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Fuzzy n-fold KU-ideals of KU-algebras

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Abstract. In this paper, we introduce the concept of fuzzy n-fold KU-ideal in KU-algebras, which is a generalization of fuzzy KU-ideal of KU-algebras and we obtain a few properties that is similar to the properties of fuzzy KU-ideal in KU-algebras, see [8]. Furthermore, we construct some algorithms for folding theory applied to KU-ideals in KU-algebras.

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1. Introduction

Prabpayak and Leerawat [11, 12] constructed a new algebraic structure which is called KU-algebras and introduced the concept of homomorphisms for such algebras and investigated some related properties. Zadeh [14] introduced the notion of fuzzy sets. At present this concept has been applied to many mathematical branches, such as groups, functional analysis, probability theory and topology. Mostafa et al [8] introduced the notion of fuzzy KU-ideals of KU-algebras and then they investigated several basic properties which are related to fuzzy KU-ideals. Akram et al and Yaqoob et al [2, 13] introduced the notion of cubic sub-algebras and ideals in KU-algebras. They discussed relationship between a cubic subalgebra and a cubic KU-ideal. Muhiuddin [10] applied the bipolar-valued fuzzy set theory to KU-algebras, and introduced the notions of bipolar fuzzy KU-subalgebras and bipolar fuzzy KU-ideals in KU-algebras. He considered the specifications of a bipolar fuzzy KU-subalgebra, a bipolar fuzzy KU-ideal in KU-algebras and discussed the relations between a bipolar fuzzy KU-subalgebra and a bipolar fuzzy KU-ideal and provided conditions for a bipolar fuzzy KU-subalgebra to be a bipolar fuzzy KU-ideal. Gulistan et al. [4] studied (α, β)-fuzzy KU-ideals in KU-algebras and discussed some special properties. Jun and Dudek [6] introduced n-fold BCC-ideals and obtained some related results. Jun [5] introduced n-fold fuzzy BCC-ideals and gave a relation between an n-fold fuzzy BCC-ideal and a fuzzy BCK-ideal. Mostafa and Kareem [9] introduced n-fold KU-ideals and obtained some related results. Akram et al. [1] introduced the notion of interval-valued $(\tilde{\theta}, \tilde{\delta})$-fuzzy KU-ideals of KU-algebras and obtained some related properties. In this paper, we introduce a generalization of fuzzy KU-ideal of KU-algebras. Therefore, a few properties similar to the properties of fuzzy KU-ideal in KU-algebras can be obtained. Also, a few results of fuzzy n-fold KU-ideals of KU-algebra under homomorphism have been discussed. Moreover, some algorithms for folding theory have been constructed.

2. Preliminaries

In this section, we submit some concepts related to KU-algebra from the literature. Meanwhile, some comments and results are obtained.

Definition 2.1 ([11, 12])**.** An algebra $(X, *, 0)$ of type (2, 0) is said to be a KU -algebra, if for all $x, y, z \in X$, the following axioms are obtained:
$(ku_1)(x * y) * [(y * z) * (x * z)] = 0$,
$(ku_2)x * 0 = 0$,
$(ku_3)0 * x = x$,
$(ku_4)x * y = 0$ and $y * x = 0$ implies $x = y$,
$(ku_5)x * x = 0$,
On a KU-algebra $(x, *, 0)$ we can define a binary relation $\leq$ on X by putting:
$x \leq y \Leftrightarrow y * x = 0$.
Thus a KU-algebra X satisfies the conditions:
$(ku_{1'}) : (y * z) * (x * z) \leq (x * y)$
$(ku_{2'}) : 0 \leq x$
$(ku_{3'}) : x \leq y, y \leq x$ implies $x = y$,
$(ku_{4'}) : y * x \leq x$.

Theorem 2.2 ([8])**.** *In a KU-algebra* $(X, *, 0)$*, the following axioms are satisfied:*
For all $x, y, z \in X$,
(1): $x \leq y$ *implies* $y * z \leq x * z$,
(2): $x * (y * z) = y * (x * z)$,
(3): $((y * x) * x) \leq y$.

Definition 2.3 ([8, 11])**.** A non-empty subset S of a KU-algebra $(X, *, 0)$ is called a KU-sub algebra of X if $x * y \in S$ whenever $x, y \in S$.

Definition 2.4. [11] A non-empty subset I of a KU-algebra $(X, *, 0)$ is called an ideal of X if for any $x, y \in X$,
(i) $0 \in I$,
(ii) $x * y, x \in I$ imply $y \in I$.
We will refer to X is a KU-algebra unless otherwise indicated.

Lemma 2.5. *In a KU-algebra X, any ideal is a KU-sub algebra.*

Proof. Let I be an ideal. Then $0 \in I$ and $y * (x * y) = 0$ for all $x, y \in X$. Thus for $x, y \in I$ we have $y * (x * y) \in I$, which implies $x * y \in I$. □

Definition 2.6 ([8, 11]). Let I be a non empty subset of a KU-algebra X. Then I is said to be an KU- ideal of X, if
$(I_1) 0 \in I$
$(I_2) \forall\, x, y, z \in X$, if $x * (y * z) \in I$ and $y \in I$, imply $x * z \in I$.

