Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 8, No. 6, (December 2014), pp. 1027–1044
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

On L-fuzzy soft semigroups

Muhammad Shabir, Mamoona Ghafoor

Received 3 December 2013; Accepted 1 Jun 2014

Abstract. In this paper we introduced the concepts of L-fuzzy soft left (right, two-sided) ideals of a semigroup over a universe U, where L is a complete bounded distributive lattice. We also studdied some properties of L-fuzzy soft left (right, two-sided) ideals of a semigroup over a universe U. Regular and intra-regular semigroups are characterized by the properties of theses L-fuzzy soft left (right) ideals.

2010 AMS Classification: 20M12, 20M17

Keywords: L-fuzzy soft subsemigroup, L-fuzzy soft left (right, two-sided) ideal, simple semigroups, regular semigroups.

Corresponding Author: Muhammad Shabir (`mshabirbhatti@yahoo.co.uk`)

1. Introduction

The concept of soft set was initiated by Molodtsov [11] in 1999, to handle the ambiguousness and uncertainty, that were not handled by old classical methods. He has given a number of applications of soft sets in the field of economics, engineering, social science and medical science etc. Maji et al. [9] introduced some basic operations of soft sets. Ali et al. [1] also worked on the operations of soft sets. They improved some already defined operations and introduced some new operations. Sezgin and Atagün [13] and Ali et al. [3] also worked on the operations of soft set. Feng et al. [4] and Feng et al. [5] worked on the combination of fuzzy sets, rough sets and soft sets. Sezgin et al. [14] introduced the notions of soft-int ideals and soft-int bi-ideal of a ring. Maji et al. [10] initiated the study of fuzzy soft set by combining the concepts of fuzzy sets and soft sets. Many authors worked on fuzzy soft sets, e.g. [2, 12, 6, 15, 16].

Goguen [7] was the first who gave the concept of L-fuzzy sets by generalizing Zadeh's fuzzy set. Recently Li, Zheng and Hao worked on L-fuzzy soft sets based on complete Boolean lattice [8]. They discussed topological and algebraic structures of L-fuzzy soft sets.

In this paper we defined L-fuzzy soft subsemigroup, L-fuzzy soft left (right, two-sided) ideal of semigroups over a universe U and studied some properties of L-fuzzy soft subsemigroups and L-fuzzy soft left (right, two-sided) ideals of semigroups over a universe U. We characterized different classes of semigroups by the properties of these L-fuzzy soft ideals.

2. Preliminaries

An algebraic system $(S, \cdot)$ consisting of a non-empty set S together with an associative binary operation "$\cdot$" is called a semigroup. By a subsemigroup of a semigroup S we mean a non-empty subset A of S such that $A^2 \subseteq A$. A non-empty subset A of a semigroup S is called a left (right) ideal of S if $SA \subseteq A$ $(AS \subseteq A)$. A non-empty subset A of S is called a two-sided ideal or simply an ideal of S if it is both a left and a right ideal of S.

A partially ordered set (poset) $(L, \leq)$ is called

1) a lattice, if $a \vee b \in L$, $a \wedge b \in L$ for any $a, b \in L$.

2) a complete lattice, if $\vee N \in L$, $\wedge N \in L$ for any $N \subseteq L$.

3) distributive, if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for any $a, b, c \in L$.

Let L be a lattice with top element 1_L and bottom element 0_L and let a, $b \in L$. Then b is called a complement of a, if $a \vee b = 1_L$ and $a \wedge b = 0_L$. If $a \in L$ has complement element, then it is unique. It is denoted by $a^{'}$.

A lattice L is called a Boolean lattice, if

(i) L is distributive,

(ii) L has 0_L and 1_L,

(iii) each $a \in L$ has the complement $a^{'} \in L$.

Let X be a non-empty set. A fuzzy set A in X is a function, $A : X \to [0, 1]$ and $A(x)$ is interpreted as the degree of membership of element x in the fuzzy set A for each $x \in X$.

In [7], Goguen generalized the concept of fuzzy set and introduced L-fuzzy set as: An L-fuzzy set A in a non-empty set X is a function $A : X \to L$, where L is a complete distributive lattice with 1 and 0. We denote by L^X the set of all L-fuzzy sets in X.

Let $A, B \in L^X$. Then their union and intersection are L-fuzzy sets in X, defined as

$(A \cup B)(x) = A(x) \vee B(x)$ and $(A \cap B)(x) = A(x) \wedge B(x)$ for all $x \in X$.

$A \subseteq B$ if and only if $A(x) \leq B(x)$ for all $x \in X$.

The L-fuzzy sets $\widehat{0}$ and $\widehat{1}$ of X are defined as $\widehat{0}(x) = 0$ and $\widehat{1}(x) = 1$ for all $x \in X$. Obviously $\widehat{0} \subseteq A \subseteq \widehat{1}$ for all $A \in L^X$.

A pair (F, E) is called a soft set (over U) if F is a mapping of E into the power set of U, that is $F : E \to P(U)$.

In other words, the soft set is a parametrized family of subsets of the set U [11].

Definition 2.1 ([8])**.** Let E be a set of parameters, U be an initial universe, L be a complete Boolean lattice and $A \subseteq E$. An L-fuzzy soft set f_A over U is a mapping $f_A : E \to L^U$ such that $f_A(e) = \widehat{0}$ for all $e \notin A$.

The following operations on L-fuzzy soft sets are defined in [8],

1) Let f_A and g_B be two L-fuzzy soft sets over U. Then f_A is contained in g_B denoted by $f_A \widetilde{\subseteq} g_B$ if $f_A(e) \subseteq g_B(e)$ for all $e \in E$, that is $(f_A(e))(u) \leq (g_B(e))(u)$ for all $u \in U$.

Two L-fuzzy soft sets f_A and g_B over U are said to be equal, denoted by $f_A \widetilde{=} g_B$ if $f_A \widetilde{\subseteq} g_B$ and $f_A \widetilde{\supseteq} g_B$.

2) Let f_A and g_B be two L-fuzzy soft sets over U. Then their union $f_A \widetilde{\cup} g_B \widetilde{=} h_{A\cup B}$, where $h_{A\cup B}(e) = f_A(e) \cup g_B(e)$ for all $e \in E$.

3) Let f_A and g_B be two L-fuzzy soft sets over U. Then their intersection $f_A \widetilde{\cap} g_B \widetilde{=} h_{A\cap B}$, where $h_{A\cap B}(e) = f_A(e) \cap g_B(e)$ for all $e \in E$.

Proposition 2.2 ([8]). *Let A, B, $C \subseteq E$ and f_A, g_B, h_C are L-fuzzy soft sets over U. Then the following holds:*

1) $f_A \widetilde{\cup} f_A \widetilde{=} f_A$, $f_A \widetilde{\cap} f_A \widetilde{=} f_A$

2) $f_A \widetilde{\cup} g_B \widetilde{=} g_B \widetilde{\cup} f_A$, $f_A \widetilde{\cap} g_B \widetilde{=} g_B \widetilde{\cap} f_A$

3) $(f_A \widetilde{\cup} g_B) \widetilde{\cup} h_C \widetilde{=} f_A \widetilde{\cup} (g_B \widetilde{\cup} h_C)$, $(f_A \widetilde{\cap} g_B) \widetilde{\cap} h_C \widetilde{=} f_A \widetilde{\cap} (g_B \widetilde{\cap} h_C)$

4) $(f_A \widetilde{\cup} g_B) \widetilde{\cap} h_C \widetilde{=} (f_A \widetilde{\cap} h_C) \widetilde{\cup} (g_B \widetilde{\cap} h_C)$, $(f_A \widetilde{\cap} g_B) \widetilde{\cup} h_C \widetilde{=} (f_A \widetilde{\cup} h_C) \widetilde{\cap} (g_B \widetilde{\cup} h_C)$.

