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# A new contraction mapping principle in partially ordered fuzzy metric spaces

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Abstract. In this article, we define a new type of coupled contraction mapping in fuzzy metric spaces having a partially ordering and obtain a coupled coincidence point theorem by using a *Hadžić* type t-norm. The two mappings considered here are assumed to be compatible. Several corollaries are derived from our theorem. The main theorem of this paper is illustrated with an example which shows that the corollaries are actually contained in the theorem. By an application of the coincidence point theorem in fuzzy metric spaces, a corresponding result is obtained in metric spaces. An example is discussed in the metric space context. Our work extends some existing results.

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## 1. Introduction

In this paper we consider a coupled coincidence problem in fuzzy metric spaces. There are several independent definitions of fuzzy metric spaces in the literature as, for instances [9, 11, 18, 20] out of which we consider here the definition given by George and Veeramani [9]. On this fuzzy metric space fixed point theory has a developed literature. The theory is not a mere extension of the ordinary fixed point theory. In fact, due to the inherent flexibility of the fuzzy concepts, the fuzzy metric space is sometimes more versatile than the ordinary metric and, in most of the cases, a theory built in its context naturally has more freedom than the corresponding theory in metric spaces. For some inherent issues of fuzzy metric spaces we refer to [1, 6, 7, 14, 15, 16, 17, 23, 24, 25, 28, 29, 33]. There are also applications of fuzzy metric concepts to real life situations as, for instance, in [4, 26] an application has been made to colour delection problems.

Coupled fixed point was introduced by Guo et al [12]. A coupled contraction mapping principle was established by Bhaskar et al [2] in partially ordered metric spaces. The result was extended to coincidence point problems by Ciric et al [8] under two different set of conditions. In [5], the well known concept of compatible mappings was extended to the context of coupled and single mappings. Afterwards, there have appeared a large number of articles on this chapter of fixed point theory, some of there are noticed in [3, 22, 27, 31].

In fuzzy metric spaces Zhu et al [34] were first to correctly work out a fuzzy fixed point theorem. Afterwards, a coupled coincidence point result was established by Choudhury et al [6], Hu [16]. The purpose of this paper is to prove a coincidence point result for compatible mappings in a fuzzy metric space which has *Hadžić* type t-norm under the assumption of a new inequality.

In a separate section we apply the theorem to obtain coupled coincidence point results in metric spaces. Some existing results on coupled coincidence points are extended, as instances [2, 3, 5]. The results of this paper are illustrated with examples. One example shows that extensions are actual improvement of the above mentioned work.

## 2. Preliminaries

**Definition 2.1** ([13, 32])**.** A binary operation $* : [0,1]^2 \longrightarrow [0,1]$ is called a $t$-norm if the following properties are satisfied:
(i) $*$ is associative and commutative,
(ii) $a * 1 = a$ for all $a \in [0,1]$,
(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.

Some examples of continuous $t$-norm are $a *_M b = \min\{a, b\}$, $a *_P b = ab$. Several aspects of the theory of t-norms with examples are given comprehensively by Klement et al in their book [19].

George and Veeramani in their paper [9] introduced the following definition of fuzzy metric space. We will be concerned only with this definition of fuzzy metric space.

**Definition 2.2** ([9])**.** The 3-tuple $(X, M, *)$ is called a fuzzy metric space in the sense of George and Veeramani if $X$ is a non-empty set, $*$ is a continuous $t$-norm and $M$ is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$:
(i)$M(x, y, t) > 0$,
(ii) $M(x, y, t) = 1$ if and only if $x = y$,
(iii) $M(x, y, t) = M(y, x, t)$,
(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ and
(v) $M(x, y, .) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, $0 < r < 1$, the open ball $B(x, t, r)$ with center $x \in X$ is defined by

$$B(x, t, r) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, t, r) \subset A$. Let $\tau$ denote the family of all open subsets of $X$. Then $\tau$ is a topology and is called the topology on $X$ induced by the fuzzy metric $M$. This topology is metrizable as we indicated above.

**Example 2.3** ([9]). Let $X$ be the set of all real numbers and $d$ be the Euclidean metric by any set $X$ and any metric $d$ on $X$. Let $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$. For each $t > 0$, $x, y \in X$, let

$$M(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M, *)$ is a fuzzy metric space.

