for any $x, y \in X$ from $(ku^*)$, we have $x \ast y \leq y$ and (by Lemma 2.16) $\mu$ be a fuzzy ideal of a KU-algebra $X$ then (by Lemma(2.14)) $\mu(x \ast y) \geq \mu(y)$ and (by Lemma(2.15)) $\mu(y) \geq \min\{\mu(x), \mu(y)\}$, hence $\mu(x \ast y) \geq \min\{\mu(x), \mu(y)\}$.

The following example shows that the converse of Lemma2.17 may not be true. □

Example 2.18. Let $X = \{0, 1, 2, 3, 4\}$ with $*$ defined as in Example 2.13, and $\mu$ be a fuzzy set in $X$ given by

$$
\mu(x) = \begin{cases} 
  t_1 & x \in \{0, 2, 3\} \\
  t_2 & \text{otherwise}
\end{cases}
$$

where $t_1 > t_2$ in $[0, 1]$. It is easy to see that $\mu$ is a fuzzy sub-algebra of $X$ (by using the algorithms in Appendix A). But $\mu$ is not a fuzzy KU-ideal of $X$ because $\mu(0 \ast 1) = \mu(1) = t_2 < t_1 = \min\{\mu(0 \ast (3 \ast 1)), \mu(3)\}$.

Definition 2.19 [12]. Let $(X, \ast, 0)$ and $(X', \ast', 0')$ be two KU-algebras, a homomorphism is a map $f : X \rightarrow X'$ satisfying $f(x \ast y) = f(x) \ast' f(y)$ for all $x, y \in X$.

Theorem 2.20 [112]. Let $f$ be a homomorphism of a KU-algebra $X$ into a KU-algebra $Y$, then

(i) If $\theta$ is the identity in $X$, then $f(\theta)$ is the identity in $Y$.
(ii) If $S$ is a KU-subalgebra of $X$, then $f(S)$ is a KU-subalgebra of $Y$.
(iii) If $I$ is an n-fold KU-ideal of $X$, then $f(I)$ is an n-fold KU-ideal in $Y$.
(iv) If $S$ is a KU-subalgebra of $Y$, then $f^{-1}(S)$ is a KU-algebra of $X$.
(v) If $B$ is an n-fold KU-ideal in $f(X)$, then $f^{-1}(B)$ is an n-fold KU-ideal in $X$.

Definition 2.21 [8]. A fuzzy $\mu$ is called a fuzzy relation on any set $X$, if $\mu$ is a fuzzy subset $\mu : X \times X \rightarrow [0, 1]$.

Definition 2.22 [8]. If $\mu$ is a fuzzy relation on a set $X$ and $\beta$ is a fuzzy subset of $X$, then $\mu$ is a fuzzy relation on $\beta$ if $\mu(x, y) \leq \min\{\beta(x), \beta(y)\}, \forall x, y \in X$.

Definition 2.23 [8]. Let $\mu$ and $\beta$ be two fuzzy subsets of a set $X$, the product of $\mu$ and $\beta$ are defined by $(\mu \ast \beta)(x, y) = \min\{\mu(x), \beta(y)\}, \forall x, y \in X$.

Lemma 2.24 [8]. Let $\mu$ and $\beta$ be two fuzzy subsets of a set $X$, then

(i) $\mu \ast \beta$ is a fuzzy relation on $X$.
(ii) $(\mu \ast \beta)_t = \mu_t \ast \beta_t$ for all $t \in [0, 1]$.

Definition 2.25 [8]. If $\beta$ is a fuzzy subset of a set $X$, the strongest fuzzy relation on $X$, that is, a fuzzy relation on $\beta$ is $\mu_\beta$ given by $\mu_\beta(x, y) = \min\{\beta(x), \beta(y)\}, \forall x, y \in X$.

Lemma 2.26 [9]. For a given fuzzy subset $\beta$ of a set $X$, let $\mu_\beta$ be the strongest fuzzy relation on $X$, then for $t \in [0, 1]$, we have $(\mu_\beta)_t = \beta_t \ast \beta_t$.

Remark 2.27 [9]. Let $X$ and $Y$ be two KU-algebras, we define $*$ on $X \times Y$ by:

For every $(x, y), (u, v) \in X \times Y, (x, y) \ast (u, v) = (x \ast u, y \ast v)$, then clearly $(X \times Y, \ast, (0, 0))$ is a KU-algebra.
3. Major Section

For any elements \(x\) and \(y\) of a KU-algebra \(X\), \(x^n \ast y\), denotes \(x \ast (x \ast \ldots \ast (x \ast y))\), where \(x\) occurs \(n\) times.