3. L-FUZZY SOFT SETS OF SEMIGROUPS

In this section we define product of L-fuzzy soft sets of a semigroup S over U and study some properties of this product. Throughout this paper L is a complete bounded distributive lattice and U is the initial universe and the set of parameters is a semigroup S.

Definition 3.1. Let A be a non-empty subset of a semigroup S. Define an L-fuzzy soft set C_A of S over U by

$$C_A(x) = \begin{cases} \widehat{1} & \text{if } x \in A \\ \widehat{0} & \text{if } x \notin A \end{cases}$$

for all $x \in S$. We shall call this L-fuzzy soft set the L-fuzzy soft characteristic function of A.

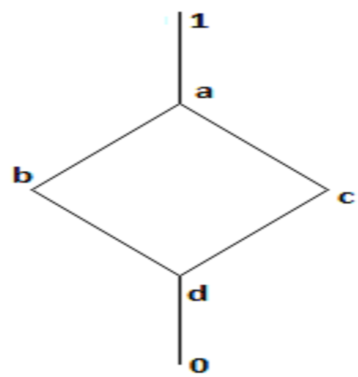
Definition 3.2. Let f_A and g_B be two L-fuzzy soft sets of a semigroup S over U. Then their product $f_A \circledcirc g_B$ is an L-fuzzy soft set of S over U and is defined as

$$(f_A \circledcirc g_B)(x) = \begin{cases} \cup_{x=yz}\{f_A(y) \cap g_B(z)\}, & if\ \exists\ y, z \in S \text{ such that } x = yz \\ \widehat{0} & \text{otherwise} \end{cases}$$

We explain this concept with the help of an example.

Example 3.3. Let $S = \{x, y, z\}$ be a semigroup , $L = \{0, a, b, c, d, 1\}$ be a complete bounded distributive lattice, $U = \{p, q\}$ and $A = \{x, y\}$, $B = \{x, z\}$ are subsets of S.

$*$	x	y	z
x	x	x	x
y	y	y	y
z	z	z	z



Let f_A, g_B be L-fuzzy soft sets of S over U, defined by

$$f_A(x) = \{\frac{p}{b}, \frac{q}{d}\}, f_A(y) = \{\frac{p}{a}, \frac{q}{b}\}, f_A(z) = \{\frac{p}{0}, \frac{q}{0}\},$$

$$g_B(x) = \{\frac{p}{1}, \frac{q}{0}\}, g_B(y) = \{\frac{p}{0}, \frac{q}{0}\}, g_B(z) = \{\frac{p}{b}, \frac{q}{1}\}.$$

Now for $x \in S$, we have

$$\begin{aligned}(f_A \circledcirc g_B)(x) &= \cup_{x=bc}[f_A(b) \cap g_B(c)] \\ &= \cup\{f_A(x) \cap g_B(x), f_A(x) \cap g_B(y), f_A(x) \cap g_B(z)\} \\ &= \cup\{\{\frac{p}{b}, \frac{q}{d}\} \cap \{\frac{p}{1}, \frac{q}{0}\}, \{\frac{p}{b}, \frac{q}{d}\} \cap \{\frac{p}{0}, \frac{q}{0}\}, \{\frac{p}{b}, \frac{q}{d}\} \cap \{\frac{p}{b}, \frac{q}{1}\}\} \\ &= \cup\{\{\frac{p}{b}, \frac{q}{0}\}, \{\frac{p}{0}, \frac{q}{0}\}, \{\frac{p}{b}, \frac{q}{d}\}\}.\end{aligned}$$

$\Longrightarrow (f_A \circledcirc g_B)(x) = \{\frac{p}{b}, \frac{q}{d}\}$.

For $y \in S$, we have

$$\begin{aligned}(f_A \circledcirc g_B)(y) &= \cup_{y=bc}[f_A(b) \cap g_B(c)] \\ &= \cup\{f_A(y) \cap g_B(x), f_A(y) \cap g_B(y), f_A(y) \cap g_B(z)\} \\ &= \cup\{\{\frac{p}{a}, \frac{q}{b}\} \cap \{\frac{p}{1}, \frac{q}{0}\}, \{\frac{p}{a}, \frac{q}{b}\} \cap \{\frac{p}{0}, \frac{q}{0}\}, \{\frac{p}{a}, \frac{q}{b}\} \cap \{\frac{p}{b}, \frac{q}{1}\}\} \\ &= \cup\{\{\frac{p}{a}, \frac{q}{0}\}, \{\frac{p}{0}, \frac{q}{0}\}, \{\frac{p}{b}, \frac{q}{b}\}\}.\end{aligned}$$

$\Longrightarrow (f_A \circledcirc g_B)(y) = \{\frac{p}{a}, \frac{q}{b}\}$.

For $z \in S$, we have

$$\begin{aligned}(f_A \circledcirc g_B)(z) &= \cup_{z=bc}[f_A(b) \cap g_B(c)] \\ &= \cup\{f_A(z) \cap g_B(x), f_A(z) \cap g_B(y), f_A(z) \cap g_B(z)\} \\ &= \cup\{\{\frac{p}{0}, \frac{q}{0}\} \cap \{\frac{p}{1}, \frac{q}{0}\}, \{\frac{p}{0}, \frac{q}{0}\} \cap \{\frac{p}{0}, \frac{q}{0}\}, \{\frac{p}{0}, \frac{q}{0}\} \cap \{\frac{p}{b}, \frac{q}{1}\}\} \\ &= \cup\{\{\frac{p}{0}, \frac{q}{0}\}, \{\frac{p}{0}, \frac{q}{0}\}, \{\frac{p}{0}, \frac{q}{0}\}\}.\end{aligned}$$

$\Longrightarrow (f_A \circledcirc g_B)(z) = \{\frac{p}{0}, \frac{q}{0}\}$.

Lemma 3.4. *Let A and B be non-empty subsets of a semigroup S. Then*
(1) $C_A \widetilde{\cap} C_B \widetilde{=} C_{A\cap B}$
(2) $C_A \circledcirc C_B \widetilde{=} C_{AB}$.

Proof. (1) Let $a \in S$. If $a \in A \cap B$ then $C_{A\cap B}(a) = \widehat{1}$. On the other hand $a \in A$ and $a \in B$, so $C_A(a) = \widehat{1}$ and $C_B(a) = \widehat{1}$. Thus $\left(C_A \widetilde{\cap} C_B\right)(a) = C_A(a) \cap C_B(a) = \widehat{1} \cap \widehat{1} = \widehat{1}$. Hence $C_A \widetilde{\cap} C_B \widetilde{=} C_{A\cap B}$.

If $a \notin A\cap B$ then $C_{A\cap B}(a) = \widehat{0}$. On the other hand $a \notin A$ or $a \notin B$, so $C_A(a) = \widehat{0}$ or $C_B(a) = \widehat{0}$. Thus $\left(C_A \widetilde{\cap} C_B\right)(a) = C_A(a) \cap C_B(a) = \widehat{0} \cap \widehat{0} = \widehat{0}$. Hence in any case $C_A \widetilde{\cap} C_B \widetilde{=} C_{A\cap B}$.

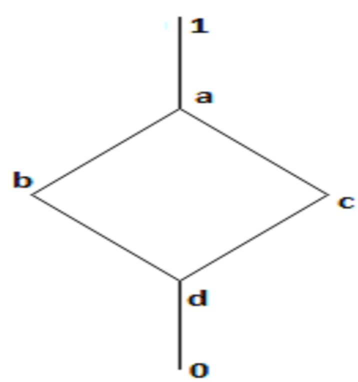
(2) Let $a \in S$. If $a \in AB$ then $a = xy$ for some $x \in A$ and $y \in B$. So we have $(C_{AB})(a) = \widehat{1}$. On the other hand $(C_A \circledcirc C_B)(a) = \cup_{a=uv}\left[C_A(u) \cap C_B(v)\right] \supseteq C_A(x) \cap C_B(y) = \widehat{1} \cap \widehat{1} = \widehat{1}$.

