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Fuzzy rough topological transformation groups

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Abstract. The purpose of this paper is to introduce the concepts of fuzzy rough topological spaces, fuzzy rough continuous functions, fuzzy rough compact spaces, fuzzy rough topological groups, fuzzy rough topological transformation groups and fuzzy rough points are introduced and studied. Some interesting properties are also discussed.

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1. Introduction

The concept of fuzzy sets introduced by L. A. Zadeh [12]. Fuzzy sets have applications in many fields such as information [9] and control [10]. The theory of fuzzy topological spaces was introduced and developed by C. L. Chang [2]. Z. Pawlak [8] introduced the definition of rough set. R. Biswas and S. Nanda [1] introduced the notion of rough groups and rough subgroups. S. Nanda and S. Majumdar [6] studied the concept of fuzzy rough set. The concept of fuzzy rough set and rough fuzzy set first proposed by D. Dubois and H. Prade [3]. W. H. Xu , Q. R. Wang and X. T. Zhang [11] introduced the concepts of multi-granulation fuzzy rough sets in fuzzy tolerance approximation spaces. The concepts of topology approach to multigranulation rough sets was introduced by G. P. Lin, J. Y. Liang and Y. H. Qian [4]. The concept of fuzzy rough topological space was introduced by S. Padmapriya, M. K. Uma and E. Roja [7]. In this paper, the concepts of fuzzy rough topological spaces, fuzzy rough continuous functions, fuzzy rough compact spaces, fuzzy rough subgroups, fuzzy rough topological groups, fuzzy rough topological transformation groups, fuzzy rough homomorphisms and fuzzy rough points are introduced and studied. Some interesting properties are also discussed.

2. Preliminaries

Definition 2.1 ([6]). Let U be a set and $\mathscr{B}$ be a Boolean subalgebra of the Boolean algebra of all subsets of U. Let L be a lattice. Let X be a rough set. Then $X = (X_L, X_U) \in \mathscr{B}^2$ with $X_L \subset X_U$.

A fuzzy rough set $A = (A_L, A_U)$in X is characterized by a pair of maps $\mu_{A_L} : X_L \to L$ and $\mu_{A_U} : X_U \to L$ with the property that $\mu_{A_L}(x) \leq \mu_{A_U}(x)$ for all $x \in X_U$.

Note 2.2 ([6]). In particular, L could be the closed interval $[0, 1]$.

Definition 2.3 ([6]). For any two fuzzy rough sets $A = (A_L, A_U)$ and $B = (B_L, B_U)$ in X we define

(i) $A = B$ iff

$$\mu_{A_L}(x) = \mu_{B_L}(x) \text{ for every } x \in X_L \text{ and}$$
$$\mu_{A_U}(x) = \mu_{B_U}(x) \text{ for every } x \in X_U.$$

(ii) $A \subseteq B$ iff

$$\mu_{A_L}(x) \leq \mu_{B_L}(x) \text{ for every } x \in X_L \text{ and}$$
$$\mu_{A_U}(x) \leq \mu_{B_U}(x) \text{ for every } x \in X_U.$$

(iii) $C = A \cup B$ iff

$$\mu_{C_L}(x) = max\{\mu_{A_L}(x), \mu_{B_L}(x)\} \text{ for all } x \in X_L,$$
$$\mu_{C_U}(x) = max\{\mu_{A_U}(x), \mu_{B_U}(x)\} \text{ for all } x \in X_U.$$

(iv) $D = A \cap B$ iff

$$\mu_{D_L}(x) = min\{\mu_{A_L}(x), \mu_{B_L}(x)\} \text{ for all } x \in X_L,$$
$$\mu_{D_U}(x) = min\{\mu_{A_U}(x), \mu_{B_U}(x)\} \text{ for all } x \in X_U.$$

More generally, if L is complete lattice, then for any index set I, if $\{A_i : i \in I\}$ is a family of fuzzy rough sets we have $E = \cup_i A_i$ iff

$$\mu_{E_L}(x) = sup_{i\in I}\mu_{A_{L_i}}(x) \text{ for every } x \in X_L$$
$$\text{and } \mu_{E_U}(x) = sup_{i\in I}\mu_{A_{U_i}}(x) \text{ for every } x \in X_U.$$

Similarly $F = \cap_i A_i$ iff

$$\mu_{F_L}(x) = inf_{i\in I}\mu_{A_{L_i}}(x) \text{ for every } x \in X_L$$
$$\text{and } \mu_{F_U}(x) = inf_{i\in I}\mu_{A_{U_i}}(x) \text{ for every } x \in X_U.$$

We define the complement A' of A by the ordered pair (A'_L, A'_U) of membership functions where

$$\mu_{A'_L}(x) = 1 - \mu_{A_L}(x) \text{ for all } x \in X_L$$
$$\text{and } \mu_{A'_U}(x) = 1 - \mu_{A_U}(x) \text{ for all } x \in X_U.$$

Definition 2.4 ([5]). The null fuzzy rough set and whole fuzzy rough set in X are defined by $\tilde{0} = (0_L, 0_U)$ and $\tilde{1} = (1_L, 1_U)$.

