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Generalized int-soft subsemigroups

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Abstract. Further properties of int-soft subsemigroups are investigated, and then generalizations of int-soft subsemigroups are discussed. The notion of θ-generalized int-soft subsemigroups in semigroups is introduced, and several properties are investigated. Characterizations of a θ-generalized int-soft subsemigroup are considered, and a condition for a special set to be a subsemigroup is provided. The soft union and soft intersection of two θ-generalized int-soft subsemigroup over U are dealt with, and the soft pre-image and soft image of a θ-generalized int-soft subsemigroup under the homomorphism are discussed.

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1. Introduction

Molodtsov [12] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [11] described the application of soft set theory to a decision making problem. Maji et al. [10] also studied several operations on the theory of soft sets. Many algebraic properties of soft sets are studied (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 13, 15]). Song et al. [14] introduced the notion of an int-soft subsemigroup in a semigroup, and investigated their properties.

In this paper, we first discuss further properties of int-soft subsemigroups, and then we consider generalizations of int-soft subsemigroups. We introduce the notion of θ-generalized int-soft subsemigroups in semigroups, and investigate several properties. We consider characterizations of a θ-generalized int-soft subsemigroup. We provide a condition for a special set to be a subsemigroup. We show that the soft intersection of two θ-generalized int-soft subsemigroup over U is a θ-generalized int-soft subsemigroup over U. We discuss the soft pre-image and soft image of a θ-generalized int-soft subsemigroup under the homomorphism.

2. Preliminaries

Let S be a semigroup. Let A and B be subsets of S. Then the multiplication of A and B is defined as follows:

$$AB = \{ab \in S \mid a \in A \text{ and } b \in B\}.$$

A nonempty subset A of S is called a *subsemigroup* of S if $AA \subseteq A$, that is, $ab \in A$ for all $a, b \in A$.

Molodtsov [12] defined the soft set in the following way: Let U be an initial universe set and E be a set of parameters. Let $\mathscr{P}(U)$ denotes the power set of U and $A \subset E$.

A pair $(\tilde{f}, A)$ is called a *soft set* (see [12]) over U, where $\tilde{f}$ is a mapping given by

$$\tilde{f} : A \to \mathscr{P}(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $\tilde{f}(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set $(\tilde{f}, A)$. Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [12].

The *soft union* of $(\tilde{f}, S)$ and $(\tilde{g}, S)$, denoted by $(\tilde{f}, S) \tilde{\cup} (\tilde{g}, S)$, is defined to be the soft set $\left(\tilde{f} \tilde{\cup} \tilde{g}, S\right)$ of S (over U) in which $\tilde{f} \tilde{\cup} \tilde{g}$ is defined by

$$\left(\tilde{f} \tilde{\cup} \tilde{g}\right)(x) = \tilde{f}(x) \cup \tilde{g}(x) \text{ for all } x \in S.$$

The *soft intersection* of $(\tilde{f}, S)$ and $(\tilde{g}, S)$, denoted by $(\tilde{f}, S) \tilde{\cap} (\tilde{g}, S)$, is defined to be the soft set $\left(\tilde{f} \tilde{\cap} \tilde{g}, S\right)$ of S (over U) in which $\tilde{f} \tilde{\cap} \tilde{g}$ is defined by

$$\left(\tilde{f} \tilde{\cap} \tilde{g}\right)(x) = \tilde{f}(x) \cap \tilde{g}(x) \text{ for all } x \in S.$$

3. Inclusive sets

For a soft set $(\tilde{f}, A)$ over U and a subset γ of U, the γ-*inclusive set* of $(\tilde{f}, A)$, denoted by $(\tilde{f}, A)^{\supseteq}_{\gamma}$, is defined to be the set

$$(\tilde{f}, A)^{\supseteq}_{\gamma} := \left\{x \in A \mid \gamma \subseteq \tilde{f}(x)\right\}.$$

The *proper* γ-*inclusive set* of $(\tilde{f}, A)$, denoted by $(\tilde{f}, A)^{\supsetneq}_{\gamma}$, is defined to be the set

$$(\tilde{f}, A)^{\supsetneq}_{\gamma} := \left\{x \in A \mid \gamma \subsetneq \tilde{f}(x)\right\}.$$

Proposition 3.1. *For a soft set $(\tilde{f}, A)$ over U, we have*

(1) $(\forall x \in A)\left(\tilde{f}(x) = \cap\left\{\gamma \in \mathscr{P}(U) \mid x \in (\tilde{f}, A)^{\supseteq}_{\gamma}\right\}\right)$.

(2) $(\tilde{f}, A)^{\supsetneq}_{\gamma} \subseteq (\tilde{f}, A)^{\supseteq}_{\gamma}$.

(3) $(\tilde{f}, A)^{\supsetneq}_{\gamma} = (\tilde{f}, A)^{\supseteq}_{\gamma}$ *if and only if there is no $x \in A$ such that $\tilde{f}(x) = \gamma$, that is, $\tilde{f}(x) \neq \gamma$ for all $x \in A$.*

(4) $(\forall \gamma_1, \gamma_2 \in \mathscr{P}(U)) \left(\gamma_1 \subseteq \gamma_2 \Rightarrow (\tilde{f}, A)^{\supsetneq}_{\gamma_1} \supseteq (\tilde{f}, A)^{\supsetneq}_{\gamma_2},\ (\tilde{f}, A)^{\supseteq}_{\gamma_1} \supseteq (\tilde{f}, A)^{\supseteq}_{\gamma_2} \right)$.

(5) $(\forall \gamma_1, \gamma_2 \in \mathrm{Im}(\tilde{f}, A)) \left(\gamma_1 \neq \gamma_2 \Rightarrow (\tilde{f}, A)^{\supseteq}_{\gamma_1} \neq (\tilde{f}, A)^{\supseteq}_{\gamma_2} \right)$.

(6) *If* $\gamma_1 \subset \gamma_2$ *in* $\mathscr{P}(U) \setminus \mathrm{Im}(\tilde{f}, A)$, *then* $(\tilde{f}, A)^{\supsetneq}_{\gamma_1} \supseteq (\tilde{f}, A)^{\supsetneq}_{\gamma_2}$ *and* $(\tilde{f}, A)^{\supseteq}_{\gamma_1} \supseteq (\tilde{f}, A)^{\supseteq}_{\gamma_2}$.

Proof. (1) $\sim$ (5) Straightforward.

(6) Let $\gamma_1 \subset \gamma_2$ in $\mathscr{P}(U) \setminus \mathrm{Im}(\tilde{f}, A)$. If $x \in (\tilde{f}, A)^{\supsetneq}_{\gamma_2}$, then $\tilde{f}(x) \supsetneq \gamma_2 \supsetneq \gamma_1$, and so $x \in (\tilde{f}, A)^{\supsetneq}_{\gamma_1}$. Hence $(\tilde{f}, A)^{\supsetneq}_{\gamma_1} \supseteq (\tilde{f}, A)^{\supsetneq}_{\gamma_2}$. Since $\gamma_1, \gamma_2 \notin \mathrm{Im}(\tilde{f}, A)$, it follows from (3) that $(\tilde{f}, A)^{\supseteq}_{\gamma_1} = (\tilde{f}, A)^{\supsetneq}_{\gamma_1} \supseteq (\tilde{f}, A)^{\supsetneq}_{\gamma_2} = (\tilde{f}, A)^{\supseteq}_{\gamma_2}$. □

4. Int-soft subsemigroups

In what follows, we take a semigroup S as a set of parameters, and let $\mathscr{P}^*(U) = \mathscr{P}(U) \setminus \{\varnothing\}$ unless otherwise specified.