**Definition 3.1.** A nonempty subset \(I\) of a KU-algebra \(X\) is called an \(n\)-fold KU-ideal of \(X\) if

(I) \(0 \in I\)

(II) \(\forall x, y, z \in X\) there exists a natural number \(n\) such that \(x^n \ast z \in I\) whenever \(x^n \ast (y \ast z) \in I\) and \(y \in I\).

For a KU-algebra \(X\), obviously \(\{0\}\) and \(X\) itself are \(n\)-fold KU-ideal of \(X\) for every positive integer \(n\).

**Example 3.2.** Let \(X = \{0, 1, 2, 3, 4\}\) with \(*\) defined as Example 2.13. By using the algorithms in Appendix A, it is easy to check that \(I = \{0, 1, 2, 3\}\) is an \(n\)-fold KU-ideal of \(X\) for every positive integer \(n\).

**Proposition 3.3.** Let \(X\) be a KU-algebra, a nonempty subset \(I\) of a KU-algebra \(X\) is an \(n\)-fold KU-ideal of \(X\) if and only if \(I\) is an ideal of \(X\).

**Proof.** Let \(I\) be an \(n\)-fold KU-ideal in \(X\); it is clear that \(0 \in I\). Since for any \(x, y, z \in X\), \((x^n \ast (y \ast z)) \in I\), \(y \in I \Rightarrow (x^n \ast z) \in I\), then by setting \(x = 0\), we obtain \((y \ast z) \in I, y \in I \Rightarrow z \in I\). Hence \(I\) is an ideal.

Conversely, let \(I\) be an ideal of \(X\), then \(0 \in I\). Now, if \((x^n \ast (y \ast z)) \in I\), \(y \in I\) then \((by \text{Th.}2.2(2)) (y \ast (x^n \ast z)) \in I\) and \(y \in I\), since \(I\) is an ideal of \(X\), thus \((x^n \ast z) \in I\), therefore \(I\) is an \(n\)-fold KU-ideal of \(X\).

**Proposition 3.4.** Let \(I\) be an ideal of a KU-algebra \(X\), if \(\forall x, y, z \in X, x^n \ast (y \ast z) \in I\), then \(I\) is an \(n\)-fold KU-ideal.

**Proof.** Let \(x, y, z \in X\), such that \((x^n \ast (y \ast z)) = y \ast (x^n \ast z) \in I\) and \(y \in I\), since \(I\) is an ideal and \(y \in I\), we easily obtain \(x^n \ast z \in I\). Hence \(I\) is an \(n\)-fold KU-ideal.

**Proposition 3.5.** If \(I\) is an \(n\)-fold KU-ideal of a KU-algebra \(X\), then for any \(x, y, z \in X, x^n \ast z \in I \Rightarrow x^n \ast (y \ast z) \in I\).

**Proof.** If we assume that for any \(n \in N\), we have

\[
\begin{align*}
(x^n \ast z) \ast (x^n \ast (y \ast z)) &= x^n \ast ((x^n \ast z) \ast (y \ast z)) = x^n \ast (y \ast ((x^n \ast z) \ast (x^n \ast z))) = \\
&= y \ast (x^n \ast ((x^n \ast z) \ast (x^n \ast z))) = y \ast ((x^n \ast z) \ast (x^n \ast z)) = y \ast 0 = 0 \in I
\end{align*}
\]

since \(I\) is an ideal and \(x^n \ast z \in I\), hence \(x^n \ast (y \ast z) \in I\).

**Definition 3.6.** A fuzzy set \(\mu\) in a KU-algebra \(X\) is called an \(n\)-fold fuzzy KU-ideal of \(X\) if

\((F_1)\mu(0) \geq \mu(x)\) for all \(x \in X\).

\((F_2)\forall x, y, z \in X\), there exists a natural number \(n\) such that \(\mu(x^n \ast z) \geq \min\{\mu(x \ast (y \ast z)), \mu(y)\}\).

**Remark 3.7.** The 1-fold fuzzy KU-ideal is precisely a fuzzy KU-ideal.
Example 3.8. Let $X = \{0, 1, 2, 3, 4\}$ with $*$ defined as in Example 2.13, define a fuzzy set $\mu$ in $X$ by $\mu(4) = 0.2$ and $\mu(x) = 0.7$ for all $x \neq 4$. Then $\mu$ is an $n$-fold fuzzy KU-ideal of $X$. By using the algorithms at the end of this paper, many examples of $n$-fold and fuzzy $n$-fold KU-ideals can be given.