Definition 2.5 ([5]). Let $A = (A_L, A_U)$ be a fuzzy rough set in X. Then the complement A' of A is defined by ordered pairs (A'_L, A'_U) of membership functions where

$$A'_L(x) = 1 - A_L(x) \text{ for every } x \in X_L$$
$$\text{and } A'_U(x) = 1 - A_U(x) \text{ for every } x \in X_U.$$

Definition 2.6 ([5])**.** Let $(V, \mathcal{B})$ and $(V_1, \mathcal{B}_1)$ be two rough universes and $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$. Let $A = (A_L, A_U)$ be a fuzzy rough set in X. Then $Y = f(X) \in \mathcal{B}_1^2$ and $Y_L = f(X_L), Y_U = f(X_U)$. The image of A under f, denoted by

$$f(A) = (f(A_L), f(A_U)$$

is defined by

$$f(A_L)(y) = \vee\{A_L(x) : x \in X_L \cap f^{-1}(y)\} \text{ for every } y \in Y_L, \text{ and}$$
$$f(A_U)(y) = \vee\{A_U(x) : x \in X_U \cap f^{-1}(y)\} \text{ for every } y \in Y_U.$$

Definition 2.7 ([5])**.** Let $B = (B_L, B_U)$ be a fuzzy rough set in Y where $Y = (Y_L, Y_U) \in \mathcal{B}'_1$ is a rough set. Then $X = f^{-1}(Y) \in \mathcal{B}_1^2$, where $X_L = f^{-1}(Y_L), X_U = f^{-1}(Y_U)$. Then the inverse image of B under f, denoted by

$$f^{-1}(B) = (f^{-1}(B_L), f^{-1}(B_U))$$

is defined by

$$f^{-1}(B_L)(x) = B_L(f(x)) \text{ for every } x \in X_L \text{ and}$$
$$f^{-1}(B_U)(x) = B_U(f(x)) \text{ for every } x \in X_U.$$

Proposition 2.8 ([5])**.** *If $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Then for all FRSs $A, A_i \in Y$, $i \in J$, we have*

(i) $f^{-1}(\cup_i B_i) = \cup_i f^{-1}(B_i)$,
(ii) $f^{-1}(\cap_i B_i) = \cap_i f^{-1}(B_i)$,
(iii) $f(\cup_i B_i) = \cup_i f(B_i)$,
(iv) $f(\cap_i B_i) \subset \cap_i f(B_i)$.

Proposition 2.9 ([5])**.** *If $f : (V, \mathcal{B}) \to (V_1, \mathcal{B}_1)$ be a mapping such that $f^{-1} : (V_1, \mathcal{B}_1) \to (V, \mathcal{B})$. Then for all FRS A in X and B in Y, we have*

(i) $B = f(f^{-1}(B))$,
(ii) $A \subset f^{-1}(f(A))$.

3. Fuzzy rough topological space

Definition 3.1 ([7])**.** A fuzzy rough topology (in short, FRT) is a family τ of fuzzy rough sets in $X = (X_L, X_U)$ satisfying the following axioms:

(i) $\tilde{0}, \tilde{1} \in \tau$,
(ii) $A \cap B \in \tau$ for any $A, B \in \tau$,
(iii) $\cup A_i \in \tau$ for any arbitrary family $\{A_i : i \in J\} \subseteq \tau$.

In this case the pair (X, τ) is called a fuzzy rough topological space (in short, FRTS) and any fuzzy rough set in τ is known as fuzzy rough open set (in short, FROS) in X.

Example 3.2. Let $X = \{a, b, c\}$ be a non-empty set and $\mathfrak{B}$ a Boolean subalgebra of the Boolean algebra of all subsets of X. Let X be a rough set. Then $X = (X_L, X_U) \in \mathfrak{B}^2$ with $X_L \subset X_U$, where $X_L = \{a, b\}$ and $X_U = \{a, b, c\}$. Let $A = ((\frac{a}{0.2}, \frac{b}{0.3}), (\frac{a}{0.3}, \frac{b}{0.3}), \frac{c}{0.3})$ and $B = ((\frac{a}{0.3}, \frac{b}{0.3}), (\frac{a}{0.3}, \frac{b}{0.4}, \frac{c}{0.3}))$ be fuzzy rough sets of X. Then the family $\tau = \{\tilde{0}, \tilde{1}, A, B\}$ is a fuzzy rough topology on X. Clearly, (X, τ) is a fuzzy rough topological space.

Definition 3.3 ([7]). A fuzzy rough set is a fuzzy rough closed set (in short, FRCS) if and only if its complement is a fuzzy rough open set.

Definition 3.4. Let (X, τ) be a fuzzy rough topological space and $A = (A_L, A_U)$ be a fuzzy rough set in $X = (X_L, X_U)$. Then the fuzzy rough interior of A is denoted by $FRint(A)$ and is defined by

$$FRint(A) = \cup\{B : B = (B_L, B_U) \text{ is a fuzzy rough open set in } X \text{ and } B \subseteq A\}.$$

Example 3.5. Let $X = \{a, b, c\}$ be a non-empty set and $\mathfrak{B}$ a Boolean subalgebra of the Boolean algebra of all subsets of X. Let X be a rough set. Then $X = (X_L, X_U) \in \mathfrak{B}^2$ with $X_L \subset X_U$, where $X_L = \{a, b\}$ and $X_U = \{a, b, c\}$. Let $A = ((\frac{a}{0.1}, \frac{b}{0.2}), (\frac{a}{0.2}, \frac{b}{0.3}, \frac{c}{0.3}))$ and $B = ((\frac{a}{0.4}, \frac{b}{0.2}), (\frac{a}{0.4}, \frac{b}{0.3}, \frac{c}{0.4}))$ be fuzzy rough sets of X. Then the family $\tau = \{\tilde{0}, \tilde{1}, A, B\}$ is a fuzzy rough topology on X. Clearly, (X, τ) is a fuzzy rough topological space. Let $C = \{(\frac{a}{0.4}, \frac{b}{0.3}), (\frac{a}{0.5}, \frac{b}{0.3}, \frac{c}{0.3})\}$ be a fuzzy rough set of X. Now, $FRint(C) = ((\frac{a}{0.4}, \frac{b}{0.2}), (\frac{a}{0.4}, \frac{b}{0.3}, \frac{c}{0.4}))$.