Definition 4.1 ([14])**.** A soft set $(\tilde{f}, S)$ over U is called an *int-soft subsemigroup* over U if it satisfies:

$$(\forall x, y \in S) \left(\tilde{f}(xy) \supseteq \tilde{f}(x) \cap \tilde{f}(y) \right). \tag{4.1}$$

Lemma 4.2 ([14])**.** *A soft set $(\tilde{f}, S)$ over U is an int-soft subsemigroup over U if and only if the nonempty γ-inclusive set of $(\tilde{f}, S)$ is a subsemigroup of S for all $\gamma \in \mathscr{P}(U)$.*

Example 4.3. Let $S = \{e, a, b, c\}$ be a semigroup with the following Cayley table:

$\cdot$	e	a	b	c
e	e	e	e	e
a	e	a	e	a
b	e	e	b	b
c	e	a	b	c

Let $(\tilde{f}, S)$ be a soft set over $U = \{0, 1, 2, 3\}$ defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} U & \text{if } x = e, \\ \{0, 1\} & \text{if } x \in \{a, c\} \\ \{0, 1, 3\} & \text{if } x = b \end{cases}$$

Then $(\tilde{f}, S)$ is an int-soft subsemigroup over U.

Theorem 4.4. *If $(\tilde{f}, S)$ and $(\tilde{g}, S)$ are int-soft subsemigroups over U, then the soft intersection $(\tilde{f}, S) \tilde{\cap} (\tilde{g}, S)$ is also an int-soft subsemigroup over U.*

Proof. For any $x, y \in S$, we have

$$\begin{aligned} \left(\tilde{f} \tilde{\cap} \tilde{g} \right)(xy) &= \tilde{f}(xy) \cap \tilde{g}(xy) \\ &\supseteq \left(\tilde{f}(x) \cap \tilde{f}(y) \right) \cap (\tilde{g}(x) \cap \tilde{g}(y)) \\ &= \left(\tilde{f}(x) \cap \tilde{g}(x) \right) \cap \left(\tilde{f}(y) \cap \tilde{g}(y) \right) \\ &= \left(\tilde{f} \tilde{\cap} \tilde{g} \right)(x) \cap \left(\tilde{f} \tilde{\cap} \tilde{g} \right)(y). \end{aligned}$$

Therefore $(\tilde{f}, S) \tilde{\cap} (\tilde{g}, S)$ is an int-soft subsemigroup over U. □

The following example shows that the converse of Theorem 4.4 is not true in general.

Example 4.5. Let $S = \{a, b, c, d\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	a
c	a	a	c	c
d	a	a	d	d

Let $(\tilde{f}, S)$ and $(\tilde{g}, S)$ be soft sets over $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{1, 2, 4, 8\} & \text{if } x = a, \\ \{2, 3, 4, 5\} & \text{if } x = b, \\ \{2, 4, 6, 8\} & \text{if } x = c, \\ \{1, 2, 5, 10\} & \text{if } x = d, \end{cases}$$

and

$$\tilde{g} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{1, 2, 3, 5\} & \text{if } x = a, \\ \{1, 2, 3, 6\} & \text{if } x = b, \\ \{1, 2, 3, 4\} & \text{if } x = c, \\ \{2, 5, 8\} & \text{if } x = d, \end{cases}$$

respectively. Then the soft intersection $(\tilde{f}, S) \tilde{\cap} (\tilde{g}, S)$ of $(\tilde{f}, S)$ and $(\tilde{g}, S)$ is given as follows:

$$\tilde{f} \tilde{\cap} \tilde{g} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{1, 2\} & \text{if } x = a, \\ \{2, 3\} & \text{if } x = b, \\ \{2, 4\} & \text{if } x = c, \\ \{2, 5\} & \text{if } x = d, \end{cases}$$

and it is an int-soft subsemigroup over U. But $(\tilde{f}, S)$ is not an int-soft subsemigroup over U since

$$\tilde{f}(bd) = \tilde{f}(a) = \{1, 2, 4, 8\} \not\supseteq \{2, 5\} = \tilde{f}(b) \cap \tilde{f}(d).$$

In general, we know that the soft union of two int-soft subsemigroups over U is not an int-soft subsemigroup over U as seen in the following example.

Example 4.6. Let $S = \{a, b, c\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c
a	a	a	a
b	a	b	a
c	c	c	c

Let $(\tilde{f}, S)$ and $(\tilde{g}, S)$ be soft sets over $U = \{0, 1, 2, 3\}$ defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{2, 3\} & \text{if } x = a, \\ \{0, 3\} & \text{if } x = b, \\ \{1, 2, 3\} & \text{if } x = c \end{cases}$$

and

$$\tilde{g} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{3\} & \text{if } x = a, \\ \{1,3\} & \text{if } x = b, \\ \{0,3\} & \text{if } x = c \end{cases}$$

respectively. Then the soft union $(\tilde{f}, S) \tilde{\cup} (\tilde{g}, S)$ of $(\tilde{f}, S)$ and $(\tilde{g}, S)$ is given as follows:

$$\tilde{f} \tilde{\cup} \tilde{g} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{2,3\} & \text{if } x = a, \\ \{0,1,3\} & \text{if } x = b, \\ U & \text{if } x = c \end{cases}$$

We note that $(\tilde{f}, S)$ and $(\tilde{g}, S)$ are int-soft subsemigroups over U, but $(\tilde{f}, S) \tilde{\cup} (\tilde{g}, S)$ is not an int-soft subsemigroup over U since

$$\left(\tilde{f} \tilde{\cup} \tilde{g}\right)(bc) = \left(\tilde{f} \tilde{\cup} \tilde{g}\right)(a) = \{2,3\} \not\supseteq \{0,1,3\} = \left(\tilde{f} \tilde{\cup} \tilde{g}\right)(b) \cap \left(\tilde{f} \tilde{\cup} \tilde{g}\right)(c).$$

We know that there exist soft sets $(\tilde{f}, S)$ and $(\tilde{g}, S)$ over U such that the soft union $(\tilde{f}, S) \tilde{\cup} (\tilde{g}, S)$ of $(\tilde{f}, S)$ and $(\tilde{g}, S)$ is an int-soft subsemigroup over U, but $(\tilde{f}, S)$ is not an int-soft subsemigroup over U or $(\tilde{g}, S)$ is not an int-soft subsemigroup over U.