Theorem 2.7. *In a KU-algebra X, any KU-ideal is an ideal.*

Proof. Indeed, by putting $x = 0$ in Definition 2.6 (I_2), we obtain the result. □

Combining Lemma 2.5 and Theorem 2.7, we have the following corollary.

Corollary 2.8. *Any KU-ideal of a KU-algebra X is a KU -sub algebra.*

Now, we review some fuzzy logic concepts.

Definition 2.9 ([14]). Let X be a set, a fuzzy set μ in X is a function $\mu : X \to [0, 1]$. For a fuzzy set μ in X and $t \in [0, 1]$ define $U(\mu, t)$ to be the set $U(\mu, t) = \{x \in X : \mu(x) \geq t\}$, which is called a level set of μ.

Definition 2.10 ([8]). A fuzzy set μ in a KU-algebra X is called a fuzzy sub-algebra of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

Definition 2.11. Let X be a KU-algebra, a fuzzy set μ in X is called a fuzzy ideal of X if it satisfies the following conditions:
$(F_1) \mu(0) \geq \mu(x)$ for all $x \in X$.
$(F_2) \forall\, x, y \in X, \mu(y) \geq \min\{\mu(x * y), \mu(x)\}$.

Definition 2.12 ([8]). Let X be a KU-algebra, a fuzzy set μ in X is called a fuzzy KU-ideal of X if it satisfies the following conditions:
$(FI_1) \mu(0) \geq \mu(x)$ for all $x \in X$.
$(FI_2) \forall\, x, y, z \in X, \mu(x * z) \geq \min\{\mu(x * (y * z)), \mu(y)\}$.

Example 2.13. Let $X = \{0, 1, 2, 3, 4\}$ with $*$ is defined by the following table

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	3	4
2	0	1	0	3	4
3	0	0	0	0	4
4	0	0	0	0	0

Using the algorithms in Appendix A, we can prove that $(X, *, 0)$ is KU-algebra. We define $\mu : X \to [0, 1]$ in X by
$\mu(0) = t_0$, $\mu(1) = \mu(2) = t_1$, $\mu(3) = \mu(4) = t_2$, where $t_0, t_1, t_2 \in [0, 1]$ with $t_0 > t_1 > t_2$. By routine calculations, we know that μ is a fuzzy KU-ideal of KU-algebra X.

Lemma 2.14 ([8]). *If μ is a fuzzy ideal of KU-algebra X and if $x \leq y$, then $\mu(x) \geq \mu(y)$.*

Lemma 2.15 ([8]). *Let μ be a fuzzy KU-ideal of KU-algebra X, if the inequality $x * y \leq z$ hold in X. Then $\mu(y) \geq \min\{\mu(x), \mu(z)\}$*

Lemma 2.16. *Any fuzzy KU- ideal of KU- algebras X is a fuzzy ideal.*

Proof. clear. □

Lemma 2.17. *In a KU-algebra X any fuzzy KU-ideal is a fuzzy sub-algebra.*

Proof. let μ be a fuzzy KU-ideal of a KU-algebra X, for any $x, y \in X$ from $(ku_{4'})$, we have $x * y \leq y$ and (by Lemma 2.16) μ be a fuzzy ideal of a KU-algebra X then (by Lemma(2.14)) $\mu(x * y) \geq \mu(y)$ and (by Lemma(2.15)) $\mu(y) \geq \min\{\mu(x), \mu(y)\}$, hence $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$.
The following example shows that the converse of Lemma2.17 may not be true. □

Example 2.18. Let $X = \{0, 1, 2, 3, 4\}$ with $*$ defined as in Example 2.13, and μ be a fuzzy set in X given by

$$\mu(x) = \begin{cases} t_1 & x \in \{0, 2, 3\} \\ t_2 & otherwise \end{cases}$$

where $t_1 > t_2$ in [0, 1]. It is easy to see that μ is a fuzzy sub-algebra of X (by using the algorithms in Appendix A). But μ is not a fuzzy KU-ideal of X because

$$\mu(0 * 1) = \mu(1) = t_2 < t_1 = \min\{\mu(0 * (3 * 1)), \mu(3)\}.$$

Definition 2.19 ([12]). Let $(X, *, 0)$ and $(X', *', 0')$ be two KU-algebras, a homomorphism is a map $f : X \to X'$ satisfying $f(x * y) = f(x) *' f(y)$ for all $x, y \in X$.

Theorem 2.20 ([12]). *Let f be a homomorphism of a KU-algebra X into a KU-algebra Y, then*
(i) If 0 is the identity in X, then $f(0)$ is the identity in Y.
(ii) If S is a KU-subalgebra of X, then $f(S)$ is a KU-subalgrbra of Y.
(iii) If I is an n-fold KU-ideal of X, then $f(I)$ is an n-fold KU-ideal in Y.
(iv) If S is a KU-subalgebra of Y, then $f^{-1}(S)$ is a KU-algebra of X.
(v) If B is an n-fold KU-ideal in $f(X)$, then $f^{-1}(B)$ is an n-fold KU-ideal in X.

Definition 2.21 ([3]). A fuzzy μ is called a fuzzy relation on any set X, if μ is a fuzzy subset $\mu : X \times X \to [0, 1]$.

