Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 8, No. 5, (November 2014), pp. 803–814
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

@FMI

On generalized fuzzy weakly interior ideals of ordered semigroups

G. Mohanraj, D. Krishnaswamy, R. Hema

Received 4 Januay 2014; Revised 7 May 2014; Accepted 5 June 2014

Abstract. In this paper, we introduce generalized fuzzy weakly interior ideals of an ordered semigroup. We generalize fuzzy interior ideals of an ordered semigroup. We establish necessary and sufficient condition for a fuzzy set to be a generalized fuzzy weakly interior ideal. We establish three equivalent conditions for an ordered semigroup to be regular. We characterize semisimple ordered semigroups by generalized fuzzy weakly interior ideals and generalized fuzzy right ideals of ordered semigroups. We prove that every generalized fuzzy weakly interior ideal coincides with generalized fuzzy ideal in semisimple ordered semigroups. We establish necessary and sufficient conditions for an ordered semigroup to be right weakly regular in terms of generalized fuzzy weakly interior ideals and generalized fuzzy generalized bi-ideals.

2010 AMS Classification: 06F05, 08A72, 20N25

Keywords: Fuzzy algebra, (λ, μ)-fuzzy weakly interior ideal, Semisimple ordered semigroup, Right weakly regular ordered semigroup.

Corresponding Author: G. Mohanraj (gmohanraaj@gmail.com)

1. Introduction

The application of fuzzy technology in information processing is important and its significance will certainly increase in importance in the future. Our aim is to promote research and development in fuzzy technology by studying the fuzzy ordered semigroups. The goal is to explain new methodological developments in fuzzy ordered semigroups which is of growing importance in the future. The important concept of a fuzzy set put forth by Zadeh in 1965 [14] has opened up keen insights and applications in a wide range of scientific fields. Since then, many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, groupoids, topology, etc. A theory of fuzzy sets on ordered semigroups has been recently developed. Fuzzy sets in ordered semigroup were first studied by Kehayopulu and Tsingelis in [7], then they defined fuzzy analogies for several notions, which have proven useful in the theory of ordered semigroups. Jun and Song [5] discussed general forms of fuzzy interior ideals in semigroups. A. Khan, Y.B.Jun and M. Z. Abbas [1] gave the characterization of ordered semigroups in terms of $(\in, \in \vee q)-$fuzzy interior-ideals. Kazanci and Yamak [6] introduced the concept of a generalized fuzzy bi-ideal in semigroups and gave some properties of fuzzy bi-ideals in terms of $(\in, \in \vee q)$-fuzzy bi-ideals. Jun. et al. [4] gave the concept of weakly fuzzy bi-ideals in ordered semigroups and characterized regular ordered semigroups.

P. Dheena and G. Mohanraj introduced $(\lambda, \mu)-$fuzzy ideals in semirings [2] and $(\lambda, \mu)-$fuzzy prime ideals in semirings [3]. G. Mohanraj, D. Krishnaswamy and R. Hema [9] redefined generalized fuzzy ideal of an ordered semigroup and introduced fuzzy m-systems and n-systems of ordered semigroups (see [10]). G. Mohanraj, D. Krishnaswamy and R. Hema [8] introduced $(\bar{\in}, \bar{\in} \vee \bar{q})-$fuzzy bi-ideals of an ordered semigroup. A.Khan, Y.B.Jun and M.Z.Abbas [1] proved that $(\in, \in \vee q)$-fuzzy interior ideals coincide with $(\in, \in \vee q)$-fuzzy ideals in regular and semisimple ordered semigroups.

In this paper, fuzzy interior ideal of an ordered semigroup based on [2],[3], which is the generalization of existing $(\in, \in \vee q)$-fuzzy interior ideal of S is generalized. We introduce (λ, μ)-fuzzy weakly interior ideal. We establish necessary and sufficient condition for a fuzzy set to be a (λ, μ)-fuzzy weakly interior ideal. We establish three equivalent conditions for an ordered semigroup to be regular. We prove that every (λ, μ)-fuzzy weakly interior ideal coincides with (λ, μ)-fuzzy ideal in semisimple ordered semigroups. We characterize semisimple ordered semigroups through (λ, μ)-fuzzy weakly interior ideals and (λ, μ)-fuzzy right ideals of an ordered semigroup. We establish necessary and sufficient conditions for an ordered semigroup to be right weakly regular in terms of (λ, μ)-fuzzy weakly interior ideals and (λ, μ)-fuzzy generalized bi-ideals.