**Definition 2.4** ([10]). Let $(X, M, *)$ be a fuzzy metric space.
(i) A sequence $\{x_n\}$ in $X$ is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.
(ii) A sequence $\{x_n\}$ in $X$ is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists a positive integer $n_0$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$.
(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

The following lemma was proved by Grabiec [10] for fuzzy metric spaces defined by Kramosil et al. The proof is also applicable to the fuzzy metric space given in definition 2.2.

**Lemma 2.5** ([10]). *Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, .)$ is non-decreasing for all $x, y \in X$.*

**Lemma 2.6** ([30]). *$M$ is a continuous function on $X^2 \times (0, \infty)$.*

Our purpose in this paper is to prove a coupled coincidence point theorem for two mappings in a complete fuzzy metric space which has a partial order defined on it.

Let $(X, \preceq)$ be a partially ordered set and $F$ be a mapping from $X$ to itself. The mapping $F$ is said to be non-decreasing if for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1) \preceq F(x_2)$ and non-increasing if for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1) \succeq F(x_2)$[2].

**Definition 2.7** ([2]). Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. The mapping $F$ is said to have the mixed monotone property if $F$ is non-decreasing in its first argument and is non-increasing in its second argument, that is, if, for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1, y) \preceq F(x_2, y)$, for fixed $y \in X$ and, for all $y_1, y_2 \in X$, $y_1 \preceq y_2$ implies $F(x, y_1) \succeq F(x, y_2)$, for fixed $x \in X$.

**Definition 2.8** ([2]). Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ be two mappings. The mapping $F$ is said to have the mixed $g$-monotone property if $F$ is monotone $g$-non-decreasing in its first argument and is monotone $g$-non-increasing in its second argument, that is, if, for all $x_1, x_2 \in X$, $g(x_1) \preceq g(x_2)$ implies $F(x_1, y) \preceq F(x_2, y)$, for any $y \in X$ and, for all $y_1, y_2 \in X$, $g(y_1) \preceq g(y_2)$ implies $F(x, y_1) \succeq F(x, y_2)$, for any $x \in X$.

**Definition 2.9** ([2]). Let $X$ be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$$F(x, y) = x,\ \ F(y, x) = y.$$

Further Lakshmikantham and Ćirić have introduced the concept of coupled coincidence point.

**Definition 2.10** ([21]). Let $X$ be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mappings $F : X \times X \to X$ and $g : X \to X$ if

$$g(x) = F(x, y),\ \ g(y) = F(y, x).$$

**Definition 2.11** ([21]). Let $X$ be a nonempty set and the mappings $F : X \times X \to X$ and $g : X \to X$ are commuting if for all $x, y \in X$

$$g(F(x, y)) = F(g(x), g(y)).$$

Compatibility between two mappings $F : X \times X \to X$ and $g : X \to X$, where $(X, d)$ is a metric space, was defined in [5]. It is an extension of the commuting condition. Compatibility was used to obtain a coupled coincidence point result in the same work.

**Definition 2.12** ([5]). Let $(X, d)$ be a metric space. The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be compatible if

$$\lim_{n\to\infty} d(g(F(x_n, y_n)), F(g(x_n), g(y_n))) = 0$$

and

$$\lim_{n\to\infty} d(g(F(y_n, x_n)), F(g(y_n), g(x_n))) = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ for some $x, y \in X$.

Intuitively we can think that the functions $F$ and $g$ commute in the limit in the situations where the functional values tend to the same point.

The following is the notion of compatibility in the fuzzy metric spaces.

**Definition 2.13** ([6, 16]). Let $(X, M, *)$ be a fuzzy metric space. The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be compatible if for all $t > 0$

$$\lim_{n\to\infty} M(g(F(x_n, y_n)), F(g(x_n), g(y_n), t) = 1$$

and

$$\lim_{n\to\infty} M(g(F(y_n, x_n)), F(g(y_n), g(x_n), t) = 1,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ for some $x, y \in X$.

In the following lemma we establish that the compatibility in a metric space implies that the compatibility in the corresponding fuzzy metric space of example 2.3. We use it to obtain a result in metric spaces in section 4.

**Lemma 2.14** ([6]). *Let $(X, d)$ be a metric space. If the mappings $F$ and $g$ where $F : X \times X \to X$ and $g : X \to X$ are compatible in $(X, d)$, then $F$ and $g$ are also compatible in the corresponding fuzzy metric space $(X, M, *)$.*

We use continuous *Hadžić* type t-norm in our theorem.