Definition 2.22 ([3]). If μ is a fuzzy relation on a set X and β is a fuzzy subset of X, then μ is a fuzzy relation on β if $\mu(x, y) \leq \min\{\beta(x), \beta(y)\}, \forall x, y \in X$.

Definition 2.23 ([3]). Let μ and β be two fuzzy subsets of a set X, the product of μ and β are define by $(\mu \times \beta)(x, y) = \min\{\mu(x), \beta(y)\}, \forall x, y \in X$.

Lemma 2.24 ([3]). *Let μ and β be two fuzzy subsets of a set X, then*
(i) $\mu \times \beta$ is a fuzzy relation on X.
(ii) $(\mu \times \beta)_t = \mu_t \times \beta_t$ for all $t \in [0, 1]$.

Definition 2.25 ([3]). If β is a fuzzy subset of a set X, the strongest fuzzy relation on X, that is, a fuzzy relation on β is μ_β given by $\mu_\beta(x, y) = \min\{\beta(x), \beta(y)\}, \forall x, y \in X$.

Lemma 2.26 ([9]). *For a given fuzzy subset β of a set X, let μ_β be the strongest fuzzy relation on X, then for $t \in [0, 1]$, we have $(\mu_\beta)_t = \beta_t \times \beta_t$.*

Remark 2.27 ([9]). Let X and Y be two KU-algebras, we define $*$ on $X \times Y$ by: For every $(x, y), (u, v) \in X \times Y, (x, y) * (u, v) = (x * u, y * v)$, then clearly $(X \times Y, *, (0, 0))$ is a KU-algebra .

3. Major section

For any elements x and y of a KU-algebra $X, x^n * y$, denotes $x * (x * \ldots (x * y)$, where x occurs n times.

Definition 3.1. A nonempty subset I of a KU-algebra X is called an n-fold KU-ideal of X if
(I) $0 \in I$
(II) $\forall x, y, z \in X$ there exists a natural number n such that $x^n * z \in I$ whenever $x^n * (y * z) \in I$ and $y \in I$.
For a KU-algebra X, obviously $\{0\}$ and X itself are n-fold KU-ideal of X for every positive integer n.

Example 3.2. Let $X = \{0, 1, 2, 3, 4\}$ with $*$ defined as Example 2.13. By using the algorithms in Appendix A, it is easy to check that $I = \{0, 1, 2, 3\}$ is an n-fold KU-ideal of X for every positive integer n.

Proposition 3.3. *Let X be a KU-algebra, a nonempty subset I of a KU-algebra X is an n-fold KU-ideal of X if and only if I is an ideal of X.*

Proof. Let I be an n-fold KU-ideal in X; it is clear that $0 \in I$. Since for any $x, y, z \in X$, $(x^n * (y * z)) \in I, y \in I \Rightarrow (x^n * z) \in I$, then by setting $x = 0$, we obtain $(y * z) \in I, y \in I \Rightarrow z \in I$. Hence I is an ideal.
Conversely, let I be an ideal of X, then $0 \in I$. Now, if $(x^n * (y * z)) \in I, y \in I$ then (by Th.2.2 (2)) $(y * (x^n * z)) \in I$ and $y \in I$, since I is an ideal of X, thus $(x^n * z) \in I$, therefore I is an n-fold KU-ideal of X. □

Proposition 3.4. *Let I be an ideal of a KU-algebra X, if $\forall x, y, z \in X, x^n * (y * z) \in I$, then I is an n-fold KU-ideal.*

Proof. Let $x, y, z \in X$, such that $x^n * (y * z) = y * (x^n * z) \in I$ and $y \in I$, since I is an ideal and $y \in I$, we easily obtain $x^n * z \in I$. Hence I is an n-fold KU-ideal. □

Proposition 3.5. *If I is an n-fold KU-ideal of a KU-algebra X, then for any $x, y, z \in X, x^n * z \in I \Rightarrow x^n * (y * z) \in I$.*

Proof. If we assume that for any $n \in N$, we have

$$\overbrace{\left\{\begin{array}{ll} (x^n * z) * (x^n * (y * z)) & = x^n * ((x^n * z) * (y * z))) = x^n * (y * ((x^n * z)) = \\ & = y * (x^n * ((x^n * z) * z) = y * ((x^n * z) * (x^n * z)) \\ & = y * 0 = 0 \in I \end{array}\right.}^{by (2), Th.2.2, ku_3}$$

since I is an ideal and $x^n * z \in I$, hence $x^n * (y * z) \in I$. □

Definition 3.6. A fuzzy set μ in a KU-algebra X is called an n-fold fuzzy KU-ideal of X if
$(F_1)\mu(0) \geq \mu(x)$ for all $x \in X$.
$(F_2)\forall x, y, z \in X$, there exists a natural number n such that
$\mu(x^n * z) \geq \min\{\mu(x^n * (y * z)), \mu(y)\}$.

Remark 3.7. The 1-fold fuzzy KU-ideal is precisely a fuzzy KU-ideal.