2. Preliminaries

Definition 2.1. By an ordered semigroup(po-semigroup), we mean a structure $(S, ., \leq)$ in which the following conditions are satisfied:
(OS1) $(S, .)$is a semigroup.
(OS2) $(S, \leq)$is a poset.
(OS3) $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for all $a, b, x \in S$.

For $A \subseteq S$, we denote $(A] := \{t \in S | t \leq h \ for \ some \ h \in A\}$. If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. For non-empty subsets A, B of S, we denote $AB := \{ab | a \in A, b \in B\}$.
Let $(S, ., \leq)$ be an ordered semigroup. A non-empty subset A of S is called a sub-semigroup of S if $A^2 \subseteq A$.

Definition 2.2. Let $(S, ., \leq)$ be an ordered semigroup. A non-empty subset A of S is called a right (left) ideal of S if

(1) For all $a \in S$, for all $b \in A$ and $a \leq b$ implies $a \in A$.
(2) $AS \subseteq A$ $(SA \subseteq A)$.

If A is both a right and left ideal of S, then it is called an ideal of S.

By a fuzzy set f of an ordered semigroup S, we mean a mapping $f : S \longrightarrow [0,1]$.

Definition 2.3. An ordered semigroup S is called regular if for every $a \in S$, there exists $x \in S$ such that $a \leq axa$.

Definition 2.4. Let f be a fuzzy set of an ordered semigroup S. The level set of f, denoted by f_t on S for $t \in (0,1]$ is defined as:

$$f_t = \{x \in S | f(x) \geq t\}.$$

Definition 2.5. For $a \in S$, we define $A_a = \{(y,z) \in S \times S | a \leq yz\}$.

Definition 2.6. Let f and g be any two fuzzy sets of an ordered semigroup S. Then the fuzzy product of f and g denoted by $f \circ g$ is defined as follows:

$$(f \circ g)(a) = \begin{cases} \bigvee\limits_{(y,z) \in A_a} \{f(y) \wedge g(z)\} & if\ A_a \neq \varnothing \\ 0 & if\ A_a = \varnothing. \end{cases}$$

For an ordered semigroup S, the fuzzy set "1" is defined as follows:
$1 : S \to [0,1]$ by $1(x) = 1$, for all $x \in S$.

Definition 2.7 ([12])**.** A fuzzy set f of S is a (λ,μ)-fuzzy generalized bi-ideal if
1. $x \leq y$ implies $f(x) \vee \lambda \geq f(y) \wedge \mu$
2. $f(xyz) \vee \lambda \geq f(x) \wedge f(z) \wedge \mu$, for all $x, y, z \in S$.

Remark 2.8. 1.If $\lambda = 0$, $\mu = 1$ in Definition 2.7, then f is a fuzzy generalized bi-ideal of S.
2. If $\lambda = 0$, $\mu = 0.5$ in Definition 2.7, then f is a $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S.
3. If $\lambda = 0.5$, $\mu = 1$ in Definition 2.7, then f is a $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy generalized bi-ideal of S.

Definition 2.9 ([12])**.** A (λ,μ)-fuzzy generalized bi-ideal of S is a (λ,μ)-fuzzy bi-ideal if $f(xy) \vee \lambda \geq f(x) \wedge f(y) \wedge \mu$, for all $x, y \in S$.

Lemma 2.10 ([9])**.** *For any two fuzzy sets f and g of S,*
(1) $f \subseteq \vee q\, g$ if and only if $g(x) \geq f(x) \wedge 0.5$, for all $x \in S$.
(2)$f \supseteq \vee \bar{q}\, g$ if and only if $f(x) \vee 0.5 \geq g(x)$, for all $x \in S$.

Lemma 2.11 ([13])**.** *Let A, B be any non-empty subsets of an ordered semigroup S. Then the following statements are true:*
1.$\chi_A \circ \chi_B = \chi_{(AB]}$.
2.$\chi_A \cap \chi_B = \chi_{A \cap B}$.
3.$\chi_A \cup \chi_B = \chi_{A \cup B}$.

3. Generalized fuzzy weakly interior ideals

Let $0 \leq \lambda < \mu \leq 1$. Hereafter S denotes an ordered semigroup unless otherwise specified.

