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On strongly semi-continuous intuitionistic fuzzy multifunctions

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Abstract. The aim of this paper is introduce the concepts of upper and lower strongly semi-continuous intuitionistic fuzzy multifunctions and obtain some properties of semi-continuous Intuitionistic fuzzy multifunctions.

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1. Introduction

After the introduction of fuzzy sets by Zadeh [24] in 1965 and fuzzy topology by Chang [5] in 1967, several researches were conducted on the generalizations of the notions of fuzzy sets and fuzzy topology. The concept of intuitionistic fuzzy sets was introduced by Atanassov [1, 2, 3] as a generalization of fuzzy sets. In the last 27 years various concepts of fuzzy mathematics have been extended for intuitionistic fuzzy sets. In 1997 Coker [6] introduced the concept of intuitionistic fuzzy topological spaces as a generalization of fuzzy topological spaces. Recently many fuzzy topological concepts such as fuzzy compactness [8], fuzzy connectedness [22],connectedness between fuzzy sets [20], fuzzy separation axioms [4, 12], fuzzy nets and filters [11], fuzzy metric spaces [21], fuzzy continuity and it's weak forms [9, 10, 16, 19], fuzzy generalized closed sets [18] and fuzzy topological groups [14, 23], have been generalized for intuitionistic fuzzy topological spaces. The section 2 of this paper review the concepts of intuitionistic fuzzy sets and intuitionistic fuzzy topology. The section 3 explores the some properties of Lower and Upper semi-continuous intuitionistic fuzzy multifunctions introduced by Ozbakir and Coker[13] and the last section introduces and studies stronger classes of Upper and Lower semi continuous intuitionistic fuzzy multifunctions.

2. Preliminaries

Throughout this paper (X,τ) and (Y,Γ) represents a topological space and an intuitionistic fuzzy topological space respectively.

Definition 2.1 ([1, 2, 3])**.** Let Y be a nonempty fixed set. An intuitionistic fuzzy set $\tilde{A}$ in Y is an object having the form

$$\tilde{A} = \{< y, \mu_{\tilde{A}}(y), \nu_{\tilde{A}}(y) >: y \in Y\}$$

where the functions $\mu_{\tilde{A}}(y) : Y \to I$ and $\nu_{\tilde{A}}(y) : Y \to I$ where $I = [0,1]$, denotes the degree of membership (namely $\mu_{\tilde{A}}(y)$) and the degree of non membership (namely $\nu_{\tilde{A}}(y)$) of each element $y \in Y$ to the set $\tilde{A}$ respectively, and $0 \leq \mu_{\tilde{A}}(y) + \mu_{\tilde{A}}(y) \leq 1$ for each $y \in Y$.

Definition 2.2 ([1, 2, 3])**.** Let Y be a nonempty set and the intuitionistic fuzzy sets $\tilde{A}$ and $\tilde{B}$ be in the form $\tilde{A} = \{< y, \mu_{\tilde{A}}(y), \nu_{\tilde{A}}(y) >: y \in Y\}$, $\tilde{B} = \{< y, \mu_{\tilde{B}}(y), \nu_{\tilde{B}}(y) >: y \in Y\}$ and let $\{\tilde{A}_\alpha : \alpha \in \Lambda\}$ be an arbitrary family of intuitionistic fuzzy sets in Y. Then:

(a) $\tilde{A} \subseteq \tilde{B}$ if $\forall y \in Y$ $[\mu_{\tilde{A}}(y) \leq \mu_{\tilde{B}}(y)$ and $\nu_{\tilde{A}}(y) \geq \nu_{\tilde{B}}(y)]$
(b) $\tilde{A} = \tilde{B}$ if $\tilde{A} \subseteq \tilde{B}$ and $\tilde{B} \subseteq \tilde{A}$;
(c) $\tilde{A}^c = \{< y, \nu_{\tilde{A}}(y), \mu_{\tilde{A}}(y) >: y \in Y\}$;
(d) $\tilde{0} = \{< y, 0, 1 >: y \in Y\}$ and $\tilde{1} = \{< y, 1, 0 >: y \in Y\}$
(e) $\cap \tilde{A}_\alpha = \{< y, \wedge \mu_{\tilde{A}}(y), \vee \nu_{\tilde{A}}(y) >: y \in Y\}$
(f) $\cup \tilde{A}_\alpha = \{< y, \vee \mu_{\tilde{A}}(y), \wedge \nu_{\tilde{A}}(y) >: y \in Y\}$

