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The χ^{2I} of fuzzy numbers over p-metric spaces defined by Musielak modulus functions

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Abstract. In this article we introduce the sequence spaces

$$\left[\chi_f^{2I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$$

and $\left[\Lambda_f^{2I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$, and study some basic topological and algebraic properties of these spaces. Also we investigate the relations related to these spaces and some of their properties like solidity, symmetricity, convergence free etc., and also investigate some inclusion relations related to these spaces.

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1. Introduction

Throughout $w, \Gamma, \Lambda, \mathbb{N}$ and $\mathbb{R}$ denote the classes of all, entire and analytic scalar valued single sequences, the set of all positive integers and the set of all real numbers respectively.

We write w^2 for the set of all complex double sequences (x_{mn}), where $m, n \in \mathbb{N}$. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [1]. Later on, they were investigated by Hardy [2], Moricz [3], Moricz and Rhoades [4], Basarir and Solankan [5], Tripathy [6], Türkmenoglu [7], and many others.

We procure the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1 \, for\, some\, l \in \mathbb{C}\right\},$$

$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1 \right\},$$
$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$
$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}; \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [8,9] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [10] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [11] and Tripathy have independently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Altay and Başar [12] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Başar and Sever [13] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [14] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.

The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [15] as an extension of the definition of strongly Cesàro summable sequences. Connor [16] further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [17] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [18]-[19], and [20] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a + b)^p \leq a^p + b^p. \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$.

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence

$x = (x_{mn})$ is called double gai sequence if $((m+n)! \, |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{finite\ sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous.

Let M and Φ are mutually complementary modulus functions. Then, we have:

(i) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), (Young's\, inequality)[See[21]] \tag{1.2}$$

(ii) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \tag{1.3}$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u) \tag{1.4}$$

Lindenstrauss and Tzafriri [22] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \, for\, some\, \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p \, (1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

A sequence $f = (f_{mn})$ of modulus function is called a Musielak-modulus function. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|\, u - (f_{mn})(u) : u \geq 0\}, m, n = 1, 2, \cdots$$

is called the complementary function of a Musielak-modulus function f. For a given Musielak modulus function f, the Musielak-modulus sequence space t_f is defined as follows

$$t_f = \left\{ x \in w^2 : M_f(|x_{mn}|)^{1/m+n} \to 0\, as\, m, n \to \infty \right\},$$

where M_f is a convex modular defined by

$$M_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}(|x_{mn}|)^{1/m+n}, x = (x_{mn}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x,y) = sup_{mn} \left\{ inf\left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn}\left(\frac{|x_{mn}|^{1/m+n}}{mn} \right) \right) \leq 1 \right\}.$$

If X is a sequence space, we give the following definitions:

(i)$X^{'}$= the continuous dual of X;

(ii)$X^{\alpha} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \, for\, each\, x \in X\right\};$

(iii)$X^{\beta} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn}\, is\, convegent, \, for\, each\, x \in X\right\};$

(iv)$X^{\gamma} = \left\{a = (a_{mn}) : sup_{mn\geq 1} \left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, for\, eachx \in X\right\};$

(v)$let\, X\, be\, an\, FK - space \supset \phi; \, then\, X^{f} = \left\{f(\Im_{mn}) : f \in X^{'}\right\};$

(vi)$X^{\delta} = \left\{a = (a_{mn}) : sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty, \, for\, each\, x \in X\right\};$

$X^{\alpha}, X^{\beta}, X^{\gamma}$ are called $\alpha - (or K\ddot{o}the - Toeplitz)$dual of $X, \beta - (or\, generalized - K\ddot{o}the - Toeplitz)\, dual\, of X, \gamma - dual\, of\, X, \, \delta - dual\, of X\, respectively. X^{\alpha}$ is defined by Gupta and Kamptan . It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\},$$

for $Z = c, c_0$ and ℓ_{∞}, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.
Here c, c_0 and ℓ_{∞} denote the classes of convergent,null and bounded scalar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Başar and Altay and in the case $0 < p < 1$ by Altay and Başar. The spaces $c(\Delta), c_0(\Delta), \ell_{\infty}(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\},$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.