Definition 3.1 ([9]). Let f be a fuzzy set of an ordered semigroup S. Then f is said to be a (λ, μ)-fuzzy right (left) ideal of S if
1. $x \leq y$ implies $f(x) \vee \lambda \geq f(y) \wedge \mu$.
2. $f(xy) \vee \lambda \geq f(x) \wedge \mu (f(xy) \vee \lambda \geq f(y) \wedge \mu)$, for all $x, y \in S$.

Definition 3.2 ([11]). Let f be a fuzzy set of an ordered semigroup S. Then f is said to be a (λ, μ)-fuzzy ideal of S if
1. $x \leq y$ implies $f(x) \vee \lambda \geq f(y) \wedge \mu$.
2. $f(xy) \vee \lambda \geq [f(x) \vee f(y)] \wedge \mu$, for all $x, y \in S$.

Definition 3.3 ([11]). Let f and g be two fuzzy sets of S. If $g(x) \vee \lambda \geq f(x) \wedge \mu$, for all $x \in S$, then we write $f \subseteq_{\mu}^{\lambda} g$.

Definition 3.4. Let $(S, \cdot, \leq)$ be an ordered semigroup. A non-empty subset I of S is called a weakly interior ideal of S if
1. $a \leq b$, $b \in I$ and $a \in S$ implies $a \in I$.
2. $xay \in I$, for every $a \in I$ and for every $x, y \in S$.

Definition 3.5. A weakly interior ideal I of S is said to be an interior ideal of S if $xy \in I$, for every $x, y \in I$.

Definition 3.6. A fuzzy set f of S is called a fuzzy interior ideal of S if
1. $x \leq y$ implies $f(x) \geq f(y)$.
2. $f(xy) \geq f(x) \wedge f(y)$.
3. $f(xyz) \geq f(y)$, for all $x, y, z \in S$.

Theorem 3.7 ([1]). *Let f be a fuzzy set of S. Then f is an $(\in, \in \vee q)$-fuzzy interior ideal of S if and only if*
1. $x \leq y$ *implies* $f(x) \geq f(y) \wedge 0.5$.
2. $f(xy) \geq f(x) \wedge f(y) \wedge 0.5$.
3. $f(xyz) \geq f(y) \wedge 0.5$, *for all* $x, y, z \in S$.

Definition 3.8. A fuzzy set f of S is called a (λ, μ)-fuzzy weakly interior ideal of S if
1. $x \leq y$ implies $f(x) \vee \lambda \geq f(y) \wedge \mu$.
2. $f(xyz) \vee \lambda \geq f(y) \wedge \mu$, for all $x, y, z \in S$.

Definition 3.9. A (λ, μ)-fuzzy weakly interior ideal f of S is a (λ, μ)-fuzzy interior ideal if $f(xy) \vee \lambda \geq f(x) \wedge f(y) \wedge \mu$, for all $x, y, z \in S$.

Remark 3.10. 1.If $\lambda = 0$, $\mu = 1$ in Definition 3.9, then f is a fuzzy interior ideal of S.
2.If $\lambda = 0$, $\mu = 0.5$ in Definition 3.9, then f is a $(\in, \in \vee q)$-fuzzy interior ideal of S.
3. If $\lambda = 0.5$, $\mu = 1$ in Definition 3.9, then f is a $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy interior ideal of S.

Remark 3.11. 1. Every fuzzy weakly interior ideal is a (λ, μ)-fuzzy weakly interior ideal of S.
2. Every (λ, μ)-fuzzy weakly interior ideal need not be a fuzzy weakly interior ideal of S.
3. Every (λ, μ)-fuzzy weakly interior ideal need not be a (λ, μ)-fuzzy interior ideal as shown in the following example.

Example 3.12. Let $(S, \cdot, \leq)$ be an ordered semigroup where $S = \{a, b, c, d\}$. The order relation "$\leq$" is given by

$$\leq := \{(a,a), (b,b), (c,c), (d,d), (a,b)\}$$

and the multiplication is given as follows:

·	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$$g(x) = \begin{cases} 0.8 & \text{if } x = a \\ 0.5 & \text{if } x = c \\ 0.3 & \text{if } x = d \\ 0.2 & \text{otherwise} \end{cases}$$