**Definition 2.15** ([13]). A t-norm $*$ is said to be *Hadžić* type t-norm if the family $\{*^p\}_{p\geq 0}$ of its iterates defined for each $s \in [0,1]$ by

$*^0(s) = 1$, $*^{p+1}(s) = *(*^p(s), s)$ for all $p \geq 0$, is equi-continuous at $s = 1$, that is, given $\lambda > 0$ there exists $\eta(\lambda) \in (0,1)$ such that

$1 \geq s > \eta(\lambda) \Rightarrow *^{(p)}(s) > 1 - \lambda$ for all $p \geq 0$.

For an example of a non trivial *Hadžić* type t-norm, we refer to [13].

We will require the result of the following lemma to establish our main theorem.

**Lemma 2.16** ([6]). *Let $(X, M, *)$ be a fuzzy metric space with a Hadžić type t-norm $*$ such that $M(x, y, t) \to 1$ as $t \to \infty$, for all $x, y \in X$. If the sequences $\{x_n\}$ and $\{y_n\}$ in $X$ are such that, for all $n \geq 1$, $t > 0$,*

$$M(x_n, x_{n+1}, t) * M(y_n, y_{n+1}, t) \geq M(x_{n-1}, x_n, \tfrac{t}{k}) * M(y_{n-1}, y_n, \tfrac{t}{k})$$

*where $0 < k < 1$, then the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences.*

## 3. Major Section

**Theorem 3.1.** *Let $(X, M, *)$ be a complete fuzzy metric space with a Hadžić type t-norm such that $M(x, y, t) \to 1$ as $t \to \infty$, for all $x, y \in X$. Let $\preceq$ be a partial order defined on $X$. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that $F$ has mixed $g$-monotone property and satisfies the following conditions:*

*(i) $F(X \times X) \subseteq g(X)$,*

*(ii) $g$ is continuous and monotonic increasing,*

*(iii) $(g, F)$ is a compatible pair,*

*(iv)*
$$M(F(x, y), F(u, v), kt) * M(F(y, x), F(v, u), kt) \geq M(g(x), g(u), t) * M(g(y), g(v), t), \tag{3.1}$$

*for all $x, y, u, v \in X$, $t > 0$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$, where $0 < k < 1$. Also suppose either*

*(a) $F$ is continuous or*

*(b) $X$ has the following properties:*

*(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \geq 0$, (3.2)*

*(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all $n \geq 0$. (3.3)*

*If there are $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0, y_0)$, $g(y_0) \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is, $g$ and $F$ have a coupled coincidence point in $X$.*

*Proof.* Starting with $x_0, y_0$ in X, we define the sequences $\{x_n\}$ and $\{y_n\}$ in X as follows:

$$g(x_1) = F(x_0, y_0) \text{ and } g(y_1) = F(y_0, x_0),$$
$$g(x_2) = F(x_1, y_1) \text{ and } g(y_2) = F(y_1, x_1),$$

and in general, for all $n \geq 0$,

$$g(x_{n+1}) = F(x_n, y_n) \text{ and } g(y_{n+1}) = F(y_n, x_n). \tag{3.4}$$

This construction is possible by the condition (i) of the theorem.

Next, we prove that for all $n \geq 0$,

$$g(x_n) \preceq g(x_{n+1}) \tag{3.5}$$

and

$$g(y_n) \succeq g(y_{n+1}). \tag{3.6}$$

From the conditions on $x_0, y_0$, we have $g(x_0) \preceq F(x_0, y_0) = g(x_1)$ and $g(y_0) \succeq F(y_0, x_0) = g(y_1)$. Therefore (3.5) and (3.6) hold for $n = 0$.
Let (3.5) and (3.6) hold for some $n = m$. As $F$ has the mixed $g$-monotone property and $g(x_m) \preceq g(x_{m+1})$, $g(y_m) \succeq g(y_{m+1})$, it follows that

$$g(x_{m+1}) = F(x_m, y_m) \preceq F(x_{m+1}, y_m) \text{ and } F(y_{m+1}, x_m) \preceq F(y_m, x_m) = g(y_{m+1}). \tag{3.7}$$

Also, for the same reason, we have

$$\begin{aligned} &F(x_{m+1}, y_m) \preceq F(x_{m+1}, y_{m+1}) = g(x_{m+2}) \\ &\text{and } g(y_{m+2}) = F(y_{m+1}, x_{m+1}) \preceq F(y_{m+1}, x_m). \end{aligned} \tag{3.8}$$