Lemma 3.9. In a KU-algebra $X$, every fuzzy $n$-fold KU-ideal is a fuzzy ideal.

Proof. Let $\mu$ be an $n$-fold fuzzy KU-ideal of a KU-algebra $X$. By taking $x = 0$ in $(F_2)$ and using $(ku_3)$, we get

$\mu(z) = \mu(0^n * z) = \min\{\mu(0^n * (y * z)), \mu(y)\}$

Hence $\mu$ is a fuzzy ideal of $X$. □

Lemma 3.10. Let $\mu$ be a fuzzy $n$-fold KU-ideal of a KU-algebra $X$, if the inequality $x^n * y \leq z$ holds in $X$. Then $\mu(y) = \min\{\mu(x^n), \mu(z)\}$.

Proof. Assume that the inequality $x^n * y \leq z$ holds in $X$, then $z * (x^n * y) = 0$ and $\mu((x^n * y)) = \min\{\mu(x^n * (z * y)), \mu(z)\}$

by $(F_2)$

$\mu((x^n * y)) = \min\{\mu(x^n * (z * y)), \mu(z)\} = \min\{\mu(z * (x^n * y)), \mu(z)\}$

Hence $\mu$ is a fuzzy ideal of $X$. □

Proposition 3.11. If $\mu$ is a fuzzy $n$-fold KU-ideal of $X$, then

$\mu(x^n * (x^n * y) \geq \mu(y)$

Proof. By taking $z = x^n * y$ in $(F_2)$ and using $(ku_2)$ and $(F_1)$, we get

$\mu(x^n * (x^n * y) \geq \min\{\mu(x^n * (y * (x^n * y))), \mu(y)\}$

$\mu(x^n * (x^n * y) \geq \min\{\mu(x^n * (x^n * (y * y))), \mu(y)\}$

$\mu(x^n * (x^n * y) \geq \min\{\mu(x^n * 0), \mu(y)\}$

$\mu(x^n * (x^n * y) \geq \min\{\mu(0), \mu(y)\}$

The proof is completed. □

Proposition 3.12. If $\mu$ is a fuzzy $n$-fold KU-ideal, then

$\mu(x^n * (y * z)) \geq \mu(x^n * z)$

Proof. Since

by (2), Th.2.2, $ku_3$

$\begin{cases}
(x^n * z) * (x^n * (y * z)) = x^n * ((x^n * z) * (y * z)) \\
= x^n * (y * ((x^n * z) * (y * z))) \\
= y * (x^n * ((x^n * z) * (y * z))) \\
= y * ((x^n * z) * (x^n * z)) \\
= y * 0 = 0
\end{cases}$
we have $x^n \ast (y \ast z) \leq (x^n \ast z)$, by Lemma 2.14, we get

$$\mu(x^n \ast (y \ast z)) \geq \mu(x^n \ast z).$$

The proof is completed. □

**Proposition 3.13.** Let $A$ be a nonempty subset of a KU-algebra $X$ and $\mu$ be a fuzzy set in $X$ defined by

$$\mu(x) = \begin{cases} t_1 & x \in A \\ t_2 & \text{otherwise} \end{cases},$$

where $t_1 > t_2$ in $[0, 1]$. Then $\mu$ is an n-fold fuzzy KU-ideal of $X$ if and only if $A$ is an n-fold fuzzy KU-ideal of $X$.

Moreover, $X_\mu = A$ where $X_\mu = \{x \in X : \mu(x) = \mu(0)\}$.

**Proof.** Assume that $\mu$ is an n-fold fuzzy KU-ideal of $X$. Since $\mu(0) \geq \mu(x)$ for all $x \in X$, we have $\mu(0) = t_1$ and so $0 \in A$. For any $x, y, z \in X$ such that $x^n \ast (y \ast z) \in A$ and $y \in A$. Using $(F_2)$, we know that $\mu(x^n \ast z) \geq \min(\mu(x^n \ast (y \ast z)), \mu(y)) = t_1$ and thus $\mu(x^n \ast z) = t_1$. Hence $x^n \ast z \in A$, and $A$ is an n-fold KU-ideal of $X$.