Clearly g is a $(0.3, 0.5)-$fuzzy weakly interior ideal. But g is not a $(0.3, 0.5)-$fuzzy interior ideal, since $g(c \cdot c) \vee 0.3 = g(b) \vee 0.3 = 0.3 \ngeq g(c) \wedge g(c) \wedge 0.5 = 0.5$, g is not a $(\in, \in \vee q)$-fuzzy interior ideal, since $g(d \cdot d) = g(b) = 0.2 \ngeq g(d) \wedge g(d) \wedge 0.5 = 0.3 \wedge 0.5 = 0.3$ and g is not a fuzzy interior ideal, since $g(d \cdot d) = g(b) = 0.2 \ngeq g(d) \wedge g(d) = 0.3$

Theorem 3.13. *For every $a \in S$, $\chi_{(a \cup SaS]}$ is a fuzzy weakly interior-ideal of S.*

Proof. Let $x \leq y$. If $y \notin (a \cup SaS]$, then $\chi_{(a\cup SaS]}(y) = 0$. Therefore $\chi_{(a\cup SaS]}(x) \geq \chi_{(a\cup SaS]}(y) = 0$. If $y \in (a \cup SaS]$ and $x \leq y$, then $x \in (a \cup SaS]$. Therefore $\chi_{(a\cup SaS]}(x) = 1 \geq \chi_{(a\cup SaS]}(y)$. Thus $x \leq y$ implies $\chi_{(a\cup SaS]}(x) \geq \chi_{(a\cup SaS]}(y)$. Let $x, y, z \in S$. If $y \notin (a \cup SaS]$, then $\chi_{(a\cup SaS]}(xyz) \geq \chi_{(a\cup SaS]}(y) = 0$. If $y \in (a \cup SaS]$, then $xyz \in (SaS]$ implies $\chi_{(a\cup SaS]}(xyz) = 1 \geq \chi_{(a\cup SaS]}(y)$. Therefore $\chi_{(a\cup SaS]}$ is a fuzzy weakly interior-ideal of S. □

Theorem 3.14. *For every $a \in S$, $\chi_{(a \cup aSa]}$ is a fuzzy generalized bi-ideal of S.*

Proof. Proof is similar to the proof of the Theorem 3.13. □

Theorem 3.15. *A fuzzy set f is a (λ, μ)-fuzzy weakly interior ideal of S if and only if the level set f_t is a weakly interior ideal of S for all $t \in (\lambda, \mu]$ whenever non-empty.*

Proof. Let f be a (λ, μ)-fuzzy weakly interior ideal of S. Let $y \in f_t$, $t \in (\lambda, \mu]$ and $x \leq y$. Then $f(x) \vee \lambda \geq f(y) \wedge \mu \geq t \wedge \mu = t > \lambda$ implies $f(x) \geq t$. Thus $x \in f_t$. Let $y \in f_t$ and $x, z \in S$. Then $f(xyz) \vee \lambda \geq f(y) \wedge \mu \geq t \wedge \mu = t > \lambda$. Then $f(xyz) \geq t$ implies $xyz \in f_t$. Therefore f_t is a weakly interior ideal of S for all $t \in (\lambda, \mu]$, whenever non-empty.

Conversely, let $x \leq y$. Take $f(y) \wedge \mu = t$. If $t > \lambda$, then f_t is an interior ideal of S. Now, $x \leq y$ and $y \in f_t$ imply $x \in f_t$. Thus $f(x) \vee \lambda \geq t \vee \lambda = t = f(y) \wedge \mu$. If $t \leq \lambda$, then $f(y) \wedge \mu = t \leq \lambda \leq f(x) \vee \lambda$. Therefore $x \leq y$ implies $f(x) \vee \lambda \geq f(y) \wedge \mu$. If there exist $x, y, z \in S$ such that $f(xyz) \vee \lambda < f(y) \wedge \mu$, then choose a $t \in (\lambda, \mu)$ such that $f(xyz) \vee \lambda < t < f(y) \wedge \mu$. Then $y \in f_t$ and $t \in (\lambda, \mu]$ but $xyz \notin f_t$, which is a contradiction. Therefore f is a $(\lambda, \mu)-$fuzzy weakly interior ideal of S. □

Corollary 3.16. *A fuzzy set f is a (λ, μ)-fuzzy interior ideal of S if and only if the level set f_t is an interior ideal in S for all $t \in (\lambda, \mu]$ whenever non-empty.*

Proof. Let f be a (λ, μ)-fuzzy interior ideal of S. By Theorem 3.15, f_t is a weakly interior ideal in S for all $t \in (\lambda, \mu]$, whenever non-empty. Now $x, y \in f_t$ for $t \in (\lambda, \mu]$ implies $f(x) \wedge f(y) \geq t$. Thus $f(xy) \vee \lambda \geq f(x) \wedge f(y) \wedge \mu \geq t \wedge \mu = t > \lambda$. Therefore $xy \in f_t$. Hence f_t is an interior ideal in S.