Definition 2.3 ([7])**.** Two Intuitionistic fuzzy sets $\tilde{A}$ and $\tilde{B}$ of Y are said to be quasi-coincident ($\tilde{A} q \tilde{B}$ for short) if $\exists y \in Y$ such that
$\mu_{\tilde{A}}(y) > \nu_{\tilde{B}}(y)$ or $\nu_{\tilde{A}}(y) < \mu_{\tilde{B}}(y)$

Lemma 2.4 ([7])**.** *For any two intuitionistic fuzzy sets $\tilde{A}$ and $\tilde{B}$ of Y, $\rceil(\tilde{A} q \tilde{B}) \Leftrightarrow \tilde{A} \subset \tilde{B}^c$.*

Definition 2.5 ([6])**.** An intuitionistic fuzzy topology on a non empty set Y is a family Γ of intuitionistic fuzzy sets in Y which satisfies the following axioms:
O_1 $\tilde{0}, \tilde{1} \in \Gamma$,
O_2 $\tilde{A}_1 \cap \tilde{A}_2 \in \Gamma$ for any $\tilde{A}_1, \tilde{A}_2 \in \Gamma$,
O_3 $\cup \tilde{A}_\alpha \in \Gamma$ for arbitrary family $\{\tilde{A}_\alpha : \alpha \in \Lambda\} \in \Gamma$.

In this case the pair (Y,Γ) is called an intuitionistic fuzzy topological space and each intuitionistic fuzzy set in Γ is known as an intuitionistic fuzzy open set in Y. The complement $\tilde{B}^c$ of an intuitionistic fuzzy open set $\tilde{B}$ is called an intuitionistic fuzzy closed set in Y.

Definition 2.6 ([7])**.** Let Y be a nonempty set and $c \in Y$ a fixed element in Y. If $\alpha \in (0,1]$ and $\beta \in [0,1)$ are two real numbers such that $\alpha + \beta < 1$ then,

(a) $c(\alpha,\beta) =< y, c_\alpha, c_{1-\beta} >$ is called an intuitionistic fuzzy point(IFP in short) in Y, where α denotes the degree of membership of $c(\alpha,\beta)$, and β denotes the degree of non membership of $c(\alpha,\beta)$.

(b) $c(\beta) =< y, 0, 1-c_{1-\beta} >$ is called a vanishing intuitionistic fuzzy point (VIFP in short) in Y, where β denotes the degree of non membership of $c(\beta)$.

Definition 2.7 ([6]). Let (Y,Γ) be an intuitionistic fuzzy topological space and $\tilde{A}$ be an intuitionistic fuzzy set in Y. Then the interior and closure of $\tilde{A}$ are defined by:

$cl(\tilde{A})$=$\cap\{\tilde{K} : \tilde{K}$ is an intuitionistic fuzzy closed set in Y and $\tilde{A} \subseteq \tilde{K}\}$,
$Int(\tilde{A})$=$\cup\{\tilde{G} : \tilde{G}$ is an intuitionistic fuzzy open set in Y and $\tilde{G} \subseteq \tilde{A}\}$.

Lemma 2.8 ([5]). *For any intuitionistic fuzzy set $\tilde{A}$ in (Y,Γ) we have:*
(a) $\tilde{A}$ is an intuitionistic fuzzy closed set in $Y \Leftrightarrow Cl(\tilde{A}) = \tilde{A}$
(b) $\tilde{A}$ is an intuitionistic fuzzy open set in $Y \Leftrightarrow Int(\tilde{A}) = \tilde{A}$
(c) $Cl(\tilde{A}^c) = (Int\tilde{A})^c$
(d) $Int(\tilde{A}^c) = (Cl\tilde{A})^c$

Definition 2.9 ([15]). Let X and Y are two non empty sets. A function $F : (X,\tau) \to (Y,\Gamma)$ is called intuitionistic fuzzy multifunction if $F(x)$ is an intuitionistic fuzzy set in Y, $\forall x \in X$.