Example 3.8. Let $X = \{0, 1, 2, 3, 4\}$ with $*$ defined as in Example 2.13, define a fuzzy set μ in X by $\mu(4) = 0.2$ and $\mu(x) = 0.7$ for all $x \neq 4$. Then μ is an n-fold fuzzy KU-ideal of X. By using the algorithms at the end of this paper, many examples of n-fold and fuzzy n-fold KU-ideals can be given.

Lemma 3.9. *In a KU-algebra X, every fuzzy n-fold KU-ideal is a fuzzy ideal.*

Proof. Let μ be an n-fold fuzzy KU-ideal of a KU-algebra X. By taking $x = 0$ in (F_2) and using (ku_3), we get

$$\mu(z) = \mu(0^n * z) \geq \min\{\mu(0^n * (y * z)), \mu(y)\} = \min\{\mu(y * z), \mu(y)\}, \text{ for all } y, z \in X.$$

Hence μ is a fuzzy ideal of X. □

Lemma 3.10. *Let μ be a fuzzy n-fold KU-ideal of KU-algebra X, if the inequality $x^n * y \leq z$ holds in X. Then $\mu(y) \geq \min\{\mu(x^n), \mu(z)\}$.*

Proof. Assume that the inequality $x^n * y \leq z$ holds in X, then $z * (x^n * y) = 0$ and by (F_2)

$$\begin{aligned} \mu((x^n * y)) \geq \min\{\mu(x^n * (z * y)), \mu(z)\} &= \min\{\mu(z * (x^n * y)), \mu(z)\} \\ &= \min\{\mu(0), \mu(z)\} = \mu(z) - -(I) \end{aligned}$$

but

$$\begin{aligned} \mu(0 * y) = \mu(y) \geq \min\{\mu(0 * (x^n * y)), \mu(x^n)\} &= \min\{\mu(x^n * y), \mu(x^n)\} \\ &\geq \min\{\mu(z), \mu(x^n)\} \ \ (by (I)) \end{aligned}$$

i.e. $\mu(y) \geq \min\{\mu(x^n), \mu(z)\}$. □

Proposition 3.11. *If μ is a fuzzy n-fold KU - ideal of X, then*

$$\mu(x^n * (x^n * y) \geq \mu(y)$$

Proof. By taking $z = x^n * y$ in (F_2) and using (ku_2) and (F_1), we get

$$\begin{aligned} \mu(x^n * (x^n * y)) &\geq \min\{\mu(x^n * (y * (x^n * y))), \mu(y)\} \\ &= \min\{\mu(x^n * (x^n * (y * y))), \mu(y)\} \\ &= \min\{\mu(x^n * (x^n * 0)), \mu(y)\} \\ &= \min\{\mu(x^n * 0), \mu(y)\} \\ &= \min\{\mu(0), \mu(y)\} = \mu(y). \end{aligned}$$

The proof is completed. □

Proposition 3.12. *If μ is a fuzzy n-fold KU-ideal, then*

$$\mu(x^n * (y * z)) \geq \mu(x^n * z)$$

Proof. Since

$$\overbrace{\begin{cases} (x^n * z) * (x^n * (y * z)) & = x^n * ((x^n * z) * (y * z)) \\ & = x^n * (y * ((x^n * z) * z)) \\ & = y * (x^n * ((x^n * z) * z)) \\ & = y * ((x^n * z) * (x^n * z)) \\ & = y * 0 = 0, \end{cases}}^{by(2), Th.2.2, ku_3}$$

we have $x^n * (y * z) \leq (x^n * z)$, by Lemma 2.14, we get

$$\mu(x^n * (y * z)) \geq \mu(x^n * z).$$

The proof is completed. □

Proposition 3.13. *Let A be a nonempty subset of a KU-algebra X and μ be a fuzzy set in X defined by*

$$\mu(x) = \begin{cases} t_1 & x \in A \\ t_2 & otherwise \end{cases},$$

where $t_1 > t_2$ in [0, 1]. Then μ is an n-fold fuzzy KU-ideal of X if and only if A is an n-fold fuzzy KU-ideal of X.
Moreover, $X_\mu = A$ where $X_\mu = \{x \in X : \mu(x) = \mu(0)\}$.

Proof. Assume that μ is an n-fold fuzzy KU-ideal of X. Since $\mu(0) \geq \mu(x)$ for all $x \in X$, we have $\mu(0) = t_1$ and so $0 \in A$. For any $x, y, z \in X$ such that $x^n * (y * z) \in A$ and $y \in A$. Using (F_2), we know that $\mu(x^n * z) \geq \min\{\mu(x^n * (y * z)), \mu(y)\} = t_1$ and thus $\mu(x^n * z) = t_1$. Hence $x^n * z \in A$, and A is an n-fold KU-ideal of X.