Definition 3.6. Let (X, τ) be a fuzzy rough topological space and $A = (A_L, A_U)$ be a fuzzy rough set in $X = (X_L, X_U)$. Then the fuzzy rough closure of A is denoted by $FRcl(A)$ and is defined by

$$FRcl(A) = \cap\{B : B = (B_L, B_U) \text{ is a fuzzy rough closed set in } X \text{ and } B \supseteq A\}.$$

Example 3.7. Let $X = \{a, b, c\}$ be a non-empty set and $\mathfrak{B}$ a Boolean subalgebra of the Boolean algebra of all subsets of X. Let X be a rough set. Then $X = (X_L, X_U) \in \mathfrak{B}^2$ with $X_L \subset X_U$, where $X_L = \{a\}$ and $X_U = \{a, b\}$. Let $A = ((\frac{a}{0.1}), (\frac{a}{0.3}, \frac{b}{0.3}))$ and $B = ((\frac{a}{0.2}), (\frac{a}{0.3}, \frac{b}{0.4}))$ be fuzzy rough sets of X. Then the family $\tau = \{\tilde{0}, \tilde{1}, A, B\}$ is a fuzzy rough topology on X. Clearly, (X, τ) is a fuzzy rough topological space. Let $D = \{(\frac{a}{0}), (\frac{a}{0.2}, \frac{b}{0.3})\}$ be a fuzzy rough set of X. Now, $FRcl(D) = ((\frac{a}{0.8}), (\frac{a}{0.7}, \frac{b}{0.6}))$.

Remark 3.8.
(i) $FRint(A) \subseteq A \subseteq FRcl(A)$.
(ii) $FRint(FRint(A)) = FRint(A)$.
(iii) $FRcl(FRcl(A)) = FRcl(A)$.

Definition 3.9 ([7]). A function f from a fuzzy rough topological space (X, τ) to a fuzzy rough topological space (Y, σ) is said to be a fuzzy rough continuous function if $f^{-1}(A)$ is a fuzzy rough open (resp. closed) set in X for each fuzzy rough open (resp. closed) set A in Y.

Proposition 3.10. *Let $f : X \to Y$ be a surjective function. If $f : (X, \tau) \to (Y, \sigma)$ is a fuzzy rough continuous function. Then $f(FRcl(A)) \subseteq FRcl(f(A))$, for each fuzzy rough set $A = (A_L, A_U)$ of X.*

Definition 3.11. Let X be a fuzzy rough topological space and $A = (A_L, A_U)$ be a fuzzy rough set of X. A fuzzy rough open cover of A is a collection $\{A_i : i \in J\}$ of fuzzy rough open sets of X such that $A \subseteq \cup\{A_i : i \in J\}$. If every fuzzy rough open cover of A and $\varepsilon > 0$ there exists a finite subcollection $\{A_j : j = 1, 2, ..., n\}$ such that $\cup\{A_j : j = 1, 2, ..., n\} \supseteq A - \varepsilon$ then A is said to be fuzzy rough compact.

Definition 3.12. Let $(G,+)$ be a rough group. Then a fuzzy rough set $A = (A_L, A_U)$ is said to be a fuzzy rough subgroup of $G = (G_L, G_U)$ if

(i) $A_L(x+y) \geq min\{A_L(x), A_L(y)\}$ and $A_U(x+y) \geq min\{A_U(x), A_U(y)\}$,

(ii) $A_L(-x) = A_L(x)$ and $A_U(-x) = A_U(x)$.

Definition 3.13. Let (X,τ) be a fuzzy rough topological space. A fuzzy rough set $\overline{A} = (\overline{A}_L, \overline{A}_U)$, where $\overline{A}_L(x) = \alpha$ and $\overline{A}_U(x) = \beta$ such that $\overline{A}_L(x) \leq \overline{A}_U(x)$, for every $x \in X_U$. Then (X,τ) is called a fully stratified fuzzy rough topological space if for every $\alpha, \beta \in [0,1]$, $\overline{A} \in \tau$.

Definition 3.14. Let G be a rough group and let (X,τ) be a fully stratified fuzzy rough topological space. Then (G,τ) is called a fuzzy rough topological group if it satisfies the following conditions:

(i) The mapping $f : (x,y) \to x+y$ of $(G,\tau) \times (G,\tau)$ into (G,τ) is fuzzy rough continuous.

(ii) The mapping $g : x \to (-x)$ of (G,τ) into (G,τ) is fuzzy rough continuous.

4. Fuzzy rough topological transformation groups

In this section, the concept of fuzzy rough topological transformation group is introduced and some interesting properties are also studied.

Notation 4.1. (i) X, Y, Z denote fuzzy rough topological spaces.

(ii) G denotes fuzzy rough topological group.

Definition 4.2. Let X be a fuzzy rough topological space and G be a fuzzy rough topological group. If $\Psi : G \times X \to X$ satisfies the following conditions:

(i) $\Psi(0,x) = x$, where 0 is the identity element of G.

(ii) $\Psi(u, \Psi(v,x)) = \Psi(u+v, x)$.

(iii) Ψ is fuzzy rough continuous.

then (Ψ, G, X) is called a fuzzy rough topological transformation group.

Definition 4.3. Let $v \in G$, then the v-transition of (Ψ, G, X) denoted by Ψ^v is the mapping $\Psi^v : X \to X$ such that

$$\Psi^v(x) = \Psi(v,x).$$

Definition 4.4. Let $(G,+)$ and $(G',+)$ are two rough groups. A fuzzy rough homomorphism $\Psi^v : (G,+) \to (G',+)$ is a map such that

$$\Psi^v(x+y) = \Psi^v(x) + \Psi^v(y),$$

for all x, y.

