Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 8, No. 6, (December 2014), pp. 941–963
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

A new type of generalized fuzzy quasi-ideals of ordered semigroups

Jian Tang, Yanfeng Luo

Received 6 March 2014; Accepted 12 June 2014

Abstract. In this paper, the concept of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals of an ordered semigroup S is introduced by the ordered fuzzy points of S, which is a generalization of fuzzy quasi-ideal of S, and related properties are investigated. Furthermore, we introduce the concepts of prime and completely semiprime $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals of ordered semigroups, and characterize bi-regular ordered semigroups in terms of completely semiprime $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals. Moreover, characterizations of regular ordered semigroups and intra-regular ordered semigroups by the properties of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideals, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideals and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals are given. Finally, we investigate the characterizations and decompositions of left and right simple ordered semigroups by means of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals.

2010 AMS Classification: 06F05, 08A72

Keywords: Ordered semigroup, Strongly convex fuzzy subset, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal, Prime $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal, Completely semiprime $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal.

Corresponding Author: Jian Tang (tangjian0901@126.com)

1. Introduction

The theory of fuzzy sets, proposed by Zadeh in 1965, has provided a useful mathematical tool for describing the behavior of systems that are too complex to admit precise mathematical analysis by classical methods and tools. Extensive applications of fuzzy set theory have been found in various fields such as artificial intelligence, computer science, control engineering, expert systems, management science, operations research, pattern recognition, robotics, and others. It soon invoked a natural question concerning a possible connection between fuzzy sets and algebraic systems. In [27] Rosenfeld inspired the fuzzification of algebraic structures and introduced the notion of fuzzy subgroups. Kuroki initiated the theory of fuzzy semigroups in his papers [22]. A systematic exposition of fuzzy semigroups by Mordeson et al. appeared in [26], where one can find theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy languages and fuzzy finite state machines. Since then, a new type of fuzzy subgroup, i.e., the (α, β)-fuzzy subgroup, was introduced in an earlier paper of Bhakat and Das ([1], [2]) by using the combined notions of "belongs to" relation ($\in$) and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup. In particular, the concept of an $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup [27]. In [4] Davvaz defined $(\in, \in \vee q)$-fuzzy subnearring and ideals of a nearring. In [5] Jun and Song initiated the study of (α, β)-fuzzy interior ideals of a semigroup. Ma and Zhan [24] studied $(\in, \in \vee q)$-fuzzy h-bi-ideals and h-quasi-ideals of a hemiring. Generalizing the concept of the quasi-coincident of a fuzzy point with a fuzzy subset Jun ([6], [7]) defined $(\in, \in \vee q_k)$-fuzzy subgroups and $(\in, \in \vee q_k)$-fuzzy subalgebras in BCK/BCI-algebras, respectively. In [29] $(\in, \in \vee q_k)$-fuzzy ideals of semigroups are defined and investigated. In [12], Kehayopulu and Tsingelis applied the concept of fuzzy sets to the theory of ordered semigroups. Then they defined "fuzzy" analogous of several notations, which appeared to be useful in the theory of ordered semigroups. The theory of fuzzy sets on ordered semigroups has been recently developed. For more details, the reader is referred to ([8], [13], [14], [15], [16], [17], [19], [20], [21], [25], [30], [31], [32], [33]). Especially, Xie and Tang [32] introduced the concept of ordered fuzzy points of an ordered semigroup, and studied prime fuzzy ideals of ordered semigroups. In [30] Tang and Xie introduced the concepts of $(\in, \in \vee q_k)$-fuzzy ideals and $(\in, \in \vee q_k)$-fuzzy (generalized) bi-ideals of an ordered semigroup, and gave some characterizations of regular ordered semigroups by the properties of $(\in, \in \vee q_k)$-fuzzy ideals and $(\in, \in \vee q_k)$-fuzzy (generalized) bi-ideals.

As we know, quasi-ideals play an important role in the study of ring, semigroup and ordered semigroup structures. The concept of a quasi-ideal in rings and semigroups was studied by Stienfeld in [28]. Furthermore, Kehayopulu and Tsingelis extended the concept of quasi-ideals in ordered semigroups as a non-empty subset Q of an ordered semigroup S such that: (1) $(QS] \cap (SQ] \subseteq Q$ and (2) If $a \in Q$ and $S \ni b \leq a$, then $b \in Q$ (see [15]). The fuzzy quasi-ideals in ordered semigroups were studied in [15] and [30], where the basic properties of ordered semigroups in terms of fuzzy quasi-ideals are given. As a continuation of these papers, we define and study a new sort of fuzzy quasi-ideals in ordered semigroups called $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals in the present paper. The rest of this paper is organized as follows. In Section 2, we recall some basic definitions and results of ordered semigroups which will be used throughout this paper. In Section 3, we introduce the concept of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals of an ordered semigroup S by the ordered fuzzy points of S, and investigate some related properties. In Section 4, we introduce the concept of prime $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals of ordered semigroups, and give some characterizations of them. Some characterizations of regular ordered semigroups and intra-regular ordered semigroups in terms of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideals, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideals and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals are given in Section 5. In Section 6, we introduce the notion of completely semiprime $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals of ordered semigroups and characterize bi-regular ordered semigroups in terms of completely semiprime $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals. In Section 7, we investigate the characterizations and decompositions of left and right simple ordered semigroups by means of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals. In particular, we prove that an ordered semigroup S is left and right simple if and only if every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S is a constant function. Some conclusions are given in the last Section.

2. Preliminaries and some notations

Throughout this paper, unless stated otherwise S stands for an ordered semigroup, i.e., a semigroup S with an order relation " $\leq$ " such that $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for any $x \in S$. We denote by Z^+ the set of all positive integers. A function f from S to the real closed interval [0,1] is a *fuzzy subset* of S. The ordered semigroup S itself is a fuzzy subset of S such that $S(x) \equiv 1$ for all $x \in S$ (the fuzzy subset S is also denoted by 1 in [16]). Let f and g be two fuzzy subsets of S. Then the inclusion relation $f \subseteq g$ is defined by $f(x) \leq g(x)$ for all $x \in S$, and $f \cap g$, $f \cup g$ are defined by

$$(f \cap g)(x) = \min\{f(x), g(x)\} = f(x) \wedge g(x),$$
$$(f \cup g)(x) = \max\{f(x), g(x)\} = f(x) \vee g(x)$$

for all $x \in S$, respectively. The set of all fuzzy subsets of S is denoted by $F(S)$. One can easily show that $(F(S), \subseteq, \cap, \cup)$ forms a complete lattice with the maximum element S and the minimum element 0, which is a mapping from S into $[0, 1]$ defined by

$$0: \ S \to [0, 1], \ x \mapsto 0(x) := 0, \ \forall x \in S.$$

Let $(S, \cdot, \leq)$ be an ordered semigroup. For $x \in S$, we define $A_x := \{(y, z) \in S \times S |\ x \leq yz\}$. The product $f \circ g$ of f and g is defined by

$$(f \circ g)(x) = \begin{cases} \bigvee\limits_{(y,z) \in A_x} [f(y) \wedge g(z)], & \text{if } A_x \neq \varnothing \ , \\ 0, & \text{if } A_x = \varnothing, \end{cases}$$

for all $x \in S$. It is well known (see Theorem of [13]) that this operation "$\circ$" is associative and $(F(S), \circ, \subseteq)$ forms an ordered semigroup.

Let S be an ordered semigroup. For $H \subseteq S$, we define

$$(H] := \{t \in S \mid t \leq h \ \text{ for some } \ h \in H\}.$$

For $H = \{a\}$, we write $(a]$ instead of $(\{a\}]$. For two subsets A, B of S, we have: (1) $A \subseteq (A]$; (2) If $A \subseteq B$, then $(A] \subseteq (B]$; (3) $(A](B] \subseteq (AB]$; (4) $((A]] = (A]$; (5) $((A](B]] = (AB]$; (6) $((A] \cap (B]] = (A] \cap (B]$; (7) $(A] \cup (B] = (A] \cup (B]$ (see [11]).

By a *subsemigroup* of S, we mean a nonempty subset A of S such that $A^2 \subseteq A$. A nonempty subset A of an ordered semigroup S is called a *left* (resp. *right*) *ideal* of S if (1) $SA \subseteq A$ (resp. $AS \subseteq A$) and (2) If $a \in A$ and $S \ni b \leq a$, then $b \in A$. If A is both a left and a right ideal of S, then it is called an *(two-sided) ideal* of S (see [11]). Clearly, every left (right) ideal of S is a quasi-ideal of S. A subsemigroup B of an ordered semigroup S is called a *bi-ideal* of S if (1) $BSB \subseteq B$ and (2) If $a \in B$ and $S \ni b \leq a$, then $b \in B$ (see [17]). An quasi-ideal Q of S is called *prime* if for any two elements x, y of S such that $xy \in Q$, then $x \in Q$ or $y \in Q$. We denote by $Q(a)$ the quasi-ideal of S generated by a $(a \in S)$. Then $Q(a) = (a \cup Sa] \cap (a \cup aS]$ (see [3]).

Let A be a nonempty subset of S. We denote by f_A the characteristic function of A, that is the mapping of S into $[0,1]$ defined by

$$f_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Then f_A is a fuzzy subset of S.

Let S be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy left* (resp. *right*) *ideal* of S if

(1) $x \leq y \Rightarrow f(x) \geq f(y)$, and

(2) $f(xy) \geq f(y)$ (resp. $f(xy) \geq f(x)$) $\forall x, y \in S$. Equivalently, $S \circ f \subseteq f$ (resp. $f \circ S \subseteq f$).

A fuzzy subset f of S is called a *fuzzy ideal* of S if it is both a fuzzy left and a fuzzy right ideal of S (see [12], [14]). A fuzzy subset f of S is called a *fuzzy quasi-ideal* of S if

(1) $x \leq y \Rightarrow f(x) \geq f(y)$, and

(2) $(f \circ S) \cap (S \circ f) \subseteq f$ (see [15]).

We denote by a_λ an *ordered fuzzy point* of an ordered semigroup S, where

$$a_\lambda(x) = \begin{cases} \lambda, & \text{if } x \in (a], \\ 0, & \text{if } x \notin (a], \end{cases}$$

for any $x \in S$. It is easy to see that an ordered fuzzy point of an ordered semigroup S is a fuzzy subset of S. For any fuzzy subset f of S, we also denote $a_\lambda \subseteq f$ by $a_\lambda \in f$ in the sequel (see [32]).

Definition 2.1 ([32])**.** Let f be a fuzzy subset of S, we define $(f]$ by the rule that

$$(f](x) = \bigvee_{y \geq x} f(y)$$

for all $x \in S$. A fuzzy subset of S is called *strongly convex* if $f = (f]$.