Conversely, suppose that A is an n-fold KU-ideal of X. Since $0 \in A$, it follows that $\mu(0) = t_1 \geq \mu(x)$ for all $x \in X$. Let $x, y, z \in X$. If $y \notin A$ and $x^n * z \in A$, then clearly $\mu(x^n * z) \geq \min\{\mu(x^n * (y * z)), \mu(y)\}$. Assume that $y \in A$ and $x^n * z \notin A$. Then by (II), we have $x^n * (y * z) \notin A$. Therefore

$$\mu(x^n * z) = t_2 = \min\{\mu(x^n * (y * z)), \mu(y)\}.$$

Finally we have that $X_\mu = \{x \in X : \mu(x) = \mu(0)\} = \{x \in X : \mu(x) = t_1\} = A$. □

Theorem 3.14. *Let μ be a fuzzy set in KU-algebra X and n a positive integer. Then μ is an n-fold fuzzy KU-ideal of X if and only if the nonempty level set $U(\mu, t)$ of μ is an n-fold KU-ideal of X. We then call $U(\mu, t)$ the level n-fold KU-ideal of μ.*

Proof. Suppose that μ is an n-fold fuzzy KU-ideal of X and $U(\mu, t) \neq \phi$ for any $t \in [0, 1]$, there exists $x \in U(\mu, t)$ and so $\mu(x) \geq t$. It follows from (F_1) that $\mu(0) \geq \mu(x) \geq t$ so that $0 \in U(\mu, t)$. Let $x, y, z \in X$ be such that $x^n * (y * z) \in U(\mu, t)$ and $y \in U(\mu, t)$. Using (F_2), we know that

$$\mu(x^n * z) \geq \min\{\mu(x^n * (y * z)), \mu(y)\} \geq \min\{t, t\} = t$$

and thus $x^n * z \in U(\mu, t)$. Hence $U(\mu, t)$ is an n-fold KU-ideal of X.
Conversely, suppose that $U(\mu, t) \neq \phi$ is an n-fold KU-ideal of X for every $t \in [0, 1]$. For any $x \in X$, let $\mu(x) = t$. Then $x \in U(\mu, t)$. Since $0 \in U(\mu, t)$, it follows that $\mu(0) \geq t = \mu(x)$ so that $\mu(0) \geq \mu(x)$ for all $x \in X$. Now, we need to show that μ satisfies (F_2). If not, then there exist $a, b, c \in X$ such that $\mu(a^n * c) \geq \min\{\mu(a^n * (b * c)), \mu(b)\}$.
By taking $t_0 = \frac{1}{2}(\mu(a^n * c) + \min\{\mu((a^n * (b * c)), \mu(b))\}$ then we have

$$\mu(a^n * c) < t_0 < \min\{\mu(a^n * (b * c)), \mu(b)\}.$$

Hence $(a^n * (b * c)) \in U(\mu, t_0)$ and $b \in U(\mu, t_0)$, but $a^n * c \notin U(\mu, t_0)$, which means that $U(\mu, t_0)$ is not an n-fold KU-ideal of X. This is contradiction. Therefore μ is a fuzzy n-fold KU-ideal of X. □

Lemma 3.15. *Let μ be a fuzzy n-fold KU-ideal of a KU-algebra X and $t_1, t_2 \in [0,1]$ with $t_1 > t_2$. Then*
(i) $U(\mu, t_1) \subseteq U(\mu, t_2)$,
(ii) Whenever $t_1, t_2 \in Im(\mu)$, where $Im(\mu) = \{t_i : i \in \Lambda\}$ then $U(\mu, t_1) \neq U(\mu, t_2)$,
(iii) $U(\mu, t_1) = U(\mu, t_2)$ if and only if there does not exist $x \in X$ such that $t_1 \leq \mu(x) < t_2$.

Proof. Clear. □

Theorem 3.16. *Let μ be a fuzzy n-fold KU-ideal of a KU-algebra X with $Im(\mu) = \{t_i : i \in \Lambda\}$ and $\Omega = \{U(\mu, t_i) : i \in \Lambda\}$ where Λ is an arbitrary index set. Then*
(i) There exists a unique $i_0 \in \Lambda$ such that $t_{i_0} \geq t_i$ for all $i \in \Lambda$.
(ii) $X_\mu = \bigcap_{i \in \Lambda} U(\mu, t_i) = U(\mu, t_{i_0})$,
(iii) $X = \bigcup_{i \in \Lambda} U(\mu, t_i)$,

Proof. (i) since $\mu(0) \in \mathrm{Im}(\mu)$, there exists a unique $i_0 \in \Lambda$ such that $\mu(0) = t_{i_0}$. Hence by (F_1), we get $\mu(x) \leq \mu(0) = t_{i_0}$ for all $x \in X$, and so $t_{i_0} \geq t_i$ for all $i \in \Lambda$.

(ii) We have that

$$\begin{aligned} U(\mu, t_{i_0}) &= \{x \in X : \mu(x) \geq t_{i_0}\} \\ &= \{x \in X : \mu(x) = t_{i_0}\} \\ &= \{x \in X : \mu(x) = \mu(0)\} = X. \end{aligned}$$

Note that $U((\mu, t_{i_0}) \subseteq U(\mu, t_i)$ for all $i \in \Lambda$, so that $U(\mu, t_{i_0}) \subseteq \bigcap_{i \in \Lambda} U(\mu, t_i)$. Since $i_0 \in \Lambda$, it follows that $X_\mu = U(\mu, t_{i_0}) = \bigcap_{i \in \Lambda} U(\mu, t_i)$.