On the other hand, by Theorem 3.15, f is a (λ, μ)-fuzzy weakly interior ideal of S. If there exist $x, y \in S$ such that $f(xy) \vee \lambda < f(x) \wedge f(y) \wedge \mu$, then choose a $t \in (\lambda, \mu)$ such that $f(xy) \vee \lambda < t < f(x) \wedge f(y) \wedge \mu$. Then $x, y \in f_t$ and $t \in (\lambda, \mu)$ but $xy \notin f_t$, which is a contradiction. Therefore f is a (λ, μ)-fuzzy interior ideal of S. □

Corollary 3.17. *A fuzzy set f is a fuzzy interior ideal of S if and only if the level set f_t is an interior ideal in S for all $t \in (0, 1]$ whenever non-empty.*

Proof. By taking $\lambda = 0$, $\mu = 1$ in Corollary 3.16, then we get the result. □

Corollary 3.18. *Let f be fuzzy set of S. Then f is a $(\in, \in \vee q)$-fuzzy interior ideal of S if and only if the level set f_t is a interior ideal of S for all $t \in (0, 0.5]$ whenever nonempty.*

Proof. By taking $\lambda = 0$, $\mu = 0.5$ in Corollary 3.16, then we get the result. □

Now, $(\lambda,\ \mu)$-fuzzy weakly interior ideal is a generalization of fuzzy interior ideal, $(\in, \in \vee q)$-fuzzy weakly interior ideal and $(\bar{\in}, \bar{\in} \vee \bar{q})$-fuzzy weakly interior ideal of S as shown in the following example.

Example 3.19. Let $(S, \cdot, \leq)$ be an ordered semigroup where $S = \{a, b, c, d, e\}$. The order relation " $\leq$ " is given by

$$\leq := \{(a, a), (a, c), (a, d), (a, e), (b, b), (b, d), (b, e), (c, c), (c, e), (d, d), (d, e), (e, e)\}$$

and the multiplication is given as follows:

·	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

$$g(x) = \begin{cases} 0.8 & \text{if } x \in \{b, d\} \\ 0.7 & \text{if } x = a \\ 0.3 & \text{if } x = c \\ 0.2 & \text{if } x = e \end{cases}$$

$$g_t = \begin{cases} \varnothing & \text{if } t \in (0.8, 1] \\ \{b, d\} & \text{if } t \in (0.7, 0.8] \\ \{a, b, d\} & \text{if } t \in (0.3, 0.7] \\ \{a, b, d, c\} & \text{if } t \in (0.2, 0.3] \\ S & \text{if } t \in (0, 0.2] \end{cases}$$

Clearly g is a $(0.3, 0.7)$-fuzzy weakly interior ideal. But g is not a fuzzy weakly interior ideal since $a \leq d$ implies $g(a) = 0.7 \not\geq g(d) = 0.8$ and g is not a $(\in, \in \vee q)$-fuzzy weakly interior ideal since $g(c \cdot c \cdot e) = g(e) = 0.2 \not\geq g(c) \wedge 0.5 = 0.3 \wedge 0.5 = 0.3$.

4. Regular ordered semigroup

Theorem 4.1. *If S is a regular ordered semigroup, then every (λ, μ)-fuzzy weakly interior ideal is a (λ, μ)-fuzzy interior ideal of S.*

Proof. Let f be a (λ, μ)-fuzzy weakly interior ideal of regular ordered semigroup S. Let $x, y \in S$. Then for every $y \in S$, there exists $z \in S$ such that $y \leq yzy$. Thus $y \leq yzy$ implies $xy \leq xyzy$. Now,

$$\begin{aligned} f(xy) \vee \lambda &= (f(xy) \vee \lambda) \vee \lambda \\ &\geq (f(xy(zy)) \wedge \mu) \vee \lambda \\ &= (f(xy(zy)) \vee \lambda) \wedge \mu \\ &\geq (f(y) \wedge \mu) \wedge \mu \\ &= f(y) \wedge \mu \\ &\geq f(x) \wedge f(y) \wedge \mu. \end{aligned}$$