2. Preliminaries

Let $n \in \mathbb{N}$ and X be a real vector space of dimension m, where $n \leq m$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1), \ldots, d_n(x_n))\|_p$ on X satisfying the following four conditions:
(i) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p = 0$ if and and only if $d_1(x_1), \ldots, d_n(x_n)$ are linearly dependent,
(ii) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p$ is invariant under permutation,
(iii) $\|(\alpha d_1(x_1), \ldots, \alpha d_n(x_n))\|_p = |\alpha| \, \|(d_1(x_1), \ldots, d_n(x_n))\|_p, \alpha \in \mathbb{R}$
(iv) $d_p((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)) = (d_X(x_1, x_2, \cdots x_n)^p + d_Y(y_1, y_2, \cdots y_n)^p)^{1/p}$

for $1 \leq p < \infty$; (or)
(v) $d\left((x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)\right) := \sup\left\{d_X(x_1, x_2, \cdots x_n), d_Y(y_1, y_2, \cdots y_n)\right\}$,
for $x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\|(d_1(x_1), \ldots, d_n(x_n))\|_E = sup\left(|det(d_{mn}\left(x_{mn}\right))|\right) =$$

$$sup\left(\begin{vmatrix} d_{11}\left(x_{11}\right) & d_{12}\left(x_{12}\right) & ... & d_{1n}\left(x_{1n}\right) \\ d_{21}\left(x_{21}\right) & d_{22}\left(x_{22}\right) & ... & d_{2n}\left(x_{1n}\right) \\ . & & & \\ . & & & \\ . & & & \\ d_{n1}\left(x_{n1}\right) & d_{n2}\left(x_{n2}\right) & ... & d_{nn}\left(x_{nn}\right) \end{vmatrix}\right)$$

where $x_i = (x_{i1}, \cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots n$.
If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

The notion of ideal convergence was introduced first by Kostyrko et al.[24] as a generalization of statistical convergence which was further studied in topological spaces by Kumar et al.[25,26] and also more applications of ideals can be deals with various authors by B.Hazarika [27-39] and B.C.Tripathy and B. Hazarika [40-43].

Definition 2.1. A family $I \subset 2^Y$ of subsets of a non empty set Y is said to be an ideal in Y if
(1) $\phi \in I$,
(2) $A, B \in I$ implies $A \bigcup B \in I$,
(3) $A \in I, B \subset A$ implies $B \in I$.

While an admissible ideal I of Y further satisfies $\{x\} \in I$ for each $x \in Y$. Given $I \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a non trivial ideal in $\mathbb{N} \times \mathbb{N}$. A sequence $(x_{mn})_{m,n \in \mathbb{N}\times\mathbb{N}}$ in X is said to be $I-$ convergent to $0 \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \{m, n \in \mathbb{N} \times \mathbb{N} : \|(d_1(x_1), \ldots, d_n(x_n)) - 0\|_p \geq \epsilon\}$ belongs to I.

Definition 2.2. A non-empty family of sets $F \subset 2^X$ is a filter on X if and only if
(1) $\phi \notin F$,
(2) $A, B \in F$ implies $A \bigcap B \in F$,
(3) $A \in F, A \subset B$, implies $B \in F$.

Definition 2.3. An ideal I is called non-trivial ideal if $I \neq \phi$ and $X \notin I$. Clearly $I \subset 2^X$ is a non-trivial ideal if and only if $F = F(I) = \{X - A : A \in I\}$ is a filter on X.

Definition 2.4. A non-trivial ideal $I \subset 2^X$ is called (i) admissible if and only if $\{\{x\} : x \in X\} \subset I$. (ii) maximal if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset.

If we take $I = I_f = \{A \subseteq \mathbb{N} \times \mathbb{N} : A \; is \; a \; finite \; subset\}$. Then I_f is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincides with the usual convergence. If we take $I = I_\delta = \{A \subseteq \mathbb{N} \times \mathbb{N} : \delta(A) = 0\}$ where $\delta(A)$ denote the asyptotic density of the set A. Then I_δ is a non-trivial admissible ideal of $\mathbb{N} \times \mathbb{N}$ and the corresponding convergence coincides with the statistical convergence.