If $a \notin AB$ then there does not exist $x \in A$ and $y \in B$ such that $a = xy$. Thus $C_{AB}(a) = \widehat{0}$ and $(C_A \circledcirc C_B)(a) = \cup_{a=uv}\left[C_A(u) \cap C_B(v)\right] = \cup_{a=uv}\left[\widehat{0} \cap \widehat{0}\right] = \widehat{0}$. Hence in any case $C_A \circledcirc C_B \widetilde{=} C_{AB}$. □

Next we show that the operation $\circledcirc$ is not commutative.

Example 3.5. Let $S = \{x, y, z\}$ be a semigroup, $L = \{0, a, b, c, d, 1\}$ be a complete bounded distributive lattice, $U = \{p, q\}$ and $A = \{x, y\}$, $B = \{x, z\}$ are subsets of S.

$*$	x	y	z
x	x	x	x
y	y	y	y
z	z	z	z



Let f_A, g_B be L-fuzzy soft sets of S over U defined by,

$$\begin{aligned} f_A(x) &= \{\frac{p}{b}, \frac{q}{d}\}, f_A(y) = \{\frac{p}{a}, \frac{q}{b}\}, f_A(z) = \{\frac{p}{0}, \frac{q}{0}\}, \\ g_B(x) &= \{\frac{p}{1}, \frac{q}{0}\}, g_B(y) = \{\frac{p}{0}, \frac{q}{0}\}, g_B(z) = \{\frac{p}{1}, \frac{q}{b}\}. \end{aligned}$$

Now for $x \in S$ we have

$$\begin{aligned}(f_A \circledcirc g_B)(x) &= \cup_{x=yz}[f_A(y) \cap g_B(z)] \\ &= \cup\{f_A(x) \cap g_B(x), f_A(x) \cap g_B(y), f_A(x) \cap g_B(z)\}\end{aligned}$$

$\Longrightarrow (f_A \circledcirc g_B)(x) = \{\frac{p}{b}, \frac{q}{d}\}$.

On the other hand

$$\begin{aligned}(g_B \circledcirc f_A)(x) &= \cup_{x=yz}[g_B(y) \cap f_A(z)] \\ &= \cup\{g_B(x) \cap f_A(x), g_B(x) \cap f_A(y), g_B(x) \cap f_A(z)\}\end{aligned}$$

$\Longrightarrow (g_B \circledcirc f_A)(x) = \{\frac{p}{a}, \frac{q}{0}\}$.

This shows that $f_A \circledcirc g_B \widetilde{\neq} g_B \circledcirc f_A$.

Lemma 3.6. *Let f_A, g_B and h_C be L-fuzzy soft sets of a semigroup S over U. Then the following hold:*

(1) $f_A \circledcirc (g_B \circledcirc h_C) \widetilde{=} (f_A \circledcirc g_B) \circledcirc h_C$

(2) $f_A \circledcirc \left(g_B \widetilde{\cup} h_C\right) \widetilde{=} (f_A \circledcirc g_B) \widetilde{\cup} (f_A \circledcirc h_C)$

(3) $\left(g_B \widetilde{\cup} h_C\right) \circledcirc f_A \widetilde{=} (g_B \circledcirc f_A) \widetilde{\cup} (h_C \circledcirc f_A)$

(4) $f_A \circledcirc \left(g_B \widetilde{\cap} h_C\right) \widetilde{\subseteq} (f_A \circledcirc g_B) \widetilde{\cap} (f_A \circledcirc h_C)$

(5) $\left(f_A \widetilde{\cap} g_B\right) \circledcirc h_C \widetilde{\subseteq} (f_A \circledcirc h_C) \widetilde{\cap} (g_B \circledcirc h_C)$.

Proof. (1) Let $x \in S$. Then

$$\begin{aligned}[f_A \circledcirc (g_B \circledcirc h_C)](x) &= \cup_{x=yz}\{f_A(y) \cap (g_B \circledcirc h_C)(z)\} \\ &= \cup_{x=yz}\{f_A(y) \cap [\cup_{z=pq} g_B(p) \cap h_C(q)]\} \\ &= \cup_{x=yz} \cup_{z=pq} \{f_A(y) \cap [g_B(p) \cap h_C(q)]\} \\ &= \cup_{x=yz} \cup_{z=pq} \{[f_A(y) \cap g_B(p)] \cap h_C(q)\} \\ &\subseteq \cup_{x=lm}\{\cup_{l=ab}[f_A(a) \cap g_B(b)] \cap h_C(m)\} \\ &= \cup_{x=lm}\{(f_A \circledcirc g_B)(l) \cap h_C(m)\} \\ &= [(f_A \circledcirc g_B) \circledcirc h_C](x).\end{aligned}$$

This implies that $f_A \circledcirc (g_B \circledcirc h_C) \widetilde{\subseteq} (f_A \circledcirc g_B) \circledcirc h_C$.

Similarly we can show that

$$(f_A \circledcirc g_B) \circledcirc h_C \widetilde{\subseteq} f_A \circledcirc (g_B \circledcirc h_C).$$

Hence $f_A \circledcirc (g_B \circledcirc h_C) \widetilde{=} (f_A \circledcirc g_B) \circledcirc h_C$.

(2) Let $x \in S$. If x is not expressible as $x = yz$ for $y, z \in S$, then

$$\left(f_A \circledcirc \left(g_B \widetilde{\cup} h_C\right)\right)(x) = \widehat{0} = (f_A \circledcirc g_B)(x) \cup (f_A \circledcirc h_C)(x).$$

Otherwise

$$\begin{aligned}\left(f_A \circledcirc \left(g_B \widetilde{\cup} h_C\right)\right)(x) &= \cup_{x=yz}\{f_A(y) \cap \left(g_B \widetilde{\cup} h_C\right)(z)\} \\ &= \cup_{x=yz}\{f_A(y) \cap [g_B(z) \cup h_C(z)]\} \\ &= \cup_{x=yz}\{[f_A(y) \cap g_B(z)] \cup [f_A(y) \cap h_C(z)]\} \\ &= \{\cup_{x=yz}[f_A(y) \cap g_B(z)]\} \cup \{\cup_{x=yz}[f_A(y) \cap h_C(z)]\} \\ &= (f_A \circledcirc g_B)(x) \cup (f_A \circledcirc h_C)(x).\end{aligned}$$

Hence $f_A \circledcirc \left(g_B \widetilde{\cup} h_C\right) \widetilde{=} \left(f_A \circledcirc g_B\right) \widetilde{\cup} \left(f_A \circledcirc h_C\right)$.

Similarly we can prove (3).

(4) Let $x \in S$. If x is not expressible as $x = yz$ for $y, z \in S$, then

$$\left(f_A \circledcirc \left(g_B \widetilde{\cap} h_C\right)\right)(x) = \widehat{0} = \left(f_A \circledcirc g_B\right)(x) \cap \left(f_A \circledcirc h_C\right)(x).$$

Otherwise

$$\begin{aligned} \left(f_A \circledcirc \left(g_B \widetilde{\cap} h_C\right)\right)(x) &= \cup_{x=yz}\{f_A(y) \cap \left(g_B \widetilde{\cap} h_C\right)(z)\} \\ &= \cup_{x=yz}\{f_A(y) \cap [g_B(z) \cap h_C(z)]\} \\ &= \cup_{x=yz}\{[f_A(y) \cap g_B(z)] \cap [f_A(y) \cap h_C(z)]\} \\ &\subseteq \{\cup_{x=yz}[f_A(y) \cap g_B(z)]\} \cap \{\cup_{x=yz}[f_A(y) \cap h_C(z)]\} \\ &= \left(f_A \circledcirc g_B\right)(x) \cap \left(f_A \circledcirc h_C\right)(x). \end{aligned}$$

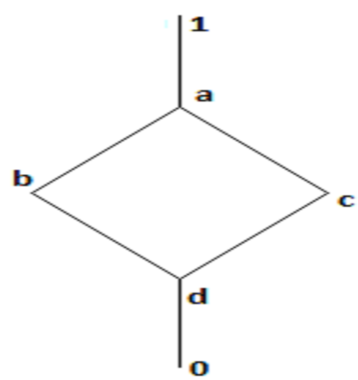
Hence $f_A \circledcirc \left(g_B \widetilde{\cap} h_C\right) \widetilde{\subseteq} \left(f_A \circledcirc g_B\right) \widetilde{\cap} \left(f_A \circledcirc h_C\right)$.