Hence f is a (λ, μ)-fuzzy interior ideal of S. □

Theorem 4.2. *If S is a regular ordered semigroup, then every (λ, μ)-fuzzy weakly interior ideal is a (λ, μ)-fuzzy ideal of S.*

Proof. Let f be a (λ, μ)-fuzzy weakly interior ideal of a regular ordered semigroup S. Let $x, y \in S$. Then for every $y \in S$, there exists $z \in S$ such that $y \leq yzy$. Thus $y \leq yzy$ implies $xy \leq xyzy$. Now,

$$\begin{aligned} f(xy) \vee \lambda &= (f(xy) \vee \lambda) \vee \lambda \\ &\geq (f(xy(zy)) \wedge \mu) \vee \lambda \\ &= (f(xy(zy)) \vee \lambda) \wedge \mu \\ &\geq (f(y) \wedge \mu) \wedge \mu \\ &= f(y) \wedge \mu. \end{aligned}$$

Also, for every $x \in S$, there exists $a \in S$ such that $x \leq xax$. Thus $x \leq xax$ implies $xy \leq xaxy$. Now,

$$\begin{aligned} f(xy) \vee \lambda &= (f(xy) \vee \lambda) \vee \lambda \\ &\geq (f((xa)xy) \wedge \mu) \vee \lambda \\ &= (f((xa)xy)) \vee \lambda) \wedge \mu \\ &\geq (f(x) \wedge \mu) \wedge \mu \\ &= f(x) \wedge \mu. \end{aligned}$$

Therefore $f(xy) \vee \lambda \geq (f(x) \wedge \mu) \vee (f(y) \wedge \mu) = (f(x) \vee f(y)) \wedge \mu$. Hence f is a (λ, μ)-fuzzy ideal of S. □

Lemma 4.3. *Every (λ, μ)-fuzzy right (left) ideal is a (λ, μ)-fuzzy bi-ideal of S.*

Proof. Let f and g be (λ,μ)-fuzzy right and left ideal of S respectively. Let $x,y,z \in S$. Then

$$\begin{aligned} f(x(yz)) \vee \lambda &\geq f(x) \wedge \mu \\ &\geq f(x) \wedge f(z) \wedge \mu \\ g((xy)z) \vee \lambda &\geq g(z) \wedge \mu \\ &\geq g(x) \wedge g(z) \wedge \mu. \end{aligned}$$

Therefore $f(xyz) \vee \lambda \geq f(x) \wedge f(z) \wedge \mu$ and $g(xyz) \vee \lambda \geq g(x) \wedge g(z) \wedge \mu$. Hence f and g are (λ,μ)-fuzzy bi-ideals of S. □

Theorem 4.4. *Let S be an ordered semigroup. Then the following statements are equivalent.*
1. S is regular.
2. $f \cap g \cap h \subseteq_\mu^\lambda f \circ g \circ h$ for every (λ,μ)-fuzzy weakly interior ideal g and for every (λ,μ)-fuzzy generalized bi-ideals f and h of S.
3. $f \cap g \cap h \subseteq_\mu^\lambda f \circ g \circ h$ for every (λ,μ)-fuzzy right ideal f, for every (λ,μ)-fuzzy weakly interior ideal g and for every (λ,μ)-fuzzy generalized bi-ideal h of S.
4. $f \cap g \cap h \subseteq_\mu^\lambda f \circ g \circ h$ for every (λ,μ)-fuzzy right ideal f, for every (λ,μ)-fuzzy weakly interior ideal g and for every (λ,μ)-fuzzy left ideal h of S.
5. $\chi_{(a)_r} \cap \chi_{(a)_{(a\cup SaS]}} \cap \chi_{(a)_l} \subseteq_\mu^\lambda \chi_{(a)_r} \circ \chi_{(a\cup SaS]} \circ \chi_{(a)_l}$, for all $a \in S$, where $(a)_r$ is the right ideal generated by a and $(a)_l$ is the left ideal generated by a.