By Definition 2.1, for an ordered fuzzy point a_λ and a strongly convex fuzzy subset f of S, we have $a_\lambda \in f$ if and only if $f(a) \geq \lambda$.

Lemma 2.2 ([31])**.** *Let f be a fuzzy subset of an ordered semigroup S. Then f is a strongly convex fuzzy subset of S if and only if $x \leq y \Rightarrow f(x) \geq f(y)$, for all $x, y \in S$.*

Definition 2.3. An ordered fuzzy point a_λ of an ordered semigroup S is said to be *not belong to* (resp. *not k-quasi-coincident with*) a fuzzy subset f of S, written as $a_\lambda \overline{\in} f$ (resp. $a_\lambda \overline{q_k} f$) if $f(a) < \lambda$ (resp. $f(a) + \lambda + k \leq 1$), where $k \in [0,1)$. If $a_\lambda \overline{\in} f$ or $a_\lambda \overline{q_k} f$, then we write $a_\lambda \overline{\in} \vee \overline{q_k} f$.

Lemma 2.4 ([32])**.** *Let a_λ, b_μ $(\lambda \neq 0, \mu \neq 0)$ be ordered fuzzy points of S, and f, g fuzzy subsets of S. Then the following statements are true:*

(1) *$a_\lambda \circ b_\mu = (ab)_{\lambda \wedge \mu}$ for all ordered fuzzy points a_λ and b_μ of S. In particular, $a_\lambda \circ a_\lambda = (a^2)_\lambda$.*

(2) *If f, g are fuzzy ideals of S, then $f \circ g$, $f \cup g$ are fuzzy ideals of S.*

(3) *If $f \subseteq g$, and $h \in F(S)$, then $f \circ h \subseteq g \circ h$, $h \circ f \subseteq h \circ g$.*

The reader is referred to [32], [34] and [35] for notation and terminology not defined in this paper.

3. $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals of ordered semigroups

In what follows, let k denote an arbitrary element of $[0, 1)$ unless otherwise specified. In this section, we define a generalization form of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideals of an ordered semigroup S given in [19] and introduce $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals of S by the ordered fuzzy points of S.

Definition 3.1. A fuzzy subset f of an ordered semigroup S is called an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$*-fuzzy quasi-ideal* of S, if for all $t, r \in (0, 1]$ and $x, y, z, u, v \in S$, the following conditions hold:

(1) $x \le y \Rightarrow f(x) \ge f(y)$.
(2) $x \le yu, x \le vz$ and $x_{t\wedge r}\overline{\in} f \Rightarrow y_t \overline{\in} \vee \overline{q_k} f$ or $z_r \overline{\in} \vee \overline{q_k} f$.

Theorem 3.2. *Let S be an ordered semigroup and f a fuzzy subset of S. Then f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S if and only if f satisfies that*

(1) $x \le y \Rightarrow f(x) \ge f(y)$.
(2) $f(x) \vee \frac{1-k}{2} \ge (f \circ S)(x) \wedge (S \circ f)(x)$ *for all* $x \in S$.

Proof. Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S and $x, y \in S$. Then, by Definition 3.1, the condition (1) holds. Furthermore, $f(x) \vee \frac{1-k}{2} \ge (f \circ S)(x) \wedge (S \circ f)(x)$ for all $x \in S$. Indeed, if $f(x) \vee \frac{1-k}{2} < (f \circ S)(x) \wedge (S \circ f)(x)$ for some $x \in S$, then there exists $t \in (\frac{1-k}{2}, 1]$ such that $f(x) \vee \frac{1-k}{2} < t < (f \circ S)(x) \wedge (S \circ f)(x)$. Since $(f \circ S)(x) > t > 0, (S \circ f)(x) > t > 0$, there exist $y, z, u, v \in S$ such that $x \le yu$ and $x \le vz$, and $t < f(y) \wedge S(u) = f(y), t < S(v) \wedge f(z) = f(z)$. Then $y_t \in f, z_t \in f$ and $x_t \overline{\in} f$. Thus, by Definition 3.1, we have $y_t \overline{q_k} f$ or $z_t \overline{q_k} f$. Then ($f(y) \ge t$ and $f(y) + t + k \le 1$) or ($f(z) \ge t$ and $f(z) + t + k \le 1$). It follows that $t \le \frac{1-k}{2}$, which is a contradiction with $t \in (\frac{1-k}{2}, 1]$. Therefore, $f(x) \vee \frac{1-k}{2} \ge (f \circ S)(x) \wedge (S \circ f)(x)$ for all $x \in S$.

Conversely, assume that the conditions (1) and (2) hold. Let $x, y, z, u, v \in S$ and $t, r \in (0, 1]$ be such that $x \le yu, x \le vz$ and $x_{t\wedge r}\overline{\in} f$. Then $f(x) < t \wedge r$, and

$$\begin{aligned} f(x) \vee \frac{1-k}{2} &\ge (f \circ S)(x) \wedge (S \circ f)(x) \\ &= \Big(\bigvee_{(s,t)\in A_x} [f(s) \wedge S(t)]\Big) \wedge \Big(\bigvee_{(p,q)\in A_x} [S(p) \wedge f(q)]\Big) \\ &\ge (f(y) \wedge S(u)) \wedge (S(v) \wedge f(z)) = f(y) \wedge f(z). \end{aligned}$$

We consider the following two cases:

Case 1. If $f(x) \ge f(y) \wedge f(z)$, then $f(y) \wedge f(z) < t \wedge r$, and $f(y) < t$ or $f(z) < r$. It thus follows that $y_t \overline{\in} f$ or $z_r \overline{\in} f$, which implies that $y_t \overline{\in} \vee \overline{q_k} f$ or $z_r \overline{\in} \vee \overline{q_k} f$.

Case 2. If $f(x) < f(y) \wedge f(z)$, then, by hypothesis, we have $f(y) \wedge f(z) \le \frac{1-k}{2}$. If $y_t \overline{\in} f$ or $z_r \overline{\in} f$, then $y_t \overline{\in} \vee \overline{q_k} f$ or $z_r \overline{\in} \vee \overline{q_k} f$. Now let $y_t \in f$ and $z_r \in f$. Then $t \le f(y) \le \frac{1-k}{2}$ or $r \le f(z) \le \frac{1-k}{2}$. It implies that $y_t \overline{q_k} f$ or $z_r \overline{q_k} f$, and so $y_t \overline{\in} \vee \overline{q_k} f$ or $z_r \overline{\in} \vee \overline{q_k} f$.

Thus, in both cases, we have $y_t \overline{\in} \vee \overline{q_k} f$ or $z_r \overline{\in} \vee \overline{q_k} f$. Therefore, f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S. $\square$

Remark 3.3. If we take $k = 0$ in Theorem 3.2, then we have the characterization of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideals of S. Thus the concept of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals of an ordered semigroup S is a generalization of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideals of S.

From Theorem 3.2, we can observe that the $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal is characterized in term of the multiplications $f \circ S$ and $S \circ f$. A natural question is if an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal f can be defined using only the fuzzy subset f itself. The theorems below give the answer.

Theorem 3.4. *Let S be an ordered semigroup. Then a fuzzy subset f of S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S if and only if the following conditions are satisfied:*
(1) $x \leq y \Rightarrow f(x) \geq f(y)$.
(2) $x \leq yu$ *and* $x \leq vz \Rightarrow f(x) \vee \frac{1-k}{2} \geq f(y) \wedge f(z)$ *for all* $x, y, z, u, v \in S$.

Proof. $\Rightarrow$. Let $x, y, z, u, v \in S$ be such that $x \leq yu$ and $x \leq vz$. Since f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S and $x \in S$, by Theorem 3.2 we have

$$f(x) \vee \tfrac{1-k}{2} \geq (f \circ S)(x) \wedge (S \circ f)(x).$$

Since $x \leq yu$, we have $(y, u) \in A_x$, then

$$(f \circ S)(x) = \bigvee_{(s,t) \in A_x} [f(s) \wedge S(t)] \geq f(y) \wedge S(u) = f(y).$$

By $x \leq vz$ in a similar way we can get $(S \circ f)(x) \geq f(z)$. It thus follows that $f(x) \vee \frac{1-k}{2} \geq f(y) \wedge f(z)$.

$\Leftarrow$. Let $x \in S$. Then $f(x) \vee \frac{1-k}{2} \geq (f \circ S)(x) \wedge (S \circ f)(x)$. Indeed, if $A_x = \varnothing$, then $(f \circ S)(x) \wedge (S \circ f)(x) = 0$, and

$$(f \circ S)(x) \wedge (S \circ f)(x) = 0 \leq f(x) \vee \tfrac{1-k}{2}.$$

Let $A_x \neq \varnothing$. Then

$$(f \circ S)(x) = \bigvee_{(s,t) \in A_x} [f(s) \wedge S(t)], \qquad (3.1)$$

$$(S \circ f)(x) = \bigvee_{(p,q) \in A_x} [S(p) \wedge f(q)].$$

We consider the following two cases:

Case 1. If $f(x) \vee \frac{1-k}{2} \geq (f \circ S)(x)$, then $f(x) \vee \frac{1-k}{2} \geq (f \circ S)(x) \geq (f \circ S)(x) \wedge (S \circ f)(x)$.

Case 2. Let $f(x) \vee \frac{1-k}{2} < (f \circ S)(x)$. Then, by (3.1), there exists $(y, u) \in A_x$ such that $f(x) \vee \frac{1-k}{2} < f(y) \wedge S(u)$. Since $f(y) \wedge S(u) = f(y)$, we have

$$f(x) \vee \tfrac{1-k}{2} < f(y). \qquad (3.2)$$

So we can show that $f(x) \vee \frac{1-k}{2} \geq S(v) \wedge f(z)$ for any $(v, z) \in A_x$. In fact, since $(y, u) \in A_x$, we have $y, u \in S$ and $x \leq yu$. Since $(v, z) \in A_x$, we have $v, z \in S$ and $x \leq vz$. Since $x, y, z, u, v \in S$ such that $x \leq yu$ and $x \leq vz$, by hypothesis, we have

$$f(x) \vee \tfrac{1-k}{2} \geq f(y) \wedge f(z).$$

If $f(y) \wedge f(z) = f(y)$, then $f(x) \vee \frac{1-k}{2} \geq f(y)$, which is impossible by (3.2). Hence we have

$$f(x) \vee \tfrac{1-k}{2} \geq f(z) = S(v) \wedge f(z)$$

for any $(v, z) \in A_x$. Then we have

$$f(x) \vee \frac{1-k}{2} \geq \bigvee_{(v,z) \in A_x} [S(v) \wedge f(z)] = (S \circ f)(x) \geq (f \circ S)(x) \wedge (S \circ f)(x),$$

and the proof is completed by Theorem 3.2. □

Example 3.5. We consider the ordered semigroup $S := \{a, b, c, d\}$ defined by the following multiplication " $\cdot$ " and the order " $\leq$ ":

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$$\leq := \{(a, a), (a, b), (b, b), (c, c), (d, d)\}.$$

Let f be a fuzzy subset of S such that $f(a) = 0.6, f(b) = 0.4, f(c) = 0.3, f(d) = 0.3$. Then, by Theorem 3.4, we can easily show that f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S for any $k \in [0, 1)$.