Then, from (3.7) and (3.8),

$$g(x_{m+1}) \preceq g(x_{m+2}) \text{ and } g(y_{m+1}) \succeq g(y_{m+2}).$$

Then, by induction, (3.5) and (3.6) hold for all $n \geq 0$.
Due to (3.4), (3.5) and (3.6), from (3.1), for all $t > 0$, $n \geq 1$, we have

$$\begin{aligned} &M(g(x_n), g(x_{n+1}), kt) * M(g(y_n), g(y_{n+1}), kt) \\ &\quad = M(F(x_{n-1}, y_{n-1}), F(x_n, y_n), kt) * M(F(y_{n-1}, x_{n-1}), F(y_n, x_n), kt) \\ &\quad \geq M(g(x_{n-1}), g(x_n), t) * M(g(y_{n-1}), g(y_n), t). \text{ (by (3.1))} \end{aligned} \tag{3.9}$$

From (3.9), by an application of Lemma 2.16, we conclude that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences. Since $X$ is complete, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} g(x_n) = x \text{ and } \lim_{n\to\infty} g(y_n) = y. \tag{3.10}$$

Therefore, $\lim_{n\to\infty} g(x_{n+1}) = \lim_{n\to\infty} F(x_n, y_n) = x$ and $\lim_{n\to\infty} g(y_{n+1}) = \lim_{n\to\infty} F(y_n, x_n) = y$.
Since $(g, F)$ is a compatible pair, using continuity of $g$, we have

$$g(x) = \lim_{n\to\infty} g(g(x_{n+1})) = \lim_{n\to\infty} g(F(x_n, y_n)) = \lim_{n\to\infty} F(g(x_n), g(y_n)) \tag{3.11}$$

and

$$g(y) = \lim_{n\to\infty} g(g(y_{n+1})) = \lim_{n\to\infty} g(F(y_n, x_n)) = \lim_{n\to\infty} F(g(y_n), g(x_n)). \tag{3.12}$$

Now assume that(a) holds. Then by continuity of $F$, from (3.11), (3.12) and by using (3.10), we have
$g(x) = \lim_{n\to\infty} g(F(x_n, y_n)) = \lim_{n\to\infty} F(g(x_n), g(y_n)) = F(\lim_{n\to\infty} g(x_n), \lim_{n\to\infty} g(y_n)) = F(x, y)$
and $g(y) = \lim_{n\to\infty} g(F(y_n, x_n)) = \lim_{n\to\infty} F(g(y_n), g(x_n)) = F(\lim_{n\to\infty} g(y_n), \lim_{n\to\infty} g(x_n)) = F(y, x)$.
which implies that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.
Next we assume that (b) holds.
By (3.5), (3.6) and (3.10), we have that $\{g(x_n)\}$ is a non-decreasing sequence with $g(x_n) \to x$ and $\{g(y_n)\}$ is a non-increasing sequence with $g(y_n) \to y$ as $n \to \infty$. Then, by (3.2) and (3.3), it follows that, for all $n \geq 0$,

$$g(x_n) \preceq x \text{ and } g(y_n) \succeq y.$$

Since $g$ is monotonic increasing,

$$g(g(x_n)) \preceq g(x) \text{ and } g(g(y_n)) \succeq g(y). \tag{3.13}$$

Now, for all $t > 0$, $n \geq 0$, we have

$$\begin{aligned} M(F(x, y), g(F(x_n, y_n)), t) \geq\ & M(F(x, y), g(g(x_{n+1})), kt) \\ & * M(g(g(x_{n+1})), g(F(x_n, y_n)), (t - kt)). \end{aligned} \tag{3.14}$$

$$\begin{aligned} M(F(y, x), g(F(y_n, x_n)), t) \geq\ & M(F(y, x), g(g(y_{n+1})), kt) \\ & * M(g(g(y_{n+1})), g(F(y_n, x_n)), (t - kt)). \end{aligned} \tag{3.15}$$

From (3.14) and (3.15), for all $t > 0$, we have

$$\begin{aligned} &M(F(x,y), g(F(x_n,y_n)), t) * M(F(y,x), g(F(y_n,x_n)), t) \\ &\quad \geq [M(F(x,y), g(g(x_{n+1})), kt) * M(g(g(x_{n+1})), g(F(x_n,y_n)), (t-kt))] \\ &\quad * [M(F(y,x), g(g(y_{n+1})), kt) * M(g(g(y_{n+1})), g(F(y_n,x_n)), (t-kt))]. \end{aligned}$$