Let D denote the set of all closed and bounded intervals $X = [x_1, x_2]$ on the real line $\mathbb{R} \times \mathbb{N}$. For $X, Y \in D$, we define $X \leq Y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$, $d(X,Y) = max\{|x_1 - y_1|, |x_2 - y_2|\}$, where $X = [x_1, x_2]$ and $Y = [y_1, y_2]$.

Then it can be easily seen that d defines a metric on D and (D, d) is a complete metric space. Also the relation $\leq$ is a partial order on D. A fuzzy number X is a fuzzy subset of the real line $\mathbb{R} \times \mathbb{R}$ that is a mapping $X : R \to J$ where $J = [0,1]$ associating each real number t with its grade of membership $X(t)$.

Definition 2.5. A fuzzy number X is said to be (i) convex if $X(t) \geq X(s) \wedge X(r) = min\{X(s), X(r)\}$, where $s < t < r$. (ii) normal if there exists $t_0 \in \mathbb{R} \times \mathbb{R}$ such that $X(t_0) = 1$. (iii) upper semi-continuous if for each $\epsilon > 0, X^{-1}([0, a + \epsilon])$ for all $a \in [0,1]$ is open in the usual topology of $\mathbb{R} \times \mathbb{R}$.

Let $R(J)$ denote the set of all fuzzy numbers which are upper semicontinuous and have compact support, that is if $X \in \mathbb{R}(J) \times \mathbb{R}(J)$ the for any $\alpha \in [0,1], [X]^\alpha$ is compact, where $[X]^\alpha = \{t \in \mathbb{R} \times \mathbb{R} : X(t) \geq \alpha, \ if \ \alpha \in [0,1]\}$, $[X]^0 = (\{t \in \mathbb{R} \times \mathbb{R} : X(t) > \alpha, if \alpha = 0\})$.

The set $\mathbb{R}$ of real numbers can be embedded $\mathbb{R}(J)$ if we define $\bar{r} \in \mathbb{R}(J) \times \mathbb{R}(J)$ by

$$\bar{r}(t) = \begin{cases} 1, & \text{if } t = r : \\ 0, & \text{if } t \neq r \end{cases}$$

The absolute value, $|X|$ of $X \in \mathbb{R}(J)$ is defined by

$$|X|(t) = \begin{cases} max\{X(t), X(-t)\}, & \text{if } t \geq 0; \\ 0, & \text{if } t < 0 \end{cases}$$

Define a mapping $\bar{d} : \mathbb{R}(J) \times \mathbb{R}(J) \to \mathbb{R}^+ \cup \{0\}$by

$$\bar{d}(X,Y) = sup_{0 \leq \alpha \leq 1} d\left([X]^\alpha, [Y]^\alpha\right).$$

It is known that $\left(\mathbb{R}(J), \bar{d}\right)$ is a complete metric space.

Definition 2.6. A metric on $\mathbb{R}(J)$ is said to be translation invariant if

$$\bar{d}(X + Z, Y + Z) = \bar{d}(X, Y)$$

for $X, Y, Z \in \mathbb{R}(J)$.

Definition 2.7. A sequence $X = (X_{mn})$ of fuzzy numbers is said to be convergent to a fuzzy number X_0 if for every $\epsilon > 0$, there exists a positive integer n_0 such that $\bar{d}(X_{mn}, X_0) < \epsilon$ for all $m, n \geq n_0$.

Definition 2.8. A sequence $X = (X_{mn})$ of fuzzy numbers is said to be (i) I-convergent to a fuzzy number X_0 if for each $\epsilon > 0$ such that

$$A = \left\{m, n \in \mathbb{N} : \bar{d}(X_{mn}, X_0) \geq \epsilon\right\} \in I.$$

The fuzzy number X_0 is called I-limit of the sequence (X_{mn}) of fuzzy numbers and we write $I - limX_{mn} = X_0$. (ii) I-bounded if there exists $M > 0$ such that

$$\{m, n \in \mathbb{N} : d(X_{mn}, \bar{0}) > M\} \in I.$$

Definition 2.9. Let a sequence space E_F of fuzzy numbers. Then, (i) E_F is said to be solid (or normal) if $(Y_{mn}) \in E_F$ whenever $(X_{mn}) \in E_F$ and $\bar{d}\left(Y_{mn}, \bar{0}\right) \leq \bar{d}\left(X_{mn}, \bar{0}\right)$ for all $m, n \in \mathbb{N}$. (ii) E_F is said to be symmetric if $(X_{mn}) \in E_F$ implies $\left(X_{\pi(mn)}\right) \in E_F$ where π is a permutation of $\mathbb{N} \times \mathbb{N}$.