(iii) for any $x \in X$ we have $\mu(x) \in \mathrm{Im}(\mu)$ and so there exists $i(x) \in \Lambda$ such that $\mu(x) = t_{i(x)}$. This implies $x \in U(\mu, t_{i(x)}) \subseteq \bigcup_{i \in \Lambda} U(\mu, t_i)$. Hence $X = \bigcup_{i \in \Lambda} U(\mu, t_i)$ □

4. Image (pre-image) of fuzzy n-fold KU- ideals under homomorphism

Definition 4.1. Let f be a mapping from the set X to the set Y. If μ is a fuzzy subset of X, then the fuzzy subset B of Y defined by

$$f(\mu)(y) = B(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x), & \text{if } f^{-1}(y) = \{x \in X, f(x) = y\} \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

Is said to be the image of μ under f.
Similarly if β is a fuzzy subset of Y, then the fuzzy subset $\mu = \beta \circ f$ in X (i.e. the fuzzy subset defined by $\mu(x) = \beta(f(x))$ for all $x \in X$) is called the pre-image of β under f.

Theorem 4.2. *An onto homomorphic pre-image of a fuzzy n-fold KU-ideal is also a fuzzy n-fold KU-ideal.*

Proof. Let $f : X \to X'$ be an onto homomorphism of KU-algebras, β be a fuzzy n-fold KU-ideal of X' and μ be the pre-image of β under f, then $\mu(x) = \beta(f(x))$, for all $x \in X$. Let $x \in X$, then $\mu(0) = \beta(f(0)) \geq \beta(f(x)) = \mu(x)$. Now let $x, y, z \in X$ then

$$\begin{aligned}\mu(x^n * z) &= \beta(f(x^n * z) = \beta(f(x^n) *' f(z)) \\ &\geq \min\{\beta(f(x^n) *' (f(y) *' f(z)), \beta(f(y))\} \\ &= \min\{\beta(f(x^n * (y * z))), \beta(f(y))\} \\ &= \min\{\mu(x^n * (y * z)), \mu(y)\},\end{aligned}$$

the proof is completed. □

Definition 4.3. A fuzzy subset μ of X has sup property if for any subset T of X, there exist $t_0 \in T$ such that $\mu(t_0) = \underset{t \in T}{SUP}\ \mu(t)$.

Theorem 4.4. *Let $f : X \to X'$ be a homomorphism between KU-algebras X and X'. For every fuzzy n-fold KU-ideal μ in $X, f(\mu)$ is a fuzzy n-fold KU-ideal of X'.*

Proof. By definition

$$B(y') = f(\mu)(y') := \sup_{x \in f^{-1}(y')} \mu(x)$$

for all $y' \in X'$ and $\sup \phi := 0$. We have to prove that

$$B((x')^n * z') \geq \min\{B((x')^n * (y' * z'), B(y')\}, \forall\, x', y', z' \in X'.$$

Let $f : X \to X'$ be an onto a homomorphism of KU-algebras, μ be a fuzzy n-fold KU-ideal of X with sup property and β be the image of μ under f, since μ is a fuzzy n-fold KU-ideal of X, we have $\mu(0) \geq \mu(x)$ for all $x \in X$. Note that $0 \in f^{-1}(0')$, where $0, 0'$ are the zero of X and X' respectively, Thus, $B(0') = \sup\limits_{t \in f^{-1}(0')} \mu(t) = \mu(0) \geq \mu(x)$ for all $x \in X$, which implies that $B(0') \geq \sup\limits_{t \in f^{-1}(x')} \mu(t) = B(x')$ for any $x' \in X'$.

For any $x', y', z' \in X'$, Let

$$x_0 \in f^{-1}(x'), y_0 \in f^{-1}(y'), z_0 \in f^{-1}(z')$$

be such that

$$\mu((x_0)^n * z_0) = \sup_{t \in f^{-1}((x')^n * z')} \mu(t), \mu(y_0) = \sup_{t \in f^{-1}(y')} \mu(t)$$

and

$$\begin{aligned}\mu((x_0)^n * (y_0 * z_0) &= B\{f((x_0)^n * (y_0 * z_0)\} = B((x')^n * (y' * z')) \\ &= \sup\nolimits_{((x_0)^n * (y_0 * z_0) \in f^{-1}((x')^n * (y' * z')} \mu((x_0)^n * (y_0 * z_0)\} \\ &= \sup\nolimits_{t \in f^{-1}((x')^n * (y' * x')} \mu(t).\end{aligned}$$

Then

$$\begin{aligned}B((x')^n * z') &= \sup\nolimits_{t \in f^{-1}((x')^n * z')} \mu(t) = \mu((x_0)^n * z_0) \\ &\geq \min\{\mu((x_0)^n * (y_0 * z_0), \mu(y_0)\} \\ &= \min\{\sup\nolimits_{t \in f^{-1}((x')^n * (y' * z'))} \mu(t), \sup\nolimits_{t \in f^{-1}(y')} \mu(t)\} \\ &= \min\{B((x')^n * (y' * z')), B(y')\}.\end{aligned}$$

Hence B is a fuzzy n-fold KU-ideal of Y. □

Proposition 4.5. *For a given fuzzy subset β of a KU-algebra X, let μ_β be the strongest fuzzy relation on X. If μ_β is a fuzzy n-fold KU-ideal of $X \times X$, then $\beta(x) \leq \beta(0)$ for all $x \in X$.*