Definition 2.10 ([17]). Let $F :(X,\tau) \to (Y,\Gamma)$ be an intuitionistic fuzzy multifunction and A be a subset of X. Then $F(A) = \cup_{x\in A} F(x)$.

Lemma 2.11 ([17]). *Let $F :(X,\tau) \to (Y,\Gamma)$ be an intuitionistic fuzzy multifunction. Then*
(a) $A \subseteq B \Rightarrow F(A) \subseteq F(B)$ for any subsets A and B of X.
(b) $F(A \cap B) \subseteq F(A) \cap F(B)$ for any subsets A and B of X.
(c) $F(\cup_{\alpha\in\Lambda} A_\alpha) = \cup\{F(A_\alpha) : \alpha \in \Lambda\}$ for any family of subsets $\{A_\alpha : \alpha \in \Lambda\}$ in X.

Definition 2.12 ([15]). Let $F : (X,\tau) \to (Y,\Gamma)$ is an intuitionistic fuzzy multifunction. Then the upper inverse $F^+(\tilde{A})$ and lower inverse $F^-(\tilde{A})$ of an intuitionistic fuzzy set $\tilde{A}$ in Y are defined as follows:

$$F^+(\tilde{A}) = \{x \in X : F(x) \subseteq \tilde{A}\}$$

$$F^-(\tilde{A}) = \{x \in X : F(x) q \tilde{A}\}$$

Lemma 2.13 ([17]). *Let $F : (X,\tau) \to (Y,\Gamma)$ be an intuitionistic fuzzy multifunction and $\tilde{A}, \tilde{B}$ be intuitionistic fuzzy sets in Y. Then:*
(a) $F^+(\tilde{1}) = F^-(\tilde{1}) = X$
(b) $F^+(\tilde{A}) \subseteq F^-(\tilde{A})$
(c) $[F^-(\tilde{A})]^c = [F^+(\tilde{A})^c]$
(d) $[F^+(\tilde{A})]^c = [F^-(\tilde{A})^c]$
(e) If $\tilde{A} \subseteq \tilde{B}$, then $F^+(\tilde{A}) \subseteq F^+(\tilde{B})$
(f) If $\tilde{A} \subseteq \tilde{B}$, then $F^-(\tilde{A}) \subseteq F^-(\tilde{B})$.

3. Some properties of upper and lower semi-continuous intuitionistic fuzzy multifunctions

In 1999, Ozbakir and Coker [13] introduced the concept intuitionistic fuzzy multifunctions and studied their lower and upper intuitionistic fuzzy semi continuity from a topological space to an intuitionistic fuzzy topological space.In this section we obtain some characterizations and properties of lower and upper semi-continuous intuitionistic fuzzy multifunctions

Definition 3.1 ([13]). An Intuitionistic fuzzy multifunction $F : (X, \tau) \to (Y, \Gamma)$ is said to be:

(a) Intuitionistic fuzzy upper semi continuous at a point $x_0 \in X$, if for any intuitionistic fuzzy open set $\tilde{W} \subset Y$ such that $F(x_0) \subset \tilde{W}$ there exists an open set $U \subset X$ containing x_0 such that $F(U) \subset \tilde{W}$.

(b) Intuitionistic fuzzy lower semi continuous at a point $x_0 \in X$, if for any intuitionistic fuzzy open set $\tilde{W} \subset Y$ such that $F(x_0)q\tilde{W}$ there exists an open set $U \subset X$ containing x_0 such that $F(x)q\tilde{W}$, $\forall x \in U$.

(c) Intuitionistic fuzzy upper semi-continuous (intuitionistic fuzzy lower semi-continuous) if it is intuitionistic fuzzy upper semi-continuous (Intuitionistic fuzzy lower semi-continuous) at each point of X.