Similarly we can prove (5). □

Now we show that equality does not hold in (4) and (5).

Example 3.7. Let $S = \{\alpha, \beta, \gamma\}$ be a semigroup, $L = \{0, a, b, c, d, 1\}$ be a complete bounded distributive lattice, $U = \{p, q\}$ and $A = S$, $B = \{\beta, \gamma\}$ and $C = \{\alpha, \gamma\}$ are subsets of S.

$*$	α	β	γ
α	α	α	α
β	β	β	β
γ	γ	γ	γ



Let f_A, g_B and h_C be L-fuzzy soft sets of S over U.

$$\begin{aligned} f_A(\alpha) &= \{\frac{p}{a}, \frac{q}{0}\}, f_A(\beta) = \{\frac{p}{a}, \frac{q}{b}\}, f_A(\gamma) = \{\frac{p}{c}, \frac{q}{a}\}, \\ g_B(\alpha) &= \{\frac{p}{0}, \frac{q}{0}\}, g_B(\beta) = \{\frac{p}{1}, \frac{q}{b}\}, g_B(\gamma) = \{\frac{p}{c}, \frac{q}{b}\}, \\ h_C(\alpha) &= \{\frac{p}{d}, \frac{q}{1}\}, h_C(\beta) = \{\frac{p}{0}, \frac{q}{0}\}, h_C(\gamma) = \{\frac{p}{0}, \frac{q}{b}\}. \end{aligned}$$

Now for $\gamma \in S$, we have

$$\begin{aligned}\left[f_A \circledcirc \left(g_B \widetilde{\cap} h_C\right)\right](\gamma) &= \cup_{\gamma=\alpha\beta}\left[f_A(\alpha) \cap \left(g_B(\gamma) \cap h_C(\gamma)\right)\right] \\ &= \cup\{f_A(\gamma) \cap \left(g_B(\alpha) \cap h_C(\alpha)\right), \\ &\quad f_A(\gamma) \cap \left(g_B(\beta) \cap h_C(\beta)\right), f_A(\gamma) \cap \left(g_B(\gamma) \cap h_C(\gamma)\right)\}.\end{aligned}$$

Simple calculations show that $\left[f_A \circledcirc \left(g_B \widetilde{\cap} h_C\right)\right](\gamma) = \{\frac{p}{0}, \frac{q}{b}\}$. Now

$$\begin{aligned}\left(f_A \circledcirc g_B\right)(\gamma) &= \cup_{\gamma=\alpha\beta}\left[f_A(\alpha) \cap g_B(\beta)\right] \\ &= \cup\{f_A(\gamma) \cap g_B(\alpha), f_A(\gamma) \cap g_B(\beta), f_A(\gamma) \cap g_B(\gamma)\}.\end{aligned}$$

Simple calculations show that $\left(f_A \circledcirc g_B\right)(\gamma) = \{\frac{p}{c}, \frac{q}{b}\}$. Also

$$\begin{aligned}\left(f_A \circledcirc h_C\right)(\gamma) &= \cup_{\gamma=\alpha\beta}\left[f_A(\alpha) \cap h_C(\beta)\right] \\ &= \cup\{f_A(\gamma) \cap h_C(\alpha), f_A(\gamma) \cap h_C(\beta), f_A(\gamma) \cap h_C(\gamma)\}.\end{aligned}$$

Simple calculations show that $\left(f_A \circledcirc h_C\right)(\gamma) = \{\frac{p}{d}, \frac{q}{a}\}$. Then

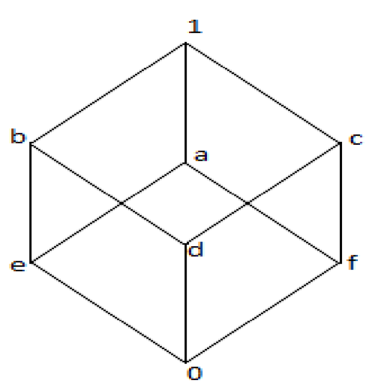
$$\left[\left(f_A \circledcirc g_B\right) \widetilde{\cap} \left(f_A \circledcirc h_C\right)\right](\gamma) = \{\frac{p}{d}, \frac{q}{b}\},$$

which shows that

$$f_A \circledcirc \left(g_B \widetilde{\cap} h_C\right) \widetilde{\neq} \left(f_A \circledcirc g_B\right) \widetilde{\cap} \left(f_A \circledcirc h_C\right).$$

Example 3.8. Let $S = \{x, y, z, p, q\}$ be a semigroup, $L = \{0, a, b, c, d, e, f, 1\}$ be a complete Boolean lattice, $U = \{p, q\}$ and $A = \{x, y, z, p\}$, $B = \{x, y, p\}$, $C = \{y, p, q\}$ are subsets of S.

$*$	x	y	z	p	q
x	x	x	x	x	x
y	x	x	x	y	z
z	x	y	z	x	x
p	x	x	x	p	q
q	x	p	q	x	x



Let f_A, g_B and h_C be L-fuzzy soft sets of S over U.

$$\begin{aligned} f_A(x) &= \{\frac{l}{1}, \frac{m}{a}\}, f_A(y) = \{\frac{l}{a}, \frac{m}{b}\}, f_A(z) = \{\frac{l}{c}, \frac{m}{a}\}, f_A(p) = \{\frac{l}{c}, \frac{m}{f}\}, \\ & f_A(q) = \{\frac{l}{0}, \frac{m}{0}\} \\ g_B(x) &= \{\frac{l}{1}, \frac{m}{d}\}, g_B(y) = \{\frac{l}{d}, \frac{m}{b}\}, g_B(z) = \{\frac{l}{0}, \frac{m}{0}\}, g_B(p) = \{\frac{l}{e}, \frac{m}{f}\}, \\ & g_B(q) = \{\frac{l}{0}, \frac{m}{0}\} \\ h_C(x) &= \{\frac{p}{0}, \frac{q}{0}\}, h_C(y) = \{\frac{p}{b}, \frac{q}{0}\}, h_C(z) = \{\frac{p}{0}, \frac{q}{0}\}, h_C(p) = \{\frac{p}{d}, \frac{q}{f}\}, \\ & h_C(q) = \{\frac{p}{c}, \frac{q}{d}\}. \end{aligned}$$

Now for $x \in S$ we have

$$\begin{aligned} \left[\left(f_A \widetilde{\cap} g_B\right) \circledcirc h_C\right](y) &= \cup_{y=xz} \left[\left(f_A(x) \cap g_B(x)\right) \cap h_C(z)\right] \\ &= \cup\{\left(f_A(y) \cap g_B(y)\right) \cap h_C(p), \left(f_A(z) \cap g_B(z)\right) \cap h_C(y)\}. \end{aligned}$$