Taking $n \to \infty$ on the both sides of the above inequality, for all $t > 0$,

$$\begin{aligned} &\lim_{n\to\infty} [M(F(x,y), g(F(x_n,y_n)), t) * M(F(y,x), g(F(y_n,x_n)), t)] \\ &\quad \geq \lim_{n\to\infty} [M(F(x,y), g(g(x_{n+1})), kt) * M(g(g(x_{n+1})), g(F(x_n,y_n)), (t-kt))] \\ &\quad * \lim_{n\to\infty} [M(F(y,x), g(g(y_{n+1})), kt) * M(g(g(y_{n+1})), g(F(y_n,x_n)), (t-kt))], \end{aligned}$$

that is,

$$\begin{aligned} &M(F(x,y), \lim_{n\to\infty} g(F(x_n,y_n)), t) * M(F(y,x), \lim_{n\to\infty} g(F(y_n,x_n)), t) \\ &\geq [M(F(x,y), \lim_{n\to\infty} g(g(x_{n+1})), kt) * M(\lim_{n\to\infty} g(g(x_{n+1})), \lim_{n\to\infty} g(F(x_n,y_n)), (t-kt))] \\ &* [M(F(y,x), \lim_{n\to\infty} g(g(y_{n+1})), kt) * M(\lim_{n\to\infty} g(g(y_{n+1})), \lim_{n\to\infty} g(F(y_n,x_n)), (t-kt))], \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by lemma 2.6)} \end{aligned}$$

that is, $M(F(x,y), g(x), t) * M(F(y,x), g(y), t)$

$$\begin{aligned} &\geq [M(F(x,y), \lim_{n\to\infty} g(F(x_n,y_n)), kt) * M(g(x), g(x), (t-kt))] \\ &* [M(F(y,x), \lim_{n\to\infty} g(F(y_n,x_n)), kt) * M(g(y), g(y), (t-kt))], \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (3.11))} \end{aligned}$$

that is, $M(F(x,y), g(x), t) * M(F(y,x), g(y), t)$

$$\begin{aligned} &\geq \lim_{n\to\infty} [M(F(x,y), g(F(x_n,y_n)), kt) * 1] \\ &* \lim_{n\to\infty} [M(F(y,x), g(g(F(y_n,x_n))), kt) * 1]. \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by lemma 2.6)} \\ &\geq \lim_{n\to\infty} [M(F(g(x_n), g(y_n)), F(x,y), kt) * M(F(g(y_n), g(x_n)), F(y,x), kt)] \\ &\geq \lim_{n\to\infty} [M(g(g(x_n)), g(x), t) * M(g(g(y_n)), g(y), t)] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (3.1) and using (3.13))} \\ &= M(\lim_{n\to\infty} g(g(x_n)), g(x), t) * M(\lim_{n\to\infty} g(g(y_n)), g(y), t) \\ &= M(g(x), g(x), t) * M(g(y), g(y), t) \quad \text{(by (3.11))} \\ &= M(g(x), g(x), t) * M(g(y), g(y), t) \\ &= 1 * 1 \\ &= 1, \end{aligned}$$

that is, $M(F(x,y), g(x), t) * M(F(y,x), g(y), t) \geq 1$.
Therefore $M(F(x,y), g(x), t) = 1$ and $M(F(y,x), g(y), t) = 1$,
which implies that $g(x) = F(x,y)$ and $g(y) = F(y,x)$.
This completes the proof of the theorem. □

**Corollary 3.2.** *Let $(X, M, *)$ be a complete fuzzy metric space with a Hadžić type t-norm such that $M(x,y,t) \to 1$ as $t \to \infty$, for all $x, y \in X$. Let $\preceq$ be a partial order defined on $X$. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that $F$ has mixed $g$-monotone property and satisfies the following conditions:*
*(i) $F(X \times X) \subseteq g(X)$,*
*(ii) $g$ is continuous and monotonic increasing,*
*(iii)$(g, F)$ is a commuting pair,*
*(iv)*$M(F(x,y), F(u,v), kt) * M(F(y,x), F(v,u), kt)$
$$\geq M(g(x), g(u), t) * M(g(y), g(v), t),$$

*for all $x, y, u, v \in X$, $t > 0$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$, where $0 < k < 1$. Also suppose either*
*(a) $F$ is continuous or*
*(b) $X$ has the following properties:*
*(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \geq 0$,*
*(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all $n \geq 0$.*
*If there are $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0, y_0)$, $g(y_0) \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is, $g$ and $F$ have a coupled coincidence point in $X$.*

*Proof.* Since a commuting pair is also a compatible pair, the result of the corollary 3.2 follows from theorem 3.1. □

Later, by an example, we will show that the corollary 3.2 is properly contained in theorem 3.1.