Conversely, suppose that $A$ is an n-fold KU-ideal of $X$. Since $0 \in A$, it follows that $\mu(0) = t_1 \geq \mu(x)$ for all $x \in A$. Let $x, y, z \in X$. If $y \notin A$ and $x^n \ast z \in A$, then clearly $\mu(x^n \ast z) \geq \min\{\mu(x^n \ast (y \ast z)), \mu(y)\}$. Assume that $y \in A$ and $x^n \ast z \notin A$. Then by $(II)$, we have $x^n \ast (y \ast z) \notin A$. Therefore

$$\mu(x^n \ast z) = t_2 = \min(\mu(x^n \ast (y \ast z)), \mu(y)).$$

Finally we have that $X_\mu = \{x \in X : \mu(x) = \mu(0)\} = \{x \in X : \mu(x) = t_1\} = A$. □

**Theorem 3.14.** Let $\mu$ be a fuzzy set in KU-algebra $X$ and $n$ a positive integer. Then $\mu$ is an n-fold fuzzy KU-ideal of $X$ if and only if the nonempty level set $U(\mu, t)$ of $\mu$ is an n-fold KU-ideal of $X$. We then call $U(\mu, t)$ the level n-fold KU-ideal of $\mu$.

**Proof.** Suppose that $\mu$ is an n-fold fuzzy KU-ideal of $X$ and $U(\mu, t) \neq \phi$ for any $t \in [0, 1]$, there exists $x \in U(\mu, t)$ and so $\mu(x) \geq t$. It follows from $(F_1)$ that $\mu(0) \geq \mu(x) \geq t$ so that $0 \in U(\mu, t)$. Let $x, y, z \in X$ be such that $x^n \ast (y \ast z) \in U(\mu, t)$ and $y \in U(\mu, t)$. Using $(F_2)$, we know that

$$\mu(x^n \ast z) \geq \min\{\mu(x^n \ast (y \ast z)), \mu(y)\} \geq \min\{t, t\} = t$$

and thus $x^n \ast z \in U(\mu, t)$. Hence $U(\mu, t)$ is an n-fold KU-ideal of $X$.

Conversely, suppose that $U(\mu, t) \neq \phi$ is an n-fold KU-ideal of $X$ for every $t \in [0, 1]$. For any $x \in X$, let $\mu(x) = t$. Then $x \in U(\mu, t)$. Since $0 \in U(\mu, t)$, it follows that $\mu(0) \geq t = \mu(x)$ so that $\mu(0) \geq \mu(x)$ for all $x \in X$. Now, we need to show that $\mu$ satisfies $(F_2)$. If not, then there exist $a, b, c \in X$ such that $\mu(a^n \ast c) \geq \min\{\mu(a^n \ast (b \ast c)), \mu(b)\}$. By taking $t_0 = \frac{1}{2}(\mu(a^n \ast c) + \min\{\mu((a^n \ast (b \ast c)), \mu(b))\}$ then we have

$$\mu(a^n \ast c) < t_0 < \min(\mu(a^n \ast (b \ast c)), \mu(b)).$$

Hence $(a^n \ast (b \ast c)) \in U(\mu, t_0)$ and $b \in U(\mu, t_0)$, but $a^n \ast c \notin U(\mu, t_0)$, which means that $U(\mu, t_0)$ is not an n-fold KU-ideal of $X$. This is contradiction. Therefore $\mu$ is a fuzzy n-fold KU-ideal of $X$. □
Lemma 3.15. Let \( \mu \) be a fuzzy \( n \)-fold KU-ideal of a KU-algebra \( X \) and \( t_1, t_2 \in [0, 1] \) with \( t_1 > t_2 \). Then
(i) \( U(\mu, t_1) \subseteq U(\mu, t_2) \),
(ii) Whenever \( t_1, t_2 \in \text{Im}(\mu) \), where \( \text{Im}(\mu) = \{ t_i : i \in \Lambda \} \) then \( U(\mu, t_1) \neq U(\mu, t_2) \),
(iii) \( U(\mu, t_1) = U(\mu, t_2) \) if and only if there does not exist \( x \in X \) such that \( t_1 \leq \mu(x) < t_2 \).