Simple calculations show that $\left[\left(f_A \widetilde{\cap} g_B\right) \circledcirc h_C\right](y) = \{\frac{p}{0}, \frac{q}{0}\}$. Now

$$\begin{aligned} \left(f_A \circledcirc h_C\right)(y) &= \cup_{y=xz} \left[f_A(x) \cap h_C(z)\right] \\ &= \cup\{f_A(y) \cap h_C(p), f_A(z) \cap h_C(y)\}. \end{aligned}$$

Simple calculations show that $\left(f_A \circledcirc h_C\right)(y) = \{\frac{p}{d}, \frac{q}{0}\}$. Also

$$\begin{aligned} \left(g_B \circledcirc h_C\right)(y) &= \cup_{y=xz} \left[g_B(x) \cap h_C(z)\right] \\ &= \cup\{g_B(y) \cap h_C(p), g_B(z) \cap h_C(y)\}. \end{aligned}$$

Simple calculations show that $\left(g_B \circledcirc h_C\right)(y) = \{\frac{p}{d}, \frac{q}{0}\}$. Thus

$$\left[\left(f_A \circledcirc h_C\right) \widetilde{\cap} \left(g_B \circledcirc h_C\right)\right](y) = \{\tfrac{p}{d}, \tfrac{q}{0}\},$$

which shows that $\left(f_A \widetilde{\cap} g_B\right) \circledcirc h_C \widetilde{\not=} \left(f_A \circledcirc h_C\right) \widetilde{\cap} \left(g_B \circledcirc h_C\right)$.

Lemma 3.9. *Let f_A, g_B and h_C be L-fuzzy soft sets of a semigroup S over U. If $f_A \widetilde{\subseteq} g_B$, then $f_A \circledcirc h_C \widetilde{\subseteq} g_B \circledcirc h_C$ and $h_C \circledcirc f_A \widetilde{\subseteq} h_C \circledcirc g_B$.*

Proof. Let $x \in S$. If $x \neq yz$ for $y, z \in S$, then $\left(f_A \circledcirc h_C\right)(x) = \widehat{0} = \left(g_B \circledcirc h_C\right)(x)$. Otherwise

$$\begin{aligned} \left(f_A \circledcirc h_C\right)(x) &= \cup_{x=yz} \left[f_A(y) \cap h_C(z)\right] \\ & \widetilde{\subseteq} \cup_{x=yz} \left[g_B(y) \cap h_C(z)\right], (\text{because } f_A \widetilde{\subseteq} g_B) \\ &= \left(g_A \circledcirc h_A\right)(x). \end{aligned}$$

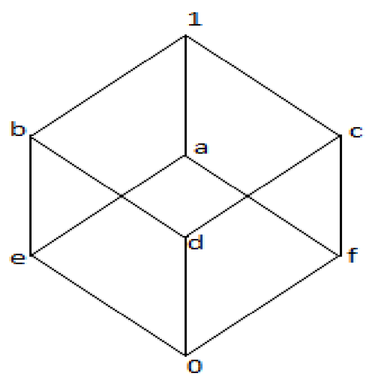
Similarly it can be shown that $h_C \circledcirc f_A \widetilde{\subseteq} h_C \circledcirc g_B$. □

4. L-fuzzy soft ideals of semigroups

Definition 4.1. An L-fuzzy soft set f_A of a semigroup S over U is called an L-fuzzy soft subsemigroup of S over U if for all $x, y \in S$, $f_A(xy) \supseteq f_A(x) \cap f_A(y)$. That is $[f_A(xy)](u) \geq [f_A(x)](u) \wedge [f_A(y)](u)$, for all $u \in U$.

Example 4.2. Let $S = \{0, x, y, z\}$ be a semigroup, $L = \{0, a, b, c, d, e, f, 1\}$ be a complete Boolean lattice, $U = \{l, m\}$ and $G = \{0, x, y\} \subseteq S$.

$*$	0	x	y	z
0	0	0	0	0
x	0	x	y	0
y	0	0	0	0
z	0	z	0	0



Let f_G be an L-fuzzy soft set of S over U defined by,

$$f_G(0) = \{\frac{l}{1}, \frac{m}{a}\}, f_G(x) = \{\frac{l}{b}, \frac{m}{1}\}, f_G(y) = \{\frac{l}{c}, \frac{m}{e}\}, f_G(z) = \{\frac{l}{0}, \frac{m}{0}\}.$$

Simple calculations show that f_G is an L-fuzzy soft subsemigroup of S over U.

Lemma 4.3. *The intersection of two L-fuzzy soft subsemigroups of a semigroup S over U is again an L-fuzzy soft subsemigroup of S over U.*

Proof. Let f_A and g_B be two L-fuzzy soft subsemigroups of a semigroup S over U and $x, y \in S$. Then

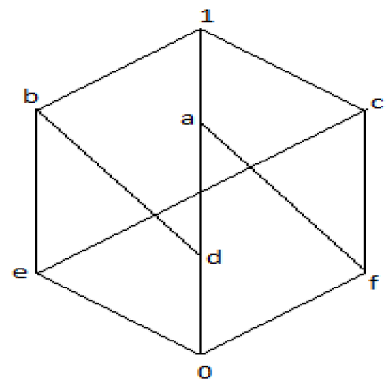
$$\begin{aligned} \left(f_A \tilde{\cap} g_B\right)(xy) &= f_A(xy) \cap g_B(xy) \\ &\supseteq (f_A(x) \cap f_A(y)) \cap (g_B(x) \cap g_B(y)) \\ &= (f_A(x) \cap g_B(x)) \cap (f_A(y) \cap g_B(y)) \\ &= \left(f_A \tilde{\cap} g_B\right)(x) \cap \left(f_A \tilde{\cap} g_B\right)(y). \end{aligned}$$

□

Next we show that the union of two L-fuzzy soft subsemigroups of a semigroup is not necessarily an L-fuzzy soft subsemigroup.

Example 4.4. Let $S = \{x, y, z, p, q\}$ be a semigroup, $L = \{0, a, b, c, d, e, f, 1\}$ be a complete Boolean lattice, $U = \{l, m\}$ and A, $B = S$.

$*$	x	y	z	p	q
x	x	x	x	x	x
y	x	x	x	y	z
z	x	y	z	x	x
p	x	x	x	p	q
q	x	p	q	x	x



Let f_A, g_B be L-fuzzy soft sets of S over U defined by,

$$\begin{aligned} f_A(x) &= \{\frac{l}{1}, \frac{m}{a}\}, f_A(y) = \{\frac{l}{b}, \frac{m}{0}\}, f_A(z) = \{\frac{l}{e}, \frac{m}{f}\}, \\ & f_A(p) = \{\frac{l}{c}, \frac{m}{f}\}, f_A(q) = \{\frac{l}{0}, \frac{m}{e}\} \\ g_B(x) &= \{\frac{l}{1}, \frac{m}{1}\}, g_B(y) = \{\frac{l}{1}, \frac{m}{c}\}, g_B(z) = \{\frac{l}{b}, \frac{m}{a}\}, \\ & g_B(p) = \{\frac{l}{e}, \frac{m}{f}\}, g_B(q) = \{\frac{l}{0}, \frac{m}{f}\}. \end{aligned}$$

Simple calculations show that f_A and g_A are L-fuzzy soft subsemigroups of S over U. Now

$$\begin{aligned} (f_A \tilde{\cup} g_B)(x) &= \{\frac{l}{1}, \frac{m}{1}\}, (f_A \tilde{\cup} g_B)(y) = \{\frac{l}{1}, \frac{m}{c}\}, \\ (f_A \tilde{\cup} g_B)(z) &= \{\frac{l}{b}, \frac{m}{a}\}, (f_A \tilde{\cup} g_B)(p) = \{\frac{l}{c}, \frac{m}{f}\}, \\ (f_A \tilde{\cup} g_B)(q) &= \{\frac{l}{0}, \frac{m}{1}\}. \end{aligned}$$

Now,

$$\begin{aligned} (f_A \tilde{\cup} g_B)(yq) &= (f_A \tilde{\cup} g_B)(z) = \{\frac{l}{b}, \frac{m}{a}\}. \\ (f_A \tilde{\cup} g_B)(y) \cap (f_A \tilde{\cup} g_B)(q) &= \{\frac{l}{0}, \frac{m}{c}\}, \text{ but } a \ngeq c. \end{aligned}$$

Hence $f_A \tilde{\cup} g_B$ is not an L-fuzzy soft subsemigroup of S over U.