Remark 3.2. The concepts of intuitionistic fuzzy upper semi-continuous and intuitionistic fuzzy lower semi-continuous intuitionistic fuzzy multifunctions are independent. For,

Example 3.3. Let $X = \{a, b\}, Y = [0, 1]$ and let $\tau = \{\phi, \{b\}, X\}$ and

$$\Gamma = \{\tilde{0}, \tilde{1}, C_{(1/3,2/3)}, C_{(5/6,1/6)}\}$$

are topology and intuitionistic fuzzy topology on X and Y respectively. We use the notion $C_{(\alpha,\beta)}(0 \leq \alpha \leq 1, 0 \leq \beta \leq 1$ and $0 \leq \alpha + \beta \leq 1)$ to denote the constant intuitionistic fuzzy sets such that $C_{(\alpha,\beta)}(y) = \{< y, \alpha, \beta >; \forall y \in Y\}$. Then the intuitionistic fuzzy multifunction $F : (X, \tau) \to (Y, \Gamma)$ defined by $F(a) = C_{(5/6,1/6)}$ and $F(b) = C_{(1/2,1/2)}$ is intuitionistic fuzzy upper semi-continuous but not intuitionistic fuzzy lower semi-continuous.

Example 3.4. Let $X = \{a, b, c\}, Y = [0, 1]$ and let $\tau = \{\phi, \{a, c\}, X\}$ and

$$\Gamma = \{\tilde{0}, \tilde{1}, C_{(1/2,1/2)}, C_{(1/3,2/3)}\}$$

are topology and intuitionistic fuzzy topology on X and Y respectively. We use the notion $C_{(\alpha,\beta)}(0 \leq \alpha \leq 1, 0 \leq \beta \leq 1$ and $0 \leq \alpha + \beta \leq 1)$ to denote the constant intuitionistic fuzzy sets such that $C_{(\alpha,\beta)}(y) = \{< y, \alpha, \beta >; \forall y \in Y\}$. Then the intuitionistic fuzzy multifunction $F : (X, \tau) \to (Y, \Gamma)$ defined by $F(a) = C_{(5/6,1/6)}$ and $F(b) = C_{(1/2,1/2)}$ and $F(c) = C_{(3/4,1/4)}$ is intuitionistic fuzzy lower semi-continuous but not intuitionistic fuzzy upper semi-continuous.

Definition 3.5. An Intuitionistic fuzzy multifunction $F : (X, \tau) \to (Y, \Gamma)$, then the Intuitionistic fuzzy multifunction $ClF : (X, \tau) \to (Y, \Gamma)$ is defined by $(ClF)(x) = Cl(F(x))$ for every $x \in X$

Theorem 3.6. *For a intuitionistic fuzzy multifunction $F : (X, \tau) \to (Y, \Gamma)$, it follows that $(Cl(F))^{-}(\tilde{V}) = F^{-}(\tilde{V})$, for each intuitionistic fuzzy open set $\tilde{V}$ of Y.*

Proof. Let $\tilde{V}$ be any intuitionistic fuzzy open set of Y and $x \in (Cl(F))^{-}(\tilde{V})$. Then $Cl(F(x))q\tilde{V}$ and hence $F(x)q\tilde{V}$ because $\tilde{V}$ is intuitionistic fuzzy open set of Y Therefore, we obtain $x \in F^{-}(\tilde{V})$, this shows that $(Cl(F))^{-}(\tilde{V}) \subseteq F^{-}(\tilde{V})$. Conversely let, $x \in F^{-}(\tilde{V})$. Then we have $F(x)q\tilde{V}$. which implies that $Cl(F(x))q\tilde{V}$ and hence $x \in (Cl(F))^{-}(\tilde{V})$. This shows that $F^{-}(\tilde{V}) \subseteq (Cl(F))^{-}(\tilde{V})$. Consequently, we obtain $(Cl(F))^{-}(\tilde{V}) = F^{-}(\tilde{V})$. □

Theorem 3.7. *An intuitionistic fuzzy multifunction $F : (X, \tau) \to (Y, \Gamma)$ is intuitionistic fuzzy lower semi continuous if and only if $Cl(F) : (X, \tau) \to (Y, \Gamma)$ is intuitionistic fuzzy lower semi continuous.*

Proof. $Necessity$, Suppose that F is intuitionistic fuzzy lower semi continuous. Let $x \in X$ and let $\tilde{V}$ be any intuitionistic fuzzy open set of Y such that $Cl(F(x))q\tilde{V}$. By theorem 3.6 , we have $x \in (Cl(F))^{-}(\tilde{V}) = F^{-}(\tilde{V})$ and hence $F(x)q\tilde{V}$. Since F is intuitionistic fuzzy lower semi continuous, there exists an open set U of X containing x such that $F(u)q\tilde{V}\forall u \in U$. Hence $Cl(F)(u)q\tilde{V}$ for each $u \in U$. This shows that $Cl(F)$ is intuitionistic fuzzy lower semi continuous.