Theorem 3.6. *Let S be an ordered semigroup and f a fuzzy subset of S. Then f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S if and only if the following conditions are satisfied:*

(1) $x \leq y \Rightarrow f(x) \geq f(y)$.

(2) $x \leq yu$ *and* $x \leq vz \Rightarrow f(x) \vee \frac{1-k}{2} \geq \max\{f(y) \wedge f(z), f(v) \wedge f(u)\}$ *for all* $x, y, z, u, v \in S$.

Proof. Since the proof is straightforward verification by Theorem 3.4, we omit it. □

Definition 3.7. A fuzzy subset f of an ordered semigroup S is called an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-*fuzzy left* (resp. *right*) *ideal* of S, if for all $t \in (0, 1]$ and $x, y \in S$, the following conditions hold:

(1) $x \leq y \Rightarrow f(x) \geq f(y)$, and

(2) $(xy)_t \overline{\in} f$(resp. $(yx)_t \overline{\in} f) \Rightarrow y_t \overline{\in} \vee \overline{q_k} f$.

A fuzzy subset f of an ordered semigroup S is called an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-*fuzzy ideal* if it is both an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal and an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal of S.

Definition 3.8. A fuzzy subset f of an ordered semigroup S is called an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-*fuzzy bi-ideal* of S, if for all $t, r \in (0, 1]$ and $x, y, z \in S$, the following assertions are satisfied:

(1) $x \leq y \Rightarrow f(x) \geq f(y)$.

(2) $(xy)_{t \wedge r} \overline{\in} f \Rightarrow x_t \overline{\in} \vee \overline{q_k} f$ or $y_r \overline{\in} \vee \overline{q_k} f$.

(3) $(xyz)_{t \wedge r} \overline{\in} f \Rightarrow x_t \overline{\in} \vee \overline{q_k} f$ or $z_r \overline{\in} \vee \overline{q_k} f$.

Similar to Theorem 3.4, we have the following two theorems, the proofs of which are similar to that of Theorem 3.4.

Theorem 3.9. *Let S be an ordered semigroup and f a fuzzy subset of S. Then f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (resp. right) ideal of S if and only if f satisfies*

(1) $x \leq y \Rightarrow f(x) \geq f(y)$, *and*

(2) $f(xy) \vee \frac{1-k}{2} \geq f(y)$ *(resp.* $f(xy) \vee \frac{1-k}{2} \geq f(x)$*) for all* $x, y \in S$.

Theorem 3.10. *Let S be an ordered semigroup and f a fuzzy subset of S. Then f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S if and only if f satisfies that*

(1) $x \leq y \Rightarrow f(x) \geq f(y)$.

(2) $f(xy) \vee \frac{1-k}{2} \geq f(x) \wedge f(y)$ *for all* $x, y \in S$.

(3) $f(xyz) \vee \frac{1-k}{2} \geq f(x) \wedge f(z)$ *for all* $x, y, z \in S$.

Theorem 3.11. *Let S be an ordered semigroup. Then every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S.*

Proof. Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S and $x, y, z \in S$. Since $xy \leq xy$ and $xy \leq xy$, by Theorem 3.4 we have $f(xy) \vee \frac{1-k}{2} \geq f(x) \wedge f(y)$. Also, since $xyz \leq x(yz)$ and $xyz \leq (xy)z$, by Theorem 3.4 we have $f(xyz) \vee \frac{1-k}{2} \geq f(x) \wedge f(z)$. Thus f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S by Theorem 3.10. □

The converse of Theorem 3.11 is not true in general. We can illustrate it by the following example:

Example 3.12. We consider the ordered semigroup $S := \{a, b, c, d\}$ defined by the following multiplication " $\cdot$ " and the order " $\leq$ ":

·	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

$$\leq := \{(a, a), (a, b), (a, c), (a, d), (b, b), (b, d), (c, c), (d, d)\}.$$

Let f be a fuzzy subset of S such that $f(a) = 0.8, f(b) = 0.3, f(c) = 0.6, f(d) = 0.2$. Then, by Theorem 3.10, we can easily show that f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S for any $k \in [0, 1)$. But f is not an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S for any $k \in [0, 1)$. In fact, since $b \leq b = cd$ and $b \leq b = dc$, while $f(b) \vee \frac{1-k}{2} = 0.3 \vee \frac{1-k}{2} \leq 0.5 < 0.6 = f(c) \wedge f(c)$. By Theorem 3.4, f is not an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S.

An ordered semigroup $(S, \cdot, \leq)$ is called *regular* if, for each element a of S, there exists an element x in S such that $a \leq axa$. **Equivalent definition:** $a \in (aSa], \forall a \in S$ (see [9]).

Theorem 3.13. *Every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of a regular ordered semigroup S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S.*

Proof. Assume that f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of a regular ordered semigroup S. Let $x, y, z, u, v \in S$ such that $x \leq yu$ and $x \leq vz$. Then, since S is regular, there exists $x_1 \in S$ such that $x \leq xx_1x$, and we have

$$x \leq xx_1x \leq (yu)x_1(vz) = y(ux_1v)z.$$

Then, by Theorem 3.10, we have

$$f(x) \vee \tfrac{1-k}{2} \geq f(y(ux_1v)z) \vee \tfrac{1-k}{2} \geq f(y) \wedge f(z).$$

Therefore, f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S by Theorem 3.4. □

Theorem 3.14. *Let S be an ordered semigroup and f a strongly convex fuzzy subset of S. Then f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S if and only if the level set*

$$f_t := \{x \in S \mid f(x) \geq t\}$$

is a quasi-ideal of S for all $t \in (\frac{1-k}{2}, 1]$.

Proof. Assume that f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S. Let $x \in (f_tS] \cap (Sf_t]$ for $t \in (\frac{1-k}{2}, 1]$. Then $x \in (f_tS]$ and $x \in (Sf_t]$, and there exist $y, z \in f_t$ and $u, v \in S$ such that $x \leq yu, x \leq vz$. Then $f(y) \geq t$ and $f(z) \geq t$. It follows from Theorem 3.4

that

$$f(x) \vee \tfrac{1-k}{2} \geq f(y) \wedge f(z) \geq t \wedge t = t.$$

Note that $t \in (\frac{1-k}{2}, 1]$, and we conclude that $f(x) \geq t$, i.e., $x \in f_t$. Furthermore, let $y \in f_t, S \ni x \leq y$. Then $x \in f_t$. Indeed, since $y \in f_t, f(y) \geq t$, and f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S, we have $f(x) \geq f(y) \geq t$, so $x \in f_t$. Therefore, f_t is a quasi-ideal of S.

Conversely, suppose that f_t is a quasi-ideal of S for all $t \in (\frac{1-k}{2}, 1]$, and let $x, y, z, u, v \in S$ such that $x \leq yu$ and $x \leq vz$. If there exist $x, y, z, u, v \in S$ such that $x \leq yu, x \leq vz$, and $f(x) \vee \frac{1-k}{2} < f(y) \wedge f(z) = r$, then $r \in (\frac{1-k}{2}, 1], f(x) < r$ and $y, z \in f_r$. Hence $x \in (f_r S]$ and $x \in (S f_r]$, and $x \in (f_r S] \cap (S f_r]$. Since f_r is a quasi-ideal of S for any $r \in (\frac{1-k}{2}, 1]$, we have $x \in (f_r S] \cap (S f_r] \subseteq f_r$, and $f(x) \geq r$, which contradicts with $f(x) < r$. Hence $f(x) \vee \frac{1-k}{2} \geq f(y) \wedge f(z)$. Moreover, if $x \leq y$, then $f(x) \geq f(y)$. Indeed, since f is a strongly fuzzy subset of S and $x \leq y$, we have $f(x) \geq f(y)$ by Lemma 2.2. Therefore, f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S by Theorem 3.4. □

Theorem 3.15. *Let $\{f_i \mid i \in I\}$ be a family of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals of an ordered semigroup S. Then $f := \bigcap_{i \in I} f_i$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S, where $(\bigcap_{i \in I} f_i)(x) = \bigwedge_{i \in I} (f_i(x))$.*

Proof. Let $x, y, z, u, v \in S$ such that $x \leq yu$ and $x \leq vz$. Then, since each f_i $(i \in I)$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S, we have

$$\begin{aligned} f(x) \vee \frac{1-k}{2} &= (\bigcap_{i \in I} f_i)(x) \vee \frac{1-k}{2} = \bigwedge_{i \in I} (f_i(x)) \vee \frac{1-k}{2} \\ &= \bigwedge_{i \in I} (f_i(x) \vee \frac{1-k}{2}) \geq \bigwedge_{i \in I} (f_i(y) \wedge f_i(z)) \\ &= (\bigcap_{i \in I} f_i)(y) \wedge (\bigcap_{i \in I} f_i)(z) = f(x) \wedge f(z). \end{aligned}$$

Furthermore, if $x \leq y$, then $f(x) \geq f(y)$. Indeed, since each $f_i (i \in I)$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S, so $f_i(x) \geq f_i(y)$ for all $i \in I$. Thus

$$f(x) = (\bigcap_{i \in I} f_i)(x) = \bigwedge_{i \in I} (f_i(x)) \geq \bigwedge_{i \in I} (f_i(y)) = (\bigcap_{i \in I} f_i)(y) = f(y).$$

Therefore, f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S by Theorem 3.4. □

Suppose $\{f_i \mid i \in I\}$ is a family of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals of an ordered semigroup S. Is it true that $\bigcup_{i \in I} f_i$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S? where $(\bigcup_{i \in I} f_i)(x) = \bigvee_{i \in I} (f_i(x))$. The following example gives a negative answer to the above question.