Lemma 4.5. *Let A be a non-empty subset of a semigroup S. Then A is a subsemigroup of S if and only if C_A is an L-fuzzy soft subsemigroup of S over U.*

Proof. Suppose A is a subsemigroup of S and $x, y \in S$. If $x, y \in A$ then $xy \in A$. This implies $C_A(xy) = \widehat{1} = \widehat{1} \cap \widehat{1} = C_A(x) \cap C_A(y)$. If one of x, y doesn't belong to A then $C_A(x) \cap C_A(y) = \widehat{0} \subseteq C_A(xy)$. Hence in any case $C_A(xy) \supseteq C_A(x) \cap C_A(y)$.

Conversely, assume that C_A is an L-fuzzy soft subsemigroup of S over U. Let x, $y \in A$. Then $C_A(x) = \widehat{1}$ and $C_A(y) = \widehat{1} \Longrightarrow C_A(x) \cap C_A(y) = \widehat{1} \cap \widehat{1} = \widehat{1}$. But $C_A(xy) \supseteq C_A(x) \cap C_A(y) = \widehat{1}$. Hence $C_A(xy) = \widehat{1}$. This implies $xy \in A$, that is A is a subsemigroup of S. □

Lemma 4.6. *Let f_A be an L-fuzzy soft set of a semigroup S. Then f_A is an L-fuzzy soft subsemigroup of S if and only if $f_A \circledcirc f_A \widetilde{\subseteq} f_A$.*

Proof. Let f_A be an L-fuzzy soft subsemigroup of S over U. Let $x \in S$. If $(f_A \circledcirc f_A)(x) = \widehat{0}$, then clearly $(f_A \circledcirc f_A)(x) \subseteq (f_A)(x)$. Otherwise there exist $y, z \in S$ such that $x = yz$. In this case

$$\begin{aligned} (f_A \circledcirc f_A)(x) &= \cup_{x=yz} (f_A(y) \cap f_A(z)) \\ &\subseteq \cup_{x=yz} f_A(yz) = \cup_{x=yz} f_A(x) = f_A(x). \end{aligned}$$

Hence $f_A \circledcirc f_A \widetilde{\subseteq} f_A$.

Conversely, for $x, y \in S$

$$\begin{aligned} f_A(xy) &\supseteq (f_A \circledcirc f_A)(xy) \\ &= \cup_{xy=pq} f_A(p) \cap f_A(q) \supseteq f_A(x) \cap f_A(y). \end{aligned}$$

Hence f_A is an L-fuzzy soft subsemigroup of S over U. □

Definition 4.7. Let $\alpha \in L^U$ and h_G be an L-fuzzy soft set of S over U. Then α-cut of h_G is denoted by h_G^{α} and is given by,

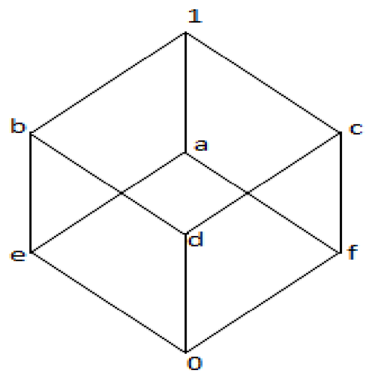
$$h_G^{\alpha} = \{s \in S : h_G(s) \supseteq \alpha\}$$

that is

$$h_G^{\alpha} = \{s \in S : [h_G(s)](u) \geq \alpha(u) \, \forall u \in U\}.$$

Example 4.8. Let $S = \{x, y, z, t\}$ be a semigroup, $L = \{0, a, b, c, d, e, f, 1\}$ be a complete Boolean lattice, $U = \{l, m\}$ and $G = \{x, y, z\} \subseteq S$.

$*$	x	y	z	t
x	x	x	x	x
y	x	x	x	x
z	x	x	y	x
t	x	x	y	y



Let h_G be an L-fuzzy soft set of S over U defined by,

$$h_G(x) = \{\frac{p}{1}, \frac{q}{c}\}, h_G(y) = \{\frac{p}{a}, \frac{q}{c}\}, h_G(z) = \{\frac{p}{e}, \frac{q}{d}\}, h_G(t) = \{\frac{p}{0}, \frac{q}{0}\}.$$

Let $\alpha(p) = a$, $\alpha(q) = f$. Then $h_G^{\alpha} = \{s \in S : h_G(s) \supseteq \alpha\} = \{s \in S : [h_G(s)](u) \geq \alpha(u)\ \forall\ u \in U\} = \{x, y\}$.

Theorem 4.9. *An L-fuzzy soft set f_G of a semigroup S over U is an L-fuzzy soft subsemigroup of S over U if and only if each α-cut of f_G is a subsemigroup of S.*

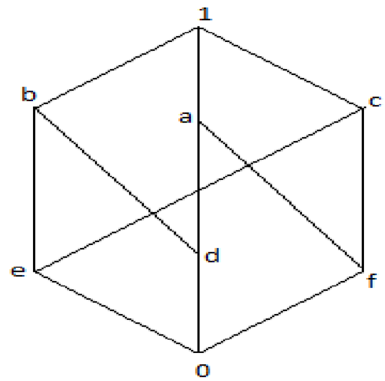
Proof. Let f_G be a non zero L-fuzzy soft subsemigroup of S over U. Let $x, y \in f_G^{\alpha} \Longrightarrow f_G(x) \supseteq \alpha$ and $f_G(y) \supseteq \alpha \Longrightarrow f_G(xy) \supseteq f_G(x) \cap f_G(y) \supset \alpha \cap \alpha = \alpha$. Thus $xy \in f_G^{\alpha}$. This implies $f_G^{\alpha} = \{s \in S : f_G(s) \supseteq \alpha\}$ is a subsemigroup of S.

Conversely, suppose that there exist $x, y \in S$ such that $f_G(xy) \subset f_G(x) \cap f_G(y)$. This implies that there exist $\beta \in L^U$ such that $f_G(xy) \subset \beta \subseteq f_G(x) \cap f_G(y)$. As $f_G(x) \cap f_G(y) \supseteq \beta \Longrightarrow x \in f_G^{\beta}$ and $y \in f_G^{\beta}$. But $xy \notin f_G^{\beta}$, because $f_G(xy) \subset \beta$. This shows that f_G^{β} is not a subsemigroup of S, which is a contradiction. Hence $f_G(xy) \supseteq f_G(x) \cap f_G(y)$ for all $x, y \in S$. □

Definition 4.10. An L-fuzzy soft set f_G of a semigroup S over U is called an L-fuzzy soft left (right) ideal of S over U if for all $x, y \in S$, $f_G(xy) \supseteq f_G(y)$ $(f_G(xy) \supseteq f_G(x))$. An L-fuzzy soft set f_G of S over U is called an L-fuzzy soft two-sided ideal of S over U if it is both an L-fuzzy soft left and an L-fuzzy soft right ideal of S over U.

Example 4.11. Let $S = \{0, x, y, z\}$ be a semigroup, $L = \{0, a, b, c, d, e, f, 1\}$ be a complete Boolean lattice, $U = \{l, m, n\}$ and $A = \{0, x, z\} \subseteq S$.