Proposition 4.5. (i) *Ψ^0 is the identity mapping of X.*

(ii) *$\Psi^u\Psi^v = \Psi^{u+v}$ for $u, v \in G$.*

(iii) *Ψ^v is one-to-one mapping of X onto X and $-(\Psi^v) = \Psi^{-v}$.*

(iv) *For $v \in G$, Ψ^v is a fuzzy rough homomorphism of X onto X.*

Proof. **(i)** $\Psi^0(x) = \Psi(0,x) = x$

(ii)
$$\begin{aligned}\Psi^{u+v}(x) &= \Psi(u+v, x)\\ &= \Psi(u, \Psi(v,x))\\ &= \Psi^u(\Psi(v,x))\end{aligned}$$

$= \Psi^u(\Psi^v(x))$
$= \Psi^u\Psi^v(x)$

This implies that $\Psi^{u+v} = \Psi^u\Psi^v$, for $u, v \in G$.

(iii) $\Psi^{-v}(x) = \Psi(-v, x)$
$= -\Psi(v, x)$
$= -\Psi^v(x)$

This implies that $\Psi^{-v} = -(\Psi^v)$, for $v \in G$.

(iv) $\Psi^v(x + y) = \Psi(v, x + y)$
$= \Psi(v, x) + \Psi(v, y)$
$= \Psi^v(x) + \Psi^v(y)$

This implies that $\Psi^v(x + y) = \Psi^v(x) + \Psi^v(y)$ for $v \in G$. Hence Ψ^v is a fuzzy rough homomorphism of X onto X. □

Definition 4.6. The transition group of (Ψ, G, X) is the set $G' = \{\Psi^v : v \in G\}$. The transition projection of (Ψ, G, X) is the mapping $\theta : G \to G'$ defined as $\theta(v) = \Psi^v$.

Definition 4.7. (Ψ, G, X) is said to be effective if $v \in G$ with $v \neq 0$ implies that $\Psi^v(x) \neq x$ for some x.

Definition 4.8. A fuzzy rough homeomorphism is a fuzzy rough continuous one-to-one function of a fuzzy rough topological space X onto a fuzzy rough topological space Y such that the inverse of the map is also fuzzy rough continuous.

Proposition 4.9. *Let X and Y be two fuzzy rough topological spaces. If $f : X \to Y$ is a fuzzy rough homeomorphism function and $A = (A_L, A_U)$ is a fuzzy rough set in Y. Then $f(FRcl(A)) = FRcl(f(A))$.*

Proposition 4.10. (i) *G is a rough group of fuzzy rough homeomorphisms of X onto X.*
(ii) *θ is a rough group homomorphism of G onto G'.*
(iii) *θ is one-to-one if and only if (Ψ, G, X) is effective.*

Definition 4.11. Let $x \in X$, then the x-motion of (Ψ, G, X) is the mapping $\Psi_x : G \to X$ such that $\Psi_x(u) = \Psi(u, x)$.

Proposition 4.12. *Ψ_x is a fuzzy rough continuous function of G into X.*

Definition 4.13. Let X be a rough set and $r, t \in (0, 1]$ such that $r \leq t$. If

$$x_{[r,t]}(y) = \begin{cases} [r, t] & \text{if} \quad y = x \\ [0, 0] & \text{if} \quad y \neq x \end{cases} \quad \text{where } y = (y_L, y_U) \in X.$$

Then $x_{[r,t]}$ is called a fuzzy rough point on X where r denotes the degree of lower approximation $x_{[r,t]}$ and t denotes the degree of upper approximation $x_{[r,t]}$ and $x = (x_L, x_U) \in X$ is called the rough support of $x_{[r,t]}$. The fuzzy rough point $x_{[r,t]}$ is contained in the fuzzy rough set A (that is, $x_{[r,t]} \in A$) if and only if $r < A_L(x)$, $t < A_U(x)$.

Definition 4.14. The fuzzy rough point $x_{[r,t]}$ is said to be contained in a fuzzy rough set $A = (A_L, A_U)$, or to belong to $A = (A_L, A_U)$, denoted by $x_{[r,t]} \in A$, if and only if $r < A_L(x)$ for all $x \in X_L$ and $t < A_U(x)$ for all $x \in X_U$. Evidently, every fuzzy rough set A can be expressed as the union of all the fuzzy rough points which belong to A.

Definition 4.15. Let X be a fuzzy rough topological space. A fuzzy rough set $A = (A_L, A_U)$ in X is called a fuzzy rough neighbourhood of a fuzzy rough point $x_{[r,t]}$ if there exists an fuzzy rough open set $B = (B_L, B_U)$ with $B \subseteq A$ and $x_{[r,t]} \in B \subseteq A$.

Definition 4.16. Let X and Y be any two non-empty sets. If $A = (A_L, A_U)$ is a fuzzy rough set of X and $B = (B_L, B_U)$ is a fuzzy rough set of Y, then the fuzzy rough set $A \times B = (A_L \times B_L, A_U \times B_U)$ on $X \times Y$ is defined by

$$A_L \times B_L(x, y) = min\{A_L(x), B_L(y)\} \text{ for every } (x, y) \in X_L \times Y_L \text{ and}$$
$$A_U \times B_U(x, y) = min\{A_U(x), B_U(y)\} \text{ for every } (x, y) \in X_U \times Y_U.$$

Definition 4.17. Let (X, τ) and (Y, ϕ) be any two fuzzy rough topological spaces. The fuzzy rough product topological space of (X, τ) and (Y, ϕ) is the cartesian product $(X, \tau) \times (Y, \phi)$ of sets (X, τ) and (Y, ϕ) together with the fuzzy rough topology $\tau \times \phi$ generated by the family $\{ p_1^{-1}(A) , p_2^{-1}(B) \mid A \in \tau$, $B \in \phi$, where p_1 and p_2 are projections of $(X, \tau) \times (Y, \phi)$ onto (X, τ) and (Y, ϕ), respectively$\}$.

Notation 4.18. $y_{[p,q]}$ denotes the fuzzy rough point in X.