Proof. (1) ⇒ (2) Let S be a regular ordered semigroup. Then for every $a \in S$, there exists $x \in S$ such that $a \leq axa$. Then $a \leq axa \leq (axa)xa = axaxa \leq (axa)(xax)(axa)$. Let f and h be (λ,μ)-fuzzy weakly bi-ideals and g be a (λ,μ)-fuzzy weakly interior ideal of S. Thus $(axa,(xax)(axa)) \in A_a$ implies $(f \circ g \circ h)(a) \neq 0$ and

$$\begin{aligned} (f \circ g \circ h)(a) &= \bigvee_{(y,z)\in A_a} f(y) \wedge (g \circ h)(z) \\ &\geq f(axa) \wedge (g \circ h)((xax)(axa)) \\ &= f(axa) \wedge [\bigvee_{(s,t)\in A_{(xax)(axa)}} g(s) \wedge h(t)] \\ &\geq f(axa) \wedge (g(xax) \wedge h(axa)). \end{aligned}$$

Therefore,

$$\begin{aligned} (f \circ g \circ h)(a) \vee \lambda &\geq [f(axa) \wedge g(xax) \wedge h(axa)] \vee \lambda \\ &= [f(axa) \vee \lambda] \wedge [g(xax) \vee \lambda] \wedge [h(axa) \vee \lambda] \\ &\geq (f(a) \wedge \mu) \wedge (g(a) \wedge \mu) \wedge (h(a) \wedge \mu) \\ &= (f(a) \wedge g(a) \wedge h(a)) \wedge \mu \\ &= (f \cap g \cap h)(a) \wedge \mu. \end{aligned}$$

Hence $f \cap g \cap h \subseteq_\mu^\lambda f \circ g \circ h$.

(2) ⇒ (3) Let f be a (λ,μ)-fuzzy right ideal of S. Then by Lemma 4.3, f is a (λ,μ)-fuzzy bi-ideal. Therefore by (2), $f \cap g \cap h \subseteq_\mu^\lambda f \circ g \circ h$.

(3) $\Rightarrow$ (4) Let h be a (λ,μ)-fuzzy left ideal of S. Then by Lemma 4.3, h is a (λ,μ)-fuzzy bi-ideal. Therefore by (3), $f\cap g\cap h\subseteq_{\mu}^{\lambda} f\circ g\circ h$.

(4) $\Rightarrow$ (5) By Theorem 3.13, $\chi_{(a\cup SaS]}$ is a $(\lambda,\mu)-$fuzzy weakly interior ideal of S, for all $a\in S$. Thus $\chi_{(a)_r}\cap\chi_{(a\cup SaS]}\cap\chi_{(a)_l}\subseteq_{\mu}^{\lambda}\chi_{(a)_r}\circ\chi_{(a\cup SaS]}\circ\chi_{(a)_l}$, for all $a\in S$.

(5) $\Rightarrow$ (1) For any $a\in S$, $\mu=1\wedge\mu=(\chi_{(a)_r}\cap\chi_{(a\cup SaS]}\cap\chi_{(a)_l})(a)\wedge\mu\leq(\chi_{(a)_r}\circ\chi_{(a\cup SaS]}\circ\chi_{(a)_l})(a)\vee\lambda$. Therefore $(\chi_{(a)_r}\circ\chi_{(a\cup SaS]}\circ\chi_{(a)_l})(a)=1$. Then $a\in((a)_r\cdot(a\cup SaS]\cdot(a)_l]$ implies $a\in(((a\cup aS])\cdot((a\cup SaS])\cdot((a\cup Sa])]$. Thus $a\in(a^3]$ or $a\in(aSa]$. Therefore $a\leq a\cdot a\cdot a$ or $a\leq axa$. Hence S is regular. □

5. Semisimple ordered semigroup

Definition 5.1 ([1])**.** An ordered semigroup $(S,\cdot,\leq)$ is called semisimple if for every $a\in S$, there exist $x,y,z\in S$ such that $a\leq xayaz$.

Theorem 5.2. *Let S be a semisimple ordered semigroup. Then every (λ,μ)-fuzzy weakly interior ideal is a (λ,μ)-fuzzy ideal of S.*

Proof. Let f be a (λ,μ)-fuzzy weakly interior ideal of S and S be semisimple ordered semigroup. Then for every $x\in S$, there exist $a,b,c\in S$ such that $x\leq axbxc$. Thus $x\leq axbxc$ implies $xy\leq axbxcy$. Now,

$$\begin{aligned}f(xy)\vee\lambda &= (f(xy)\vee\lambda)\vee\lambda\\ &\geq (f((axb)x(cy))\wedge\mu)\vee\lambda\\ &= (f((axb)x(cy))\vee\lambda)\wedge\mu\\ &\geq (f(x)\wedge\mu)\wedge\mu\\ &= f(x)\wedge\mu.\end{aligned}$$