Example 4.7. Let $S = \{a, b, c\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c
a	b	a	a
b	a	b	b
c	a	b	c

Let $(\tilde{f}, S)$ and $(\tilde{g}, S)$ be soft sets over $U = \{0, 1, 2, 3\}$ defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{0\} & \text{if } x = a, \\ \{2,3\} & \text{if } x = b, \\ \{3\} & \text{if } x = c \end{cases}$$

and

$$\tilde{g} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{2,3\} & \text{if } x = a, \\ \{0,1,2\} & \text{if } x = b, \\ \{1,3\} & \text{if } x = c \end{cases}$$

respectively. Then the soft union $(\tilde{f}, S) \tilde{\cup} (\tilde{g}, S)$ of $(\tilde{f}, S)$ and $(\tilde{g}, S)$ is given as follows:

$$\tilde{f} \tilde{\cup} \tilde{g} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{0,2,3\} & \text{if } x = a, \\ U & \text{if } x = b, \\ \{1,3\} & \text{if } x = c \end{cases}$$

and it is an int-soft subsemigroup over U. But $(\tilde{f}, S)$ is not an int-soft subsemigroup over U since

$$\tilde{f}(aa) = \tilde{f}(b) = \{2,3\} \not\supseteq \{0\} = \tilde{f}(a) \cap \tilde{f}(a).$$

Also $(\tilde{g}, S)$ is not an int-soft subsemigroup over U since

$$\tilde{g}(aa) = \tilde{g}(b) = \{0,1,2\} \not\supseteq \{2,3\} = \tilde{g}(a) \cap \tilde{g}(a).$$

Theorem 4.8. *For any soft set $(\tilde{f}, S)$ over U and $\gamma \in \mathscr{P}(U)$, let $(\tilde{f}_\gamma, S)$ be a soft set defined as follows:*

$$\tilde{f}_\gamma : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \tilde{f}(x) & \text{if } x \in (\tilde{f}, S)^{\supseteq}_{\gamma}, \\ \tau & \text{otherwise} \end{cases}$$

where $\tau \in \mathscr{P}(U)$ with $\tau \subsetneq \tilde{f}(x)$. If $(\tilde{f}, S)$ is an int-soft subsemigroup over U, then so is $(\tilde{f}_\gamma, S)$.

Proof. Assume that $(\tilde{f}, S)$ is an int-soft subsemigroup over U. Then $(\tilde{f}, S)^{\supseteq}_{\gamma}$ is a subsemigroup of S for all $\gamma \in \mathscr{P}(U)$ with $(\tilde{f}, S)^{\supseteq}_{\gamma} \neq \varnothing$ by Lemma 4.2. Let $x, y \in S$. If $x, y \in (\tilde{f}, S)^{\supseteq}_{\gamma}$, then $xy \in (\tilde{f}, S)^{\supseteq}_{\gamma}$ and so

$$\tilde{f}_\gamma(xy) = \tilde{f}(xy) \supseteq \tilde{f}(x) \cap \tilde{f}(y) = \tilde{f}_\gamma(x) \cap \tilde{f}_\gamma(y).$$

If $x \notin (\tilde{f}, S)^{\supseteq}_{\gamma}$ or $y \notin (\tilde{f}, S)^{\supseteq}_{\gamma}$, then $\tilde{f}_\gamma(x) = \tau$ or $\tilde{f}_\gamma(y) = \tau$. Hence

$$\tilde{f}_\gamma(xy) \supseteq \tau = \tilde{f}_\gamma(x) \cap \tilde{f}_\gamma(y).$$

Therefore $(\tilde{f}_\gamma, S)$ is an int-soft subsemigroup over U. □

Theorem 4.8 is illustrated by the following example.

Example 4.9. Consider the semigroup $S = \{a, b, c\}$ in Example 4.6 and let $(\tilde{f}, S)$ be a soft set over $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ defined by

$$\tilde{f} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} U & \text{if } x = a, \\ \{1, 2, 5, 7\} & \text{if } x = b, \\ \{1, 3, 6, 9\} & \text{if } x = c \end{cases}$$

Then $(\tilde{f}, S)$ is an int-soft subsemigroup over U. If we take $\gamma := \{2, 7\}$, then $(\tilde{f}, S)^{\supseteq}_{\gamma} = \{a, b\}$ and so we can make a soft set $(\tilde{f}_\gamma, S)$ over U as follows

$$\tilde{f}_\gamma : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} U & \text{if } x = a, \\ \{1, 2, 5, 7\} & \text{if } x = b, \\ \{1\} & \text{if } x = c \end{cases}$$

which is an int-soft subsemigroup over U.

The following example shows that the converse of Theorem 4.8 is not true in general, that is, for a soft set $(\tilde{f}, S)$ over U there exist $\gamma \in \mathscr{P}(U)$ such that $(\tilde{f}_\gamma, S)$ is an int-soft subsemigroup over U, but $(\tilde{f}, S)$ is not an int-soft subsemigroup over U.

Example 4.10. Let $S = \{a, b, c, d\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	a
c	c	c	c	c
d	c	c	c	d

Let $(\tilde{f}, S)$ be a soft set over $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} U & \text{if } x = a, \\ \{2, 4, 6, 8, 10\} & \text{if } x = b, \\ \{3, 6, 9\} & \text{if } x = c, \\ \{2, 6, 10\} & \text{if } x = d, \end{cases}$$

Then $(\tilde{f}, S)$ is not an int-soft subsemigroup over U since

$$\tilde{f}(db) = \tilde{f}(c) = \{3, 6, 9\} \not\supseteq \{2, 6, 10\} = \tilde{f}(d) \cap \tilde{f}(b).$$

For a subset $\gamma = \{3, 6, 9\}$ of U, we know that $(\tilde{f}, S)^{\supseteq}_{\gamma} = \{a, c\}$ is a subsemigroup of S. Hence the soft set $(\tilde{f}_{\gamma}, S)$ over U which is given by

$$\tilde{f}_{\gamma} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} U & \text{if } x = a, \\ \{3, 6, 9\} & \text{if } x = c, \\ \{6\} & \text{if } x \in \{b, d\}. \end{cases}$$

is an int-soft subsemigroup over U.

For any soft set $(\tilde{f}, S)$ over U and $\delta \in \mathscr{P}^*(U)$, let $\left(\tilde{f}_{\cap\delta}, S\right)$ be a soft set over U where

$$\tilde{f}_{\cap\delta} : S \to \mathscr{P}(U), \quad x \mapsto \tilde{f}(x) \cap \delta.$$

Theorem 4.11. *If $(\tilde{f}, S)$ is an int-soft subsemigroup over U, then so is $\left(\tilde{f}_{\cap\delta}, S\right)$ for all $\delta \in \mathscr{P}^*(U)$.*

Proof. For any $x, y \in S$ and $\delta \in \mathscr{P}^*(U)$, we have

$$\begin{aligned} \tilde{f}_{\cap\delta}(xy) &= \tilde{f}(xy) \cap \delta \supseteq \tilde{f}(x) \cap \tilde{f}(y) \cap \delta \\ &= \left(\tilde{f}(x) \cap \delta\right) \cap \left(\tilde{f}(y) \cap \delta\right) \\ &= \tilde{f}_{\cap\delta}(x) \cap \tilde{f}_{\cap\delta}(y). \end{aligned}$$

Hence $\left(\tilde{f}_{\cap\delta}, S\right)$ is an int-soft subsemigroup over U for all $\delta \in \mathscr{P}^*(U)$. $\square$

We pose a question as follows.