Proof. Clear.

\[ \square \]

Theorem 3.16. Let \( \mu \) be a fuzzy \( n \)-fold KU-ideal of a KU-algebra \( X \) with \( \text{Im}(\mu) = \{ t_i : i \in \Lambda \} \) and \( \Omega = \{ U(\mu, t_i) : i \in \Lambda \} \) where \( \Lambda \) is an arbitrary index set. Then
(i) There exists a unique \( i_0 \in \Lambda \) such that \( t_{i_0} \geq t_i \) for all \( i \in \Lambda \).
(ii) \( X_\mu = \bigcap_{i \in \Lambda} U(\mu, t_i) = U(\mu, t_{i_0}) \),
(iii) \( X = \bigcup_{i \in \Lambda} U(\mu, t_i) \).

Proof. (i) since \( \mu(0) \in \text{Im}(\mu) \), there exists a unique \( i_0 \in \Lambda \) such that \( \mu(0) = t_{i_0} \).

Hence by \((F_1)\), we get \( \mu(x) \leq \mu(0) = t_{i_0} \) for all \( x \in X \), and so \( t_{i_0} \geq t_i \) for all \( i \in \Lambda \).

(ii) We have that
\[
U(\mu, t_{i_0}) = \{ x \in X : \mu(x) \geq t_{i_0} \} \\
= \{ x \in X : \mu(x) = t_{i_0} \} \\
= \{ x \in X : \mu(x) = \mu(0) \} = X.
\]

Note that \( U(\mu, t_{i_0}) \subseteq U(\mu, t_i) \) for all \( i \in \Lambda \), so that \( U(\mu, t_{i_0}) \subseteq \bigcap_{i \in \Lambda} U(\mu, t_i) \). Since \( i_0 \in \Lambda \), it follows that \( X_\mu = U(\mu, t_{i_0}) = \bigcap_{i \in \Lambda} U(\mu, t_i) \).

(iii) for any \( x \in X \) we have \( \mu(x) \in \text{Im}(\mu) \) and so there exists \( i(x) \in \Lambda \) such that \( \mu(x) = t_{i(x)} \). This implies \( x \in U(\mu, t_{i(x)}) \subseteq \bigcup_{i \in \Lambda} U(\mu, t_i) \). Hence \( X = \bigcup_{i \in \Lambda} U(\mu, t_i) \) \( \square \)

4. Image (pre-image) of fuzzy \( n \)-fold KU-ideals under homomorphism

Definition 4.1. Let \( f \) be a mapping from the set \( X \) to the set \( Y \). If \( \mu \) is a fuzzy subset of \( X \), then the fuzzy subset \( B \) of \( Y \) defined by
\[
f(\mu)(y) = B(y) = \begin{cases} 
\sup_{x \in f^{-1}(y)} \mu(x), & \text{if } f^{-1}(y) = \{ x \in X, f(x) = y \} \neq \emptyset \\
0 & \text{otherwise}
\end{cases}
\]

Is said to be the image of \( \mu \) under \( f \).

Similarly if \( \beta \) is a fuzzy subset of \( Y \), then the fuzzy subset \( \mu = \beta \circ f \) in \( X \) (i.e. the fuzzy subset defined by \( \mu(x) = \beta(f(x)) \) for all \( x \in X \)) is called the pre-image of \( \beta \) under \( f \).

Theorem 4.2. An onto homomorphic pre-image of a fuzzy \( n \)-fold KU-ideal is also a fuzzy \( n \)-fold KU-ideal.

Proof. Let \( f : X \to X' \) be an onto homomorphism of KU-algebras, \( \beta \) be a fuzzy \( n \)-fold KU-ideal of \( X' \) and \( \mu \) be the pre-image of \( \beta \) under \( f \), then \( \mu(x) = \beta(f(x)) \), for
all $x \in X$. Let $x \in X$, then $\mu(0) = \beta((0)) \geq \beta(f(x)) = \mu(x)$. Now let $x, y, z \in X$ then
\[
\mu(x^n \ast z) = \beta(f(x^n \ast z) = \beta(f(x^n) \ast f(z)) \\
\geq \min\{\beta(f(x^n)) \ast (f(y) \ast f(z)), \beta(f(y))\} \\
= \min\{\beta(f(x^n \ast (y \ast z))), \beta(f(y))\} \\
= \min\{\mu(x^n \ast (y \ast z)), \mu(y)\},
\]
the proof is completed. □

**Definition 4.3.** A fuzzy subset $\mu$ of $X$ has sup property if for any subset $T$ of $X$, there exist $t_0 \in T$ such that $\mu(t_0) = SUP_{t \in T} \mu(t)$.

**Theorem 4.4.** Let $f : X \rightarrow X'$ be a homomorphism between KU-algebras $X$ and $X'$. For every fuzzy $n$-fold KU-ideal $\mu$ in $X$, $f(\mu)$ is a fuzzy $n$-fold KU-ideal of $X'$.