Example 3.16. We consider the ordered semigroup $S := \{a, b, c, d\}$ defined by the following multiplication " $\cdot$ " and the order " $\leq$ ":

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	d	a
c	a	a	a	a
d	a	a	a	a

$$\leq := \{(a, a), (a, d), (b, b), (c, c), (d, d)\}.$$

Let f_1 and f_2 be two fuzzy subsets of S such that

$$f_1(a) = 0.6,\ f_1(b) = 0.6,\ f_1(c) = 0,\ f_1(d) = 0;$$
$$f_2(a) = 0.6,\ f_2(b) = 0,\ f_2(c) = 0.6,\ f_2(d) = 0.$$

Then, by Theorem 3.4, f_1 and f_2 are both $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals of S for any $k \in [0, 1)$. But $f_1 \cup f_2$ is not an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S for any $k \in [0, 1)$. In fact, since $d \leq d = bc$ and $d \leq d = bc$, while

$$\begin{aligned} (f_1 \cup f_2)(d) \vee \frac{1-k}{2} &= f_1(d) \vee f_2(d) \vee \frac{1-k}{2} = \frac{1-k}{2} \\ &< 0.6 \wedge 0.6 = (f_1 \cup f_2)(b) \wedge (f_1 \cup f_2)(c). \end{aligned}$$

By Theorem 3.4, $f_1 \cup f_2$ is not an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S for any $k \in [0, 1)$.

The following theorem can be obtained under the assumption of an additional condition.

Theorem 3.17. *Let $\{f_i \mid i \in I\}$ be a family of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals of an ordered semigroup S such that $f_i \subseteq f_j$ or $f_j \subseteq f_i$ for all $i, j \in I$. Then $f := \bigcup\limits_{i \in I} f_i$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S, where $(\bigcup\limits_{i \in I} f_i)(x) = \bigvee\limits_{i \in I} (f_i(x))$.*

Proof. Let $x, y, z, u, v \in S$ such that $x \leq yu$ and $x \leq vz$. Then we have

$$\begin{aligned} f(x) \vee \frac{1-k}{2} &= (\bigcup_{i \in I} f_i)(x) \vee \frac{1-k}{2} = \bigvee_{i \in I} (f_i(x) \vee \frac{1-k}{2}) \\ &\geq \bigvee_{i \in I} (f_i(y) \wedge f_i(z)) = (\bigvee_{i \in I} f_i)(y) \wedge (\bigvee_{i \in I} f_i)(z) \qquad (*) \\ &\quad \text{(Since } f_i \text{ is an } (\overline{\in}, \overline{\in} \vee \overline{q_k})\text{-fuzzy quasi-ideal of } S \text{ for all } i \in I) \\ &= f(y) \wedge f(z). \end{aligned}$$

In the following, we show that Eq. $(*)$ holds. It is obvious that $\bigvee\limits_{i \in I} (f_i(x) \wedge f_i(y)) \leq (\bigvee\limits_{i \in I} f_i)(x) \wedge (\bigvee\limits_{i \in I} f_i)(y)$. Assume that $\bigvee\limits_{i \in I} (f_i(x) \wedge f_i(y)) \neq (\bigvee\limits_{i \in I} f_i)(x) \wedge (\bigvee\limits_{i \in I} f_i)(y)$. Then there exists $r \in (0, 1)$ such that $\bigvee\limits_{i \in I} (f_i(x) \wedge f_i(y)) < r < (\bigvee\limits_{i \in I} f_i)(x) \wedge (\bigvee\limits_{i \in I} f_i)(y)$. Since $f_i \subseteq f_j$ or $f_j \subseteq f_i$ for all $i, j \in I$, there exists $m \in I$ such that $r < f_m(x) \wedge f_m(y)$. On the other hand, $f_i(x) \wedge f_i(y) < r$ for all $i \in I$, which is a contradiction. Hence $\bigvee\limits_{i \in I} (f_i(x) \wedge f_i(y)) = (\bigvee\limits_{i \in I} f_i)(x) \wedge (\bigvee\limits_{i \in I} f_i)(y)$.

Furthermore, if $x \leq y$, then $f(x) \geq f(y)$. Indeed, since f_i is an $(\in, \in \vee \overline{q_k})$-fuzzy quasi-ideal of S for all $i \in I$, so $f_i(x) \geq f_i(y)$ for all $i \in I$. Thus

$$f(x) = (\bigcup_{i\in I} f_i)(x) = \bigvee_{i\in I}(f_i(x)) \geq \bigvee_{i\in I}(f_i(y)) = (\bigcup_{i\in I} f_i)(y) = f(y).$$

Therefore f is an $(\in, \in \vee \overline{q_k})$-fuzzy quasi-ideal of S by Theorem 3.4. □

4. Prime $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals of ordered semigroups

In this section, we study mainly the prime $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals of ordered semigroups, and investigate their related properties.

Definition 4.1. An $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal f of an ordered semigroup S is called *prime* if for all $x, y \in S$ and $t \in (0, 1]$,

$$x_t \overline{\in} f \text{ and } y_t \overline{\in} f \Rightarrow (xy)_t \overline{\in} \vee \overline{q_k} f.$$

Example 4.2. Consider the ordered semigroup given in Example 3.5, and define a fuzzy subset f of S by $f(a) = 0.4, f(b) = 0.3, f(c) = 0.2, f(d) = 0.3$. By routine calculations, f is a prime $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S for any $k \in [0, 0.2)$.

Theorem 4.3. *Let S be an ordered semigroup and f an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S. Then f is prime if and only if f satisfies*

$$f(x) \vee f(y) \vee \tfrac{1-k}{2} \geq f(xy)$$

for all $x, y \in S$.

Proof. Let f be a prime $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S and $x, y \in S$. Then $f(x) \vee f(y) \vee \frac{1-k}{2} \geq f(xy)$. Indeed, if $f(x) \vee f(y) \vee \frac{1-k}{2} < f(xy)$ for some $x, y \in S$, then there exists $t \in (0, 1]$ such that $f(x) \vee f(y) \vee \frac{1-k}{2} < t < f(xy)$. Then $x_t \overline{\in} f$ and $y_t \overline{\in} f$, but $f(xy) > t$ and $f(xy) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, so $(xy)_t \in f$ and $(xy)_t q_k f$, which is a contradiction. Hence $f(x) \vee f(y) \vee \frac{1-k}{2} \geq f(xy)$ for all $x, y \in S$.

Conversely, assume that $f(x) \vee f(y) \vee \frac{1-k}{2} \geq f(xy)$ for all $x, y \in S$. Let $x_t \overline{\in} f$ and $y_t \overline{\in} f$. Then $f(x) < t, f(y) < t$ and so $f(xy) \leq f(x) \vee f(y) \vee \frac{1-k}{2} \leq t \vee \frac{1-k}{2}$. We consider the following two cases:

Case 1. If $t > \frac{1-k}{2}$, then $f(xy) \leq t$, i.e., $(xy)_t \overline{\in} f$. Thus $(xy)_t \overline{\in} \vee \overline{q_k} f$.

Case 2. If $t \leq \frac{1-k}{2}$, then $f(xy) \leq \frac{1-k}{2}$. Hence, $f(xy) + t + k \leq \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, i.e., $(xy)_t \overline{q_k} f$. Therefore, $(xy)_t \overline{\in} \vee \overline{q_k} f$. This proves that f is prime. □

Theorem 4.4. *Let S be an ordered semigroup and f a strongly convex fuzzy subset of S. Then f is a prime $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S if and only if the level set*

$$f_t := \{x \in S \mid f(x) \geq t\}$$

is a prime quasi-ideal of S for all $t \in (\frac{1-k}{2}, 1]$.

Proof. Let f be a prime $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S and $t \in (\frac{1-k}{2}, 1]$. Then, by Theorem 3.14, f_t is a quasi-ideal of S for all $t \in (\frac{1-k}{2}, 1]$. Let $xy \in f_t$. By Theorem 4.3, we have

$$f(x) \vee f(y) \vee \tfrac{1-k}{2} \geq f(xy) \geq t.$$

Note that $t \in (\frac{1-k}{2}, 1]$. Then $f(x) \vee f(y) \geq t$, and so $f(x) \geq t$ or $f(y) \geq t$. Thus $x \in f_t$ or $y \in f_t$. This shows that f_t is a prime quasi-ideal of S for all $t \in (\frac{1-k}{2}, 1]$.

Conversely, assume that f_t is a prime quasi-ideal of S for all $t \in (\frac{1-k}{2}, 1]$. Then, by Theorem 3.14, f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S. Let $x_t \overline{\in} f$ and $y_t \overline{\in} f$.

Then $x \overline{\in} f_t$ and $y \overline{\in} f_t$. Since f_t is prime, we have $xy \notin f_t$, i.e., $(xy)_t \overline{\in} f$. Thus, $(xy)_t \overline{\in} \vee \overline{q_k} f$. Therefore, f is a prime $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S. □

5. Characterizations of regular and intra-regular ordered semigroups

In this section we first investigate the properties of k-upper parts of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals of ordered semigroups. Furthermore, we give some characterizations of regular and intra-regular ordered semigroups by the properties of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideals, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideals and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals.

Definition 5.1. Let f be a fuzzy subset of an ordered semigroup S. Then we define the k-*upper part* f_k^+ of f as follows:

$$f_k^+(x) = f(x) \vee \tfrac{1-k}{2}$$

for all $x \in S$.

For any $f, g \in F(S)$, the fuzzy subsets $f \cap^k g$ and $f \circ^k g$ of S are defined as follows:

$$(f \cap^k g)(x) = (f \cap g)(x) \vee \tfrac{1-k}{2} \text{ and } (f \circ^k g)(x) = (f \circ g)(x) \vee \tfrac{1-k}{2}$$

for all $x \in S$.

Lemma 5.2. *Let f and g be fuzzy subsets of an ordered semigroup S. Then the following statements are true:*

(1) $(f_k^+)_k^+ = f_k^+, f_k^+ \supseteq f$.

(2) *If $f \subseteq g$, and $h \in F(S)$, then $f \circ^k h \subseteq g \circ^k h,\ h \circ^k f \subseteq h \circ^k g$.*

(3) $f \cap^k g = f_k^+ \cap g_k^+$.

(4) *$f \circ^k g \supseteq f_k^+ \circ g_k^+$, and if $A_x \neq \varnothing$, then $(f \circ^k g)(x) = (f_k^+ \circ g_k^+)(x)$.*

(5) *$f \circ^k S \supseteq f_k^+ \circ S, S \circ^k f \supseteq S \circ f_k^+$ and $f \circ^k S \circ^k f \supseteq f_k^+ \circ S \circ f_k^+$. Furthermore, if $A_x \neq \varnothing$, then $(f \circ^k S)(x) = (f_k^+ \circ S)(x), (S \circ^k f)(x) = (S \circ f_k^+)(x)$ and $(f \circ^k S \circ^k f)(x) = (f_k^+ \circ S \circ f_k^+)(x)$.*

Proof. The proof is straightforward, we omit it. □

One can easily observe that an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of an ordered semigroup S is not necessarily a fuzzy quasi-ideal of S. In the following proposition, we show that if f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S, then the k-upper part f_k^+ of f is a fuzzy quasi-ideal of S.