Let $K = \{k_1 < k_2 < ...\} \subseteq \mathbb{N}$ and E be a sequence space. A K-step space of E is a sequence space

$$\lambda_{mn}^{E} = \left\{\left(X_{m_p n_p}\right) \in w^2 : (m_p n_p) \in E\right\}.$$

A canonical preimage of a sequence $\left\{\left(x_{m_p n_p}\right)\right\} \in \lambda_K^E$ is a sequence $\{y_{mn}\} \in w^2$ defined as

$$y_{mn} = \begin{cases} x_{mn}, & \text{if } m, n \in E \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ_K^E is a set of canonical preimages of all elements in λ_K^E, that is y is in canonical preimage of λ_K^E if and only if y is canonical preimage of some $x \in \lambda_K^E$.

Definition 2.10. A sequence space E_F is said to be monotone if E_F contains the canonical pre-images of all its step spaces.

The following well-known inequality will be used throughout the article. Let $p = (p_{mn})$ be any sequence of positive real numbers with $0 \leq p_{mn} \leq sup_{mn} p_{mn} = G, D = max\{1, 2G-1\}$ then

$$|a_{mn} + b_{mn}|^{p_{mn}} \leq D\left(|a_{mn}|^{p_{mn}} + |b_{mn}|^{p_{mn}}\right) \text{ for all } m, n \in \mathbb{N} \text{ and } a_{mn}, b_{mn} \in \mathbb{C}.$$

Also $|a_{mn}|^{p_{mn}} \leq max\left\{1, |a|^G\right\}$ for all $a \in \mathbb{C}$.

First we procure some known results; those will help in establishing the results of this article.

Lemma 2.11. *A sequence space E_F is normal implies E_F is monotone. (For the crisp set case, one may refer to Kamthan and Gupta [44], page 53).*

Lemma 2.12. *(Kostyrko et al., [24], Lemma 5.1). If $I \subset 2^{\mathbb{N}}$ is a maximal ideal, then for each $A \subset \mathbb{N}$ we have either $A \in I$ or $\mathbb{N} - A \in I$.*

3. Some new sequence spaces of fuzzy numbers

The main aim of this article to introduce the following sequence spaces and examine topological and algebraic properties of the resulting sequence spaces. Let $p = (p_{mn})$ be a sequence of positive real numbers for all $m, n \in \mathbb{N}$. $f = (f_{mn})$ be a Musielak-modulus function, $\left(X, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)$ be a $p-$metric space, and $\mu_{mn}(X) = \bar{d}\left(X_{mn}, \bar{0}\right)$ be a sequence of fuzzy numbers, we define the following sequence spaces as follows:

$$\left[\chi_f^{2I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right] = \{(X_{mn}) \in E_F\} :$$
$$\left\{\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \left[f_{mn}\left(\|\mu_{mn}(X), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] \geq \epsilon\right\} \in I\right\},$$

$$\left[\Lambda_f^{2I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right] = \{(X_{mn}) \in E_F : \exists M > 0\} \ni$$
$$\left\{\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \left[f_{mn}\left(\|\mu_{mn}(X), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] \geq M\right\} \in I\right\}.$$

Theorem 3.1. *The spaces* $\left[\chi_f^{2I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$ *and* $\left[\Lambda_f^{2I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$ *are normal and monotone*