Proposition 4.19. *Let X, Y, Z be fuzzy rough topological spaces and $f : X \times Y \to Z$ be a fuzzy rough continuous function. If a_α, b_β are fuzzy rough points of X and Y respectively and C be a fuzzy rough negihbourhood of $f(a_\alpha, b_\beta)$, then there exists fuzzy rough neighbourhoods $A = (A_L, A_U)$ and $B = (B_L, B_U)$ of a_α and b_β respectively such that $f(A \times B) \subseteq C$.*

Proof. Without loss of generality we can assume that $C = (C_L, C_U)$ is a fuzzy rough open set. As f is fuzzy rough continuous, $f^{-1}(C)$ is a fuzzy rough open set containing $x_{[r,t]} \times b_\beta$. So there exists basic fuzzy rough open sets A of $x_{[r,t]}$ and B of $y_{[p,q]}$ such that

$$x_{[r,t]} \times y_{[p,q]} \in A \times B \subseteq f^{-1}(C)$$
$$f(x_{[r,t]} \times y_{[p,q]}) \in f(A \times B) \subseteq f(f^{-1}(C))$$
$$f(x_{[r,t]} \times y_{[p,q]}) \in f(A \times B) \subseteq C.$$

This implies that $f(A \times B) \subseteq C$. □

Definition 4.20. A fuzzy rough topological space X is said to be product related to another fuzzy rough topological space Y if for a fuzzy rough set $C = (C_L, C_U)$ of X and $D = (D_L, D_U)$ of Y whenever $A' \not\supseteq C$ and $B' \not\supseteq D$ implies $(A' \times \tilde{1}) \cup (\tilde{1} \times B') \supseteq C \times D$, where $A = (A_L, A_U) \in \tau$ and $B = (B_L, B_U) \in \sigma$, then there exist $A_1 \in \tau$ and $B_1 \in \sigma$ such that $A'_1 \supseteq C$ or $B'_1 \supseteq D$ and $(A' \times \tilde{1}) \cup (\tilde{1} \times B') = (A'_1 \times \tilde{1}) \cup (\tilde{1} \times B'_1)$.

Definition 4.21. Let $X = (X_L, X_U)$ be a non empty set and $A = (A_L, A_U) \subset X = (X_L, X_U)$. Then the fuzzy rough characteristic function of G is defined by

$$\chi_{G_L}(x) = \begin{cases} 1 & \text{if} \quad x \in A_L \\ 0 & \text{if} \quad x \notin A_L \end{cases} \quad \text{for every } x \in X_L \text{ and}$$

$$\chi_{G_U}(x) = \begin{cases} 1 & \text{if} \quad x \in A_U \\ 0 & \text{if} \quad x \notin A_U \end{cases} \quad \text{for every } x \in X_U.$$

It is denoted by $\chi_G = (\chi_{G_L}, \chi_{G_U})$.

Notation 4.22. If $x \in X = (X_L, X_U)$, then the fuzzy rough point $x_1 = (x_{L_1}, x_{U_1})$ denotes simply by x.

Notation 4.23. (i) $A^* = ({A_L}^*, {A_U}^*)$ and $B^* = ({B_L}^*, {B_U}^*)$ are denoted by fuzzy rough open sets in X.
(ii) The orbit of y under B is denoted by B_y.
(iii) $\mathfrak{C}_B$ is denoted by a fuzzy rough open cover of B and $\mathfrak{D}_B$ is denoted by finite subcollection of fuzzy rough open cover in $\mathfrak{C}_B$.
(iv) $\mathfrak{C}_A$ is denoted by a fuzzy rough open cover of A and $\mathfrak{D}_A$ is denoted by finite subcollection of fuzzy rough open cover in $\mathfrak{C}_A$.

Proposition 4.24. *Let X, Y, Z be fuzzy rough topological spaces and $f : X \times Y \to Z$ be a fuzzy rough continuous function. If $A = (A_L, A_U)$, $B = (B_L, B_U)$ are fuzzy rough compact sets of X and Y respectively and $C = (C_L, C_U)$ be a fuzzy rough negihbourhood of $f(A \times B)$, then there exists fuzzy rough open sets A^* and B^* such that $A^* \supseteq A - \varepsilon$ and $B^* \supseteq B - \varepsilon$ and $f(A^* \times B^*) \subseteq C$.*

Proof. Let $\varepsilon > 0$ be arbitrary. Let $x \in X_L$ and $x \in X_U$ be arbitrary fixed and suppose $A_L(x) = \alpha$ and $A_U(x) = \alpha$, for $\alpha \in [0, 1]$. Then for any $y \in Y_L$ with $B_L(y) = \beta$ and $B_U(y) = \beta$ for any $y \in Y_L$, $\beta \in [0, 1]$. By Proposition(4.2), there exist fuzzy rough open sets $A_x \ni x_\alpha$ and $B_y \ni y_\beta$ such that

$$f(x_\alpha \times y_\beta) \in f(A_x \times B_y) \subseteq C).$$

This is true for each $y \in Y_L$ and $y \in Y_U$. Thus the collection $\mathfrak{C}_B = \{B_y : y \in Y = (Y_L, Y_U)\}$ is a fuzzy rough open cover of B. As B is fuzzy rough compact, there is a finite subcollection $\mathfrak{D}_B$ of $\mathfrak{C}_B$ satisfying $\vee\{B_y : B_y \in \mathfrak{D}_B\} \supseteq B - \varepsilon$. Let B_y denote the union of all members of $\mathfrak{D}_B$ and A_x denote the intersection of the corresponding B_y's. Then A_x is a fuzzy rough open set containing x_α and B_β is a fuzzy rough open set satisfying $B_y \supseteq B - \varepsilon$.