Proposition 5.3. *If f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of an ordered semigroup S, then the k-upper part f_k^+ of f is a fuzzy quasi-ideal of S.*

Proof. Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S and $x \in S$. Then, by Theorem 3.2, $f(x) \vee \frac{1-k}{2} \geq (f \circ S)(x) \wedge (S \circ f)(x)$, and we have

$$\begin{aligned}
f_k^+(x) &= f(x) \vee \frac{1-k}{2} = (f(x) \vee \frac{1-k}{2}) \vee \frac{1-k}{2} \\
&\geq ((f \circ S)(x) \wedge (S \circ f)(x)) \vee \frac{1-k}{2} \\
&= ((f \circ S)(x) \vee \frac{1-k}{2}) \wedge ((S \circ f)(x) \vee \frac{1-k}{2}) \\
&= (f \circ^k S)(x) \wedge (S \circ^k f)(x) \geq (f_k^+ \circ S)(x) \wedge (S \circ f_k^+)(x).
\end{aligned}$$

It implies that $(f_k^+ \circ S) \cap (S \circ f_k^+) \subseteq f_k^+$. Moreover, if $x \leq y$, then $f_k^+(x) \geq f_k^+(y)$. Indeed, since f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S, we have $f(x) \geq f(y)$, and so $f_k^+(x) = f(x) \vee \frac{1-k}{2} \geq f(y) \vee \frac{1-k}{2} = f_k^+(y)$. Therefore, f_k^+ is a fuzzy quasi-ideal of S. □

Theorem 5.4. *Let $(S, \cdot, \leq)$ be an ordered semigroup and $\varnothing \neq Q \subseteq S$. Then Q is a quasi-ideal of S if and only if the characteristic function f_Q of Q is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S.*

Proof. $(\Longrightarrow)$ Let Q be a quasi-ideal of S and $x, y, z, u, v \in S$ such that $x \leq yu$ and $x \leq vz$. Then $f_Q(x) \vee \frac{1-k}{2} \geq f_Q(y) \wedge f_Q(z)$. Indeed, if $y, z \in Q$, then, since Q is a quasi-ideal of S, $x \in (QS] \cap (SQ] \subseteq Q$, and we have $f_Q(x) \vee \frac{1-k}{2} = 1 \vee \frac{1-k}{2} = 1 = f_Q(y) \wedge f_Q(z)$. If $x \notin Q$ or $y \notin Q$, then $f_Q(y) \wedge f_Q(z) = 0$. Since $x \in S$, we have $f_Q(x) \geq 0$. Thus $f_Q(x) \vee \frac{1-k}{2} = \frac{1-k}{2} > 0 = f_Q(y) \wedge f_Q(z)$. Furthermore, let $x, y \in S, x \leq y$. Then $f_Q(x) \geq f_Q(y)$. In fact, if $y \in Q$, then $f_Q(y) = 1$. Since $S \ni x \leq y \in Q$, we have $x \in Q$, then $f_Q(x) = 1$. Thus $f_Q(x) \geq f_Q(y)$. If $y \notin Q$, then $f_Q(y) = 0$. Since $x \in S$, we have $f_Q(x) \geq 0$. Thus $f_Q(y) = 0 \leq f_Q(x)$. Therefore, f_Q is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S by Theorem 3.4.

$(\Longleftarrow)$ Assume that f_Q is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S. Let $x \in (QS] \cap (SQ]$. Then there exist $y, z \in Q$ and $u, v \in S$ such that $x \leq yu$ and $x \leq vz$. Thus, by Theorem 3.4, we have

$$f_Q(x) \vee \tfrac{1-k}{2} \geq f_Q(y) \wedge f_Q(z) = 1 \wedge 1 = 1 > 0,$$

and so $f_Q(x) = 1$, i.e., $x \in Q$. Therefore, $(QS] \cap (SQ] \subseteq Q$. Moreover, let $y \in Q, S \ni x \leq y$. Then $x \in Q$. Indeed, it is enough to prove that $f_Q(x) = 1$. Since $y \in Q, f_Q(y) = 1$. Since f_Q is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S and $x \leq y$, we have $f_Q(x) \geq f_Q(y) = 1$. Since $x \in S$, we have $f_Q(x) \leq 1$. Thus we have shown that Q is a quasi-ideal of S. □

Similarly we can prove the following two lemmas.

Lemma 5.5. *Let S be an ordered semigroup and $\varnothing \neq A \subseteq S$. Then A is a left (right) ideal of S if and only if the characteristic function f_A of A is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (right) ideal of S.*

Lemma 5.6. *Let S be an ordered semigroup and $\varnothing \neq A \subseteq S$. Then A is a bi-ideal of S if and only if the characteristic function f_A of A is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S.*

Theorem 5.7. *Let $(S, \cdot, \leq)$ be an ordered semigroup and f a fuzzy subset of S. Then f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S if and only if f satisfies that*

(1) $x \leq y \Rightarrow f(x) \geq f(y)$, *for all* $x, y \in S$.

(2) $(f \circ^k S) \cap (S \circ^k f) \subseteq f_k^+$.

Proof. It is obvious by Lemma 5.2 and Theorem 3.2. □

Lemma 5.8. *Let S be an ordered semigroup and f a fuzzy subset of S. Then f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (resp. right) ideal of S if and only if f satisfies that*

(1) $x \leq y \Rightarrow f(x) \geq f(y)$, *for all* $x, y \in S$.

(2) $S \circ^k f \subseteq f_k^+$ *(resp.* $f \circ^k S \subseteq f_k^+$*).*

Proof. The proof is straightforward, we omit it. □

Theorem 5.9. *Let S be an ordered semigroup. Then every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (right) ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S.*

Proof. Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S. Then, by Lemma 5.8, we have $S \circ^k f \subseteq f_k^+$. It follows that

$$(f \circ^k S) \cap (S \circ^k f) \subseteq S \circ^k f \subseteq f_k^+.$$

Thus f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S by Theorem 5.7.

Similarly, we can show that every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal of S is also an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S. □

An ordered semigroup $(S, \cdot, \leq)$ is called *intra-regular* if, for each element a of S, there exist $x, y \in S$ such that $a \leq xa^2y$. **Equivalent definition:** $a \in (Sa^2S], \forall a \in S$ (see [10]).

Lemma 5.10 ([23])**.** *Let S be an ordered semigroup. Then the following conditions are equivalent:*

(1) *S is regular and intra-regular.*
(2) *$B \cap Q \subseteq (BQB]$ for every bi-ideal B and every quasi-ideal Q of S.*
(3) *$B \cap L \subseteq (BLB]$ for every bi-ideal B and every left ideal L of S.*
(4) *$B \cap R \subseteq (BRB]$ for every bi-ideal B and every right ideal R of S.*
(5) *$B \cap Q \subseteq (QBQ]$ for every bi-ideal B and every quasi-ideal Q of S.*
(6) *$L \cap Q \subseteq (QLQ]$ for every left ideal L and every quasi-ideal Q of S.*
(7) *$R \cap Q \subseteq (QRQ]$ for every right ideal R and every quasi-ideal Q of S.*

Now we shall give some characterizations of an ordered semigroup which is both regular and intra-regular in terms of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideals, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideals, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideals and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals.

Theorem 5.11. *Let S be an ordered semigroup. Then the following conditions are equivalent:*

(1) *S is regular and intra-regular.*
(2) *$f \cap^k g \subseteq f \circ^k g \circ^k f$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal g of S.*
(3) *$f \cap^k g \subseteq f \circ^k g \circ^k f$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal g of S.*
(4) *$f \cap^k g \subseteq f \circ^k g \circ^k f$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal g of S.*
(5) *$f \cap^k g \subseteq g \circ^k f \circ^k g$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal g of S.*
(6) *$f \cap^k g \subseteq g \circ^k f \circ^k g$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal g of S.*
(7) *$f \cap^k g \subseteq g \circ^k f \circ^k g$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal f and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal g of S.*

Proof. (1) ⇒ (2). Let f and g be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal and an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S, respectively. Let $a \in S$. Then, since S is both regular and intra-regular, there exist $x, y, z \in S$ such that $a \leq axa$ and $a \leq ya^2z$, and we have

$$a \leq (axa)x(axa) \leq ax(ya^2z)x(ya^2z)xa = (axya)(azxyz)(azxa).$$

Since f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S, by Theorem 3.10 we have

$$\begin{aligned} f_k^+(axya) &= f(axya) \vee \frac{1-k}{2} = (f(axya) \vee \frac{1-k}{2}) \vee \frac{1-k}{2} \\ &\geq (f(a) \wedge f(a)) \vee \frac{1-k}{2} = f(a) \vee \frac{1-k}{2} = f_k^+(a) \end{aligned}$$

and

$$\begin{aligned} f_k^+(azxa) &= f(azxa) \vee \frac{1-k}{2} = (f(azxa) \vee \frac{1-k}{2}) \vee \frac{1-k}{2} \\ &\geq (f(a) \wedge f(a)) \vee \frac{1-k}{2} = f(a) \vee \frac{1-k}{2} = f_k^+(a). \end{aligned}$$

Also, since g is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S, by Theorem 5.7 we have

$$\begin{aligned} g_k^+(azxya) &\geq ((g \circ^k S) \cap (S \circ^k g))(azxya) = ((g_k^+ \circ S) \cap (S \circ g_k^+))(azxya) \\ &= (g_k^+ \circ S)(azxya) \wedge (S \circ g_k^+)(azxya) \\ &= (\bigvee_{(p,q) \in A_{azxya}} [g_k^+(p) \wedge S(q)]) \wedge (\bigvee_{(u,v) \in A_{azxya}} [S(u) \wedge g_k^+(v)]) \\ &\geq (g_k^+(a) \wedge S(zxya)) \wedge (S(azxy) \wedge g_k^+(a)) \\ &= (g_k^+(a) \wedge 1) \wedge (1 \wedge g_k^+(a)) = g_k^+(a). \end{aligned}$$

Thus we have

$$\begin{aligned} (f \circ^k g \circ^k f)(a) &= (f_k^+ \circ g_k^+ \circ f_k^+)(a) = \bigvee_{(m,n) \in A_a} [(f_k^+ \circ g_k^+)(m) \wedge f_k^+(n)] \\ &\geq (f_k^+ \circ g_k^+)(axya^2zxya) \wedge f_k^+(azxa) \\ &\geq (\bigvee_{(s,t) \in A_{axya^2zxya}} [f_k^+(s) \wedge g_k^+(t)]) \wedge f_k^+(azxa) \\ &\geq (f_k^+(axya) \wedge g_k^+(azxya)) \wedge f_k^+(azxa) \\ &\geq (f_k^+(a) \wedge g_k^+(a)) \wedge f_k^+(a) \\ &= (f_k^+ \cap g_k^+)(a) = (f \cap^k g)(a), \end{aligned}$$

which implies that $f \cap^k g \subseteq f \circ^k g \circ^k f$.