Proof. Let $X = (X_{mn})$ be any element of $\left[\chi_f^{2I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$ and $Y = (y_{mn})$ be any sequence such that
$\bar{d}\left(((m+n)!Y_{mn})^{1/m+n}, \bar{0}\right) \leq \bar{d}\left(((m+n)!X_{mn})^{1/m+n}, \bar{0}\right)$ for all $m, n \in \mathbb{N}$.
Then for all $\epsilon > 0$,
$\left\{m, n \in \mathbb{N} : \bar{d}\left(((m+n)!Y_{mn})^{1/m+n}, \bar{0}\right) \geq \epsilon\right\} \subseteq$
$\left\{m, n \in \mathbb{N} : \bar{d}\left(((m+n)!X_{mn})^{1/m+n}, \bar{0}\right) \geq \epsilon\right\}$.
Hence $y = (Y_{mn}) \in \left[\chi_f^{2I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$.
Thus the spaces $\left[\chi_f^{2I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$ is normal and hence monotone. Similarly $\left[\Lambda_f^{2I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$. □

Proposition 3.2. *If I is neither maximal nor $I = I(F)$ then the space* $\left[\chi_f^{2I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$ *are not symmetric.*

Example 3.3. Let us consider a sequence of fuzzy number defined by $X = (X_{mn})$, where

$$((m+n)!X_{mn}(t))^{1/m+n} = \begin{cases} 1+t, -1 \leq t \leq 0 \\ 1-t, 0 \leq t \leq 1. \end{cases}$$

Then for $m, n \notin A \notin I \in \left[\chi_f^{2I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$.
Let $K \subset \mathbb{N}$ be such that $K \notin I$, then it must be $\mathbb{N} - K \in I$. Let us consider a sequence space $Y = (Y_{mn})$, a rearrangement of the sequence (X_{mn}) defined by

$$((m+n)!Y_{mn})^{1/m+n} = \begin{cases} ((m+n)!X_{mn})^{1/m+n}, m, n \in K \\ 0, otherwise. \end{cases}$$

Then $(Y_{mn}) \notin \left[\chi_f^{2I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$. Hence $\left[\chi_f^{2I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$ is not symmetric.

Theorem 3.4. *The spaces* $\left[\chi_f^{2I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$ *and* $\left[\Lambda_f^{2I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$ *are sequence algebra*

Proof. Let X_{mn} and Y_{mn} be two elements of $\left[\chi_f^{2I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$.
For $\alpha \in [0,1]$, let $X_{mn}^{\alpha}, Y_{mn}^{\alpha}, \bar{0}^{\alpha}$ be the α level set of $X_{mn}, Y_{mn}, \bar{0}$ respectively. Since $d\left(((m+n)!X_{mn}^{\alpha}Y_{mn}^{\alpha})^{1/m+n}, \bar{0}^{\alpha}\right) \leq C_1 d\left(((m+n)!X_{mn}^{\alpha})^{1/m+n}, \bar{0}^{\alpha}\right) +$
$C_2 d\left(((m+n)!Y_{mn}^{\alpha})^{1/m+n}, \bar{0}^{\alpha}\right)$, therefore we have

$\bar{d}\left(((m+n)!X_{mn}Y_{mn})^{1/m+n},\bar{0}\right) \le C_1\bar{d}\left(((m+n)!X_{mn})^{1/m+n},\bar{0}\right) +$
$C_2\bar{d}\left(((m+n)!Y_{mn})^{1/m+n},\bar{0}\right)$. Let $\epsilon > 0$ be given. Then
$A\left(\frac{\epsilon}{2}\right) = \left\{m,n \in \mathbb{N} : \bar{d}\left(((m+n)!X_{mn})^{1/m+n},\bar{0}\right) \ge \frac{\epsilon}{2}\right\} \in I.$
$B\left(\frac{\epsilon}{2}\right) = \left\{m,n \in \mathbb{N} : \bar{d}\left(((m+n)!Y_{mn})^{1/m+n},\bar{0}\right) \ge \frac{\epsilon}{2}\right\} \in I.$
$C(\epsilon) = \left\{m,n \in \mathbb{N} : \bar{d}\left(((m+n)!X_{mn}Y_{mn})^{1/m+n},\bar{0}\right) \ge \epsilon\right\}.$
To prove the result it is sufficient to prove that $C(\epsilon) \subseteq A(\epsilon_1) \bigcup B(\epsilon_2)$. Now
$\left\{m,n \in \mathbb{N} : \bar{d}\left(((m+n)!X_{mn}Y_{mn})^{1/m+n},\bar{0}\right) \ge \epsilon\right\} \subseteq$
$C_1\left\{m,n \in \mathbb{N} : \bar{d}\left(((m+n)!X_{mn})^{1/m+n},\bar{0}\right) \ge \frac{\epsilon}{2}\right\} \bigcup$
$C_2\left\{m,n \in \mathbb{N} : \bar{d}\left(((m+n)!Y_{mn})^{1/m+n},\bar{0}\right) \ge \frac{\epsilon}{2}\right\},$
where $\epsilon_1 = \frac{\epsilon}{2C_1}$ and $\epsilon_2 = \frac{\epsilon}{2C_2}$.
The other results can be shown similarly. □