Question. *Let $(\tilde{f}, S)$ be a soft set over U such that $\left(\tilde{f}_{\cap\delta}, S\right)$ is an int-soft subsemigroup over U for some $\delta \in \mathscr{P}^*(U)$. Is $(\tilde{f}, S)$ an int-soft subsemigroup over U?*

The answer to the question above is No. In fact, let $(\tilde{f}, S)$ be a soft set over U which is not an int-soft subsemigroup over U. If we take $\delta = \bigcap_{x \in S} \tilde{f}(x)$, then $\left(\tilde{f}_{\cap\delta}, S\right)$ is an int-soft subsemigroup over U.

Example 4.12. Let $S = \{a, b, c, d\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	a
c	a	a	c	c
d	a	a	c	d

Let $(\tilde{f}, S)$ be a soft set over $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{1, 3, 5, 7, 9\} & \text{if } x = a \\ \{4, 7, 10\} & \text{if } x = b \\ \{1, 7, 10\} & \text{if } x = c \\ U & \text{if } x = d. \end{cases}$$

Then $(\tilde{f}, S)$ is not an int-soft subsemigroup over U since

$$\tilde{f}(cb) = \tilde{f}(a) = \{1, 3, 5, 7, 9\} \not\supseteq \{7, 10\} = \tilde{f}(c) \cap \tilde{f}(b).$$

For a subset $\delta := \{1, 7, 9\}$ of U, the soft set $\left(\tilde{f}_{\cap\delta}, S\right)$ over U is given by

$$\tilde{f}_{\cap\delta} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{1, 7, 9\} & \text{if } x \in \{a, d\} \\ \{7\} & \text{if } x = b \\ \{1, 7\} & \text{if } x = c \end{cases}$$

and it is an int-soft subsemigroup over U.

For any $A, B, C \in \mathscr{P}^*(U)$, we use the notation $[A; B, C]$ which means that the following condition holds:

$$A \cap B \subseteq A \cap C. \tag{4.2}$$

Denote by $\widetilde{\mathscr{P}(U)} := \{[A; B, C] \mid A, B, C \in \mathscr{P}^*(U)\}$.

Theorem 4.13. *Let $(\tilde{f}, S)$ be a soft set over U such that*

(1) $\mathrm{Im}(\tilde{f}, S)$ *is closed under "$\cap$".*
(2) $(\forall \delta \in \mathscr{P}^*(U)) \left(\forall A, B \in \mathrm{Im}(\tilde{f}, S)\right) \left([\delta; A, B] \in \widetilde{\mathscr{P}(U)} \;\Rightarrow\; A \subseteq B\right)$.

If $\left(\tilde{f}_{\cap\delta}, S\right)$ is an int-soft subsemigroup over U for some $\delta \in \mathscr{P}^*(U)$, then $(\tilde{f}, S)$ is an int-soft subsemigroup over U.

Proof. For any $x, y \in S$, we have

$$\begin{aligned} \tilde{f}(xy) \cap \delta &= \tilde{f}_{\cap\delta}(xy) \supseteq \tilde{f}_{\cap\delta}(x) \cap \tilde{f}_{\cap\delta}(y) \\ &= \left(\tilde{f}(x) \cap \delta\right) \cap \left(\tilde{f}(y) \cap \delta\right) \\ &= \left(\tilde{f}(x) \cap \tilde{f}(y)\right) \cap \delta \end{aligned}$$

and so $[\delta; \tilde{f}(x) \cap \tilde{f}(y), \tilde{f}(xy)] \in \widetilde{\mathscr{P}(U)}$. Theorefore $\tilde{f}(xy) \supseteq \tilde{f}(x) \cap \tilde{f}(y)$, and thus $(\tilde{f}, S)$ is an int-soft subsemigroup over U. □

We provide an example to illustrate Theorem 4.13.

Example 4.14. Let $S = \{a, b, c\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c
a	b	a	c
b	a	b	c
c	c	c	c

Let $(\tilde{f}, S)$ be a soft set over $U = \{1, 2, 3\}$ defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{3\} & \text{if } x = a, \\ \{1, 3\} & \text{if } x \in \{b, c\}. \end{cases}$$

Note that $\mathrm{Im}(\tilde{f}, S)$ is closed under "$\cap$". If we take $\delta = \{1, 3\}$, then the soft set $(\tilde{f}_{\cap\delta}, S)$ is given as follows:

$$\tilde{f}_{\cap\delta} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{3\} & \text{if } x = a, \\ \{1, 3\} & \text{if } x \in \{b, c\} \end{cases}$$

and it is an int-soft subsemigroup over U. Note that $[\delta; A, B] \in \widetilde{\mathscr{P}(U)}$ for $A, B \in \mathrm{Im}(\tilde{f})$ and $(\tilde{f}, S)$ is an int-soft subsemigroup over U.

5. θ-generalized int-soft subsemigroups

Definition 5.1. If a soft set $(\tilde{f}, S)$ over U satisfies the following assertion:

$$(\forall x, y \in S)\,(\exists \theta \in \mathscr{P}^*(U))\left(\tilde{f}(xy) \supseteq \theta \cap \tilde{f}(x) \cap \tilde{f}(y)\right), \tag{5.1}$$

then we say that $(\tilde{f}, S)$ is a *θ-generalized int-soft subsemigroup* over U.

Obviously, every int-soft subsemigroup is a θ-generalized int-soft subsemigroup for all $\theta \in \mathscr{P}^*(U)$. Also, if $\theta \in \mathscr{P}(U)$ satisfies $\tilde{f}(x) \subseteq \theta$ for all $x \in S$ then every θ-generalized int-soft subsemigroup $(\tilde{f}, S)$ over U is an int-soft subsemigroup over U.

For a soft set $(\tilde{f}, S)$ over U, we know that there exists nonempty subset θ of U such that $(\tilde{f}, S)$ is a θ-generalized int-soft subsemigroup, but not an int-soft subsemigroup as seen in the following example.

Example 5.2. Let $S = \{e, a, b\}$ be a semigroup with the following Cayley table:

$\cdot$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Let $(\tilde{f}, S)$ be a soft set over $U = \mathbb{Z}$ defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} 4\mathbb{Z} & \text{if } x = e, \\ 4\mathbb{N} & \text{if } x = a, \\ 8\mathbb{N} & \text{if } x = b \end{cases}$$

Then $(\tilde{f}, S)$ is $8\mathbb{N}$-generalized int-soft subsemigroup over U. But it is not an int-soft subsemigroup over U since $\tilde{f}(aa) = \tilde{f}(b) = 8\mathbb{N} \not\supseteq 4\mathbb{N} = \tilde{f}(a) \cap \tilde{f}(a)$.

Example 5.3. Let $S = \{a, b, c, d\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c	d
a	a	a	c	c
b	b	b	d	d
c	a	a	c	c
d	b	b	d	d

Let $(\tilde{f}, S)$ be a soft set over $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} U & \text{if } x = a, \\ \{2, 4, 6, 8, 10\} & \text{if } x = b \\ \{1, 2, 3, 4, 5\} & \text{if } x = c \\ \{2, 8\} & \text{if } x = d \end{cases}$$

Then $(\tilde{f}, S)$ is not an int-soft subsemigroup over U since

$$\tilde{f}(bc) = \tilde{f}(d) = \{2, 8\} \not\supseteq \{2, 4\} = \tilde{f}(b) \cap \tilde{f}(c).$$

But it is a θ-generalized int-soft subsemigroup over U with $\theta = \{2, 6, 10\}$.