But this is true for each $x \in X_L$ and $x \in X_U$. Thus we get the collection $\{B_x : x \in X = (X_L, X_U)\}$ of fuzzy rough open sets each satisfying $B_x \supseteq B - \varepsilon$ and another collection $\{A_x : x \in X = (X_L, X_U)\}$ of fuzzy rough open sets such that $x_\alpha \in A_x(\alpha = A_L(x) \text{and} A_U(x) = \alpha)$. Then $\mathfrak{C}_A = \{A_x : x \in X = (X_L, X_U)\}$ is a fuzzy rough open cover of A. As A is fuzzy rough compact, there exists a finite collection $\mathfrak{D}_A$ of $\mathfrak{C}_A$ satisfying $\vee\{A_x : A_x \in \mathfrak{D}_A\} \supseteq A - \varepsilon$. Let A^* denote the union of all members of $\mathfrak{D}_A$ and B^* denote the intersection of the corresponding B_x. Then A^* is a fuzzy rough open set satisfying $A^* \supseteq A - \varepsilon$ and B^* is a fuzzy rough open set satisfying $B^* \supseteq B - \varepsilon$. Further then $f(A^* \times B^*) \subseteq C$. □

Notation 4.25. $\tilde{1} - A \times B = (1_L - A_L \times B_L, 1_U - A_U \times B_U)$ denotes the fuzzy rough set.

Proposition 4.26. *If $A = (A_L, A_U)$ is a fuzzy rough set of X and $B = (B_L, B_U)$ is a fuzzy rough set of Y, then $\tilde{1} - A \times B = A' \times \tilde{1} \cup \tilde{1} \times B'$.*

Proof. Since by Definition(4.11), it follows that

$$\begin{aligned}
A_L \times B_L(x,y) &= min(A_L(x), B_L(y)), \text{ for every } (x,y) \in X_L \times Y_L \\
1_L - A_L \times B_L(x,y) &= max(1 - A_L(x), 1 - B_L(y)) \\
&= max(A'_L(x), B'_L(y)) \\
&= max((A'_L \times 1_L)(x,y), (1_L \times B'_L)(x,y)) \\
1_L - A_L \times B_L &= A'_L \times 1_L \cup 1_L \times B'_L
\end{aligned}$$

and similarly $1_U - A_U \times B_U = A'_U \times 1_U \cup 1_U \times B'_U$. This implies that

$$\tilde{1} - A \times B = A' \times \tilde{1} \cup \tilde{1} \times B'.$$

□

Proposition 4.27. *Let $A = (A_L, A_U)$ be a fuzzy rough closed set of a fuzzy rough topological space X and $B = (B_L, B_U)$ be a fuzzy rough closed set of a fuzzy rough topological space Y, then $A \times B$ is a fuzzy rough closed set of the fuzzy product topological space $X \times Y$.*

Proof. Let $A \in X, B \in Y$. From Proposition(4.4), we have $\tilde{1} - A \times B = A' \times \tilde{1} \cup \tilde{1} \times B'$. Since $A' \times \tilde{1}$ and $\tilde{1} \times B'$ are fuzzy rough open sets in X and Y respectively, hence $A' \times \tilde{1} \cup \tilde{1} \times B'$ is a fuzzy rough open set of $X \times Y$. Hence $\tilde{1} - A \times B$ is a fuzzy rough open set of $X \times Y$ and consequently $A \times B$ is a fuzzy rough closed set of $X \times Y$. □

Proposition 4.28. *If $A = (A_L, A_U)$ is a fuzzy rough set of a fuzzy rough topological space X and $B = (B_L, B_U)$ is a fuzzy rough set of a fuzzy rough topological space Y, then $FRcl(A) \times FRcl(B) \supseteq FRcl(A \times B)$.*

Proof. Since $A \subseteq FRcl(A)$ and $B \subseteq FRcl(B)$, hence $A \times B \subseteq FRcl(A) \times FRcl(B)$ implies $FRcl(A \times B) \subseteq FRcl(FRcl(A) \times FRcl(B))$ and from Proposition(4.5), we have $FRcl(A \times B) \subseteq FRcl(A) \times FRcl(B)$. □

Proposition 4.29. *Let X and Y be two fuzzy rough topological spaces such that X is product related to Y. Then, for a fuzzy rough set $A = (A_L, A_U)$ of X and a fuzzy rough set $B = (B_L, B_U)$ of Y, $FRcl(A \times B) = FRcl(A) \times FRcl(B)$.*

Note 4.30. If $t \in G = (G_L, G_U)$, then the fuzzy rough point $t_1 = (t_{L_1}, t_{U_1})$ denotes simply by t.

Definition 4.31. Let X be a fuzzy rough topological space and G be a fuzzy rough topological group. Let $\Psi : G \times X \to X$. Then $\Psi(\mathrm{t} \times B) = (\Psi(t_L \times B_L), \Psi(t_U \times B_U))$, where $\Psi : G_L \times X_L \to X_L$ and $\Psi : G_U \times X_U \to X_U$ are defined by

$$\Psi(t_L \times B_L)(u) = sup\{(t_L \times B_L)(s,x) : \Psi(s,x) = u, u \in X_L\} \quad \text{and}$$
$$\Psi(t_U \times B_U)(u) = sup\{(t_U \times B_U)(s,x) : \Psi(s,x) = u, u \in X_U\}.$$

Definition 4.32. Let X be a fuzzy rough topological space and G be a fuzzy rough topological group. Let $\Psi^{\mathrm{t}} : X \to X$, for any $t \in G$. Then $\Psi^{\mathrm{t}}(B) = (\Psi^{t_L}(B_L), \Psi^{t_U}(B_U))$, where $\Psi^{t_L} : X_L \to X_L$ and $\Psi^{t_U} : X_U \to X_U$ are defined by

$$\Psi^{t_L}(B_L)(u) = \Psi(t_L \times B_L)(u), \text{ for any } u \in X_L \quad \text{and}$$

$$\Psi^{t_U}(B_U)(u) = \Psi(t_U \times B_U)(u), \text{ for any } u \in X_U.$$

Notation 4.33. $\mathcal{I}^X$ denotes the collection of all fuzzy rough sets in X.

