On L-fuzzy soft semigroups

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ABSTRACT. In this paper we introduced the concepts of L-fuzzy soft left (right, two-sided) ideals of a semigroup over a universe $U$, where $L$ is a complete bounded distributive lattice. We also studied some properties of L-fuzzy soft left (right, two-sided) ideals of a semigroup over a universe $U$. Regular and intra-regular semigroups are characterized by the properties of these L-fuzzy soft left (right) ideals.

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1. Introduction

The concept of soft set was initiated by Molodtsov [11] in 1999, to handle the ambiguousness and uncertainty, that were not handled by old classical methods. He has given a number of applications of soft sets in the field of economics, engineering, social science and medical science etc. Maji et al. [9] introduced some basic operations of soft sets. Ali et al. [1] also worked on the operations of soft sets. They improved some already defined operations and introduced some new operations. Sezgin and Atagün [13] and Ali et al. [3] also worked on the operations of soft set. Feng et al. [4] and Feng et al. [5] worked on the combination of fuzzy sets, rough sets and soft sets. Sezgin et al. [14] introduced the notions of soft-int ideals and soft-int bi-ideal of a ring. Maji et al. [10] initiated the study of fuzzy soft set by combining the concepts of fuzzy sets and soft sets. Many authors worked on fuzzy soft sets, e.g. [2, 12, 6, 15, 16].

Goguen [7] was the first who gave the concept of L-fuzzy sets by generalizing Zadeh’s fuzzy set. Recently Li, Zheng and Hao worked on L-fuzzy soft sets based on complete Boolean lattice [8]. They discussed topological and algebraic structures of L-fuzzy soft sets.
In this paper, we defined $L$-fuzzy soft subsemigroup, $L$-fuzzy soft left (right, two-sided) ideal of semigroups over a universe $U$ and studied some properties of $L$-fuzzy soft subsemigroups and $L$-fuzzy soft left (right, two-sided) ideals of semigroups over a universe $U$. We characterized different classes of semigroups by the properties of these $L$-fuzzy soft ideals.

2. Preliminaries

An algebraic system $(S, \cdot)$ consisting of a non-empty set $S$ together with an associative binary operation $\cdot$ is called a semigroup. By a subsemigroup of a semigroup $S$ we mean a non-empty subset $A$ of $S$ such that $A^2 \subseteq A$. A non-empty subset $A$ of a semigroup $S$ is called a left (right) ideal of $S$ if $SA \subseteq A$ ($AS \subseteq A$). A non-empty subset $A$ of $S$ is called a two-sided ideal or simply an ideal of $S$ if it is both a left and a right ideal of $S$.

A partially ordered set (poset) $(L, \leq)$ is called
1) a lattice, if $a \vee b \leq L$, $a \wedge b \leq L$ for any $a, b \in L$.
2) a complete lattice, if $\forall N \in L$, $\wedge N \in L$ for any $N \subseteq L$.
3) distributive, if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for any $a, b, c \in L$.

Let $L$ be a lattice with top element $1_L$ and bottom element $0_L$ and let $a, b \in L$. Then $b$ is called a complement of $a$, if $a \vee b = 1_L$ and $a \wedge b = 0_L$. If $a \in L$ has complement element, then it is unique. It is denoted by $a'$.

A lattice $L$ is called a Boolean lattice, if
(i) $L$ is distributive,
(ii) $L$ has $0_L$ and $1_L$,
(iii) each $a \in L$ has the complement $a' \in L$.

Let $X$ be a non-empty set. A fuzzy set $A$ in $X$ is a function, $A : X \to [0, 1]$ and $A(x)$ is interpreted as the degree of membership of element $x$ in the fuzzy set $A$ for each $x \in X$.

In [7], Goguen generalized the concept of fuzzy set and introduced $L$-fuzzy set as:

An $L$-fuzzy set $A$ in a non-empty set $X$ is a function $A : X \to L$, where $L$ is a complete distributive lattice with 1 and 0. We denote by $L^X$ the set of all $L$-fuzzy sets in $X$.

Let $A, B \in L^X$. Then their union and intersection are $L$-fuzzy sets in $X$, defined as

$$(A \cup B)(x) = A(x) \vee B(x) \text{ and } (A \cap B)(x) = A(x) \wedge B(x) \text{ for all } x \in X.$$  \(A \subseteq B\) if and only if $A(x) \leq B(x)$ for all $x \in X$.

The $L$-fuzzy sets $\emptyset$ and $\top$ of $X$ are defined as $\emptyset(x) = 0$ and $\top(x) = 1$ for all $x \in X$. Obviously $\emptyset \subseteq A \subseteq \top$ for all $A \in L^X$.

A pair $(F, E)$ is called a soft set (over $U$) if $F$ is a mapping of $E$ into the power set of $U$, that is $F : E \to P(U)$.

In other words, the soft set is a parametrized family of subsets of the set $U$ [11].

**Definition 2.1 ([8]).** Let $E$ be a set of parameters, $U$ be an initial universe, $L$ be a complete Boolean lattice and $A \subseteq E$. An $L$-fuzzy soft set $f_A$ over $U$ is a mapping $f_A : E \to L^U$ such that $f_A(e) = 0$ for all $e \notin A$.

The following operations on $L$-fuzzy soft sets are defined in [8],
1) Let \( f_A \) and \( g_B \) be two \( L \)-fuzzy soft sets over \( U \). Then \( f_A \) is contained in \( g_B \) denoted by \( f_A \subseteq g_B \) if \( f_A(e) \subseteq g_B(e) \) for all \( e \in E \), that is \((f_A(e))(u) \leq (g_B(e))(u)\) for all \( u \in U \).

Two \( L \)-fuzzy soft sets \( f_A \) and \( g_B \) over \( U \) are said to be equal, denoted by \( f_A \equiv g_B \) if \( f_A \subseteq g_B \) and \( f_A \supseteq g_B \).

2) Let \( f_A \) and \( g_B \) be two \( L \)-fuzzy soft sets over \( U \). Then their union \( f_A \cup g_B \equiv h_{A\cup B} \), where \( h_{A\cup B}(e) = f_A(e) \cup g_B(e) \) for all \( e \in E \).

3) Let \( f_A \) and \( g_B \) be two \( L \)-fuzzy soft sets over \( U \). Then their intersection \( f_A \cap g_B \equiv h_{A\cap B} \), where \( h_{A\cap B}(e) = f_A(e) \cap g_B(e) \) for all \( e \in E \).