Theorem 5.4. *Let $\theta \in \mathscr{P}^*(U)$. Then a soft set $(\tilde{f}, S)$ over U is a θ-generalized int-soft subsemigroup over U if and only if the nonempty γ-inclusive set $(\tilde{f}, S)^{\supseteq}_{\gamma}$ of $(\tilde{f}, S)$ is a subsemigroup of S for all $\gamma \in \mathscr{P}(U)$ with $\gamma \subseteq \theta$.*

Proof. Assume that $(\tilde{f}, S)$ is a θ-generalized int-soft subsemigroup over U. Let $x, y \in (\tilde{f}, S)^{\supseteq}_{\gamma}$ where $\gamma \in \mathscr{P}(U)$ with $\gamma \subseteq \theta$. Then $\tilde{f}(x) \supseteq \gamma$ and $\tilde{f}(y) \supseteq \gamma$. It follows from (5.1) that

$$\tilde{f}(xy) \supseteq \theta \cap \tilde{f}(x) \cap \tilde{f}(y) \supseteq \theta \cap \gamma = \gamma.$$

Hence $xy \in (\tilde{f}, S)^{\supseteq}_{\gamma}$, and $(\tilde{f}, S)^{\supseteq}_{\gamma}$ is a subsemigroup of S for all $\gamma \in \mathscr{P}(U)$ with $\gamma \subseteq \theta$.

Conversely, suppose that the nonempty γ-inclusive set $(\tilde{f}, S)^{\supseteq}_{\gamma}$ of $(\tilde{f}, S)$ is a subsemigroup of S for all $\gamma \in \mathscr{P}(U)$ with $\gamma \subseteq \theta$. Let $x, y \in S$ be such that $\tilde{f}(x) = \gamma_x$ and $\tilde{f}(y) = \gamma_y$. Take $\gamma = \theta \cap \gamma_x \cap \gamma_y$. Then $x, y \in (\tilde{f}, S)^{\supseteq}_{\gamma}$, and so $xy \in (\tilde{f}, S)^{\supseteq}_{\gamma}$. Hence

$$\tilde{f}(xy) \supseteq \gamma = \theta \cap \gamma_x \cap \gamma_y = \theta \cap \tilde{f}(x) \cap \tilde{f}(y).$$

Therefore $(\tilde{f}, S)$ is a θ-generalized int-soft subsemigroup over U. □

Corollary 5.5. *For $\theta \in \mathscr{P}^*(U)$, if a soft set $(\tilde{f}, S)$ over U is a θ-generalized int-soft subsemigroup over U, then the nonempty $\gamma \cap \theta$-inclusive set $(\tilde{f}, S)^{\supseteq}_{\gamma \cap \theta}$ of $(\tilde{f}, S)$ is a subsemigroup of S for all $\gamma \in \mathscr{P}(U)$.*

Proposition 5.6. *Let $(\tilde{f}, S)$ be a θ-generalized int-soft subsemigroup over U. Then*

$$(\forall x, y \in S) \left(\tilde{f}(x) \supseteq \theta,\ \tilde{f}(y) \supseteq \theta \ \Rightarrow \ \tilde{f}(xy) \supseteq \theta \right). \tag{5.2}$$

Proof. Straightforward. □

The following example shows that there exists a soft set $(\tilde{f}, S)$ over U such that

(i) The condition (5.2) is valid.

(ii) $(\tilde{f}, S)$ is not a θ-generalized int-soft subsemigroup over U.

Example 5.7. Let $S = \{a, b, c, d\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	c	c	c	c
d	a	b	c	d

Let $(\tilde{f}, S)$ be a soft set over $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{2, 5, 10\} & \text{if } x = a \\ \{1, 2, 3, 4, 5\} & \text{if } x = b \\ \{2, 4, 8\} & \text{if } x = c \\ U & \text{if } x = d. \end{cases}$$

If we take $\theta = \{3, 4\}$, then the condition (5.2) is valid. But $(\tilde{f}, S)$ is not θ-generalized int-soft subsemigroup over U since

$$\tilde{f}(bc) = \tilde{f}(a) = \{2, 5, 10\} \not\supseteq \{4\} = \theta \cap \tilde{f}(b) \cap \tilde{f}(c).$$

Theorem 5.8. *If $(\tilde{f}, S)$ is a θ-generalized int-soft subsemigroup over U, then the set*

$$S_a := \left\{ x \in S \mid \tilde{f}(x) \supseteq \theta \cap \tilde{f}(a) \right\}$$

is a subsemigroup of S for all $a \in S$.

Proof. Note that $a \in S_a$ for all $a \in S$. Let $x, y \in S_a$. Then $\tilde{f}(x) \supseteq \theta \cap \tilde{f}(a)$ and $\tilde{f}(y) \supseteq \theta \cap \tilde{f}(a)$. It follows from (5.1) that

$$\tilde{f}(xy) \supseteq \theta \cap \tilde{f}(x) \cap \tilde{f}(y) \supseteq \theta \cap \tilde{f}(a).$$

Thus $xy \in S_a$, and S_a is a subsemigroup of S for all $a \in S$. □

The following example shows that there exist $a \in S$ and a soft set $(\tilde{f}, S)$ over U such that

(i) The set S_a is a subsemigroup of S.
(ii) $(\tilde{f}, S)$ is not a θ-generalized int-soft subsemigroup over U.

Example 5.9. Let $S = \{e, a, b\}$ be a semigroup with the following Cayley table:

$\cdot$	e	a	b
e	e	e	e
a	e	a	e
b	e	e	b

Let $(\tilde{f}, S)$ be a soft set over $U = \{0, 1, 2, 3, 4, 5\}$ defined by

$$\tilde{f} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{1, 3, 5\} & \text{if } x = e \\ \{0, 1, 4\} & \text{if } x = a \\ \{3, 4\} & \text{if } x = b. \end{cases}$$

If $\theta = \{1, 4\}$, then $S_e = \{e, a\}$ is a subsemigroup of S. But $(\tilde{f}, S)$ is not a θ-generalized int-soft subsemigroup over U since

$$\tilde{f}(ab) = \tilde{f}(e) = \{1, 3, 5\} \not\supseteq \{4\} = \theta \cap \tilde{f}(a) \cap \tilde{f}(b).$$

Theorem 5.10. *The soft intersection of two θ-generalized int-soft subsemigroup over U is a θ-generalized int-soft subsemigroup over U.*

Proof. Let $(\tilde{f}, S)$ and $(\tilde{g}, S)$ be θ-generalized int-soft subsemigroups over U. For any $x, y \in S$, we have

$$\begin{aligned}
\left(\tilde{f} \tilde{\cap} \tilde{g}\right)(xy) &= \tilde{f}(xy) \cap \tilde{g}(xy) \\
&\supseteq \theta \cap \tilde{f}(xy) \cap \tilde{g}(xy) \\
&\supseteq \theta \cap \left((\theta \cap \tilde{f}(x) \cap \tilde{f}(y)) \cap (\theta \cap \tilde{g}(y) \cap \tilde{g}(y))\right) \\
&= \theta \cap \left(\tilde{f}(x) \cap \tilde{f}(y)\right) \cap (\tilde{g}(x) \cap \tilde{g}(y)) \\
&= \theta \cap \left(\tilde{f}(x) \cap \tilde{g}(x)\right) \cap \left(\tilde{f}(y) \cap \tilde{g}(y)\right) \\
&= \theta \cap \left(\tilde{f} \tilde{\cap} \tilde{g}\right)(x) \cap \left(\tilde{f} \tilde{\cap} \tilde{g}\right)(y).
\end{aligned}$$

Therefore $(\tilde{f}, S) \tilde{\cap} (\tilde{g}, S)$ is a θ-generalized int-soft subsemigroup over U. □

The following example shows that the soft union of two θ-generalized int-soft subsemigroup over U is not a θ-generalized int-soft subsemigroup over U in general.