Proposition 4.34. *Let X be a fuzzy rough topological space and G be a fuzzy rough topological group. Let $\Psi : G \times X \to X$ be a fuzzy rough function. Then the following statements are hold:*

(i) *For $t \in G$ and a fuzzy rough set $B = (B_L, B_U)$ of X, $FRcl(\Psi(t \times B)) = \Psi(t \times FRcl(B))$.*
(ii) *Let G and X be product related, then for a fuzzy rough set $A = (A_L, A_U)$ of G and a fuzzy rough set $B = (B_L, B_U)$ of X, $\Psi(FRcl(A) \times FRcl(B)) \subseteq FRcl(\Psi(A \times B))$ and $FRcl(\Psi(A \times FRcl(B))) = FRcl(\Psi(A \times B))$.*
(iii) *If $A = (A_L, A_U)$ is a fuzzy rough compact set of G and $B = (B_L, B_U)$ is a fuzzy rough compact set of X and $C = (C_L, C_U)$ is a fuzzy rough negihbourhood of $\Psi(A \times B)$, then for any $\varepsilon > 0$, there exists fuzzy rough open sets A^* and B^* such that $A^* \supseteq A - \varepsilon$ and $B^* \supseteq B - \varepsilon$ such that $\Psi(A^* \times B^*) \subseteq C$.*
(iv) *$\Psi^t B = B\Psi^{-t}$ for any $t \in G$.*
(v) *$\Psi^t B' = 1 - \Psi^t B$.*

Proof. **(i)** Since Ψ^{t} is fuzzy rough homeomorphism, $FRcl(\Psi^{\mathrm{t}}(B)) = \Psi^{\mathrm{t}}(FRcl(B))$. By Definition (4.2), it follows that

$$FRcl(\Psi(\mathrm{t} \times B)) = \Psi(\mathrm{t} \times FRcl(B)).$$

(ii) Since G and X are product related, $FRcl(A) \times FRcl(B) = FRcl(A \times B)$ which implies

$$\Psi(FRcl(A) \times FRcl(B)) = \Psi(FRcl(A \times B)).$$

This implies that

$$\Psi(FRcl(A) \times FRcl(B)) = \Psi(FRcl(A \times B)) \subseteq FRcl(\Psi(A \times B)),$$

since Ψ is fuzzy rough continuous. Again

$$\Psi(A \times B) \subseteq \Psi(FRcl(A) \times B) \subseteq \Psi(FRcl(A) \times FRcl(B)) \subseteq FRcl(\Psi(A \times B)).$$

This implies that

$$FRcl(\Psi(A \times B)) \subseteq FRcl(\Psi(FRcl(A) \times B)) \subseteq FRcl(\Psi(FRcl(A) \times FRcl(B))) \subseteq FRcl(FRcl(\Psi(A \times B))) = FRcl(\Psi(A \times B)).$$

$$\Rightarrow FRcl(\Psi(A \times B)) = FRcl(\Psi(FRcl(A) \times B)) \tag{4.1}$$

and

$$\Psi(A \times B) \subseteq \Psi(A \times FRcl(B)) \subseteq \Psi(FRcl(A) \times FRcl(B)) \subseteq FRcl(\Psi(A \times B)).$$

This implies that

$$FRcl(\Psi(A \times B)) \subseteq FRcl(\Psi(A \times FRcl(B))) \subseteq FRcl(\Psi(FRcl(A) \times FRcl(B))) \subseteq FRcl(FRcl(\Psi(A \times B))) = FRcl(\Psi(A \times B)).$$

$$\Rightarrow FRcl(\Psi(A \times B)) = FRcl(\Psi(A \times FRcl(B))). \tag{4.2}$$

From (4.5) and (4.6), it follows that

$$FRcl(\Psi(FRcl(A) \times B)) = FRcl(\Psi(A \times B)) = FRcl(\Psi(A \times FRcl(B))).$$

(iii) Follows from Proposition(4.3).

(iv) We have for any $u \in X_L$,

$$\begin{aligned}
\Psi^{t_L}(B_L)(u) &= \Psi(t_L \times B_L)(u) \\
&= sup\{(t_L \times B_L)(s,x) : \Psi(s,x) = u, u \in X_L\} \\
&= sup\{t_L(s) \wedge B_L(x) : \Psi(s,x) = u\} \\
&= sup\{t_L(t) \wedge B_L(x) : \Psi(t,x) = u\}, \text{since } t_L(t) \neq 0 \text{ only when } t = s \\
&= B_L(x), \text{where } \Psi(t,x) = u \\
&= B_L(x), \text{where } \Psi^t(x) = u \\
&= B_L(\Psi^{-t})(u)
\end{aligned}$$

and similarly we have for any $u \in X_U$, $\Psi^{t_U}(B_U)(u) = B_U(\Psi^{-t})(u)$. This implies that $\Psi^{\mathrm{t}}B = B\Psi^{-\mathrm{t}}$.