$Sufficiency$, Suppose $Cl(F)$ is intuitionistic fuzzy lower semi continuous. Let $x \in X$ and let $\tilde{V}$ be any intuitionistic fuzzy open set of Y such that $F(x)q\tilde{V}$. By theorem 3.5, we have $x \in F^{-}(\tilde{V}) = (Cl(F))^{-}(\tilde{V})$ and hence $Cl(F)(x)q\tilde{V}$. Since $Cl(F)$ is intuitionistic fuzzy lower semi continuous, there exists an open set U of X containing x such that $Cl(F(u))q\tilde{V}$ for each $u \in U$. Since $\tilde{V}$ be intuitionistic fuzzy open set of Y, hence $F(u)q\tilde{V}$ for each $u \in U$. This shows that F is intuitionistic fuzzy lower semi continuous. □

4. Strongly upper and strongly lower semi-continuous intuitionistic fuzzy multifunctions

In this section we introduce the concept of strongly lower and strongly upper semi-continuous intuitionistic fuzzy multifunctions obtain some of their characterizations and properties.

Definition 4.1. An Intuitionistic fuzzy multifunction $F : (X, \tau) \to (Y, \Gamma)$ is said to be:

(a) Intuitionistic fuzzy strongly upper semi continuous at a point $x \in X$, if for any intuitionistic fuzzy set $\tilde{B}$ of Y such that $F(x) \subseteq \tilde{B}$ $\exists$ a neighborhood U of x such that $U \subseteq F^{+}(\tilde{B})$.

(b) Intuitionistic fuzzy strongly upper semi-continuous if it is intuitionistic fuzzy strongly upper semi-continuous at each point $x \in X$.

Theorem 4.2. *Let F be an intuitionistic fuzzy multifunction $F : (X, \tau) \to (Y, \Gamma)$, then following conditions are equivalent:*

(a) F is intuitionistic fuzzy strongly upper semi continuous.

(b) $F^{+}(\tilde{B})$ is open in X for every intuitionistic fuzzy set $\tilde{B}$ of Y.

(c) $F^{-}(\tilde{B})$ is closed in X for every intuitionistic fuzzy set $\tilde{B}$ of Y.

Proof. (a)⇒(b). Let $\tilde{B}$ be any intuitionistic fuzzy set of Y and let $x \in F^{+}(\tilde{B})$. Then $F(x) \subseteq \tilde{B}$. And so $\exists$ a neighborhood U of x such that $U \subseteq F^{+}(\tilde{B})$. It follows that $F^{+}(\tilde{B})$ is the union of open sets of X is open in X.

(b)⇒(a). Let $x \in X$ and $\tilde{B}$ be an intuitionistic fuzzy set of Y such that $F(x) \subseteq \tilde{B}$. Then $x \in F^{+}(\tilde{B})$. Put $U = F^{+}(\tilde{B})$.Then U is a neighbourhood of x such that $U \subseteq F^{+}(\tilde{B})$. Hence F is intuitionistic fuzzy strongly upper semi continuous.

(b)⇔(c). It follows from the fact that $[F^{-}(\tilde{A})]^{c} = F^{+}(\tilde{A}^{c})$ for every intuitionistic fuzzy set $\tilde{A}$ of Y and compliment of every open set is always closed. □

Remark 4.3. Every intuitionistic fuzzy strongly upper semi continuous multifunction is intuitionistic fuzzy upper semi continuous but the converse may not be true.For,