Example 5.11. Let $S = \{a, b, c\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c
a	a	a	a
b	a	b	a
c	c	c	c

Let $(\tilde{f}, S)$ and $(\tilde{g}, S)$ be soft sets over $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{6\} & \text{if } x = a, \\ \{2, 7\} & \text{if } x = b, \\ \{1, 3, 5\} & \text{if } x = c, \end{cases}$$

and

$$\tilde{g} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{2, 6, 10\} & \text{if } x = a, \\ \{3, 6\} & \text{if } x = b, \\ \{7, 9\} & \text{if } x = c, \end{cases}$$

respectively. Then the soft union $(\tilde{f}, S) \tilde{\cup} (\tilde{g}, S)$ of $(\tilde{f}, S)$ and $(\tilde{g}, S)$ is given by

$$\tilde{f} \tilde{\cup} \tilde{g} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{2, 6, 10\} & \text{if } x = a, \\ \{2, 3, 6, 7\} & \text{if } x = b, \\ \{1, 3, 5, 7, 9\} & \text{if } x = c. \end{cases}$$

If we take $\theta = \{7, 10\}$, then $(\tilde{f}, S)$ and $(\tilde{g}, S)$ are θ-generalized int-soft subsemigroup over U, but the soft union $(\tilde{f}, S) \tilde{\cup} (\tilde{g}, S)$ is not a θ-generalized int-soft subsemigroup over U since

$$\left(\tilde{f} \tilde{\cup} \tilde{g}\right)(bc) = \left(\tilde{f} \tilde{\cup} \tilde{g}\right)(a) = \{2, 6, 10\} \not\supseteq \{7\} = \theta \cap \left(\tilde{f} \tilde{\cup} \tilde{g}\right)(b) \cap \left(\tilde{f} \tilde{\cup} \tilde{g}\right)(c).$$

We know that there exist soft sets $(\tilde{f}, S)$ and $(\tilde{g}, S)$ over U such that the soft union $(\tilde{f}, S) \tilde{\cup} (\tilde{g}, S)$ of $(\tilde{f}, S)$ and $(\tilde{g}, S)$ is a θ-generalized int-soft subsemigroup over U, but $(\tilde{f}, S)$ is not a θ-generalized int-soft subsemigroup over U or $(\tilde{g}, S)$ is not a θ-generalized int-soft subsemigroup over U.

Example 5.12. Let $S = \{a, b, c\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c
a	b	b	c
b	b	b	c
c	c	c	c

Let $(\tilde{f}, S)$ and $(\tilde{g}, S)$ be soft sets over $U = \{1, 2, 3\}$ defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} U & \text{if } x = a, \\ \{0, 2, 3\} & \text{if } x = b, \\ \{3\} & \text{if } x = c, \end{cases}$$

and

$$\tilde{g} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{3\} & \text{if } x = a, \\ \{1, 2\} & \text{if } x = b, \\ \{0, 3\} & \text{if } x = c, \end{cases}$$

respectively. Then the soft union $(\tilde{f}, S) \,\tilde{\cup}\, (\tilde{g}, S)$ of $(\tilde{f}, S)$ and $(\tilde{g}, S)$ is given by

$$\tilde{f} \,\tilde{\cup}\, \tilde{g} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} U & \text{if } x \in \{a, b\}, \\ \{0, 3\} & \text{if } x = c. \end{cases}$$

If we take $\theta = \{1, 3\}$, then the soft union $(\tilde{f}, S) \,\tilde{\cup}\, (\tilde{g}, S)$ is a θ-generalized int-soft subsemigroup over U. But $(\tilde{f}, S)$ and $(\tilde{g}, S)$ are not θ-generalized int-soft subsemigroups over U since

$$\tilde{f}(aa) = \tilde{f}(b) = \{0, 2, 3\} \not\supseteq \{1, 3\} = \theta \cap \tilde{f}(a) \cap \tilde{f}(a),$$

and

$$\tilde{g}(aa) = \tilde{g}(b) = \{1, 2\} \not\supseteq \{3\} = \theta \cap \tilde{g}(a) \cap \tilde{g}(a).$$

Theorem 5.13. *For every $\theta, \vartheta \in \mathscr{P}^*(U)$, if $\vartheta \subseteq \theta$ then every θ-generalized int-soft subsemigroup is a ϑ-generalized int-soft subsemigroup.*

Proof. Let $\theta, \vartheta \in \mathscr{P}^*(U)$ be such that $\vartheta \subseteq \theta$. Let $(\tilde{f}, S)$ be a θ-generalized int-soft subsemigroup over U. For any $x, y \in S$, we have

$$\tilde{f}(xy) \supseteq \theta \cap \tilde{f}(x) \cap \tilde{f}(y) \supseteq \vartheta \cap \tilde{f}(x) \cap \tilde{f}(y).$$

Therefore $(\tilde{f}, S)$ is a ϑ-generalized int-soft subsemigroup over U. □

The following example shows that there exist $\theta, \vartheta \in \mathscr{P}^*(U)$ such that

(i) $\vartheta \subseteq \theta$,
(ii) $(\tilde{f}, S)$ is a ϑ-generalized int-soft subsemigroup over U.
(iii) $(\tilde{f}, S)$ is not a θ-generalized int-soft subsemigroup over U.