Proof. Since μ_β is a fuzzy n-fold KU-ideal of $X \times X$, it follows from (FI_1) that $\mu_\beta(x,x) = \min\{\beta(x), \beta(x)\} \leq \min\{\beta(0), \beta(0)\}$, then $\beta(x) \leq \beta(0)$. □

Theorem 4.6. *Let μ and β be two fuzzy n-fold KU-ideals of a KU-algebra X, then $\mu \times \beta$ is a fuzzy n-fold KU-ideal of $X \times X$.*

Proof. for any $(x,y) \in X \times X$, we have,

$$(\mu \times \beta)(0,0) = \min\{\mu(0), \beta(0)\} \geq \min\{\mu(x), \beta(x)\} = (\mu \times \beta)(x,y).$$

Now let $(x_1,x_2),(y_1,y_2),(z_1,z_2) \in X \times X$, then

$$\begin{aligned}&(\mu \times \beta)(x_1^n * z_1, x_2^n * z_2) = \min\{\mu(x_1^n, z_1), \beta(x_2^n, z_2)\}\\ &\geq \min\{\min\{\mu(x_1^n * (y_1 * z_1), \mu(y_1))\}, \min\{\beta(x_2^n * (y_2 * z_2), \beta(y_2))\}\}\\ &= \min\{\min\{\mu(x_1^n * (y_1 * z_1), \beta(x_2^n * (y_2 * z_2))\}, \min\{\mu(y_1), \beta(y_2)\}\}\\ &= \min\{(\mu \times \beta)(x_1^n * (y_1 * z_1), x_2^n * (y_2 * z_2)), \{\mu \times \beta(y_1, y_2)\}\}\end{aligned}$$

Hence $\mu \times \beta$ is a fuzzy n-fold KU-ideal of $X \times X$.
Analogous to theorem 3.2 [7], we have a similar results for n-fold KU-ideal, which can be proved in similar manner, we state the results without proof. □

Theorem 4.7. *let μ and β be two fuzzy subsets of a KU-algebra X, such that $\mu \times \beta$ is a fuzzy n-fold KU-ideal of $X \times X$, then*
(i) either $\mu(x) \leq \mu(0)$ or $\beta(x) \leq \beta(0)$ for all $x \in X$.
(ii) if $\mu(x) \leq \mu(0)$ for all $x \in X$, then either $\mu(x) \leq \beta(0)$ or $\beta(x) \leq \beta(0)$.
(iii) if $\beta(x) \leq \beta(0)$ for all $x \in X$, then either $\mu(x) \leq \mu(0)$ or $\beta(x) \leq \mu(0)$.
(iv) either μ or β is a fuzzy n-fold KU-ideal of X.

Theorem 4.8. *let β be a fuzzy subset of a KU-algebra X and μ_β be the strongest fuzzy relation on X, then β is a fuzzy n-fold KU-ideal of X if and only if μ_β is a fuzzy n-fold KU-ideal of $X \times X$.*

Proof. Assume that β is a fuzzy KU-ideal of X, we note from (FI_1) that:

$$\mu_\beta(0,0) = \min\{\beta(0), \beta(0)\} \geq \min\{\beta(x), \beta(y)\} = \mu_\beta(x,y).$$

Now, for any $(x_1,x_2),(y_1,y_2),(z_1,z_2), \in X \times X$, we have from (FI_2):

$$\begin{aligned}&\mu_\beta(x_1^n * z_1, x_2^n * z_2) = \min\{\beta(x_1^n, z_1), \beta(x_2^n, z_2)\}\\ &\geq \min\{\min\{\beta(x_1^n * (y_1 * z_1), \beta(y_1))\} \min\{\beta(x_2^n * (y_2 * z_2), \beta(y_2))\}\}\\ &= \min\{\min\{\beta(x_1^n * (y_1 * z_1)), \beta(x_2^n * (y_2 * z_2))\} \min\{\beta(y_1), \beta(y_2)\}\}\\ &= \min\{(\mu_\beta(x_1^n * (y_1 * z_1), x_2^n * (y_2 * z_2))), \mu_\beta(y_1, y_2)\}\end{aligned}$$

Hence μ_β is a fuzzy KU-ideal of $X \times X$.
Conversely: For all $(x,y) \in X \times X$, we have
$\mu_\beta(0,0) = \min\{\beta(0), \beta(0)\} \geq \min\{\beta(x), \beta(y)\} = \mu_\beta(x,y)$. It follows that $\beta(0) \geq \beta(x)$ for all $x \in X$, which prove (FI_1).
Now, let $(x_1,x_2),(y_1,y_2),(z_1,z_2), \in X \times X$, then