**Proposition 2.2** ([8]). Let \( A, B, C \subseteq E \) and \( f_A, g_B, h_C \) are \( L \)-fuzzy soft sets over \( U \). Then the following holds:

1) \( f_A \cup f_A \equiv f_A, f_A \cap f_A \equiv f_A \)
2) \( f_A \cup g_B \equiv g_B \cup f_A, f_A \cap g_B \equiv g_B \cap f_A \)
3) \( (f_A \cup g_B) \cap h_C \equiv f_A \cup (g_B \cap h_C), (f_A \cap g_B) \cap h_C \equiv f_A \cap (g_B \cap h_C) \)
4) \( (f_A \cup g_B) \cap h_C \equiv (f_A \cap h_C) \cup (g_B \cap h_C), (f_A \cap g_B) \cap h_C \equiv (f_A \cap h_C) \cap (g_B \cap h_C) \).

3. \( L \)-FUZZY SOFT SETS OF SEMIGROUPS

In this section we define product of \( L \)-fuzzy soft sets of a semigroup \( S \) over \( U \) and study some properties of this product. Throughout this paper \( L \) is a complete bounded distributive lattice and \( U \) is the initial universe and the set of parameters is a semigroup \( S \).

**Definition 3.1.** Let \( A \) be a non-empty subset of a semigroup \( S \). Define an \( L \)-fuzzy soft set \( C_A \) of \( S \) over \( U \) by

\[
C_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}
\]

for all \( x \in S \). We shall call this \( L \)-fuzzy soft set the \( L \)-fuzzy soft characteristic function of \( A \).

**Definition 3.2.** Let \( f_A \) and \( g_B \) be two \( L \)-fuzzy soft sets of a semigroup \( S \) over \( U \). Then their product \( f_A \otimes g_B \) is an \( L \)-fuzzy soft set of \( S \) over \( U \) and is defined as

\[
(f_A \otimes g_B)(x) = \begin{cases} \cup_{y=z} \{f_A(y) \cap g_B(z)\}, & \text{if } \exists \ y, z \in S \text{ such that } x = yz \\ 0 & \text{otherwise} \end{cases}
\]

We explain this concept with the help of an example.

**Example 3.3.** Let \( S = \{x, y, z\} \) be a semigroup, \( L = \{0, a, b, c, d, 1\} \) be a complete bounded distributive lattice, \( U = \{p, q\} \) and \( A = \{x, y\}, B = \{x, z\} \) are subsets of \( S \).

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Let $f_A, g_B$ be $L$-fuzzy soft sets of $S$ over $U$, defined by

$$f_A(x) = \{ \frac{p}{b}, \frac{q}{d} \}, f_A(y) = \{ \frac{p}{a}, \frac{q}{b} \}, f_A(z) = \{ \frac{p}{0}, \frac{q}{0} \},$$

$$g_B(x) = \{ \frac{p}{1}, \frac{q}{0} \}, g_B(y) = \{ \frac{p}{0}, \frac{q}{0} \}, g_B(z) = \{ \frac{p}{b}, \frac{q}{1} \}.$$ 

Now for $x \in S$, we have

$$(f_A \circledast g_B)(x) = \bigcup_{y=bc} [f_A(b) \cap g_B(c)]$$

$$= \bigcup \{ f_A(x) \cap g_B(x), f_A(x) \cap g_B(y), f_A(x) \cap g_B(z) \}$$

$$= \bigcup \{ \{ \frac{p}{b}, \frac{q}{d} \} \cap \{ \frac{p}{b}, \frac{q}{d} \}, \{ \frac{p}{a}, \frac{q}{b} \} \cap \{ \frac{p}{b}, \frac{q}{d} \}, \{ \frac{p}{0}, \frac{q}{0} \} \cap \{ \frac{p}{b}, \frac{q}{d} \} \}$$

$$= \bigcup \{ \{ \frac{p}{b}, \frac{q}{d} \}, \{ \frac{p}{a}, \frac{q}{b} \}, \{ \frac{p}{0}, \frac{q}{0} \} \}.$$ 

$$\implies (f_A \circledast g_B)(x) = \{ \frac{p}{b}, \frac{q}{d} \}.$$ 

For $y \in S$, we have

$$(f_A \circledast g_B)(y) = \bigcup_{y=bc} [f_A(b) \cap g_B(c)]$$

$$= \bigcup \{ f_A(y) \cap g_B(x), f_A(y) \cap g_B(y), f_A(y) \cap g_B(z) \}$$

$$= \bigcup \{ \{ \frac{p}{b}, \frac{q}{d} \} \cap \{ \frac{p}{1}, \frac{q}{0} \}, \{ \frac{p}{a}, \frac{q}{b} \} \cap \{ \frac{p}{1}, \frac{q}{0} \}, \{ \frac{p}{0}, \frac{q}{0} \} \cap \{ \frac{p}{1}, \frac{q}{0} \} \}$$

$$= \bigcup \{ \{ \frac{p}{b}, \frac{q}{d} \}, \{ \frac{p}{a}, \frac{q}{b} \}, \{ \frac{p}{0}, \frac{q}{0} \} \}.$$ 

$$\implies (f_A \circledast g_B)(y) = \{ \frac{p}{a}, \frac{q}{b} \}.$$ 

For $z \in S$, we have

$$(f_A \circledast g_B)(z) = \bigcup_{z=bc} [f_A(b) \cap g_B(c)]$$

$$= \bigcup \{ f_A(z) \cap g_B(x), f_A(z) \cap g_B(y), f_A(z) \cap g_B(z) \}$$

$$= \bigcup \{ \{ \frac{p}{0}, \frac{q}{0} \} \cap \{ \frac{p}{0}, \frac{q}{0} \}, \{ \frac{p}{0}, \frac{q}{0} \} \cap \{ \frac{p}{0}, \frac{q}{0} \}, \{ \frac{p}{b}, \frac{q}{1} \} \cap \{ \frac{p}{0}, \frac{q}{0} \} \}$$

$$= \bigcup \{ \{ \frac{p}{0}, \frac{q}{0} \}, \{ \frac{p}{0}, \frac{q}{0} \}, \{ \frac{p}{0}, \frac{q}{0} \} \}.$$ 

$$\implies (f_A \circledast g_B)(z) = \{ \frac{p}{b}, \frac{q}{0} \}.$$ 

1030
Lemma 3.4. Let $A$ and $B$ be non-empty subsets of a semigroup $S$. Then

1. $C_A \cap C_B = C_{A \cap B}$
2. $C_A \ominus C_B = C_{AB}$.