Example 5.14. Let $S = \{a, b, c, d\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let $(\tilde{f}, S)$ be a soft set over U defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \gamma_1 & \text{if } x = a, \\ \gamma_4 & \text{if } x = b, \\ \gamma_2 & \text{if } x = c, \\ \gamma_3 & \text{if } x = d \end{cases}$$

where γ_1, γ_2, γ_3 and γ_4 are subsets of U with $\gamma_1 \supsetneq \gamma_2 \supsetneq \gamma_3 \supsetneq \gamma_4$. Take $\theta = \gamma_2$ and $\vartheta = \gamma_4$. Then $\vartheta \subseteq \theta$ and $(\tilde{f}, S)$ is a ϑ-generalized int-soft subsemigroup over U. But $(\tilde{f}, S)$ is not a θ-generalized int-soft subsemigroup over U since

$$\tilde{f}(dc) = \tilde{f}(b) = \gamma_4 \not\supseteq \gamma_3 = \theta \cap \tilde{f}(d) \cap \tilde{f}(c).$$

Corollary 5.15. *Let $\{\theta_i \in \mathscr{P}^*(U) \mid i \in \Lambda\}$ be a family of nonempty subsets of U. If a soft set $(\tilde{f}, S)$ over U is a θ_i-generalized int-soft subsemigroup over U for all $i \in \Lambda$, then $(\tilde{f}, S)$ is a $\cap_{i\in\Lambda}\theta_i$-generalized int-soft subsemigroup over U.*

Proof. Straightforward. □

Theorem 5.16. *For a subset A of S, define a soft set $(\tilde{f}_A, S)$ over U as follows:*

$$\tilde{f}_A : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \gamma \cap \theta & \text{if } x \in A, \\ \tau & \text{otherwise} \end{cases}$$

where $\gamma, \tau \in \mathscr{P}(U)$ with $\tau \subsetneq \gamma \cap \theta$. Then $(\tilde{f}_A, S)$ is a θ-generalized int-soft subsemigroup over U if and only if A is a subsemigroup of S.

Proof. Assume that $(\tilde{f}_A, S)$ is a θ-generalized int-soft subsemigroup over U. Let $x, y \in A$. Then $\tilde{f}_A(xy) \supseteq \theta \cap \tilde{f}_A(x) \cap \tilde{f}_A(y) = \theta \cap (\gamma \cap \theta) = \gamma \cap \theta$, and so $xy \in A$. Thus A is a subsemigroup of S.

Conversely, suppose that A is a subsemigroup of S. Let $x, y \in S$. If $x, y \in A$, then $xy \in A$. Hence $\tilde{f}_A(xy) = \gamma \cap \theta = \tilde{f}_A(x) \cap \tilde{f}_A(y)$. If $x \notin A$ or $y \notin A$, then $\tilde{f}_A(x) = \tau$ or $\tilde{f}_A(y) = \tau$. Hence $\tilde{f}_A(xy) \supseteq \tau = \tilde{f}_A(x) \cap \tilde{f}_A(y)$. Therefore $(\tilde{f}_A, S)$ is an int-soft subsemigroup over U, and so a θ-generalized int-soft subsemigroup over U. □

The following example illustrate Theorem 5.16.

Example 5.17. Let $S = \{a, b, c\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c
a	a	a	a
b	a	b	a
c	c	c	c

Note that $A = \{a, b\}$ is a subsemigroup of S. If we take $\theta = \{1, 2, 6, 10\}$, $\gamma = \{2, 4, 6, 8, 10\}$ and $\tau = \{2, 10\}$, then the soft set $(\tilde{f}_A, S)$ over $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ which is given by

$$\tilde{f}_A : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{2, 6, 10\} & \text{if } x \in \{a, b\}, \\ \{2, 10\} & \text{if } x = c, \end{cases}$$

is a θ-generalized int-soft subsemigroup over U.

For any semigroups S and T, let $\mu : S \to T$ be a function and $(\tilde{f}, S)$ and $(\tilde{g}, T)$ be soft sets over U.

(1) The *soft pre-image* of $(\tilde{g}, T)$ under μ is denoted by $\mu^{-1}(\tilde{g}, T)$ and is defined to be the soft set

$$\left(\mu^{-1}(\tilde{g}), T\right) = \left\{\left(x, \mu^{-1}(\tilde{g})(x)\right) : x \in S,\, \mu^{-1}(\tilde{g})(x) \in \mathscr{P}(U)\right\},$$

where $\mu^{-1}(\tilde{g})(x) = \tilde{g}(\mu(x))$.

(2) The *soft image* of $(\tilde{f}, S)$ under μ is denoted by $\mu(\tilde{f}, S)$ and is defined to be the soft set

$$\left(\mu(\tilde{f}), S\right) = \left\{\left(y, \mu(\tilde{f})(y)\right) : y \in T,\, \mu(\tilde{f})(y) \in \mathscr{P}(U)\right\}$$

where

$$\mu(\tilde{f})(y) = \begin{cases} \bigcup\limits_{x \in \mu^{-1}(y)} \tilde{f}(x) & \text{if } \mu^{-1}(y) \neq \varnothing, \\ \varnothing & \text{otherwise.} \end{cases}$$

Theorem 5.18. *Let $\mu : S \to T$ be a homomorphism of semigroups and $(\tilde{g}, T)$ a soft set over U. If $(\tilde{g}, T)$ is a θ-generalized int-soft subsemigroup over U, then the soft pre-image $\mu^{-1}(\tilde{g}, T)$ of $(\tilde{g}, T)$ under μ is also a θ-generalized int-soft subsemigroup over U.*

Proof. For any $x_1, x_2 \in S$, we have

$$\begin{aligned} \mu^{-1}(\tilde{g})(x_1x_2) &= \tilde{g}\left(\mu(x_1x_2)\right) \\ &= \tilde{g}\left(\mu(x_1)\mu(x_2)\right) \\ &\supseteq \theta \cap \tilde{g}(\mu(x_1)) \cap \tilde{g}(\mu(x_2)) \\ &= \theta \cap \mu^{-1}(\tilde{g})(x_1) \cap \mu^{-1}(\tilde{g})(x_2) \end{aligned}$$

Hence $\mu^{-1}(\tilde{g}, T)$ is also a θ-generalized int-soft subsemigroup over U. □

Theorem 5.18 is illustrated by the following example.

Example 5.19. Consider two semigroups $S = \{a_1, a_2, a_3\}$ and $T = \{b_1, b_2, b_3, b_4\}$ with the following Cayley tables, respectively.

$\cdot$	a_1	a_2	a_3
a_1	a_1	a_1	a_1
a_2	a_1	a_2	a_3
a_3	a_3	a_3	a_3

$\cdot$	b_1	b_2	b_3	b_4
b_1	b_1	b_1	b_1	b_1
b_2	b_1	b_2	b_2	b_4
b_3	b_1	b_3	b_3	b_4
b_4	b_4	b_4	b_4	b_4

The function

$$\mu : S \to T, \quad x \mapsto \begin{cases} b_1 & \text{if } x = a_1, \\ b_3 & \text{if } x = a_2, \\ b_4 & \text{if } x = a_3, \end{cases}$$

is a homomorphism. Let $(\tilde{g}, T)$ be a soft set over $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ which is given by

$$\tilde{g} : T \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{4, 8\} & \text{if } x = b_1, \\ \{2, 4, 7\} & \text{if } x = b_2, \\ \{1, 7, 10\} & \text{if } x = b_3, \\ U & \text{if } x = b_4. \end{cases}$$

Then $(\tilde{g}, T)$ is a θ-generalized int-soft subsemigroup over U with $\theta = \{4, 7, 10\}$, and the soft pre-image $\mu^{-1}(\tilde{g}, T)$ of $(\tilde{g}, T)$ under μ is described as follows:

$$\mu^{-1}(\tilde{g}) : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{4, 8\} & \text{if } x = a_1, \\ \{1, 7, 10\} & \text{if } x = a_2, \\ U & \text{if } x = a_3, \end{cases}$$

and it is a θ-generalized int-soft subsemigroup over U with $\theta = \{4, 7, 10\}$.