**Proof.** By definition
\[
B(y') = f(\mu)(y') := \sup_{x \in f^{-1}(y')} \mu(x)
\]
for all $y' \in X'$ and sup $\phi := 0$. We have to prove that
\[
B((x')^n \ast z') \geq \min\{B((x')^n \ast (y' \ast z')), B(y')\}, \forall x', y', z' \in X'.
\]
Let $f : X \rightarrow X'$ be an onto a homomorphism of KU-algebras, $\mu$ be a fuzzy $n$-fold KU-ideal of $X$ with sup property and $\beta$ be the image of $\mu$ under $f$, since $\mu$ is a fuzzy n-fold KU-ideal of $X$, we have $\mu(0) \geq \mu(x)$ for all $x \in X$. Note that $0 \in f^{-1}(0')$, where $0, 0'$ are the zero of $X$ and $X'$ respectively, Thus, $B(0') = \sup_{t \in f^{-1}(0')} \mu(t) = \mu(0) \geq \mu(x)$ for all $x \in X$, which implies that $B(0') \geq \sup_{t \in f^{-1}(x')} \mu(t) = B(x')$ for any $x' \in X'$.

For any $x', y', z' \in X'$, Let
\[
x_0 \in f^{-1}(x'), y_0 \in f^{-1}(y'), z_0 \in f^{-1}(z')
\]
be such that
\[
\mu((x_0)^n \ast z_0) = \sup_{t \in f^{-1}((x')^n \ast z')} \mu(t), \mu(y_0) = \sup_{t \in f^{-1}(y')} \mu(t)
\]
and
\[
\mu((x_0)^n \ast (y_0 \ast z_0)) = B((x_0)^n \ast (y' \ast z')) = B((x')^n \ast (y' \ast z')) = \sup_{t \in f^{-1}((x')^n \ast (y' \ast z'))} \mu(t).
\]
Then
\[
B((x')^n \ast z') = \sup_{t \in f^{-1}((x')^n \ast z')} \mu(t) = \mu((x_0)^n \ast z_0) \\
\geq \min\{\mu((x_0)^n \ast (y_0 \ast z_0)), \mu(y_0)\} \\
= \min\{\sup_{t \in f^{-1}((x')^n \ast (y' \ast z'))} \mu(t), \sup_{t \in f^{-1}(y')} \mu(t)\} \\
= \min\{B((x')^n \ast (y' \ast z')), B(y')\}.
\]
Hence $B$ is a fuzzy n-fold KU-ideal of $Y$. □

**Proposition 4.5.** For a given fuzzy subset $\beta$ of a KU-algebra $X$, let $\mu_\beta$ be the strongest fuzzy relation on $X$. If $\mu_\beta$ is a fuzzy $n$-fold KU-ideal of $X \times X$, then $\beta(x) \leq \beta(0)$ for all $x \in X$. 

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Proof. Since \( \mu_\beta \) is a fuzzy n-fold KU-ideal of \( X \times X \), it follows from \((FI_1)\) that 
\[
\mu_\beta(x, x) = \min(\beta(x), \beta(x)) \leq \min(\beta(0), \beta(0)) \Rightarrow \beta(x) \leq \beta(0).
\]
\( \square \)

**Theorem 4.6.** Let \( \mu \) and \( \beta \) be two fuzzy n-fold KU-ideals of a KU-algebra \( X \), then \( \mu \times \beta \) is a fuzzy n-fold KU-ideal of \( X \times X \).

**Proof.** for any \((x, y) \in X \times X \), we have, 
\[
(\mu \times \beta)(0, 0) = \min(\mu(0), \beta(0)) \geq \min(\mu(x), \beta(x)) = (\mu \times \beta)(x, y).
\]
Now let \((x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X \), then
\[
(\mu \times \beta)(x_1^n \ast z_1, x_2^n \ast z_2) = \min\{\mu(x_1^n, z_1), \beta(x_2^n, z_2)\}
\]
\[
\geq \min\{\min\{\mu(x_1^n \ast (y_1 \ast z_1), \mu(y_1))\}, \min\{\beta(x_2^n \ast (y_2 \ast z_2), \beta(y_2))\}\}\}
\]
\[
= \min\{\min\{\mu(x_1^n \ast (y_1 \ast z_1), \beta(x_2^n \ast (y_2 \ast z_2))\}, \min\{\mu(y_1), \beta(y_2)\}\}\}
\]
\[
= \min\{\min\{\mu \times \beta)(x_1^n \ast (y_1 \ast z_1), x_2^n \ast (y_2 \ast z_2)), \{\mu \times \beta(y_1, y_2)\}\}\}
\]
Hence \( \mu \times \beta \) is a fuzzy n-fold KU-ideal of \( X \times X \).