Proof. (1) Let $a \in S$. If $a \notin (A \cap B)$ then $C_{A \cap B} (a) = \hat{1}$. On the other hand $a \in A$ and $a \in B$, so $C_A (a) = \hat{1}$ and $C_B (a) = \hat{1}$. Thus $(C_A \cap C_B) (a) = C_A (a) \cap C_B (a) = \hat{1} \cap \hat{1} = \hat{1}$. Hence $C_A \cap C_B = C_{A \cap B}$.

If $a \notin A \cap B$ then $C_{A \cap B} (a) = 0$. On the other hand $a \notin A$ or $a \notin B$, so $C_A (a) = 0$ or $C_B (a) = 0$. Thus $(C_A \cap C_B) (a) = C_A (a) \cap C_B (a) = 0 \cap 0 = 0$. Hence in any case $C_A \cap C_B = C_{A \cap B}$.

(2) Let $a \in S$. If $a \in AB$ then $a = xy$ for some $x \in A$ and $y \in B$. So we have $(C_{AB}) (a) = \hat{1}$. On the other hand $(C_A \circ C_B) (a) = \cup_{a=uv} [C_A (u) \cap C_B (v)] \supseteq C_A (x) \cap C_B (y) = \hat{1} \cap \hat{1} = \hat{1}$.

If $a \notin AB$ then there does not exist $x \in A$ and $y \in B$ such that $a = xy$. Thus $C_{AB} (a) = 0$ and $(C_A \circ C_B) (a) = \cup_{a=uv} [C_A (u) \cap C_B (v)] = \cup_{a=uv} [0 \cap 0] = 0$. Hence in any case $C_A \circ C_B = C_{AB}$.

□

Next we show that the operation $\circ$ is not commutative.

Example 3.5. Let $S = \{x, y, z\}$ be a semigroup, $L = \{0, a, b, c, d, 1\}$ be a complete bounded distributive lattice, $U = \{p, q\}$ and $A = \{x, y\}, B = \{x, z\}$ are subsets of $S$.

\[
\begin{array}{ccc}
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x & x & x & x \\
y & y & y & y \\
z & z & z & z \\
\end{array}
\]

Let $f_A, g_B$ be L-fuzzy soft sets of $S$ over $U$ defined by,

\[
\begin{align*}
 f_A (x) &= \{p, q\} \setminus \{a\}, f_A (y) = \{p, q\} \setminus \{b\}, f_A (z) = \{p, q\} \setminus \{c\}, \\
g_B (x) &= \{p, q\} \setminus \{d\}, g_B (y) = \{p, q\} \setminus \{e\}, g_B (z) = \{p, q\} \setminus \{f\}.
\end{align*}
\]

Now for $x \in S$ we have
\[(f_A \odot g_B)(x) = \bigcup_{y \in yz} [f_A(y) \cap g_B(z)] \]
\[= \bigcup \{f_A(x) \cap g_B(x), f_A(x) \cap g_B(y), f_A(x) \cap g_B(z)\}\]
\[\Rightarrow (f_A \odot g_B)(x) = \{\frac{x}{5}, \frac{x}{9}\}.\]

On the other hand
\[(g_B \odot f_A)(x) = \bigcup_{y \in yz} [g_B(y) \cap f_A(z)] \]
\[= \bigcup \{g_B(x) \cap f_A(x), g_B(x) \cap f_A(y), g_B(x) \cap f_A(z)\}\]
\[\Rightarrow (g_B \odot f_A)(x) = \{\frac{y}{5}, \frac{y}{9}\}.\]

This shows that \(f_A \odot g_B \neq g_B \odot f_A\).

**Lemma 3.6.** Let \(f_A, g_B, h_C\) be \(L\)-fuzzy soft sets of a semigroup \(S\) over \(U\). Then the following hold:

1. \(f_A \odot (g_B \odot h_C) \equiv (f_A \odot g_B) \odot h_C\)
2. \(f_A \odot (g_B \cup h_C) \equiv (f_A \odot g_B) \cup (f_A \odot h_C)\)
3. \((g_B \cup h_C) \odot f_A \equiv (g_B \odot f_A) \cup (h_C \odot f_A)\)
4. \(f_A \odot (g_B \cap h_C) \subseteq (f_A \odot g_B) \cap (f_A \odot h_C)\)
5. \((f_A \cap g_B) \cap h_C \subseteq (f_A \odot h_C) \cap (g_B \odot h_C)\).

**Proof.**

1. Let \(x \in S\). Then
\[
[f_A \odot (g_B \odot h_C)](x) = \bigcup_{y \in yz} \{f_A(y) \cap (g_B \odot h_C)(z)\}
\[= \bigcup_{y \in yz} \{f_A(y) \cap \{x \cap \{f_A(y) \cap g_B(p) \cap h_C(q)\}\}\}
\[= \bigcup_{y \in yz} \{f_A(y) \cap [g_B(p) \cap h_C(q)]\}
\[= \bigcup_{y \in yz} \{f_A(y) \cap g_B(p) \cap h_C(q)\} \subseteq \bigcup_{x \in \lambda} \{f_A(a) \cap g_B(b) \cap h_C(m)\}
\[= (f_A \odot g_B)(x) \cap h_C(x).\]

This implies that \(f_A \odot (g_B \odot h_C) \subseteq (f_A \odot g_B) \odot h_C\).

Similarly we can show that
\[(f_A \odot g_B) \odot h_C \subseteq f_A \odot (g_B \odot h_C).\]

Hence \(f_A \odot (g_B \odot h_C) \equiv (f_A \odot g_B) \odot h_C\).

2. Let \(x \in S\). If \(x\) is not expressible as \(x = yz\) for \(y, z \in S\), then
\[(f_A \odot (g_B \cup h_C))(x) = \emptyset = (f_A \odot g_B)(x) \cup (f_A \odot h_C)(x).\]

Otherwise
\[(f_A \odot (g_B \cup h_C))(x) = \bigcup_{y \in yz} \{f_A(y) \cap (g_B \cup h_C)(z)\}
\[= \bigcup_{y \in yz} \{f_A(y) \cap (g_B(z) \cup h_C(z))\}
\[= \bigcup_{y \in yz} \{f_A(y) \cap g_B(z) \cup f_A(y) \cap h_C(z)\}
\[= \bigcup_{y \in yz} \{f_A(y) \cap g_B(z)\} \cup \{f_A(y) \cap h_C(z)\}
\[= (f_A \odot g_B)(x) \cup (f_A \odot h_C)(x).\]
Hence $f_A \odot (g_B \triangledown h_C) \equiv (f_A \odot g_B) \tilde{\cup} (f_A \odot h_C)$.