Example 4.4. Let $X = \{a, b, c\}, Y = \{x, y, z\}$ and let $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $\Gamma = \{\tilde{0}, \tilde{1}\}$ are topology and intuitionistic fuzzy topology on X and Y respectively. Let the intuitionistic fuzzy sets $\tilde{U}$ and $\tilde{V}$ of Y are defined as follows,

$$\tilde{U} = \{< x, 0.5, 0.4 >, < y, 0.4, 0.5 >, < z, 0.3, 0.5 >\},$$

$$\tilde{V} = \{< x, 0.3, 0.6 >, < y, 0, 1 >, < z, 0, 1 >\}.$$

Then the intuitionistic fuzzy multifunction $F : (X, \tau) \to (Y, \Gamma)$, defined by $F(a) = \tilde{U}, F(b) = \tilde{V}$and $F(c) = \tilde{U}$, is intuitionistic fuzzy upper semi continuous but it is not intuitionistic fuzzy strongly upper semi continuous. For the intuitionistic fuzzy set $\tilde{A} = \{< x, 0.4, 0.5 >, < y, 0, 1 >, < z, 0, 1 >\}$ of $Y, F^{+}\{\tilde{A}\} = \{b\}$ is not open in X.

Definition 4.5. An Intuitionistic fuzzy multifunction $F : (X, \tau) \to (Y, \Gamma)$ is said to be:

(a) Intuitionistic fuzzy strongly lower semi continuous at a point $x \in X$, if for any intuitionistic fuzzy set $\tilde{B}$ of Y such that $F(x)q\tilde{B}$ $\exists$ a neighborhood U of x such that $U \subseteq F^{-}(\tilde{B})$.

(b) Intuitionistic fuzzy strongly lower semi-continuous if it is intuitionistic fuzzy strongly lower semi-continuous at each point of X.

Remark 4.6. Every intuitionistic fuzzy strongly lower semi continuous multifunction is intuitionistic fuzzy lower semi continuous but the converse may not be true. For

Example 4.7. Let $X = \{a, b, c\}, Y = \{x, y, z\}$ and let $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$ and $\Gamma = \{\tilde{0}, \tilde{1}\}$ are topology and intuitionistic fuzzy topology on X and Y respectively. Let the intuitionistic fuzzy sets $\tilde{U}$ and $\tilde{V}$ of Y are defined as follows,

$$\tilde{U} = \{< x, 0.5, 0.4 >, < y, 0.4, 0.5 >, < z, 0.3, 0.5 >\},$$

$$\tilde{V} = \{< x, 0.6, 0.4 >, < y, 0, 1 >, < z, 0, 1 >\}.$$

Then the intuitionistic fuzzy multifunction $F : (X, \tau) \to (Y, \Gamma)$, defined by $F(a) = \tilde{U}, F(b) = \tilde{V}$and $F(c) = \tilde{U}$, is intuitionistic fuzzy lower semi continuous but it is not intuitionistic fuzzy strongly lower semi continuous. For the intuitionistic fuzzy set $\tilde{A} = \{< x, 0.4, 0.5 >, < y, 0, 1 >, < z, 0, 1 >\}$ of $Y, F^{-}\{\tilde{A}\} = \{b\}$ is not open in X.

Theorem 4.8. *Let F be an intuitionistic fuzzy multifunction $F : (X, \tau) \to (Y, \Gamma)$, then following conditions are equivalent:*

(a) F is intuitionistic fuzzy strongly lower semi continuous.

(b) $F^-(\tilde{B})$ is open in X for every intuitionistic fuzzy set $\tilde{B}$ of Y.

(c) $F^+(\tilde{B})$ is closed in X for every intuitionistic fuzzy set $\tilde{B}$ of Y.

Proof. (a)⇒(b). Let $\tilde{B}$ be any intuitionistic fuzzy set of Y and Let $x \in F^-(\tilde{B})$. Then $F(x)q\tilde{B}$. Therefore ∃ a neighborhood U of x such that $U \subseteq F^-(\tilde{B})$. Hence, $F^-(\tilde{B})$ is the union of open sets of X is open in X.

(b)⇒(a). Let $x \in X$ and $\tilde{B}$ be an intuitionistic fuzzy set of Y such that $F(x)q\tilde{B}$. Then $x \in F^-(\tilde{B})$. Put $U = F^-(\tilde{B})$. Then U is a neighbourhood of x such that $U \subseteq F^-(\tilde{B})$. Hence F is intuitionistic fuzzy strongly lower semi continuous.