(v) From **(iv)** we have $\Psi^{\mathrm{t}}B = B\Psi^{-\mathrm{t}}$ for any $B = (B_L, B_U) \in \mathcal{I}^X$ and $t \in G$. Now for any $x \in X_L$, we have

$$\begin{aligned}
(\Psi^{\mathrm{t}}B'_L)(x) &= (B'_L\Psi^{-\mathrm{t}})(x) \\
&= B'_L(\Psi^{-\mathrm{t}}(x)) \\
&= 1 - (B_L\Psi^{-\mathrm{t}})(x) \\
&= (1 - B_L\Psi^{-\mathrm{t}})(x)
\end{aligned}$$

and similarly for any $x \in X_U$, we have $(\Psi^{\mathrm{t}}B'_U)(x) = (1 - B_U\Psi^{-\mathrm{t}})(x)$. Therefore $\Psi^{\mathrm{t}}B' = 1 - B\Psi^{-\mathrm{t}}$. □

Proposition 4.35. *Let $C = (C_L, C_U)$ be a constant fuzzy rough set of G and $B = (B_L, B_U) \in \mathcal{I}^X$ be a fuzzy rough open set. Then $\Psi(C \times B)$ is a fuzzy rough open set.*

Proof. We have for any $u \in X_L$,

$$\begin{aligned}
\Psi(C_L \times B_L)(u) &= sup\{\Psi(C_L \times B_L)(t,x) : \Psi(t,x) = u\} \\
&= sup\{C_L(t) \wedge B_L(x) : \Psi(t,x) = u\} \\
&= sup\{C_L \wedge B_L(x) : \Psi(t,x) = u\} \\
&= C_L \wedge sup\{B_L(x) : \Psi^t(x) = u\} \\
&= C_L \wedge sup\{B_L(\Psi^{-t}(u)) : \Psi^{-t}(u) = x\} \\
&= C_L \wedge sup\{\Psi^t B_L(u) : \Psi^{-t}(u) = x\}, \text{since } \ B_L\Psi^{-t}(u) = \Psi^t B_L(u) \\
&= C_L \wedge \{\vee\{\Psi^t B_L(u)\}, \text{where } \Psi^{-t}(u) = x\} \\
&= \{C_L \wedge \{\vee\Psi^t B_L\}(u), \text{where } \Psi^{-t}(u) = x\}
\end{aligned}$$

and similarly we have for any $u \in X_U$,

$$\Psi(C_U \times B_U)(u) = \{C_U \wedge \{\vee\Psi^t B_U\}(u), \text{ where } \Psi^{-t}(u) = x\}.$$

Thus $\Psi(C \times B) = C \wedge \{\vee(\Psi^t B)\}$. Now each Ψ^t is fuzzy rough open and B is fuzzy rough open, so $\Psi^t B$ is fuzzy rough open. Also by the definition of fuzzy rough topology C is fuzzy rough open. Consequently $C \wedge \{\vee(\Psi^t B)\}$ is fuzzy rough open. Hence $\Psi(C \times B)$ is fuzzy rough open. $\square$

Proposition 4.36. *Let $B = (B_L, B_U)$ be a fuzzy rough open set of X, then for any fuzzy rough point $x_{[r,t]}$ of G, $\Psi(x_{[r,t]} \times B)$ is a fuzzy rough open set.*

Proof. We have for any $u \in X_L$,

$$\begin{aligned}
\Psi(x_r \times B_L)(u) &= sup\{\Psi(x_r \times B_L)(s,x) : \Psi(s,x) = u\} \\
&= sup\{x_r(s) \wedge B_L(x) : \Psi(s,x) = u\} \\
&= r \wedge B_L(x) : \Psi(t,x) = u, \text{ since } x_r(s) \neq 0 \text{ only when } s = t \\
&= r \wedge B_L(x) : \Psi^t(u) = x\} \\
&= r \wedge B_L(\Psi^{-t}(u)) : \Psi^{-t}(u) = x \\
&= r \wedge \Psi^t B_L(u), \text{since } B_L \Psi^{-t}(u) = \Psi^t B_L(u) \\
&= (r \wedge \Psi^t B_L)(u)
\end{aligned}$$

and similarly we have for any $u \in X_U$, $\Psi(x_t \times B_U)(u) = (t \wedge \Psi^t B_U)(u)$, considering $x_{[r,t]}$ as a constant fuzzy rough set on X. Thus $\Psi(x_{[r,t]} \times B) = [r,t] \wedge \Psi^t B$. Now Ψ^t is fuzzy rough open and B is fuzzy rough open, so $\Psi^t B$ is fuzzy rough open. Also by the definition of fuzzy rough topology $x_{[r,t]}$ is fuzzy rough open. Consequently $x_{[r,t]} \wedge \Psi^t B\}$ is fuzzy rough open. Hence $\Psi(x_{[r,t]} \times B)$ is fuzzy rough open. $\square$

Note 4.37. Let $A = (A_L, A_U)$ be a fuzzy rough set of X. Suppose $A_L(x) = r$, for $x \in X_L$ and $A_U(x) = t$, for $x \in X_U$. Then $[r,t]$ can be expressed as union of all its fuzzy rough points, that is, $A = \vee x_{[r,t]}$.

Proposition 4.38. *Let $A = (A_L, A_U)$ be any fuzzy rough set of G and $B = (B_L, B_U) \in \mathcal{I}^X$ be fuzzy rough open, then $\Psi(A \times B)$ is fuzzy rough open.*

Proof. We have $A = \vee x_{[r,t]}$, where $r = A_L(x)$ and $t = A_U(x)$. So

$$\begin{aligned}
\Psi(A \times B) &= \Psi(\vee x_{[r,t]} \times B) \\
&= \vee \Psi(x_{[r,t]} \times B).
\end{aligned}$$

As already proved each $\Psi(x_{[r,t]} \times B)$ is fuzzy rough open and hence $\Psi(A \times B)$ is fuzzy rough open. $\square$

Proposition 4.39. *Let $B = (B_L, B_U)$ be a fuzzy rough closed set of X, then for any fuzzy rough point $x_{[r,t]}$ of G, $\Psi(x_{[r,t]} \times B)$ is a fuzzy rough closed set.*

Proposition 4.40. *Let $A = (A_L, A_U)$ be any fuzzy rough set of G and $B = (B_L, B_U) \in \mathcal{I}^X$ be a fuzzy rough closed set. If $suppA = (suppA_L, suppA_U)$ is finite, then $\Psi(A \times B)$ is a fuzzy rough closed set.*

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