The following corollary is a fixed point result.

**Corollary 3.3.** *Let $(X, \preceq)$ be a partially ordered set and let $(X, M, *)$ be a complete fuzzy metric space with a Hadžić type t-norm such that $M(x, y, t) \to 1$ as $t \to \infty$, for all $x, y \in X$. Let $\preceq$ be a partial order defined on $X$. Let $F : X \times X \to X$ be a mapping such that $F$ has mixed monotone property and satisfies the following condition:*

$$M(F(x, y), F(u, v), kt) * M(F(y, x), F(v, u), kt) \geq M(x, u, t) * M(y, v, t),$$

*for all $x, y, u, v \in X$, $t > 0$ with $x \preceq u$ and $y \succeq v$, where $0 < k < 1$. Also suppose either*
*(a) $F$ is continuous or*
*(b) $X$ has the following properties:*
*(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \geq 0$,*
*(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all $n \geq 0$.*
*If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$, $y_0 \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, $F$ has a coupled fixed point in $X$.*

*Proof.* The proof follows by putting $g = I$, the identity function, in theorem 3.1. □

**Example 3.4.** Let $(X, \preceq)$ is the partially ordered set with $X = [0, 1]$ and the natural ordering $\leq$ of the real numbers as the partial ordering $\preceq$. Let for all $t > 0$, $x, y \in X$,

$$M(x, y, t) = e^{-\frac{|x - y|}{t}}.$$

Let $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$. Then $(X, M, *)$ is a complete fuzzy metric space such that $M(x, y, t) \to 1$ as $t \to \infty$, for all $x, y \in X$.
Let the mapping $g : X \to X$ be defined as

$$g(x) = \tfrac{5}{6}x^2 \text{ for all } x \in X$$

and the mapping $F : X \times X \to X$ be defined as

$$F(x, y) = \frac{x^2 - y^2}{4}.$$

Then $F(X \times X) \subseteq g(X)$ and $F$ satisfies the mixed $g$-monotone property.
Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $X$ such that

$$\lim_{n\to\infty} F(x_n, y_n) = a, \quad \lim_{n\to\infty} g(x_n) = a,$$
$$\lim_{n\to\infty} F(y_n, x_n) = b \text{ and } \lim_{n\to\infty} g(y_n) = b.$$

Now, for all $n \geq 0$,

$$g(x_n) = \tfrac{5}{6}x_n^2, \ \ g(y_n) = \tfrac{5}{6}y_n^2,$$
$$F(x_n, y_n) = \tfrac{x_n^2 - y_n^2}{4}$$

and

$$F(y_n, x_n) = \tfrac{y_n^2 - x_n^2}{4}.$$

Then necessarily $a = 0$ and $b = 0$.
It then follows from lemma 2.6 that, for all $t > 0$,

$$\lim_{n\to\infty} M(g(F(x_n, y_n)), F(g(x_n), g(y_n), t) = 1$$

and

$$\lim_{n\to\infty} M(g(F(y_n, x_n)), F(gy_n), g(x_n)), t) = 1.$$

Therefore the mappings $F$ and $g$ are compatible in $X$.
Now we show that the condition (3.1) holds.

$$|F(x,y) - F(u,v)| \leq \frac{1}{2}|g(x) - g(u)| + \frac{1}{2}|g(y) - g(v)|,\ x \geq u, y \leq v \quad (3.16)$$

and

$$|F(y,x) - F(v,u)| \leq \frac{1}{2}|g(y) - g(v)| + \frac{1}{2}|g(x) - g(u)|,\ x \geq u, y \leq v. \quad (3.17)$$

From (3.16), for all $t > 0$ and $0 < k < 1$, we have

$$\begin{aligned} e^{-\frac{|F(x,y)-F(u,v)|}{kt}} &\geq e^{-\frac{[\frac{1}{2}|g(x)-g(u)|+\frac{1}{2}|g(y)-g(v)|]}{t}} \\ &\geq e^{-\frac{|g(x)-g(u)|}{2t}}.e^{-\frac{|g(y)-g(v)|}{2t}} \\ &= \sqrt{e^{-\frac{|g(x)-g(u)|}{t}}.e^{-\frac{|g(y)-g(v)|}{t}}} \\ &\geq \min\{e^{-\frac{|g(x)-g(u)|}{t}}, e^{-\frac{|g(y)-g(v)|}{t}}\}, \\ e^{-\frac{|F(x,y)-F(u,v)|}{kt}} &\geq \min\{M(g(x), g(u), t), M(g(y), g(v), t)\}. \end{aligned} \quad (3.18)$$