Similarly we can prove (3).

(4) Let $x \in S$. If $x$ is not expressible as $x = yz$ for $y, z \in S$, then

$$(f_A \odot (g_B \triangledown h_C))(x) = \tilde{0} = (f_A \odot g_B)(x) \cap (f_A \odot h_C)(x).$$

Otherwise

$$(f_A \odot (g_B \triangledown h_C))(x) = \cup_{x=yz} f_A(y) \cap (g_B \triangledown h_C)(z)$$

$$= \cup_{x=yz} [f_A(y) \cap g_B(z) \cap h_C(z)]$$

$$= \cup_{x=yz} (f_A(y) \cap g_B(z) \cap h_C(z))$$

$$\subseteq \{ \cup_{x=yz} [f_A(y) \cap g_B(z)] \cap \cup_{x=yz} [f_A(y) \cap h_C(z)] \}$$

$$= (f_A \odot g_B)(x) \cap (f_A \odot h_C)(x).$$

Hence $f_A \odot (g_B \triangledown h_C) \equiv (f_A \odot g_B) \tilde{\cap} (f_A \odot h_C)$.

Similarly we can prove (5).

□

Now we show that equality does not hold in (4) and (5).

Example 3.7. Let $S = \{ \alpha, \beta, \gamma \}$ be a semigroup, $L = \{ 0, a, b, c, d, 1 \}$ be a complete bounded distributive lattice, $U = \{ p, q \}$ and $A = S$, $B = \{ \beta, \gamma \}$ and $C = \{ \alpha, \gamma \}$ are subsets of $S$.

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Let $f_A$, $g_B$ and $h_C$ be $L$-fuzzy soft sets of $S$ over $U$.

$$f_A(\alpha) = \{ \frac{p}{a}, \frac{q}{0} \}, f_A(\beta) = \{ \frac{p}{a}, \frac{q}{1} \}, f_A(\gamma) = \{ \frac{p}{a}, \frac{q}{0} \},$$

$$g_B(\alpha) = \{ \frac{p}{0}, \frac{q}{1} \}, g_B(\beta) = \{ \frac{p}{0}, \frac{q}{0} \}, g_B(\gamma) = \{ \frac{p}{0}, \frac{q}{0} \},$$

$$h_C(\alpha) = \{ \frac{p}{d}, \frac{q}{1} \}, h_C(\beta) = \{ \frac{p}{0}, \frac{q}{1} \}, h_C(\gamma) = \{ \frac{p}{0}, \frac{q}{0} \}.$$
Now for $\gamma \in S$, we have
\[
[f_A \otimes (g_B \triangledown h_C)](\gamma) = \bigcup_{\gamma = \alpha \beta} [f_A(\alpha) \cap (g_B(\gamma) \cap h_C(\gamma))]
= \bigcup [f_A(\gamma) \cap (g_B(\alpha) \cap h_C(\alpha)),
\quad f_A(\gamma) \cap (g_B(\beta) \cap h_C(\beta)), f_A(\gamma) \cap (g_B(\gamma) \cap h_C(\gamma))].
\]

Simple calculations show that $[f_A \otimes (g_B \triangledown h_C)](\gamma) = \{\frac{p}{2}, \frac{q}{2}\}$. Now
\[
(f_A \otimes g_B)(\gamma) = \bigcup_{\gamma = \alpha \beta} [f_A(\alpha) \cap g_B(\beta)]
= \bigcup [f_A(\gamma) \cap g_B(\alpha), f_A(\gamma) \cap g_B(\beta), f_A(\gamma) \cap g_B(\gamma)].
\]

Simple calculations show that $(f_A \otimes g_B)(\gamma) = \{\frac{p}{2}, \frac{q}{2}\}$. Also
\[
(f_A \otimes h_C)(\gamma) = \bigcup_{\gamma = \alpha \beta} [f_A(\alpha) \cap h_C(\beta)]
= \bigcup [f_A(\gamma) \cap h_C(\alpha), f_A(\gamma) \cap h_C(\beta), f_A(\gamma) \cap h_C(\gamma)].
\]

Simple calculations show that $(f_A \otimes h_C)(\gamma) = \{\frac{p}{2}, \frac{q}{2}\}$. Then
\[
[(f_A \otimes g_B) \wedge (f_A \otimes h_C)](\gamma) = \{\frac{p}{2}, \frac{q}{2}\},
\]
which shows that
\[
f_A \otimes (g_B \triangledown h_C) \neq (f_A \otimes g_B) \wedge (f_A \otimes h_C).
\]

**Example 3.8.** Let $S = \{x, y, z, p, q\}$ be a semigroup, $L = \{0, a, b, c, d, e, f, 1\}$ be a complete Boolean lattice, $U = \{p, q\}$ and $A = \{x, y, z\}, B = \{x, y, p\}, C = \{y, p, q\}$ are subsets of $S$. 

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x & x & x & x & x & x \\
y & x & x & x & y & z \\
z & x & y & z & x & x \\
p & x & x & x & p & q \\
q & x & p & q & x & x \\
\end{array}
\]

1034
Let $f_A$, $g_B$ and $h_C$ be L-fuzzy soft sets of $S$ over $U$.

\[
\begin{align*}
Let \ f_A (x) &= \{ \frac{l}{1}, \frac{m}{a} \}, \ f_A (y) = \{ \frac{l}{a}, \frac{m}{b} \}, \ f_A (z) = \{ \frac{l}{c}, \frac{m}{a} \}, \ f_A (p) = \{ \frac{l}{c}, \frac{m}{f} \}, \\
\ f_A (q) &= \{ \frac{l}{0}, \frac{m}{0} \} \\
\ g_B (x) &= \{ \frac{l}{1}, \frac{m}{d} \}, \ g_B (y) = \{ \frac{l}{d}, \frac{m}{b} \}, \ g_B (z) = \{ \frac{l}{0}, \frac{m}{0} \}, \ g_B (p) = \{ \frac{l}{c}, \frac{m}{f} \}, \\
\ g_B (q) &= \{ \frac{l}{0}, \frac{m}{0} \} \\
\ h_C (x) &= \{ \frac{p}{1}, \frac{q}{0} \}, \ h_C (y) = \{ \frac{p}{b}, \frac{q}{0} \}, \ h_C (z) = \{ \frac{p}{0}, \frac{q}{0} \}, \ h_C (p) = \{ \frac{p}{d}, \frac{q}{0} \}, \\
\ h_C (q) &= \{ \frac{p}{c}, \frac{q}{d} \}.
\end{align*}
\]