Since every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (right) ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S, and so (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are clear.

(3) $\Rightarrow$ (1). Let B and L be a bi-ideal and a left ideal of S, respectively. Let $a \in B \cap L$. Then, by Lemmas 5.5 and 5.6, f_B and f_L are an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal and an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S, respectively. By hypothesis, we have
$(f_B \circ^k f_L \circ^k f_B)(a) \geq (f_B \cap^k f_L)(a) = (f_B(a) \wedge f_L(a)) \vee \frac{1-k}{2} = (1 \wedge 1) \vee \frac{1-k}{2} = 1$
for all $a \in B \cap L$, and $A_a \neq \varnothing$. Thus, $((f_B)_k^+ \circ (f_L)_k^+ \circ (f_B)_k^+)(a) = (f_B \circ^k f_L \circ^k f_B)(a) \geq 1$. By Definition 5.1, $((f_B)_k^+ \circ (f_L)_k^+ \circ (f_B)_k^+)(a) \leq 1$ for all $a \in S$. Thus we have
$$\bigvee_{(y,z) \in A_a} [((f_B)_k^+ \circ (f_L)_k^+)(y) \wedge (f_B)_k^+(z)] = ((f_B)_k^+ \circ (f_L)_k^+ \circ (f_B)_k^+)(a) = 1 > 0,$$
which implies that there exist $b, c \in S$ such that $a \leq bc$, $((f_B)_k^+ \circ (f_L)_k^+)(b) = 1 > 0$ and $(f_B)_k^+(c) = 1$. Then $A_b \neq \varnothing$, $c \in B$ and

$$\bigvee_{(p,q)\in A_b}[(f_B)^+_k(p)\wedge(f_L)^+_k(q)]=((f_B)^+_k\circ(f_L)^+_k)(b)=1>0,$$

which implies that there exist $d,e\in S$ such that $b\leq de$, $(f_B)^+_k(d)=1$ and $(f_L)^+_k(e)=1$. Then $d\in B$, $e\in L$ and $a\leq bc\leq dec\in BLB$, i.e., $a\in(BLB]$, and so $B\cap L\subseteq(BLB]$. By Lemma 5.10, S is both regular and intra-regular.

In the same way, we can show that $(4)\Rightarrow(1),(6)\Rightarrow(1)$ and $(7)\Rightarrow(1)$ hold.

$(1)\Rightarrow(5)$. Similar to $(1)\Rightarrow(2)$, we omit it.

By Theorems 3.9 and 3.10 we can easily observe that every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left (right) ideal of S is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy bi-ideal of S, and so $(5)\Rightarrow(6)$ and $(5)\Rightarrow(7)$ are clear. This completes the proof. □

6. Characterizations of bi-regular ordered semigroups

An ordered semigroup $(S,\cdot,\leq)$ is called left (resp. right) regular if for each $a\in S$ there exists $x\in S$ such that $a\leq xa^2$ (resp. $a\leq a^2x$), i.e., $a\in(Sa^2]$ (resp., $a\in(a^2S]$) (see [14]). An ordered semigroup $(S,\cdot,\leq)$ is called bi-regular if it is both left regular and right regular. Clearly, an ordered semigroup S is bi-regular if and only if $a\in(Sa^2]\cap(a^2S]$ for every $a\in S$.

Definition 6.1. Let S be an ordered semigroup and f an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy quasi-ideal of S. Then f is called *completely semiprime* if $f^+_k(a)\geq f^+_k(a^2)$ for any $a\in S$.

Now we shall give some characterizations of bi-regular ordered semigroups by $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy quasi-ideals.

Theorem 6.2. *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *S is bi-regular.*

(2) *For each $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy quasi-ideal f of S, $f^+_k(a)=f^+_k(a^2)$ for any $a\in S$.*

(3) *For each $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy quasi-ideal f of S, $f^+_k(a^n)=f^+_k(a^{n+1})$ for any $a\in S,n\in Z^+$.*

(4) *Every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy quasi-ideal f of S is completely semiprime.*

Proof. $(1)\Rightarrow(2)$. Let S be a bi-regular ordered semigroup and f an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy quasi-ideal of S and let $a\in S$. Since S is left and right regular, we have $a\in(Sa^2]$ and $a\in(a^2S]$. Then there exist $x,y\in S$ such that $a\leq xa^2$ and $a\leq a^2y$. Then $(x,a^2),(a^2,y)\in A_a$. Since $A_a\neq\varnothing$, by Theorem 5.7 we have

$$\begin{aligned}
f^+_k(a) &\geq ((f\circ^k S)\cap(S\circ^k f))(a)=(f\circ^k S)(a)\wedge(S\circ^k f)(a)\\
&= (f^+_k\circ S)(a)\wedge(S\circ f^+_k)(a)\\
&= (\bigvee_{(p,q)\in A_a}[f^+_k(p)\wedge S(q)])\wedge(\bigvee_{(u,v)\in A_a}[S(u)\wedge f^+_k(v)])\\
&\geq (f^+_k(a^2)\wedge S(y))\wedge(S(x)\wedge f^+_k(a^2))\\
&= (f^+_k(a^2)\wedge 1)\wedge(1\wedge f^+_k(a^2))=f^+_k(a^2).
\end{aligned}$$

On the other hand, $f_k^+(a^2) \geq f_k^+(a)$. Indeed, since f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S, by Proposition 5.3, f_k^+ is a fuzzy quasi-ideal of S, and we have

$$\begin{aligned} f_k^+(a^2) &\geq (f_k^+ \circ S)(a^2) \wedge (S \circ f_k^+)(a^2) \\ &= (\bigvee_{(p,q)\in A_a} [f_k^+(p) \wedge S(q)]) \wedge (\bigvee_{(u,v)\in A_a} [S(u) \wedge f_k^+(v)]) \\ &\geq (f_k^+(a) \wedge S(a)) \wedge (S(a) \wedge f_k^+(a)) \\ &= (f_k^+(a) \wedge 1) \wedge (1 \wedge f_k^+(a)) = f_k^+(a). \end{aligned}$$

Therefore, $f_k^+(a) = f_k^+(a^2)$ for any $a \in S$.

(2) ⇒ (3). Let $a \in S$ and $n \in Z^+$. Then, by Lemma 5.2(4), we have

$$\begin{aligned} (f \circ^k S)(a^{4n}) &= (f_k^+ \circ S)(a^{4n}) = \bigvee_{(x,y)\in A_{a^{4n}}} [f_k^+(x) \wedge S(y)] \\ &\geq f_k^+(a^{n+1}) \wedge S(a^{3n-1}) = f_k^+(a^{n+1}) \wedge 1 = f_k^+(a^{n+1}). \end{aligned}$$

Similarly, it can be shown that $(S \circ^k f)(a^{4n}) \geq f_k^+(a^{n+1})$. Thus, by (2), we have

$$\begin{aligned} f_k^+(a^n) &= f_k^+(a^{2n}) = f_k^+(a^{4n}) \geq ((f \circ^k S) \cap (S \circ^k f))(a^{4n}) \text{ (By Theorem 5.7)} \\ &= (f_k^+ \circ^k S)(a^{4n}) \wedge (S \circ^k f)(a^{4n}) \\ &\geq f_k^+(a^{n+1}) \wedge f_k^+(a^{n+1}) = f_k^+(a^{n+1}). \end{aligned}$$

On the other hand, let $a \in S$. Then we have

$$\begin{aligned} (f \circ^k S)(a^{n+1}) &= (f_k^+ \circ S)(a^{n+1}) \\ &= \bigvee_{(x,y)\in A_{a^{n+1}}} [f_k^+(x) \wedge S(y)] \\ &\geq f_k^+(a^n) \wedge S(a) = f_k^+(a^n) \end{aligned}$$

for any $n \in Z^+$. Similarly, $(S \circ^k f)(a^{n+1}) \geq f_k^+(a^n)$. Since f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S, by hypothesis we have

$$\begin{aligned} f_k^+(a^{n+1}) &\geq ((f \circ^k S) \cap (S \circ^k f))(a^{n+1}) = ((f \circ^k S)(a^{n+1}) \wedge (S \circ^k f))(a^{n+1}) \\ &\geq f_k^+(a^n) \wedge f_k^+(a^n) = f_k^+(a^n). \end{aligned}$$

(3) ⇒ (2) and (2) ⇒ (4) are obvious.

(4) ⇒ (1). Let $a \in S$. We consider the quasi-ideal $Q(a^2)$ of S generated by a^2. By Theorem 5.4, $f_{Q(a^2)}$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S. By hypothesis, $(f_{Q(a^2)})_k^+(a) \geq (f_{Q(a^2)})_k^+(a^2) = f_{Q(a^2)}(a) \vee \frac{1-k}{2} = 1 \vee \frac{1-k}{2} = 1$. By Definition 5.1, $(f_{Q(a^2)})_k^+(a) \leq 1$ for all $a \in S$. Thus $(f_{Q(a^2)})_k^+(a) = 1$, and we have

$$\begin{aligned} a &\in Q(a^2) = (a^2 \cup Sa^2] \cap (a^2 \cup a^2S] \\ &= ((a^2] \cup (Sa^2]) \cap ((a^2] \cup (a^2S]) = (a^2] \cup ((Sa^2] \cap (a^2S]) \\ &= (a^2] \cup ((Sa^2] \cap (a^2S]] = (a^2 \cup ((Sa^2] \cap (a^2S])]. \end{aligned}$$

Then $a \leq t$ for some $t \in a^2 \cup ((Sa^2] \cap (a^2S])$. If $t = a^2$, then $a \leq a^2 = aa \leq a^2a^2 = aaa^2 \leq a^2aa^2 \in a^2Sa^2 \subseteq (Sa^2] \cap (a^2S]$, i.e., $a \in (Sa^2] \cap (a^2S]$. If $t \in (Sa^2] \cap (a^2S]$, then $a \in (Sa^2] \cap (a^2S]$. Therefore, S is bi-regular. □

7. Characterizations of left and right simple ordered semigroups

An ordered semigroup S is called *left* (resp. *right*) simple if for every left (resp. right) ideal A of S, we have $A = S$ (see [17]).