(b)⇔(c). It follows from the fact that $[F^+(\tilde{A})]^c = F^-(\tilde{A}^c)$ for every intuitionistic fuzzy set $\tilde{A}$ of Y and compliment of every open set is always closed. □

Definition 4.9. An intuitionistic fuzzy multifunction $F : (X, \tau) \to (Y, \Gamma)$ is said to be:

(a) Intuitionistic fuzzy strongly semi continuous at a point $x_0 \in X$, if for any intuitionistic fuzzy sets $\tilde{G}_1, \tilde{G}_2$ of Y such that $F(x_0) \subseteq \tilde{G}_1$ and $F(x_0)q\tilde{G}_2$ ∃ neighborhoods U and V of x_0 such that $U \subseteq F^+(\tilde{G}_1)$ and $V \subseteq F^-(\tilde{G}_2)$

(b) Intuitionistic fuzzy strongly semi-continuous if it is intuitionistic fuzzy strongly semi-continuous at each point of X.

Remark 4.10. Every intuitionistic fuzzy strongly continuous intuitionistic fuzzy multifunction is intuitionistic fuzzy strongly lower semi continuous (resp. intuitionistic fuzzy strongly upper semi continuous).

Theorem 4.11. *Let F be an intuitionistic fuzzy multifunction $F : (X, \tau) \to (Y, \Gamma)$, then following conditions are equivalent:*

(a) F is intuitionistic fuzzy strongly semi continuous.

(b) $F^+(\tilde{A}) \cap F^-(\tilde{B})$ is open in X for every intuitionistic fuzzy set $\tilde{A}, \tilde{B}$ of Y.

(c) $F^+(\tilde{A}) \cup F^-(\tilde{B})$ is closed in X for every intuitionistic fuzzy set $\tilde{A}, \tilde{B}$ of Y.

Proof. (a)⇒(b). Let $\tilde{A}, \tilde{B}$ be any two intuitionistic fuzzy sets of Y and $x \in F^+(\tilde{A}) \cap F^-(\tilde{B})$, Then $F(x) \subset \tilde{A}$ and $F(x)q\tilde{B}$. Therefore there exists open sets U and V containing x such that $U \subseteq F^+(\tilde{A})$ and $V \subset F^-(\tilde{B})$. It follows that $x \in U \cap V \subset F^+(\tilde{A}) \cap F^-(\tilde{B})$ and $U \cap V$ is an open set of x. Hence $F^+(\tilde{A}) \cap F^-(\tilde{B})$ is an intersection of open sets in X is an open set in X.

(b)⇒(a). Let x be any point of X and let $\tilde{A}, \tilde{B}$ be intuitionistic fuzzy sets of Y such that $F(x) \subset \tilde{A}$ and $F(x)q\tilde{B}$. Then $x \in F^+(\tilde{A})$ and $x \in F^-(\tilde{B})$ implies $x \in F^+(\tilde{A}) \cap F^-(\tilde{B})$, Put $U = V = F^+(\tilde{A}) \cap F^-(\tilde{A})$. Then by (b) U and V are open in X containing x such that $U \subset F^+(\tilde{A})$ and $V \subset F^-(\tilde{A})$. Hence, F is intuitionistic fuzzy strongly semi continuous at x.

(b)⇔(c).Let $\tilde{A}, \tilde{B}$ be any two intuitionistic fuzzy sets of Y. By Lemma 2.13 we have

$$(F^+(\tilde{A}) \cup F^-(\tilde{A}))^c = (F^+(\tilde{A}))^c \cap (F^-(\tilde{B}))^c = F^-(\tilde{A}^c) \cap F^+(\tilde{B}^c)$$

is open in X. Therefore, $(F^+(\tilde{A}) \cup F^-(\tilde{A}))$ is closed in X. □

Theorem 4.12. *Let $\{U_\omega : \omega \in \Lambda\}$ be an open cover of a topological space (X, τ). An Intuitionistic fuzzy multifunction $F(X, \tau) \to (Y, \Gamma)$ is intuitionistic fuzzy strongly upper semi continous if and only if the restriction $F \mid U_\omega : U_\omega \to Y$ is intuitionistic fuzzy strongly upper semi continous for each $\omega \in \Lambda$.*

Proof. Necessity. Suppose that F is intuitionistic fuzzy strongly upper semi continous. Let $\omega \in \Lambda$ and $x \in U_\omega$. Let $\tilde{V}$ be any intuitionistic fuzzy set of Y such that $(F \mid U_\omega)(x) \subset \tilde{V}$. Since F is intuitionistic fuzzy strongly upper semi continous and $F(x) = (F \mid U_\omega)(x)$, there exists an open set G of X containing x such that $F(G) \subset \tilde{V}$. Put $U = G \cap U_\omega$, then U is an open set of X containing x such that $(F \mid U_\omega)(U) = F(U) \subset \tilde{V}$. Hence $(F \mid U_\omega)$ is intuitionistic fuzzy strongly upper semi continous.