Analogous to theorem 3.2 [7], we have a similar results for n-fold KU-ideal, which can be proved in similar manner, we state the results without proof. \( \square \)

**Theorem 4.7.** let \( \mu \) and \( \beta \) be two fuzzy subsets of a KU-algebra \( X \), such that \( \mu \times \beta \) is a fuzzy n-fold KU-ideal of \( X \times X \), then

(i) either \( \mu(x) \leq \mu(0) \) or \( \beta(x) \leq \beta(0) \) for all \( x \in X \).

(ii) if \( \mu(x) \leq \mu(0) \) for all \( x \in X \), then either \( \mu(x) \leq \mu(0) \) or \( \beta(x) \leq \beta(0) \).

(iii) if \( \beta(x) \leq \beta(0) \) for all \( x \in X \), then either \( \mu(x) \leq \mu(0) \) or \( \beta(x) \leq \mu(0) \).

(iv) if \( \mu \) or \( \beta \) is a fuzzy n-fold KU-ideal of \( X \).

**Theorem 4.8.** let \( \beta \) be a fuzzy subset of a KU-algebra \( X \) and \( \mu_\beta \) be the strongest fuzzy relation on \( X \), then \( \beta \) is a fuzzy n-fold KU-ideal of \( X \) if and only if \( \mu_\beta \) is a fuzzy n-fold KU-ideal of \( X \times X \).

**Proof.** Assume that \( \beta \) is a fuzzy KU-ideal of \( X \), we note from \((FI_1)\) that:
\[
\mu_\beta(0, 0) = \min(\beta(0), \beta(0)) \geq \min(\beta(x), \beta(y)) = \mu_\beta(x, y).
\]
Now, for any \((x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X \), we have from \((FI_2)\):
\[
\mu_\beta(x_1^n \ast z_1, x_2^n \ast z_2) = \min\{\beta(x_1^n, z_1), \beta(x_2^n, z_2)\}
\]
\[
\geq \min\{\min\{\beta(x_1^n \ast (y_1 \ast z_1), \beta(y_1))\}, \min\{\beta(x_2^n \ast (y_2 \ast z_2), \beta(y_2))\}\}\}
\]
\[
= \min\{\min\{\beta(x_1^n \ast (y_1 \ast z_1), \beta(x_2^n \ast (y_2 \ast z_2))\}, \min\{\beta(y_1), \beta(y_2)\}\}\}
\]
\[
= \min\{\mu_\beta(x_1^n \ast (y_1 \ast z_1), x_2^n \ast (y_2 \ast z_2)), \mu_\beta(y_1, y_2)\}\}
\]
Hence \( \mu_\beta \) is a fuzzy KU-ideal of \( X \times X \).

Conversely: For all \((x, y) \in X \times X \), we have
\[
\mu_\beta(0, 0) = \min(\beta(0), \beta(0)) \geq \min(\beta(x), \beta(y)) = \mu_\beta(x, y).
\]
It follows that \( \beta(0) \geq \beta(x) \) for all \( x \in X \), which prove \((FI_1)\).

Now, let \((x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X \), then
\[
\min\{\beta(x_1^n \ast z_1), \beta(x_2^n \ast z_2)\} = \mu_\beta(x_1^n \ast z_1, x_2^n \ast z_2)
\]
\[
\geq \min\{\min\{\beta(x_1^n \ast (y_1 \ast z_1), \beta(x_2^n \ast (y_2 \ast z_2))\}, \beta(y_1)\}\}\}
\]
\[
= \min\{\beta(x_1^n \ast (y_1 \ast z_1), x_2^n \ast (y_2 \ast z_2)), \min\{\beta(y_1), \beta(y_2)\}\}\}
\]
\[
= \min\{\mu_\beta(x_1^n \ast (y_1 \ast z_2), \beta(y_1))\}, \min\{\beta(x_2^n \ast (y_2 \ast z_2), \beta(y_2))\}\}\}
\]
In particular, if we take $x_2 = y_2 = z_2 = 0$, then
\[
\beta(x_1^n * z_1) \geq \min\{\beta(x_1^n * (y_1 * z_1)), \beta(y_1)\}.
\]
This proves (FI$_2$) and completes the proof. $\square$