Similarly from (3.17), we have

$$e^{-\frac{|F(y,x)-F(v,u)|}{kt}} \geq \min\{M(g(x), g(u), t), M(g(y), g(v), t)\}. \quad (3.19)$$

From (3.18) and (3.19), we have

$$\begin{aligned} \min\{M(F(x,y), F(u,v), kt), M(F(y,x), F(v,u), kt)\} \\ \geq \min\{M(g(x), g(u), t), M(g(y), g(v), t)\}, \end{aligned}$$

that is, $M(F(x,y), F(u,v), kt) * M(F(y,x), F(v,u), kt)$

$$\geq M(g(x), g(u), t) * M(g(y), g(v), t).$$

Hence (3.1) holds.
Thus all the conditions of Theorem 3.1 are satisfied. Then, by an application of the Theorem 3.1, we conclude that $g$ and $F$ have a coupled coincidence point. Here $(0, 0)$ is a coupled coincidence point of $g$ and $F$ in $X$.

**Remark 3.5.** In the Example 3.4, the functions $g$ and $F$ do not commute. Hence Corollary 3.2 cannot be applied to this example. This shows that Theorem 3.1 properly contains its Corollary 3.2.

## 4. Application in metric space

In this section we apply Theorem 3.1 of the previous section to obtain present a coupled coincidence point result in partially ordered metric spaces. Several existing results [2, 3, 5] are hereby extended.

**Theorem 4.1.** *Let $(X, \preceq)$ be a partially ordered set and $d$ be a metric on $X$ such that $(X, d)$ is a complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that $F$ has the mixed $g$-monotone property and satisfies the following condition:*

$$\max\{d(F(x,y), F(u,v)), d(F(y,x), F(v,u))\} \leq \tfrac{k}{2}[d(g(x), g(u)) + d(g(y), g(v))], \tag{4.1}$$

*for all $x, y, u, v \in X$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$, where $0 < k < 1$. Suppose $F(X \times X) \subseteq g(X)$, $g$ is continuous and $(g, F)$ is a compatible pair. Also suppose either*

*(a) $F$ is continuous or*

*(b) $X$ has the following properties:*

*(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \geq 0$,*

*(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all $n \geq 0$.*

*If there are $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0, y_0)$, $g(y_0) \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is, $g$ and $F$ have a coupled coincidence point in $X$.*

*Proof.* For all $x, y \in X$ and $t > 0$, we define

$$M(x, y, t) = \tfrac{t}{t+d(x,y)}$$

and $a * b = \min\{a, b\}$. Then, as noted earlier, $(X, M, *)$ is a complete fuzzy metric space.

Further, from the above definition, $M(x, y, t) \to 1$ as $t \to \infty$, for all $x, y \in X$.

Using Lemma 2.15, we conclude that $(g, F)$ is a compatible pair in this fuzzy metric space. Next we show that the inequality (4.1) implies (3.1). If otherwise, from (3.1), for some $t > 0$, $x, y, u, v \in X$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$, we have

$$\min\{\tfrac{t}{t+\frac{1}{k}d(F(x,y),F(u,v))}, \tfrac{t}{t+\frac{1}{k}d(F(y,x),F(v,u))}\} < \min\{\tfrac{t}{t+d(g(x),g(u))}, \tfrac{t}{t+d(g(y),g(v))}\},$$

Form the above inequality, we have

either $$\tfrac{t}{t+\frac{1}{k}d(F(x,y),F(u,v))} < \min\{\tfrac{t}{t+d(g(x),g(u))}, \tfrac{t}{t+d(g(y),g(v))}\} \tag{4.2}$$

or $$\tfrac{t}{t+\frac{1}{k}d(F(y,x),F(v,u))}\} < \min\{\tfrac{t}{t+d(g(x),g(u))}, \tfrac{t}{t+d(g(y),g(v))}\}. \tag{4.3}$$

From (4.2), we have

$$t + \tfrac{1}{k}d(F(x,y), F(u,v)) > t + d(g(x), g(u)) \text{ and}$$
$$t + \tfrac{1}{k}d(F(x,y), F(u,v)) > t + d(g(y), g(v)).$$

Combining the above two inequalities, we have that

$$d(F(x,y), F(u,v)) > \tfrac{k}{2}[d(g(x), g(u)) + d(g(y), g(v))]. \tag{4.4}$$

Similarly from (4.3), we have

$$d(F(y,x), F(v,u)) > \tfrac{k}{2}[d(g(y), g(v)) + d(g(x), g(u))]. \tag{4.5}$$

By (4.4) and (4.5), we have

$$\max\{d(F(x,y), F(u,v)), d(F(y,x), F(v,u))\} > \tfrac{k}{2}[d(g(x), g(u)) + d(g(y), g(v))],$$

which is a contradiction with (4.1).