Sufficiency. Let $x \in X$ and $\tilde{V}$ be any intuitionistic fuzzy set of Y, such that $F(x) \subset \tilde{V}$. Since $\{U_\omega : \omega \in \Lambda\}$ be an open cover of X, there exists $\omega \in \Lambda$ such that $x \in U_\omega$. Now $F \mid U_\omega : U_\omega \to Y$ is intuitionistic fuzzy strongly upper semi continous and $F(x) = (F \mid U_\omega)(x)$, there exists an open set U of U_ω containing x, such that $(F \mid U_\omega)(U) \subset \tilde{V}$. Since each U_ω is open in X, therefore U is open in X containing x and $F(U) \subset \tilde{V}$. Hence F is intuitionistic fuzzy strongly upper semi continous. □

Theorem 4.13. *Let $\{U_\omega : \omega \in \Lambda\}$ be an open cover of a topological space (X, τ). An Intuitionistic fuzzy multifunction $F(X, \tau) \to (Y, \Gamma)$ is intuitionistic fuzzy strongly lower semi continous if and only if the restriction $F \mid U_\omega : U_\omega \to Y$ is intuitionistic fuzzy strongly lower semi continous for each $\omega \in \Lambda$.*

Proof. Necessity. Suppose that F is intuitionistic fuzzy strongly lower semi continous. Let $\omega \in \Lambda$ and $x \in U_\omega$.Let $\tilde{V}$ be any intuitionistic fuzzy set of Y such that $(F \mid U_\omega)(x)q\tilde{V}$.We have $F(x) = (F \mid U_\omega)(x)$ and hence $F(x)q\tilde{V}$. Since F is intuitionistic fuzzy strongly lower semi continous, there exists open set G of X containing x such that $F(g)q\tilde{V}$ for each $g \in G$. Put $U = G \cap U_\omega$. Then U is an open set of X containing x, such that $(F \mid U_\omega)(U) = F(U)q\tilde{V}$. Hence $(F \mid U_\omega)$ is intuitionistic fuzzy strongly lower semi continous.

Sufficiency. Let $x \in X$ and $\tilde{V}$ be any intuitionistic fuzzy set of Y, such that $F(x)q\tilde{V}$. Since $\{U_\omega : \omega \in \Lambda\}$ be an open cover of X, there exists $\omega \in \Lambda$ such that $x \in U_\omega$. Since $F(x) = (F \mid U_\omega)(x)$, we have $(F \mid U_\omega)q\tilde{V}$. Since $F \mid U_\omega : U_\omega \to Y$ is intuitionistic fuzzy strongly lower semi continous and $F(x) = (F \mid U_\omega)(x)$, there exists an open set U of U_ω containing x such that $(F \mid U_\omega)(u)q\tilde{V}$ for each for each $u \in U$. Since each U_ω is open in X therefore U is open in X containing x and $F(U)q\tilde{V}$. Therefore F is intuitionistic fuzzy strongly lower semi continous. □

Future and scope of newly defined multifunctions

The theory of Intuitionistic fuzzy Multifunctions can be applied in functional analysis and Fixed point theory, for instance we can study fixed point of Intuitionistic fuzzy multifunctions in Intutionistic fuzzy Matric Spaces. Also,Intuitionistic fuzzy Multifunctions can play important role in the study of the convergence of Intuitionistic fuzzy Nets and filters in Intuitionistic fuzzy topological spaces. Intuitionistic fuzzy Multifunctions can be applied in many areas such as, Decision theory, Non-cooperative games, Artificial Intelligence, Economic theory, Medical Sciences, Image Processing and Information Sciences. Intuitionistic fuzzy Multifunctions can be considered as a generalization of fuzzy Multifunctions.

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