Now for $x \in S$ we have

\[
[(f_A \tilde{\ast} g_B) \ast h_C] (y) = \cup_{y = xz} [(f_A (x) \cap g_B (x)) \cap h_C (z)]
\]

\[
= \cup \{(f_A (y) \cap g_B (y)) \cap h_C (p), (f_A (z) \cap g_B (z)) \cap h_C (y)\}.
\]

Simple calculations show that \([(f_A \tilde{\ast} g_B) \ast h_C] (y) = \{ \frac{p}{b}, \frac{q}{0} \}.\]

Now

\[
(f_A \ast h_C) (y) = \cup_{y = xz} [f_A (x) \cap h_C (z)]
\]

\[
= \cup \{f_A (y) \cap h_C (p), f_A (z) \cap h_C (y)\}.
\]

Simple calculations show that \((f_A \ast h_C) (y) = \{ \frac{p}{b}, \frac{q}{0} \} \). Also

\[
(g_B \ast h_C) (y) = \cup_{y = xz} [g_B (x) \cap h_C (z)]
\]

\[
= \cup \{g_B (y) \cap h_C (p), g_B (z) \cap h_C (y)\}.
\]

Simple calculations show that \((g_B \ast h_C) (y) = \{ \frac{p}{b}, \frac{q}{0} \} \). Thus

\[
[(f_A \ast h_C) \tilde{\ast} (g_B \ast h_C)] (y) = \{ \frac{p}{b}, \frac{q}{0} \},
\]

which shows that \((f_A \tilde{\ast} g_B) \ast h_C \neq (f_A \ast h_C) \tilde{\ast} (g_B \ast h_C)\).

**Lemma 3.9.** Let $f_A, g_B$ and $h_C$ be L-fuzzy soft sets of a semigroup $S$ over $U$. If $f_A \subseteq g_B$, then $f_A \ast h_C \subseteq g_B \ast h_C$ and $h_C \ast f_A \subseteq h_C \ast g_B$.

**Proof.** Let $x \in S$. If $x = yz$ for $y, z \in S$, then $(f_A \ast h_C) (x) = \tilde{0} = (g_B \ast h_C) (x)$. Otherwise

\[
(f_A \ast h_C) (x) = \cup_{x = yz} [f_A (y) \cap h_C (z)]
\]

\[
\subseteq \cup_{x = yz} [g_B (y) \cap h_C (z)], \text{ (because } f_A \subseteq g_B)
\]

\[
= (g_B \ast h_A) (x).
\]

Similarly it can be shown that $h_C \ast f_A \subseteq h_C \ast g_B$. \[\square\]

1035
4. L-Fuzzy Soft Ideals of Semigroups

**Definition 4.1.** An L-fuzzy soft set $f_A$ of a semigroup $S$ over $U$ is called an L-fuzzy soft subsemigroup of $S$ over $U$ if for all $x, y \in S$, $f_A(xy) \supseteq f_A(x) \cap f_A(y)$. That is $[f_A(xy)](u) \supseteq [f_A(x)](u) \land [f_A(y)](u)$, for all $u \in U$.

**Example 4.2.** Let $S = \{0, x, y, z\}$ be a semigroup, $L = \{0, a, b, c, d, e, f, 1\}$ be a complete Boolean lattice, $U = \{l, m\}$ and $G = \{0, x, y\} \subseteq S$.

\[
\begin{array}{cccc}
* & 0 & x & y & z \\
0 & 0 & 0 & 0 & 0 \\
x & 0 & x & y & 0 \\
y & 0 & 0 & 0 & 0 \\
z & 0 & z & 0 & 0 \\
\end{array}
\]

Let $f_G$ be an L-fuzzy soft set of $S$ over $U$ defined by,

\[
f_G(0) = \{\frac{l}{1}, \frac{m}{a}\}, f_G(x) = \{\frac{l}{b}, \frac{m}{1}\}, f_G(y) = \{\frac{l}{c}, \frac{m}{e}\}, f_G(z) = \{\frac{l}{0}, \frac{m}{0}\}.
\]

Simple calculations show that $f_G$ is an L-fuzzy soft subsemigroup of $S$ over $U$.

**Lemma 4.3.** The intersection of two L-fuzzy soft subsemigroups of a semigroup $S$ over $U$ is again an L-fuzzy soft subsemigroup of $S$ over $U$.

**Proof.** Let $f_A$ and $g_B$ be two L-fuzzy soft subsemigroups of a semigroup $S$ over $U$ and $x, y \in S$. Then

\[
(f_A \cap g_B)(xy) = f_A(xy) \cap g_B(xy) \supseteq (f_A(x) \cap f_A(y)) \cap (g_B(x) \cap g_B(y)) = (f_A(x) \cap g_B(x)) \cap (f_A(y) \cap g_B(y)) = (f_A \cap g_B)(x) \cap (f_A \cap g_B)(y).
\]

Next we show that the union of two L-fuzzy soft subsemigroups of a semigroup is not necessarily an L-fuzzy soft subsemigroup.
Example 4.4. Let \( S = \{x, y, z, p, q\} \) be a semigroup, \( L = \{0, a, b, c, d, e, f, 1\} \) be a complete Boolean lattice, \( U = \{l, m\} \) and \( A, B = S \).

\[
\begin{array}{c|cccccc}
* & x & y & z & p & q \\
\hline
x & x & x & x & x & x \\
y & y & y & y & z \\
z & z & z & x & x \\
p & p & p & p & q \\
q & q & q & q & x & x \\
\end{array}
\]

Let \( f_A, g_B \) be \( L \)-fuzzy soft sets of \( S \) over \( U \) defined by,

\[
\begin{align*}
f_A(x) &= \{l : \frac{m}{e}, m : \frac{0}{f}\}, \quad f_A(y) = \{l : \frac{2}{b}, m : \frac{0}{c}\}, \quad f_A(z) = \{l : \frac{1}{c}, m : \frac{0}{f}\}, \\
f_A(p) &= \{l : \frac{m}{f}, m : \frac{0}{c}\}, \quad f_A(q) = \{l : \frac{0}{c}, m : \frac{1}{f}\} \\
g_B(x) &= \{l : \frac{1}{c}, m : \frac{0}{f}\}, \quad g_B(y) = \{l : \frac{1}{b}, m : \frac{0}{c}\}, \quad g_B(z) = \{l : \frac{0}{b}, m : \frac{1}{f}\}, \\
g_B(p) &= \{l : \frac{0}{f}, m : \frac{0}{c}\}, \quad g_B(q) = \{l : \frac{0}{c}, m : \frac{0}{f}\}.
\end{align*}
\]