$*$	0	x	y	z
0	0	0	0	0
x	0	x	y	0
y	0	0	0	0
z	0	z	0	0



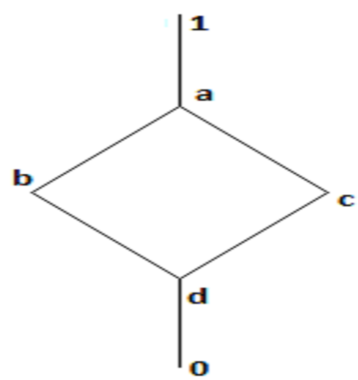
Let f_A be an L-fuzzy soft set of S over U defined by,

$$f_A(0) = \{\frac{l}{1}, \frac{m}{a}, \frac{n}{1}\}, f_A(x) = \{\frac{l}{e}, \frac{m}{d}, \frac{n}{f}\}, f_A(y) = \{\frac{l}{0}, \frac{m}{0}, \frac{n}{0}\}, f_A(z) = \{\frac{l}{b}, \frac{m}{a}, \frac{n}{c}\}.$$

Simple calculations show that f_A is an L-fuzzy soft left ideal of S over U. But f_A is not an L-fuzzy soft right ideal of S over U, because $f_A(xy) = f_A(y) = \{\frac{l}{0}, \frac{m}{0}, \frac{n}{0}\} \not\supseteq \{\frac{l}{e}, \frac{m}{d}, \frac{n}{f}\} = f_A(x)$.

Example 4.12. Let $S = \{0, x, y, z\}$ be a semigroup, $L = \{0, a, b, c, d, 1\}$ be a complete bounded distributive lattice, $U = \{p, q\}$ and $B = \{0, x, y\} \subseteq S$.

$*$	0	x	y	z
0	0	0	0	0
x	0	x	y	0
y	0	0	0	0
z	0	z	0	0



Let g_B be an L-fuzzy soft set of S over U defined by,

$$g_B(0) = \{\frac{p}{1}, \frac{q}{a}\}, g_B(x) = \{\frac{p}{c}, \frac{q}{b}\}, g_B(y) = \{\frac{p}{1}, \frac{q}{b}\}, g_B(z) = \{\frac{p}{0}, \frac{q}{0}\}.$$

Simple calculations show that g_B is an L-fuzzy soft right ideal of S over U, but it is not an L-fuzzy soft left ideal of S over U, because $g_B(zx) = g_B(z) = \{\frac{p}{0}, \frac{q}{0}\} \not\supseteq g_B(x) = \{\frac{p}{c}, \frac{q}{b}\}$.

Lemma 4.13. *Let A be a non-empty subset of a semigroup S. Then A is a left (right, two-sided) ideal of S if and only if C_A is an L-fuzzy soft left (right, two-sided) ideal of S over U.*

Proof. The proof is similar to the proof of Lemma 4.5. □

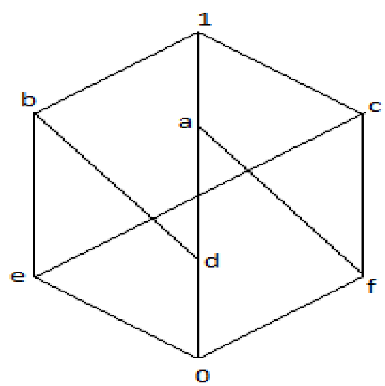
Lemma 4.14. *Every L-fuzzy soft left (right) ideal of a semigroup S over U is an L-fuzzy soft subsemigroup of S over U.*

Proof. Straightforward. □

The following example shows that the converse of the above Lemma is not true.

Example 4.15. Let $S = \{x, y, z, p, q\}$ be a semigroup, $L = \{0, a, b, c, d, e, f, 1\}$ be a Boolean lattice, $U = \{l, m\}$ and A, $B = S$.

$*$	x	y	z	p	q
x	x	x	x	x	x
y	x	x	x	y	z
z	x	y	z	x	x
p	x	x	x	p	q
q	x	p	q	x	x



Let f_A, g_B be L-fuzzy soft sets of S over U.

$$\begin{aligned} f_A(x) &= \{\frac{l}{1}, \frac{m}{a}\}, f_A(y) = \{\frac{l}{b}, \frac{m}{0}\}, f_A(z) = \{\frac{l}{e}, \frac{m}{f}\}, \\ &\quad f_A(p) = \{\frac{l}{c}, \frac{m}{f}\}, f_A(q) = \{\frac{l}{0}, \frac{m}{e}\} \\ g_B(x) &= \{\frac{l}{1}, \frac{m}{1}\}, g_B(y) = \{\frac{l}{1}, \frac{m}{c}\}, g_B(z) = \{\frac{l}{b}, \frac{m}{a}\}, \\ &\quad g_B(p) = \{\frac{l}{e}, \frac{m}{f}\}, g_B(q) = \{\frac{l}{0}, \frac{m}{f}\}. \end{aligned}$$

Simple calculations show that f_A and g_B are L-fuzzy soft subsemigroups of S over U. But neither f_A nor g_B are L-fuzzy soft left ideals of S over U, because $f_A(yp) = f_A(y) = \{\frac{l}{b}, \frac{m}{0}\} \not\supseteq f_A(p) = \{\frac{l}{c}, \frac{m}{f}\}$ and $g_B(qz) = g_B(q) = \{\frac{l}{0}, \frac{m}{f}\} \not\supseteq g_B(z) = \{\frac{l}{b}, \frac{m}{a}\}$.

Lemma 4.16. (1) *The intersection of L-fuzzy soft left (right, two-sided) ideals of a semigroup S over U is again an L-fuzzy soft left (right, two-sided) ideal of S over U.*

(2) *The union of L-fuzzy soft left (right, two-sided) ideals of a semigroup S over U is again an L-fuzzy soft left (right, two-sided) ideal of S over U.*

Proof. Straightforward. □

Lemma 4.17. *Let f_B be an L-fuzzy soft set of a semigroup S. Then f_B is an L-fuzzy soft left (right) ideal of S over U if and only if $\widetilde{1} \circledcirc f_B \widetilde{\subseteq} f_B$ ($f_B \circledcirc \widetilde{1} \widetilde{\subseteq} f_B$).*

Proof. Suppose f_B be an L-fuzzy soft left ideal of S over U. Let $x \in S$. If $x \neq yz$, then $\left(\widetilde{1} \circledcirc f_B\right)(x) = \widehat{0} \subseteq f_B(x)$. Otherwise there exist $y, z \in S$, such that $x = yz$. Then

$$\begin{aligned}\left(\widetilde{1} \circledcirc f_B\right)(x) &= \cup_{x=yz}\left(\widetilde{1}(y) \cap f_B(z)\right) \\ &\subseteq \cup_{x=yz}\left(\widehat{1} \cap f_B(yz)\right) = \cup_{x=yz}\left(\widehat{1} \cap f_B(x)\right) \\ &= \cup_{x=yz} f_B(x) = f_B(x).\end{aligned}$$

So in any case $\widetilde{1} \circledcirc f_B \widetilde{\subseteq} f_B$.

Conversely, assume that $\widetilde{1} \circledcirc f_B \widetilde{\subseteq} f_B$. Let y, $z \in S$. Then

$$\begin{aligned}f_B(yz) &\supseteq \left(\widetilde{1} \circledcirc f_B\right)(x) \\ &= \cup_{yz=ab}\left(\widetilde{1}(a) \cap f_B(b)\right) \\ &\supseteq \widetilde{1}(y) \cap f_B(z) = f_B(z).\end{aligned}$$

Hence f_B is an L-fuzzy soft left ideal of S over U. □

Theorem 4.18. *An L-fuzzy soft set f_G of a semigroup S over U is an L-fuzzy soft left (right) ideal of S over U if and only if each α-cut of f_G is a left (right) ideal of S.*

Proof. The proof is similar to the proof of Theorem 4.9. □

Next we characterize different classes of semigroups by the properties of their L-fuzzy soft ideals.