The proof is then completed by an application of Theorem 3.1. □

**Corollary 4.2.** *Let $(X, \preceq)$ be a partially ordered set and $d$ be a metric on $X$ such that $(X, d)$ is a complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that $F$ has the mixed $g$-monotone property and satisfies the following condition:*

$$[d(F(x, y), F(u, v))+d(F(y, x), F(v, u))] \leq k[d(g(x), g(u))+d(g(y), g(v))], \quad (4.6)$$

*for all $x, y, u, v \in X$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$, where $0 < k < 1$. Suppose $F(X \times X) \subseteq g(X)$, $g$ is continuous and $(g, F)$ is a compatible pair. Also suppose either*
*(a) $F$ is continuous or*
*(b) $X$ has the following properties:*
*(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \geq 0$,*
*(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all $n \geq 0$.*
*If there are $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0, y_0)$, $g(y_0) \succeq F(y_0, x_0)$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is, $g$ and $F$ have a coupled coincidence point in $X$.*

*Proof.* Since $\frac{x+y}{2} \leq \max\{x, y\}$, the proof follows from Theorem 4.1. □

**Example 4.3.** Let $(X, \preceq)$ is the partially ordered set with $X = [0, 1]$ and the natural ordering $\leq$ of the real numbers as the partial ordering $\preceq$. Let $x, y \in X$, $d(x, y) = |x - y|$.
Then $(X, d)$ is a complete metric space.
Let the mapping $g : X \to X$ be defined as

$$g(x) = \tfrac{5}{6}x^2 \text{ for all } x \in X$$

and the mapping $F : X \times X \to X$ be defined as

$$F(x, y) = \frac{x^2 - 2y^2}{4}.$$

Then $F(X \times X) \subseteq g(X)$ and $F$ satisfies the mixed $g$-monotone property.
Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $X$ such that

$$\lim_{n\to\infty} F(x_n, y_n) = a, \quad \lim_{n\to\infty} g(x_n) = a,$$
$$\lim_{n\to\infty} F(y_n, x_n) = b \text{ and } \lim_{n\to\infty} g(y_n) = b.$$

Now, for all $n \geq 0$,

$$g(x_n) = \tfrac{5}{6}x_n^2, \;\; g(y_n) = \tfrac{5}{6}y_n^2,$$
$$F(x_n, y_n) = \tfrac{x_n^2 - y_n^2}{4}$$

and

$$F(y_n, x_n) = \tfrac{y_n^2 - x_n^2}{4}.$$

Then necessarily $a = 0$ and $b = 0$.
It then follows from lemma 2.6 that, for all $t > 0$,

$$\lim_{n\to\infty} d(g(F(x_n, y_n)), F(g(x_n), g(y_n), t) = 0$$

and

$$\lim_{n\to\infty} d(g(F(y_n, x_n)), F(gy_n), g(x_n)), t) = 0.$$

Therefore the mappings $g$ and $F$ are compatible in $X$. The mappings are not commuting.
Now we show that the condition (4.6) holds.

$$|F(x, y) - F(u, v)| \leq \frac{3}{10}|g(x) - g(u)| + \frac{3}{5}|g(y) - g(v)|, \; x \geq u, y \leq v \quad (4.7)$$

and

$$|F(y,x) - F(v,u)| \leq \frac{3}{10}|g(y) - g(v)| + \frac{3}{5}|g(x) - g(u)|,\ x \geq u, y \leq v. \quad (4.8)$$

Adding (4.7) and (4.8), we get the inequality (4.6) with $k = \frac{9}{10}$.
Here $(0,0)$ is the coupled coincidence point.

**Remark 4.4.** Theorem 4.1 is a generalization of a result of Berinde [corollary 1, 2]. Hence, in theorem, generalization, a number of results in [2, 5] are made. In the above Example 4.3, the pair $(g, F)$ is compatible, but not commuting. This shows that the improvement here is actual. If $g = I$ is put in Corollary 4.2, then we have a generalization of result of Bhaskar et al [2].

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