Simple calculations show that \( f_A \) and \( g_A \) are \( L \)-fuzzy soft subsemigroups of \( S \) over \( U \). Now

\[
\begin{align*}
(f_A \square g_B)(x) &= \{l : \frac{m}{e}, m : \frac{0}{f}\}, \quad (f_A \square g_B)(y) = \{l : \frac{2}{b}, m : \frac{0}{c}\}, \\
(f_A \square g_B)(z) &= \{l : \frac{0}{f}, m : \frac{0}{c}\}, \quad (f_A \square g_B)(p) = \{l : \frac{m}{f}, m : \frac{0}{c}\}, \\
(f_A \square g_B)(q) &= \{l : \frac{m}{f}, m : \frac{0}{c}\}.
\end{align*}
\]

Now,

\[
\begin{align*}
(f_A \square g_B)(y) \cap (f_A \square g_B)(q) &= \{l : \frac{0}{f}, m : \frac{0}{c}\}, \quad \text{but } a \not\geq c.
\end{align*}
\]

Hence \( f_A \square g_B \) is not an \( L \)-fuzzy soft subsemigroup of \( S \) over \( U \).
Lemma 4.5. Let $A$ be a non-empty subset of a semigroup $S$. Then $A$ is a subsemigroup of $S$ if and only if $C_A$ is an $L$-fuzzy soft subsemigroup of $S$ over $U$.

Proof. Suppose $A$ is a subsemigroup of $S$ and $x, y \in S$. If $x, y \in A$ then $xy \in A$. This implies $C_A(xy) = \hat{1} = \hat{1} \cap \hat{1} = C_A(x) \cap C_A(y)$. If one of $x, y$ doesn’t belong to $A$ then $C_A(x) \cap C_A(y) = \emptyset \subseteq C_A(xy)$. Hence in any case $C_A(xy) \supseteq C_A(x) \cap C_A(y)$.

Conversely, assume that $C_A$ is an $L$-fuzzy soft subsemigroup of $S$ over $U$. Let $x, y \in A$. Then $C_A(x) = \hat{1}$ and $C_A(y) = \hat{1} \implies C_A(x) \cap C_A(y) = \hat{1} \cap \hat{1} = \hat{1}$. But $C_A(xy) \supseteq C_A(x) \cap C_A(y) = \hat{1}$. Hence $C_A(xy) = \hat{1}$. This implies $xy \in A$, that is $A$ is a subsemigroup of $S$. □

Lemma 4.6. Let $f_A$ be an $L$-fuzzy soft set of a semigroup $S$. Then $f_A$ is an $L$-fuzzy soft subsemigroup of $S$ if and only if $f_A \circ f_A \subseteq f_A$.

Proof. Let $f_A$ be an $L$-fuzzy soft subsemigroup of $S$ over $U$. Let $x \in S$. If $(f_A \circ f_A)(x) = \emptyset$, then clearly $(f_A \circ f_A)(x) \subseteq (f_A)(x)$. Otherwise there exist $y, z \in S$ such that $x = yz$. In this case

$$(f_A \circ f_A)(x) = \bigcup_{x=yz} (f_A(y) \cap f_A(z)) \subseteq \bigcup_{x=yz} f_A(yz) = \bigcup_{x=yz} f_A(x) = f_A(x).$$

Hence $f_A \circ f_A \subseteq f_A$.

Conversely, for $x, y \in S$

$$f_A(xy) \supseteq (f_A \circ f_A)(xy) = \bigcup_{xy=pq} f_A(p) \cap f_A(q) \supseteq f_A(x) \cap f_A(y).$$

Hence $f_A$ is an $L$-fuzzy soft subsemigroup of $S$ over $U$. □

Definition 4.7. Let $\alpha \in L'$ and $h_G$ be an $L$-fuzzy soft set of $S$ over $U$. Then $\alpha$-cut of $h_G$ is denoted by $h_G^\alpha$ and is given by,

$$h_G^\alpha = \{ s \in S : h_G(s) \geq \alpha \}$$

that is

$$h_G^\alpha = \{ s \in S : [h_G(s)](u) \geq \alpha \ \forall u \in U \}.$$

Example 4.8. Let $S = \{x, y, z, t\}$ be a semigroup, $L = \{0, a, b, c, d, e, f, 1\}$ be a complete Boolean lattice, $U = \{l, m\}$ and $G = \{x, y, z\} \subseteq S$. 

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Let $h_G$ be an $L$-fuzzy soft set of $S$ over $U$ defined by,

$$
\begin{align*}
   h_G(x) &= \{ \frac{p}{1}, \frac{q}{c} \}, \\
   h_G(y) &= \{ \frac{p}{a}, \frac{q}{c} \}, \\
   h_G(z) &= \{ \frac{p}{e}, \frac{q}{d} \}, \\
   h_G(t) &= \{ \frac{p}{0}, \frac{q}{0} \}.
\end{align*}
$$

Let $\alpha(p) = a$, $\alpha(q) = f$. Then $h_G^\alpha = \{ s \in S : h_G(s) \supseteq \alpha \}$ = \{ s \in S : [h_G(s)](u) \geq \alpha(u) \ \forall \ u \in U \} = \{ x, y \}$.

**Theorem 4.9.** An $L$-fuzzy soft set $f_G$ of a semigroup $S$ over $U$ is an $L$-fuzzy soft subsemigroup of $S$ over $U$ if and only if each $\alpha$-cut of $f_G$ is a subsemigroup of $S$.

**Proof.** Let $f_G$ be a non zero $L$-fuzzy soft subsemigroup of $S$ over $U$. Let $x, y \in f_G^\alpha \implies f_G(x) \supseteq \alpha$ and $f_G(y) \supseteq \alpha \implies f_G(xy) \supseteq f_G(x) \cap f_G(y) \supseteq \alpha \cap \alpha = \alpha$. Thus $xy \in f_G^\alpha$. This implies $f_G^\alpha = \{ s \in S : f_G(s) \supseteq \alpha \}$ is a subsemigroup of $S$.