5. Conclusion

We have studied the fuzzy foldedness of a KU-ideal in a KU-algebras. Also we discussed a few results of fuzzy n-fold KU-ideal of a KU-algebras under homomorphism, the image and the pre-image of fuzzy n-fold KU-ideals in KU-algebras are defined. How the image and the pre-image of fuzzy n-fold KU-ideals in KU-algebras become fuzzy n-fold KU-ideals are studied. Moreover, the product of fuzzy n-fold KU-ideals to product KU-algebras is established. Furthermore, we construct some algorithms for folding theory applied to KU-ideals in KU-algebras.

The main purpose of our future work is to investigate the foldedness of other types of fuzzy n-fold ideals such as a bipolar fuzzy n-fold KU-ideal of KU-algebras.

Appendix A. Algorithms

This appendix contains all necessary algorithms

**Algorithm for KU-algebras**

Input $\mathbf{X}:$ set, $\ast$: binary operation
Output "$\mathbf{X}$ is a KU-algebra or not"

Begin
If $\mathbf{X} = \emptyset$ then go to (1.);
EndIf
If $0 \notin \mathbf{X}$ then go to (1.);
EndIf
Stop: = false;
i := 1;
While $i \leq |\mathbf{X}|$ and not (Stop) do
If $x_i \ast x_i \neq 0$ then
Stop: = true;
EndIf
j := 1
While $j \leq |\mathbf{X}|$ and not (Stop) do
If $((y_j \ast x_i) \ast x_i) \neq 0$ then
Stop: = true;
EndIf
EndIf
EndIf
k := 1
While $k \leq |\mathbf{X}|$ and not (Stop) do
If $(x_i \ast y_i) \ast ((y_j \ast z_k) \ast (x_i \ast z_k)) \neq 0$ then
Stop: = true;
EndIf
EndIf
EndIf While
EndIf
EndIf While
EndIf
EndIf While

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If Stop then

(1.) Output (" X is not a KU-algebra")
Else
Output (" X is a KU-algebra")
EndIf
End

Algorithm for fuzzy subsets
Input (X : KU-algebra, A : X \to [0, 1]);
Output (" A is a fuzzy subset of X or not")
Begin
Stop: =false;
i := 1;
While i \leq |X| and not (Stop) do
If (A(x_i) < 0) or (A(x_i) > 1) then
Stop: = true;
EndIf
EndIf While
If Stop then
Output (" A is a fuzzy subset of X ")
Else
Output (" A is not a fuzzy subset of X ")
EndIf
End

Algorithm for n-fold KU-ideals
Input (X: KU-algebra, I: subset of X; n \in \mathbb{N});
Output (" I is an n-fold KU-ideal of X or not");
Begin
If I = \phi then go to (1.);
EndIf
If 0 \notin I then go to (1.);
EndIf
Stop: =false;
i := 1;
While i \leq |X| and not (Stop) do
j := 1
While j \leq |X| and not (Stop) do
k := 1
While k \leq |X| and not (Stop) do
If \((x_i^n \ast (y_j \ast z_k)) \in I\) and \(y_i \in I\) then
If \((x_i^n \ast z_k) \notin I\) then
Stop: = true;
EndIf
EndIf
EndIf
EndIf
End
EndIf While
EndIf While
EndIf While
If Stop then
Output (" I is an n-fold KU-ideal of X ")
Else
(1.) Output (" I is not an n-fold KU-ideal of X ")
EndIf
End

Algorithm for fuzzy n-fold KU-ideals
Input (X KU-algebra, *: binary operation, A: fuzzy subset of X);
Output (" A is a fuzzy n-fold KU-ideal of X or not")
Begin
Stop: =false;
i := 1;
While i ≤ |X| and not (Stop) do
If A(0) < A(x_i) then
Stop: = true;
EndIf
j := 1
While j ≤ |X| and not (Stop) do
k := 1
While k ≤ |X| and not (Stop) do
If A(x^n_i * z_k) < min(A(x^n_i * (y_j * z_k)), A(y_j)) then
Stop: = true;
EndIf
EndIf While
EndIf While
EndIf While
If Stop then
Output (" A is not a fuzzy n-fold KU-ideal of X ")
Else
 Output (" A is a fuzzy n-fold KU-ideal of X ")
EndIf
End

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