Theorem 3.5. $\left[\chi_f^{2I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$ *is a complete metric space under the metric*
$d(X,Y) = sup_{mn}\left\{\left((m+n)!\bar{d}(|X_{mn}-Y_{mn}|)^{1/m+n},\bar{0}\right) : m,n = 1,2,3,\cdots\right\},$
where $X = (X_{mn}) \in \left[\chi_f^{2I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$ *and*
$Y = (Y_{mn}) \in \left[\chi_f^{2I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$

Proof. Let $\left\{X^{(rs)}\right\}$ be a Cauchy sequence in
$\left[\chi_f^{2I(F)}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]$. Then given any $\epsilon > 0$ there exists a positive integer $N \times N$ depending on ϵ such that $d\left(X^{(rs)}, X^{(pq)}\right) < \epsilon$,
for all $r,s \ge N \times N$ and for all $p,q \ge N \times N$. Hence
$sup_{mn}\left\{\left((m+n)!\bar{d}\left(\left|X_{mn}^{(rs)} - X_{mn}^{(pq)}\right|\right)^{1/m+n},\bar{0}\right)\right\} < \epsilon$, for all $r,s \ge N \times N$
and for all $p,q \ge N \times N$. Consequently $\left((m+n)!\bar{d}(X_{mn},\bar{0})\right)^{1/m+n}$ is a Cauchy sequence in the metric space $\mathbb{C}$ of complex numbers. But $\mathbb{C}$ is complete.
So,
$\left((m+n)!\bar{d}\left(X_{mn}^{(rs)},\bar{0}\right)\right)^{1/m+n} \to \left((m+n)!\bar{d}(X_{mn},\bar{0})\right)^{1/m+n}$ as $r,s \to \infty$.
Hence there exists a positive integer r_0s_0 such that
$\left\{\left((m+n)!\bar{d}\left(\left|X_{mn}^{(rs)} - X_{mn}\right|\right)^{1/m+n},\bar{0}\right)\right\} < \epsilon$, for all $r_0,s_0 \ge N \times N$.
In particular, we have $\left\{\left((m+n)!\bar{d}\left(\left|X_{mn}^{(r_0s_0)} - X_{mn}\right|\right)^{1/m+n},\bar{0}\right)\right\} < \epsilon.$
Now
$\left((m+n)!\bar{d}(X_{mn},\bar{0})\right)^{1/m+n} \le \left((m+n)!\bar{d}\left(\left|X_{mn} - X_{mn}^{(r_0s_0)}\right|\right)^{1/m+n},\bar{0}\right) +$
$\left((m+n)!\bar{d}\left(X_{mn}^{(r_0s_0)},\bar{0}\right)\right)^{1/m+n} < \epsilon + 0$ as $m,n \to \infty$.

$\left((m+n)!\bar{d}\left(X_{mn},\bar{0}\right)\right)^{1/m+n} < \epsilon$ as $m,n \to \infty$.
That is $(X_{mn}) \in \left[\chi_f^{2I(F)}, \left\|\left(d\left(x_1\right), d\left(x_2\right), \cdots, d\left(x_{n-1}\right)\right)\right\|_p\right]$. □

Corollary 3.6. $\left[\Lambda_f^{2I(F)}, \left\|\left(d\left(x_1\right), d\left(x_2\right), \cdots, d\left(x_{n-1}\right)\right)\right\|_p\right]$ *is a complete metric space.*

Proof. Similarly the proof of Theorem 3.5. □

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