$$\begin{aligned}&\min\{\beta(x_1^n * z_1), \beta(x_2^n * z_2)\} = \mu_\beta(x_1^n * z_1, x_2^n * z_2)\\ &\geq \min\{\mu_\beta((x_1^n, x_2^n) * ((y_1, y_2) * (z_1, z_2))), \mu_\beta(y_1, y_2)\}\\ &= \min\{\min\{\mu_\beta(x_1^n * (y_1 * z_1), x_2^n * (y_2 * z_2))\}, \mu_\beta(y_1, y_2)\}\\ &= \min\{\min\{\mu_\beta(x_1^n * (y_1 * z_1), x_2^n * (y_2 * z_2))\}, \min\{\mu_\beta(y_1), \mu_\beta(y_2)\}\}\\ &= \min\{\min\{\beta(x_1^n * (y_1 * z_1), \beta(x_2^n * (y_2 * z_2)))\}, \min\{\beta(y_1), \beta(y_2)\}\}\\ &= \min\{\min\{\beta(x_1^n * (y_1 * z_2), \beta(y_1))\}, \min\{\beta(x_2^n * (y_2 * z_2), \beta(y_2))\}\}\end{aligned}$$

In particular, if we take $x_2 = y_2 = z_2 = 0$, then

$$\beta(x_1^n * z_1) \geq \min\{\beta(x_1^n * (y_1 * z_1)), \beta(y_1)\}.$$

This proves (FI_2) and completes the proof. □

5. Conclusion

We have studied the fuzzy foldedness of a KU-ideal in a KU-algebras. Also we discussed a few results of fuzzy n-fold KU-ideal of a KU-algebras under homomorphism, the image and the pre- image of fuzzy n-fold KU-ideals in KU - algebras are defined. How the image and the pre-image of fuzzy n-fold KU-ideals in KU-algebras become fuzzy n-fold KU-ideals are studied. Moreover, the product of fuzzy n-fold KU-ideals to product KU-algebras is established. Furthermore, we construct some algorithms for folding theory applied to KU-ideals in KU-algebras.

The main purpose of our future work is to investigate the foldedness of other types of fuzzy n-fold ideals such as a bipolar fuzzy n-fold KU-ideal of KU-algebras.

Appendix A. Algorithms

This appendix contains all necessary algorithms

Algorithm for KU-algebras

Input (X: set, $*$: binary operation)
Output (" X is a KU-algebra or not")
Begin
If $X = \phi$ then go to (1.);
EndIf
If $0 \notin X$ then go to (1.);
EndIf
Stop: =false;
$i := 1$;
While $i \leq |X|$ and not (Stop) do
If $x_i * x_i \neq 0$ then
Stop: = true;
EndIf
$j := 1$
While $j \leq |X|$ and not (Stop) do
If $((y_j * x_i) * x_i) \neq 0$ then
Stop: = true;
EndIf
EndIf
$k := 1$
While $k \leq |X|$ and not (Stop)do
If $(x_i * y_i) * ((y_j * z_k) * (x_i * z_k)) \neq 0$ then
Stop: = true;
EndIf
EndIf While
EndIf While
EndIf While

If Stop then

(1.) Output (" X is not a KU-algebra")
Else
Output (" X is a KU-algebra")
EndIf
End

Algorithm for fuzzy subsets

Input (X : KU-algebra, $A : X \to [0,1]$);
Output (" A is a fuzzy subset of X or not")
Begin
Stop: =false;
$i := 1$;
While $i \leq |X|$ and not (Stop) do
If $(A(x_i) < 0)$ or $(A(x_i) > 1)$ then
Stop: = true;
EndIf

EndIf While
If Stop then
Output (" A is a fuzzy subset of X ")

Else
Output (" A is not a fuzzy subset of X ")
EndIf
End

Algorithm for n-fold KU-ideals

Input (X: KU-algebra, I: subset of $X, n \in N$);
Output (" I is an n-fold KU-ideal of X or not");
Begin
If $I = \phi$ then go to (1.);
EndIf
If $0 \notin I$ then go to (1.);
EndIf
Stop: =false;
$i := 1$;
While $i \leq |X|$ and not (Stop) do
$j := 1$
While $j \leq |X|$ and not (Stop) do
$k := 1$
While $k \leq |X|$ and not (Stop) do
If $(x_i^n * (y_j * z_k)) \in I$ and $y_i \in I$ then
If $(x_i^n * z_k) \notin I$ then
Stop: = true;
EndIf
EndIf

EndIf While
EndIf While
EndIf While
If Stop then
Output (" I is an n-fold KU-ideal of X ")
Else
(1.) Output (" I is not an n-fold KU-ideal of X ")
EndIf
End

Algorithm for fuzzy n-fold KU-ideals

Input (X KU-algebra, $*$: binary operation, A: fuzzy subset of X);
Output (" A is a fuzzy n-fold KU-ideal of X or not")
Begin
Stop: =false;
$i := 1$;
While $i \leq |X|$ and not (Stop) do
If $A(0) < A(x_i)$ then
Stop: = true;
EndIf
$j := 1$
While $j \leq |X|$ and not (Stop) do
$k := 1$
While $k \leq |X|$ and not (Stop) do
If $A(x_i^n * z_k) < \min(A(x_i^n * (y_j * z_k)), A(y_j))$ then
Stop: = true;
EndIf
EndIf While
EndIf While
EndIf While
If Stop then
Output (" A is not a fuzzy n-fold KU-ideal of X ")
Else
Output (" $A$is a fuzzy n-fold KU-ideal of X ")
EndIf
End

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