5. Regular and intra-regular semigroups

Recall that a semigroup S is regular if for all $a \in S$, there exists $x \in S$ such that $a = axa$.

A semigroup S is said to be intra-regular if for each $a \in S$ there exist $y, z \in S$ such that $a = yaaz$. In general, neither regular semigroup is intra- regular nor intra-regular semigroup is regular. If S is commutative then both the concepts coincide.

Example 5.1. Let A be a countably infinite set and

$$S = \{\alpha : A \to A : \alpha \text{ is one one and } A - \alpha(A) \text{ is infinite}\}.$$

Then S is a semigroup with respect to the composition of functions and is called Baer-Levi Semigroup. It is well known that this semigroup is right cancelative, right simple without idempotents. Thus S is not regular but intra-regular.

Example 5.2. Consider the semigroup $S = \{0, 1, 2, 3, 4\}$.

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	1	2
2	0	1	2	0	0
3	0	0	0	3	4
4	0	3	4	0	0

This semigroup S is regular but not intra-regular.

It is well known that

Theorem 5.3. *A semigroup S is regular if and only if $R \cap L = RL$ for every right ideal R and left ideal L of S.*

Now we show that:

Theorem 5.4. *The following assertions are equivalent for a semigroup S.*
(1) *S is regular.*
(2) *$f_A \tilde{\cap} g_B \tilde{=} f_A \circledcirc g_B$ for every L-fuzzy soft right ideal f_A and L-fuzzy soft left ideal g_B of S over U.*

Proof. (1) $\Rightarrow$ (2) Let f_A be an L-fuzzy soft right ideal and g_B an L-fuzzy soft left ideal of S over U. Then by Lemma 4.17, $f_A \circledcirc g_B \tilde{\subseteq} f_A \circledcirc \tilde{1} \tilde{\subseteq} f_A$ and $f_A \circledcirc g_B \tilde{\subseteq} \tilde{1} \circledcirc g_B \tilde{\subseteq} g_B \Longrightarrow f_A \circledcirc g_B \tilde{\subseteq} f_A \tilde{\cap} g_B$. Now, let $x \in S$, since S is regular so there exists $a \in S$ such that $x = xax$. Thus we have

$$\begin{aligned} (f_A \circledcirc g_B)(x) &= \cup_{x=yz} (f_A(y) \cap g_B(z)) \\ &\supseteq f_A(xa) \cap g_B(x) \supseteq f_A(x) \cap g_B(x) \\ &= \left(f_A \tilde{\cap} g_B\right)(x) \end{aligned}$$

$\Longrightarrow f_A \circledcirc g_A \tilde{\supseteq} f_A \tilde{\cap} g_A$. Hence $f_A \tilde{\cap} g_A \tilde{=} f_A \circledcirc g_A$.

(2) $\Rightarrow$ (1) Let R and L be any right and left ideal of S, respectively. Then by Lemma 4.13, C_R and C_L are L-fuzzy soft right ideal and L-fuzzy soft left ideal of S over U, respectively. Then by Lemma 3.4

$$C_{RL} \tilde{=} (C_R \circledcirc C_L) \tilde{=} \left(C_R \tilde{\cap} C_L\right) \tilde{=} C_{R \cap L}.$$

Thus $R \cap L = RL$. Hence by Theorem 5.3, S is regular. □

Theorem 5.5. *For a semigroup S the following conditions are equivalent:*
(1) *S is intra-regular.*
(2) *$L \cap R \subseteq LR$ for every left ideal L and right ideal R of S.*
(3) *$f_A \tilde{\cap} g_B \tilde{\subseteq} f_A \circledcirc g_B$ for every L-fuzzy soft left ideal f_A and L-fuzzy soft right ideal g_B of S over U.*

Proof. (1) $\Leftrightarrow$ (2) It is well known.

(1) $\Rightarrow$ (3) Let f_A be an L-fuzzy soft left and g_B an L-fuzzy soft right ideals of S. Let $a \in S$. Since S is intra-regular, so there exist $x, y \in S$ such that $a = xa^2y$. Thus

$$\begin{aligned}(f_A \circledcirc g_B)(a) &= \cup_{a=pq}(f_A(p) \cap g_B(q)) \\ &\supseteq f_A(xa) \cap g_B(ay) \supseteq f_A(a) \cap g_B(a) \\ &= \left(f_A \widetilde{\cap} g_B\right)(a)\end{aligned}$$

$\Longrightarrow f_A \widetilde{\cap} g_B \widetilde{\subseteq} f_A \circledcirc g_B$.

(3) $\Rightarrow$ (2) Let L be a left and R be a right ideal of S. By Lemma 4.13, C_L and C_R are L-fuzzy soft left ideal and L-fuzzy soft right ideal of S over U, respectively. So by Lemma 3.4, $C_{LR} \widetilde{=} C_L \circledcirc C_R \widetilde{\supseteq} C_L \widetilde{\cap} C_R \widetilde{=} C_{L \cap R}$. This implies $L \cap R \subseteq LR$. □

References

[1] M. I. Ali, F. Feng, X. Liu. W. K. Min and M. Shabir, On some new operations in soft set theory, Comput. Math. Appl. 57 (2009) 1547–1553.

[2] M. I. Ali, A note on soft sets, rough soft sets and fuzzy soft sets, Appl. Soft comput. 11 (2011) 3329–3332.

[3] M. I. Ali, M. Shabir and M. Naz, Algebra structures of soft sets associated with new operations, Comput. Math. Appl. 9 (2011) 2647–2654.

[4] F. Feng, C. Li, B. Davvaz and M. I. Ali, Soft sets combined with fuzzy sets and rough sets: a tentative approach, Soft Comput. 14 (2010) 899–911.

[5] F. Feng, X. Liu, V. Leoreanu-Fotea and Y. B. Jun, Soft sets and rough sets, Inform. Sci. 181 (2011) 1125–1137.

[6] F. Feng, Y. B. Jun, X. Liu and L. Li, An adjustable approach to fuzzy soft set based decision making, J. Comput. Appl. Math. 234(1) (2010) 10–20.

[7] J. A. Goguen, L-fuzzy sets, J. Math. Anal. Appl. 18 (1967) 145–174.

[8] Z. Li, D. Zheng and J. Hao, L-fuzzy soft sets based on complete Boolean lattices, Comput. Math. Appl. 64 (2012) 2558–2574.

[9] P. K. Maji, R. Biswas and A. R. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.

[10] P. K. Maji, R. Biswas and A. R. Roy, Fuzzy soft sets, J. Fuzzy Math. 9 (2001) 589–602.

[11] D. Molodtsov, Soft set theory-first result, Comput. Math. Appl. 37 (1999) 19–31.

[12] A. R. Roy and P. K. Maji, A fuzzy soft set theoretic approach to decision making problems, J. Comput. Appl. Math. 203 (2007) 412–418.

[13] A. Sezgin and A. O. Atagün, On operations of soft sets, Comput. Math. Appl. 61(5) (2011) 1457–1467.

[14] A. Sezgin, N. Çağman and A. O. Atagün, A new view to soft intersedtion rings, ideals and bi-ideals via soft uni-int product, Submitted.

[15] B. Tanay and M. B. Kandemir, Topological structure of fuzzy soft sets, Comput. Math. Appl. 61 (2011) 2952–2957.

[16] X. Yang, T. Lin, J. Yang, Y. Li and D. Yu, Combination of interval-valued fuzzy sets and soft sets, Comput. Math. Appli. 58 (2009) 521–527.

M. Shabir (mshabirbhatti@yahoo.co.uk)
Department of Mathematics, Quaid-i-Azam University, Islamabad-45320, Pakistan

M. Ghafoor (mamoonaghafoor55@gmail.com)
Department of Mathematics, Quaid-i-Azam University, Islamabad-45320, Pakistan