Lemma 7.1 ([17]). *Let S be an ordered semigroup. Then S is left (resp. right) simple if and only if $(Sa] = S$ (resp. $(aS] = S$) for every $a \in S$.*

Lemma 7.2. *Let S be an ordered semigroup. If S is left and right simple, then S is regular.*

Proof. It is obvious by Lemma 7.1. □

Theorem 7.3. *Let S be an ordered semigroup. Then S is left and right simple if and only if every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S is a constant function.*

Proof. Suppose that S is a left and right simple ordered semigroup. Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S and $a \in S$. We consider the set

$$E_S := \{e \in S | e^2 \geq e\}.$$

Then E_S is nonempty. Indeed, by Lemma 7.2, there exists $x \in S$ such that $a \leq axa$. Then we have $(ax)^2 = (axa)x \geq ax$, and so $ax \in E_S$.

To prove the main result of this theorem, we first show that the upper part f_k^+ of f is a constant function for any $k \in [0, 1)$. Now we consider the following two steps:

(1) f_k^+ is a constant mapping on E_S. Indeed, let $t \in E_S$. Then $f_k^+(e) = f_k^+(t)$ for every $e \in E_S$. In fact, since S is left and right simple, we have $(St] = S$ and $(tS] = S$. Since $e \in S$, we have $e \in (St]$ and $e \in (tS]$, so there exist $x, y \in S$ such that $e \leq xt$ and $e \leq ty$. Hence

$$e^2 = ee \leq (xt)(xt) = (xtx)t \text{ and } e^2 = ee \leq (ty)(ty) = t(yty),$$

and we have $(xtx, t) \in A_{e^2}$ and $(t, yty) \in A_{e^2}$. Since f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S, by Theorem 5.7 we have

$$\begin{aligned}
f_k^+(e^2) &\geq ((f \circ^k S) \cap (S \circ^k f))(e^2) \\
&= ((f \circ^k S)(e^2) \wedge (S \circ^k f))(e^2) = ((f_k^+ \circ S)(e^2) \wedge (S \circ f_k^+))(e^2) \\
&= \Big(\bigvee_{(p,q) \in A_{e^2}} [f_k^+(p) \wedge S(q)]\Big) \wedge \Big(\bigvee_{(u,v) \in A_{e^2}} [S(u) \wedge f_k^+(v)]\Big) \\
&\geq (f_k^+(t) \wedge S(yty)) \wedge (S(xtx) \wedge f_k^+(t)) \\
&= (f_k^+(t) \wedge 1) \wedge (1 \wedge f_k^+(t)) = f_k^+(t).
\end{aligned}$$

Since $e \in E_S$, it follows that $e^2 \geq e$ and f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S, and we have $f_k^+(e) = f(e) \vee \frac{1-k}{2} \geq f(e^2) \vee \frac{1-k}{2} = f_k^+(e^2)$. Thus $f_k^+(e) \geq f_k^+(t)$. On the other hand, since S is left and right simple and $e \in S$, we have $(St] = S$ and $(tS] = S$. Since $t \in E_S \subseteq S$, as in the previous case, we also have $f_k^+(t) \geq f_k^+(t^2) \geq f_k^+(e)$.

(2) f_k^+ is a constant mapping on S. Indeed, let $a \in S$. Then $f_k^+(a) = f_k^+(t)$ for every $t \in E_S$. In fact, since S is left and right simple, by Lemma 7.2, S is regular, and there exists $x \in S$ such that $a \leq axa$. Then

$$(ax)^2 = (axa)x \geq ax \text{ and } (xa)^2 = x(axa) \geq xa,$$

which implies that $ax, xa \in E_S$. Then by (1) we have $f_k^+(ax) = f_k^+(t)$ and $f_k^+(xa) = f_k^+(t)$. Since $(ax)(axa) \geq axa \geq a$, and $(axa)(xa) \geq axa \geq a$, we have, $(ax, axa) \in$

A_a and $(axa, xa) \in A_a$. Since f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S, we have

$$\begin{aligned} f_k^+(a) &\geq ((f \circ^k S) \cap (S \circ^k f))(a) \\ &= ((f \circ^k S)(a) \wedge (S \circ^k f))(a) = ((f_k^+ \circ S)(a) \wedge (S \circ f_k^+))(a) \\ &= (\bigvee_{(p,q)\in A_{e^2}} [f_k^+(p) \wedge S(q)]) \wedge (\bigvee_{(u,v)\in A_{e^2}} [S(u) \wedge f_k^+(v)]) \\ &\geq (f_k^+(ax) \wedge S(axa)) \wedge (S(axa) \wedge f_k^+(xa)) \\ &= (f_k^+(t) \wedge 1) \wedge (1 \wedge f_k^+(t)) = f_k^+(t). \end{aligned}$$

On the other hand, since S is left and right simple, we have $(Sa] = S, (aS] = S$. Since $t \in S$, $t \in (Sa]$ and $t \in (aS]$. Then $t \leq ua$ and $t \leq av$ for some $u, v \in S$. Then $(u, a) \in A_t$ and $(a, v) \in A_t$. Since f is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S, we have

$$\begin{aligned} f_k^+(t) &\geq ((f \circ^k S) \cap (S \circ^k f))(t) \\ &= ((f \circ^k S)(t) \wedge (S \circ^k f))(t) = ((f_k^+ \circ S)(t) \wedge (S \circ f_k^+))(t) \\ &= (\bigvee_{(y_1,z_1)\in A_t} [f_k^+(y_1) \wedge S(z_1)]) \wedge (\bigvee_{(y_2,z_2)\in A_t} [S(y_2) \wedge f_k^+(z_2)]) \\ &\geq (f_k^+(a) \wedge S(v)) \wedge (S(u) \wedge f_k^+(a)) \\ &= (f_k^+(a) \wedge 1) \wedge (1 \wedge f_k^+(a)) = f_k^+(a). \end{aligned}$$

Hence it is shown, from (1) and (2), that f_k^+ is a constant function on S for any $k \in [0, 1)$. Therefore, f is a constant function on S by Definition 5.1.

Conversely, let $a \in S$. Since the set $(Sa]$ is a left ideal of S, and so $(Sa]$ is a quasi-ideal of S. By Theorem 5.4, the characteristic function $f_{(Sa]}$ of $(Sa]$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S. By hypothesis, $f_{(Sa]}$ is a constant function, that is, there exists $c \in \{0, 1\}$ such that $f_{(Sa]}(x) = c$ for every $x \in S$. Let $(Sa] \subset S$ and t be an element of S such that $t \notin (Sa]$. Then $f_{(Sa]}(t) = 0$. Also, since $a^2 \in (Sa]$, $f_{(Sa]}(a^2) = 1$, leading to a contradiction to the fact that $f_{(Sa]}$ is a constant function on S. Thus $(Sa] = S$. Similarly, we can prove that $(aS] = S$. Therefore, by Lemma 7.1, S is left and right simple. □

An equivalence relation ρ on an ordered semigroup S is called *congruence* if $(a, b) \in \rho$ implies $(ac, bc) \in \rho$ and $(ca, cb) \in \rho$ for every $c \in S$. A congruence ρ on S is called *semilattice congruence* on S, if $(a, a^2) \in \rho$ and $(ab, ba) \in \rho$ for all $a, b \in S$ (see [18]). An ordered semigroup S is called a semilattice of left and right simple semigroups if there exists a semilattice congruence ρ on S such that the ρ-class $(x)_\rho$ of S containing x is a left and right simple subsemigroup of S for every $x \in S$. Equivalently, there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha \in Y}$ of left and right simple subsemigroups of S such that:

(*i*) $S_\alpha \cap S_\beta = \varnothing$ for each $\alpha, \beta \in Y, \alpha \neq \beta$.
(*ii*) $S = \bigcup_{\alpha \in Y} S_\alpha$.
(*iii*) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for each $\alpha, \beta \in Y$.

Definition 7.4 ([18])**.** A subsemigroup F of an ordered semigroup S is called a *filter* of S if (1) $a, b \in S, ab \in F \;\Rightarrow\; a \in F$ and $b \in F$; (2) $a \in F$, $S \ni c \geq a \;\Rightarrow\; c \in F$.

We denote by $N(a)$ the filter of S generated by a ($a \in S$), and by "$\mathcal{N}$" the equivalence relation on S defined by $a\mathcal{N}b$ if and only if $N(a) = N(b)$. $\mathcal{N}$ is a semilattice congruence on S (see [18]).

A subset T of an ordered semigroup S is called completely semiprime if for every $a \in S$ such that $a^2 \in T$ we have $a \in T$ (see [32]).

Lemma 7.5 ([17])**.** *Let S be an ordered semigroup. Then the following statements are equivalent:*

(1) *$(x)_{\mathcal{N}}$ is a left (resp., right) simple subsemigroup of S, for every $x \in S$.*

(2) *Every left (resp., right) ideal of S is a right (resp., left) ideal of S and completely semiprime.*

Lemma 7.6 ([17])**.** *An ordered semigroup S is a semilattice of left and right simple semigroups if and only if $(A] = A$ and $(AB] = (BA]$ for all quasi-ideals A, B of S.*

Lemma 7.7 ([15])**.** *An ordered semigroup S is regular if and only if $(RL] = R \cap L$ for every right ideal R and left ideal L of S.*

Theorem 7.8. *Let S be an ordered semigroup. Then S is a semilattice of left and right simple semigroups if and only if for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal f of S, we have*

$$f_k^+(a) = f_k^+(a^2) \text{ and } f_k^+(ab) = f_k^+(ba)$$

for all $a, b \in S$.

Proof. $\Longrightarrow$. By hypothesis, there exists a semilattice Y and a family $\{S_\alpha\}_{\alpha \in Y}$ of left and right simple subsemigroups of S such that:

(*i*) $S_\alpha \cap S_\beta = \varnothing$ for each $\alpha, \beta \in Y, \alpha \neq \beta$.

(*ii*) $S = \bigcup_{\alpha \in Y} S_\alpha$.

(*iii*) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for each $\alpha, \beta \in Y$.