Conversely, suppose that there exist $x, y \in S$ such that $f_G(xy) \subset f_G(x) \cap f_G(y)$. This implies that there exist $\beta \in L^U$ such that $f_G(xy) \subset f_G(x) \cap f_G(y)$. As $f_G(x) \cap f_G(y) \supseteq \beta \implies x \in f_G^\beta$ and $y \in f_G^\beta$. But $xy \notin f_G^\beta$, because $f_G(xy) \subset \beta$. This shows that $f_G^\beta$ is not a subsemigroup of $S$, which is a contradiction. Hence $f_G(xy) \supseteq f_G(x) \cap f_G(y)$ for all $x, y \in S$. \qed

**Definition 4.10.** An $L$-fuzzy soft set $f_G$ of a semigroup $S$ over $U$ is called an $L$-fuzzy soft left (right) ideal of $S$ over $U$ if and only if each $\alpha$-cut of $f_G$ is a subsemigroup of $S$.

**Definition 4.11.** An $L$-fuzzy soft set $f_G$ of a semigroup $S$ over $U$ is called an $L$-fuzzy soft left (right) ideal of $S$ over $U$ if and only if each $\alpha$-cut of $f_G$ is a subsemigroup of $S$.

**Example 4.11.** Let $S = \{0, x, y, z\}$ be a semigroup, $L = \{0, a, b, c, d, e, f, 1\}$ be a complete Boolean lattice, $U = \{l, m, n\}$ and $A = \{0, x, z\} \subseteq S$.

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Let $f_A$ be an $L$-fuzzy soft set of $S$ over $U$ defined by,

$$f_A(0) = \{l, m_0, n_0\}, f_A(x) = \{\frac{l}{c}, m_a, n_b\},$$

$$f_A(y) = \{\frac{l}{e}, m_0, n_0\}, f_A(z) = \{\frac{l}{b}, m_a, n_c\}.$$

Simple calculations show that $f_A$ is an $L$-fuzzy soft left ideal of $S$ over $U$. But $f_A$ is not an $L$-fuzzy soft right ideal of $S$ over $U$, because $f_A(xy) = f_A(y) = \{\frac{l}{b}, m_0, n_0\} \nsubseteq \{\frac{l}{c}, m_a, n_b\} = f_A(x)$.

**Example 4.12.** Let $S = \{0, x, y, z\}$ be a semigroup, $L = \{0, a, b, c, d, 1\}$ be a complete bounded distributive lattice, $U = \{p, q\}$ and $B = \{0, x, y\} \subseteq S$.

$$
\begin{array}{c|cccc}
  & 0 & x & y & z \\
\hline
0 & 0 & 0 & 0 & 0 \\
x & 0 & x & y & 0 \\
y & 0 & 0 & 0 & 0 \\
z & 0 & z & 0 & 0 \\
\end{array}
$$

Let $g_B$ be an $L$-fuzzy soft set of $S$ over $U$ defined by,

$$g_B(0) = \{\frac{p}{a}, q_0\}, g_B(x) = \{\frac{p}{c}, q_b\}, g_B(y) = \{\frac{p}{1}, q_0\}, g_B(z) = \{\frac{p}{b}, q_0\}.$$

Simple calculations show that $g_B$ is an $L$-fuzzy soft right ideal of $S$ over $U$, but it is not an $L$-fuzzy soft left ideal of $S$ over $U$, because $g_B(xz) = g_B(z) = \{\frac{p}{b}, q_0\} \nsubseteq g_B(x) = \{\frac{p}{c}, q_b\}$.
Lemma 4.13. Let $A$ be a non-empty subset of a semigroup $S$. Then $A$ is a left (right, two-sided) ideal of $S$ if and only if $C_A$ is an $L$-fuzzy soft left (right, two-sided) ideal of $S$ over $U$.

Proof. The proof is similar to the proof of Lemma 4.5. □

Lemma 4.14. Every $L$-fuzzy soft left (right) ideal of a semigroup $S$ over $U$ is an $L$-fuzzy soft subsemigroup of $S$ over $U$.

Proof. Straightforward. □

The following example shows that the converse of the above Lemma is not true.

Example 4.15. Let $S = \{x, y, z, p, q\}$ be a semigroup, $L = \{0, a, b, c, d, e, f, 1\}$ be a Boolean lattice, $U = \{l, m\}$ and $A, B = S$.

\[
\begin{array}{c|ccccc}
* & x & y & z & p & q \\
\hline
x & x & x & x & x & x \\
y & x & x & x & y & z \\
z & x & y & z & x & x \\
p & x & x & x & p & q \\
q & x & p & q & x & x \\
\end{array}
\]

Let $f_A, g_B$ be $L$-fuzzy soft sets of $S$ over $U$.

\[
f_A(x) = \{\frac{l}{1}, \frac{m}{0}, \frac{m}{a}\}, f_A(y) = \{\frac{l}{b}, \frac{m}{0}\}, f_A(z) = \{\frac{l}{c}, \frac{m}{f}\},
\]
\[
f_A(p) = \{\frac{l}{b}, \frac{m}{a}\}, f_A(q) = \{\frac{l}{0}, \frac{m}{c}\}
\]
\[
g_B(x) = \{\frac{l}{1}, \frac{m}{c}\}, g_B(y) = \{\frac{l}{b}, \frac{m}{a}\}, g_B(z) = \{\frac{l}{0}, \frac{m}{c}\},
\]
\[
g_B(p) = \{\frac{l}{e}, \frac{m}{f}\}, g_B(q) = \{\frac{l}{0}, \frac{m}{f}\}.
\]

Simple calculations show that $f_A$ and $g_B$ are $L$-fuzzy soft subsemigroups of $S$ over $U$. But neither $f_A$ nor $g_B$ are $L$-fuzzy soft left ideals of $S$ over $U$, because $f_A(yp) = f_A(y) = \{\frac{l}{b}, \frac{m}{0}\} \nsubseteq f_A(p) = \{\frac{l}{c}, \frac{m}{f}\}$ and $g_B(qz) = g_B(q) = \{\frac{l}{0}, \frac{m}{f}\} \nsubseteq g_B(z) = \{\frac{l}{b}, \frac{m}{a}\}$. 

1041
Lemma 4.16. (1) The intersection of L-fuzzy soft left (right, two-sided) ideals of a semigroup $S$ over $U$ is again an L-fuzzy soft left (right, two-sided) ideal of $S$ over $U$.

(2) The union of L-fuzzy soft left (right, two-sided) ideals of a semigroup $S$ over $U$ is again an L-fuzzy soft left (right, two-sided) ideal of $S$ over $U$.