The following example shows that the converse of Theorem 5.18 may not be true.

Example 5.20. Consider two semigroups $S = \{a_1, a_2, a_3\}$ and $T = \{b_1, b_2, b_3, b_4\}$ with the following Cayley tables, respectively.

$\cdot$	a_1	a_2	a_3
a_1	a_1	a_1	a_1
a_2	a_2	a_2	a_2
a_3	a_3	a_3	a_3

$\cdot$	b_1	b_2	b_3	b_4
b_1	b_2	b_2	b_2	b_3
b_2	b_2	b_2	b_2	b_2
b_3	b_3	b_3	b_3	b_3
b_4	b_4	b_4	b_4	b_4

The function

$$\mu : S \to T, \quad x \mapsto \begin{cases} b_2 & \text{if } x = a_1, \\ b_3 & \text{if } x = a_2, \\ b_4 & \text{if } x = a_3, \end{cases}$$

is a homomorphism. Let $(\tilde{g}, T)$ be a soft set over $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ which is given by

$$\tilde{g} : T \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{2, 7, 10\} & \text{if } x = b_1, \\ \{2, 5\} & \text{if } x = b_2, \\ U & \text{if } x = b_3, \\ \{3, 5, 7, 10\} & \text{if } x = b_4. \end{cases}$$

Then $(\tilde{g}, T)$ is not a θ-generalized int-soft subsemigroup over U with $\theta = \{2, 5, 10\}$ since

$$\tilde{g}(b_1 b_1) = \tilde{g}(b_2) = \{2, 5\} \not\supseteq \{2, 10\} = \theta \cap \tilde{g}(b_1) \cap \tilde{g}(b_1).$$

Now the soft pre-image $\mu^{-1}(\tilde{g}, T)$ of $(\tilde{g}, T)$ under μ is described as follows:

$$\mu^{-1}(\tilde{g}) : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{2, 5\} & \text{if } x = a_1, \\ U & \text{if } x = a_2, \\ \{3, 5, 7, 10\} & \text{if } x = a_3, \end{cases}$$

and it is a θ-generalized int-soft subsemigroup over U with $\theta = \{2, 5, 10\}$.

Theorem 5.21. *Let $\mu : S \to T$ be a homomorphism of semigroups and $(\tilde{f}, S)$ a soft set over U. If $(\tilde{f}, S)$ is a θ-generalized int-soft subsemigroup over U, then the soft image $\mu(\tilde{f}, S)$ of $(\tilde{f}, S)$ under μ is also a θ-generalized int-soft subsemigroup over U.*

Proof. Let $y_1, y_2 \in T$. If at least one of $\mu^{-1}(y_1)$ and $\mu^{-1}(y_2)$ is empty, then the inclusion

$$\theta \cap \mu(\tilde{f})(y_1) \cap \mu(\tilde{f})(y_2) \subseteq \mu(\tilde{f})(y_1 y_2)$$

is clear. Assume that $\mu^{-1}(y_1) \neq \varnothing$ and $\mu^{-1}(y_2) \neq \varnothing$. Then $\mu^{-1}(y_1 y_2) \neq \varnothing$, and so

$$\begin{aligned}
\mu(\tilde{f})(y_1 y_2) &= \bigcup_{x \in \mu^{-1}(y_1 y_2)} \tilde{f}(x) \supseteq \bigcup_{\substack{x_1 \in \mu^{-1}(y_1) \\ x_2 \in \mu^{-1}(y_2)}} \left(\tilde{f}(x_1 x_2)\right) \\
&\supseteq \bigcup_{\substack{x_1 \in \mu^{-1}(y_1) \\ x_2 \in \mu^{-1}(y_2)}} \left(\theta \cap \tilde{f}(x_1) \cap \tilde{f}(x_2)\right) \\
&= \theta \cap \left(\bigcup_{x_1 \in \mu^{-1}(y_1)} \tilde{f}(x_1)\right) \cap \left(\bigcup_{x_2 \in \mu^{-1}(y_2)} \tilde{f}(x_2)\right) \\
&= \theta \cap \mu(\tilde{f})(y_1) \cap \mu(\tilde{f})(y_2).
\end{aligned}$$

Therefore $\mu(\tilde{f}, S)$ is a θ-generalized int-soft subsemigroup over U. □

Theorem 5.21 is illustrated by the following example.

Example 5.22. Consider two semigroups $S = \{a_1, a_2, a_3\}$ and $T = \{b_1, b_2, b_3\}$ with the following Cayley tables, respectively:

$\cdot$	a_1	a_2	a_3
a_1	a_2	a_2	a_2
a_2	a_2	a_2	a_2
a_3	a_2	a_2	a_3

$\cdot$	b_1	b_2	b_3
b_1	b_2	b_2	b_2
b_2	b_2	b_2	b_2
b_3	b_3	b_3	b_3

The function

$$\mu : S \to T, \quad x \mapsto \begin{cases} b_1 & \text{if } x = a_1, \\ b_2 & \text{if } x \in \{a_2, a_3\}, \end{cases}$$

is a homomorphism. Let $(\tilde{f}, S)$ be a soft set over $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ which is given by

$$\tilde{f} : S \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{7\} & \text{if } x = a_1, \\ \{2, 7\} & \text{if } x = a_2, \\ \{1, 2, 8, 10\} & \text{if } x = a_3. \end{cases}$$

Then $(\tilde{f}, S)$ is a θ-generalized int-soft subsemigroup over U with $\theta = \{7, 10\}$. Now the soft image $\mu(\tilde{f}, S)$ of $(\tilde{f}, S)$ under μ is described as follows:

$$\mu(\tilde{f}) : T \to \mathscr{P}(U), \quad x \mapsto \begin{cases} \{7\} & \text{if } y = b_1, \\ \{1, 2, 7, 8, 10\} & \text{if } y = b_2, \\ \varnothing & \text{if } y = b_3, \end{cases}$$

and it is a θ-generalized int-soft subsemigroup over U with $\theta = \{7, 10\}$.

Theorem 5.23. *If $(\tilde{f}, S)$ is a θ-generalized int-soft subsemigroup over U, then $\left(\tilde{f}_{\cap\delta}, S\right)$ is a $(\theta \cap \delta)$-generalized int-soft subsemigroup over U for all $\delta \in \mathscr{P}^*(U)$.*

Proof. Let $x, y \in S$. Then

$$\begin{aligned}\tilde{f}_{\cap\delta}(xy) &= \tilde{f}(xy) \cap \delta \supseteq \theta \cap \tilde{f}(x) \cap \tilde{f}(y) \cap \delta \\ &= (\theta \cap \delta) \cap \left(\tilde{f}(x) \cap \delta\right) \cap \left(\tilde{f}(y) \cap \delta\right) \\ &= (\theta \cap \delta) \cap \tilde{f}_{\cap\delta}(x) \cap \tilde{f}_{\cap\delta}(y)\end{aligned}$$

and thus $\left(\tilde{f}_{\cap\delta}, S\right)$ is a $(\theta \cap \delta)$-generalized int-soft subsemigroup over U for all $\delta \in \mathscr{P}^*(U)$. □

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