Let f be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S. Then we have

(1) Let $a \in S$. Then $f_k^+(a) = f_k^+(a^2)$. Indeed, by Theorem 6.2, it suffices to prove that S is bi-regular. Since $a \in S = \bigcup\limits_{\alpha \in Y} S_\alpha$, there exists $\alpha \in Y$ such that $a \in S_\alpha$. Since S_α is left and right simple, we have $S_\alpha = (S_\alpha a]$ and $S_\alpha = (aS_\alpha]$. Then we have $(aS_\alpha] = (a(S_\alpha a]] = (aS_\alpha a]$. Since $a \in S_\alpha$, we have $a \in (aS_\alpha a]$, then there exists $x \in S_\alpha$ such that $a \leq axa$. Since $x \in S_\alpha = (aS_\alpha a]$, there exists $y \in S_\alpha$ such that $x \leq aya$. Thus

$$a \leq axa \leq a(aya)a = a^2ya^2 \in a^2S_\alpha a^2 \subseteq a^2Sa^2 \subseteq (Sa^2] \cap (a^2S],$$

which implies that S is bi-regular.

(2) Let $a, b \in S$. Then $f_k^+(ab) = f_k^+(ba)$. Indeed, by (1), we have $f_k^+(ab) = f_k^+((ab)^2) = f_k^+((ab)^4)$. Moreover, we have

$$\begin{aligned}
(ab)^4 &= (aba)(babab) \in Q(aba)Q(babab) \subseteq (Q(aba)Q(babab)] \\
&= (Q(babab)Q(aba)] \text{ (By Lemma 7.6)} \\
&= ((babab \cup (bababS \cap Sbabab)](aba \cup (abaS \cap Saba)]] \\
&= ((babab \cup (bababS \cap Sbabab))(aba \cup (abaS \cap Saba))] \\
&\subseteq ((babab \cup bababS)(aba \cup Saba)] \\
&\subseteq ((baS)(Sba)] = ((baS](Sba]] \\
&= (baS] \cap (Sba] \text{ (By Lemmas 7.2 and 7.7).}
\end{aligned}$$

Then there exist $x, y \in S$ such that $(ab)^4 \leq (ba)x$ and $(ab)^4 \leq y(ba)$, i.e., $(ba, x) \in A_{(ab)^4}$ and $(y, ba) \in A_{(ab)^4}$. Since f is a fuzzy quasi-ideal of S, we have

$$\begin{aligned}
f_k^+((ab)^4) &\geq ((f \circ^k S) \cap (S \circ^k f))((ab)^4) \\
&= ((f \circ^k S)((ab)^4) \wedge (S \circ^k f))((ab)^4) \\
&= ((f_k^+ \circ S)((ab)^4) \wedge (S \circ f_k^+))((ab)^4) \\
&= (\bigvee_{(p,q) \in A_{e^2}} [f_k^+(p) \wedge S(q)]) \wedge (\bigvee_{(u,v) \in A_{e^2}} [S(u) \wedge f_k^+(v)]) \\
&\geq (f_k^+(ba) \wedge S(x)) \wedge (S(y) \wedge f_k^+(ba)) \\
&= (f_k^+(ba) \wedge 1) \wedge (1 \wedge f_k^+(ba)) = f_k^+(ba).
\end{aligned}$$

It thus follows that $f_k^+(ab) = f_k^+((ab)^4) \geq f_k^+(ba)$. In a similar way, we can prove that $f_k^+(ba) \geq f_k^+(ab)$.

$\Longleftarrow$. Since $\mathcal{N}$ is a semilattice congruence on S, by Lemma 7.5, it suffices to prove that every left (resp. right) ideal of S is an ideal of S and completely semiprime. Let L be a left ideal of S and hence a quasi-ideal of S. By Theorem 5.4, f_L is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S. Let $a \in L, b \in S$. Then, by hypothesis, we have $(f_L)_k^+(ab) = (f_L)_k^+(ba) = f_L(ba) \vee \frac{1-k}{2} = 1 \vee \frac{1-k}{2} = 1$, which implies that $ab \in L$. Thus $LS \subseteq L$ and if $a \in L, S \ni b \leq a$, then $b \in L$. Thus L is a right ideal of S. Hence L is an ideal of S. Let $x \in S$ such that $x^2 \in L$. Then $x \in L$. Indeed, since L is a quasi-ideal of S, by Theorem 5.4, f_L is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S. By hypothesis, $(f_L)_k^+(x^2) = (f_L)_k^+(x)$. Since $x^2 \in L$, we have $(f_L)_k^+(x^2) = 1$. Thus we have $(f_L)_k^+(x) = 1$, and $x \in L$. Hence L is completely semiprime. In a similar way we can prove that every right ideal of S is an ideal and completely semiprime. This completes the proof. $\square$

Acknowledgements. The authors are extremely grateful to the referees for their valuable comments and suggestions which helped a lot in improving the presentation of this paper.

This work was supported by the Natural Science Foundation of China (11371177), the Anhui Provincial Excellent Youth Talent Foundation (2012SQRL115ZD) and the University Natural Science Project of Anhui Province (KJ2013A204).

References

[1] S. K. Bhakat and P. Das, On the definition of a fuzzy subgroup, Fuzzy Sets and Systems 51(2) (1992) 235–241.
[2] S. K. Bhakat and P. Das, $(\in, \in \vee q)$-fuzzy subgroup, Fuzzy Sets and Systems 80(3) (1996) 359–368.
[3] Y. Cao, Characterizations of regular ordered semigroups by quasi-ideals, Vietnam J. Math. 30(3) (2002) 239–250.
[4] B. Davvaz, $(\in, \in \vee q)$-fuzzy subnearrings and ideals, Soft Computing 10(3) (2006) 206–211.
[5] Y. B. Jun and S. Z. Song, Generalized fuzzy interior ideals in semigroups, Inform. Sci. 176 (2006) 3079–3093.
[6] Y. B. Jun, On (α, β)-fuzzy subalgebras of BCK/BCI-algebras, Bull. Korean Math. Soc. 42(4) (2005) 703–711.
[7] Y. B. Jun, Fuzzy subalgebras of type (α, β) in BCK/BCI algebras, Kyungpook Math. J. 47(3) (2007) 403–410.

[8] Y. B. Jun, A. Khan and M. Shabir, Ordered semigroups characterized by their $(\in, \in \vee q)$-fuzzy bi-ideals, Bull. Malays. Math. Sci. Soc. 32(3) (2009) 391–408.
[9] N. Kehayopulu, On regular ordered semigroups, Math. Japonica 45(3) (1997) 549–553.
[10] N. Kehayopulu, On intra-regular ordered semigroups, Semigroup Forum 46(1) (1993) 271–278.
[11] N. Kehayopulu, On weakly prime ideals of ordered semigroups, Math. Japonica 35(6) (1990) 1051–1056.
[12] N. Kehayopulu and M. Tsingelis, Fuzzy sets in ordered groupoids, Semigroup Forum 65(1) (2002) 128–132.
[13] N. Kehayopulu and M. Tsingelis, The embedding of an ordered groupoid into a poe-groupoid in terms of fuzzy sets, Inform. Sci. 152 (2003) 231–236.
[14] N. Kehayopulu and M. Tsingelis, Left regular and intra-regular ordered semigroups in terms of fuzzy subsets, Quasigroups Related Systems 14(2) (2006) 167–178.
[15] N. Kehayopulu and M. Tsingelis, Regular ordered semigroups in terms of fuzzy subsets, Inform. Sci. 176 (2006) 3675–3693.
[16] N. Kehayopulu and M. Tsingelis, Fuzzy right, left, quasi-ideals, bi-ideals in ordered semigroups, Lobachevskii J. Math. 30(1) (2009) 17–22.
[17] N. Kehayopulu and M. Tsingelis, Fuzzy bi-ideals in ordered semigroups, Inform. Sci. 171 (2005) 13–28.
[18] N. Kehayopulu, Remarks on ordered semigroups, Math. Japonica 35(5) (1990) 1061–1063.
[19] A. Khan, Some new kinds of fuzzy quasi-ideals of ordered semigroups, Adv. Fuzzy Sets Syst. 14(2) (2013) 71–100.
[20] A. Khan, Y. B. Jun and M. Shabir, A study of generalized fuzzy ideals in ordered semigroups, Neural Comput. & Applic. 21 (2012) 69–78.
[21] A. Khan, Y. B. Jun, N. H. Sarmin and F. M. Khan, Ordered semigroups characterized by $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals, Neural Comput. & Applic. 21 (2012) 121–132.
[22] N. Kuroki, On fuzzy ideals and fuzzy bi-ideals in semigroups, Fuzzy Sets and Systems 5(2) (1981) 203–215.
[23] S. K. Lee, On characterizations of some classes of po-semigroups, Math. Japonica 47(1) (1998) 51–55.
[24] X. Ma and J. M. Zhan, Generalized fuzzy h-bi-ideals and h-quasi-ideals of hemirings, Inform. Sci. 179 (2009) 1249–1268.
[25] G. Mohanraj, D. Krishnaswamy and R. Hema, On fuzzy m-systems and n-systems of ordered semigroup, Ann. Fuzzy Math. Inform. 7(1) (2014) 173–179.
[26] J. N. Mordeson, D. S. Malik and N. Kuroki, Fuzzy Semigroups, Studies in Fuzziness and Soft Computing, vol. 131, Springer-Verlag, Berlin, 2003.
[27] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512–517.
[28] O. Stienfeld, Quasi-ideals in rings and semigroups, Akademiakiado, Budapest, 1978.
[29] M. Shabir, Y. B. Jun and Y. Nawaz, Semigroups characterized by $(\in, \in \vee q_k)$-fuzzy ideals, Comput. Math. Appl. 60(5) (2010) 1473–1493.
[30] M. Shabir and A. Khan, Fuzzy quasi-ideal in ordered semigroups, Bull. Malays. Math. Sci. Soc. 34(1) (2011) 87–102.
[31] J. Tang and X. Y. Xie, Characterizations of regular ordered semigroups by generalized fuzzy ideals, J. Intell. Fuzzy Systems 26(1) (2014) 239–252.
[32] X. Y. Xie and J. Tang, Fuzzy radicals and prime fuzzy ideals of ordered semigroups, Inform. Sci. 178 (2008) 4357–4374.
[33] X. Y. Xie and J. Tang, Regular ordered semigroups and intra-regular ordered semigroups in terms of fuzzy subsets, Iran. J. Fuzzy Syst. 7(2) (2010) 121–140.
[34] X. Y. Xie, An introduction to ordered semigroup theory, Kexue Press, Beijing, 2001.
[35] X. Y. Xie and M. F. Wu, The Theory of Fuzzy Semigroups, Kexue Press, Beijing, 2005.

Jian Tang (tangjian0901@126.com)
Department of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu, 730000, P. R. China; School of Mathematics and Statistics, Fuyang Normal College, Fuyang, Anhui, 236037, P. R. China

Yanfeng Luo (luoyf@lzu.edu.cn)
Department of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu, 730000, P. R. China