Proof. Straightforward. □

Lemma 4.17. Let $f_B$ be an L-fuzzy soft set of a semigroup $S$. Then $f_B$ is an L-fuzzy soft left (right) ideal of $S$ over $U$ if and only if $\bigcap f_B \subseteq f_B$.

Proof. Suppose $f_B$ be an L-fuzzy soft left ideal of $S$ over $U$. Let $x \in S$. If $x \neq yz$, then $(\bigcap f_B)(x) = 0 \subseteq f_B(x)$. Otherwise there exist $y, z \in S$, such that $x = yz$. Then

$$\bigcap f_B(x) = \bigcup_{x=yz} \left( \bigcap f_B(yz) \right) \subseteq \bigcup_{x=yz} \left( \bigcap f_B(x) \right) = \bigcup_{x=yz} \left( \bigcap f_B(x) \right).$$

So in any case $\bigcap f_B \subseteq f_B$.

Conversely, assume that $\bigcap f_B \subseteq f_B$. Let $y, z \in S$. Then

$$f_B(yz) \supseteq \bigcap f_B(x) = \bigcup_{y=ab} \left( \bigcap (a) \cap f_B(b) \right) \supseteq \bigcap f_B(z) = f_B(z).$$

Hence $f_B$ is an L-fuzzy soft left ideal of $S$ over $U$. □

Theorem 4.18. An L-fuzzy soft set $f_G$ of a semigroup $S$ over $U$ is an L-fuzzy soft left (right) ideal of $S$ over $U$ if and only if each $\alpha$-cut of $f_G$ is a left (right) ideal of $S$.

Proof. The proof is similar to the proof of Theorem 4.9. □

Next we characterize different classes of semigroups by the properties of their L-fuzzy soft ideals.

5. Regular and intra-regular semigroups

Recall that a semigroup $S$ is regular if for all $a \in S$, there exists $x \in S$ such that $a = axa$.

A semigroup $S$ is said to be intra-regular if for each $a \in S$ there exist $y, z \in S$ such that $a = yaa$. In general, neither regular semigroup is intra-regular nor intra-regular semigroup is regular. If $S$ is commutative then both the concepts coincide.

Example 5.1. Let $A$ be a countably infinite set and

$$S = \{ \alpha : A \rightarrow A : \alpha \text{ is one one and } A - \alpha(A) \text{ is infinite} \}.$$
Then $S$ is a semigroup with respect to the composition of functions and is called Baer-Levi Semigroup. It is well known that this semigroup is right cancelative, right simple without idempotents. Thus $S$ is not regular but intra-regular.

**Example 5.2.** Consider the semigroup $S = \{0, 1, 2, 3, 4\}$.

\[
\begin{array}{cccccc}
 & 0 & 1 & 2 & 3 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 2 \\
2 & 0 & 1 & 2 & 0 & 0 \\
3 & 0 & 0 & 0 & 3 & 4 \\
4 & 0 & 3 & 4 & 0 & 0 \\
\end{array}
\]

This semigroup $S$ is regular but not intra-regular.

It is well known that

**Theorem 5.3.** A semigroup $S$ is regular if and only if $R \cap L = RL$ for every right ideal $R$ and left ideal $L$ of $S$.

Now we show that:

**Theorem 5.4.** The following assertions are equivalent for a semigroup $S$.

1. $S$ is regular.
2. $f_A \cap g_B = f_A \circ g_B$ for every $L$-fuzzy soft right ideal $f_A$ and $L$-fuzzy soft left ideal $g_B$ of $S$ over $U$.

**Proof.** (1) $\Rightarrow$ (2) Let $f_A$ be an $L$-fuzzy soft right ideal and $g_B$ an $L$-fuzzy soft left ideal of $S$ over $U$. Then by Lemma 4.17, $f_A \circ g_B \subseteq f_A \circ 1 \subseteq f_A$ and $f_A \circ g_B \subseteq 1 \circ g_B \Rightarrow f_A \circ g_B \subseteq f_A \cap g_B$. Now, let $x \in S$, since $S$ is regular so there exists $a \in S$ such that $x = xa$. Thus we have

\[
(f_A \circ g_B)(x) = \bigcup_{z \in g_B} (f_A(y) \cap g_B(z)) \supseteq f_A(xa) \cap g_B(x) \supseteq f_A(x) \cap g_B(x) = (f_A \cap g_B)(x)
\]

$\Rightarrow f_A \cap g_A \subseteq f_A \cap g_A$. Hence $f_A \cap g_A = f_A \circ g_B$.

(2) $\Rightarrow$ (1) Let $R$ and $L$ be any right and left ideal of $S$, respectively. Then by Lemma 4.13, $C_R$ and $C_L$ are $L$-fuzzy soft right ideal and $L$-fuzzy soft left ideal of $S$ over $U$, respectively. Then by Lemma 3.4

\[
C_{RL} = (C_R \cap C_L) \supseteq (C_R \cap C_L) = C_{RL}.
\]

Thus $R \cap L = RL$. Hence by Theorem 5.3, $S$ is regular.

**Theorem 5.5.** For a semigroup $S$ the following conditions are equivalent:

1. $S$ is intra-regular.
2. $L \cap R \subseteq LR$ for every left ideal $L$ and right ideal $R$ of $S$.
3. $f_A \cap g_B \subseteq f_A \circ g_B$ for every $L$-fuzzy soft left ideal $f_A$ and $L$-fuzzy soft right ideal $g_B$ of $S$ over $U$.
Proof. (1) $\Leftrightarrow$ (2) It is well known.

(1) $\Rightarrow$ (3) Let $f_A$ be an $L$-fuzzy soft left and $g_B$ an $L$-fuzzy soft right ideals of $S$. Let $a \in S$. Since $S$ is intra-regular, so there exist $x, y \in S$ such that $a = xa^2y$. Thus

$$(f_A \otimes g_B)(a) = \bigcup_{a=pq} (f_A(p) \cap g_B(q))$$

$$(f_A(xa) \cap g_B(ay) \supseteq f_A(a) \cap g_B(a))$$

$$= (f_A \cap g_B)(a)$$

$$\Rightarrow f_A \cap g_B \subseteq f_A \otimes g_B.$$

(3) $\Rightarrow$ (2) Let $L$ be a left and $R$ be a right ideal of $S$. By Lemma 4.13, $C_L$ and $C_R$ are $L$-fuzzy soft left ideal and $L$-fuzzy soft right ideal of $S$ over $U$, respectively. So by Lemma 3.4, $C_{LR} = C_L \otimes C_R \supseteq C_L \cap C_R = C_{LR}$. This implies $L \cap R \subseteq LR$. □

References


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