Also, for every $y\in S$, there exist $d,e,g\in S$ such that $y\leq dyeyg$. Thus $y\leq dyeyg$ implies $xy\leq xdyeyg$. Now,

$$\begin{aligned}f(xy)\vee\lambda &= (f(xy)\vee\lambda)\vee\lambda\\ &\geq (f((xd)y(eyg))\wedge\mu)\vee\lambda\\ &= (f((xd)y(eyg))\vee\lambda)\wedge\mu\\ &\geq (f(y)\wedge\mu)\wedge\mu\\ &= f(y)\wedge\mu.\end{aligned}$$

Therefore $f(xy)\vee\lambda\geq(f(x)\wedge\mu)\vee(f(y)\wedge\mu)=(f(x)\vee f(y))\wedge\mu$. Hence f is a (λ,μ)-fuzzy ideal of S. □

Corollary 5.3. *Let S be a semisimple ordered semigroup. Then every (λ,μ)-fuzzy weakly interior ideal is a (λ,μ)-fuzzy interior ideal of S.*

Proof. By Theorem 5.2, f is a (λ,μ)-fuzzy ideal of S. Therefore f is a (λ,μ)-fuzzy interior ideal of S. □

Theorem 5.4. *Let S be an ordered semigroup. Then S is semisimple if and only if $f\cap g\subseteq_{\mu}^{\lambda} f\circ g$ for every (λ,μ)-fuzzy weakly interior ideals f and g of S.*

Proof. Let S be a semisimple ordered semigroup. Then for every $a \in S$, there exist $x, y, z \in S$ such that $a \leq xayaz$. Thus $a \leq xayaz$ implies $a \leq xayaz \leq (xay)(xayazz)$. Thus $((xay), (xa(yaz^2))) \in A_a$ implies $(f \circ g)(a) \neq 0$ and

$$\begin{aligned}(f \circ g)(a) &= \bigvee_{(p,q)\in A_a} f(p) \wedge g(q) \\ &\geq f(xay) \wedge g(xa(yaz^2)).\end{aligned}$$

Therefore,

$$\begin{aligned}(f \circ g)(a) \vee \lambda &\geq [f(xay) \wedge g(xa(yaz^2))] \vee \lambda \\ &= [f(xay) \vee \lambda] \wedge [g(xa(yaz^2)) \vee \lambda] \\ &\geq (f(a) \wedge \mu) \wedge (g(a) \wedge \mu) \\ &= (f(a) \wedge g(a)) \wedge \mu \\ &= (f \cap g)(a) \wedge \mu.\end{aligned}$$

Hence $f \cap g \subseteq^{\lambda}_{\mu} f \circ g$.

On the other hand, by Theorem 3.13, $\chi_{(a\cup SaS]}$ is a $(\lambda, \mu)-$fuzzy weakly interior ideal of S, for all $a \in S$. Thus $\chi_{(a\cup SaS]} \cap \chi_{(a\cup SaS]} \subseteq^{\lambda}_{\mu} \chi_{(a\cup SaS]} \circ \chi_{(a\cup SaS]}$, for all $a \in S$. For any $a \in S$, $\mu = 1 \wedge \mu = (\chi_{(a\cup SaS]} \cap \chi_{(a\cup SaS]})(a) \wedge \mu \leq (\chi_{(a\cup SaS]} \circ \chi_{(a\cup SaS]})(a) \vee \lambda$. Therefore $(\chi_{(a\cup SaS]} \circ \chi_{(a\cup SaS]})(a) = 1$. Thus $a \in ((a \cup SaS] \cdot (a \cup SaS]]$ implies $a \in (a^2]$ or $a \in (aSaS]$ or $a \in (SaSa]$ or $a \in (SaSaS]$. Therefore $a \leq a \cdot a \leq a \cdot a \cdot a \cdot a \leq a \cdot a \cdot a \cdot a \cdot a$ or $a \leq axay \leq axaxayy = (ax)axa(y^2)$ or $a \leq xaya \leq xayxaya = xa(yx)a(ya)$ or $a \leq xayaz$. Hence S is semisimple. □

Theorem 5.5. *Let S be an ordered semigroup. Then S is semisimple if and only if $f \cap g \cap h \subseteq^{\lambda}_{\mu} f \circ g \circ h$ for every (λ, μ)-fuzzy right ideal g and for every (λ, μ)-fuzzy weakly interior ideals f and h of S.*

Proof. Let S be a semisimple ordered semigroup. Then for every $a \in S$, there exist $x, y, z \in S$ such that $a \leq xayaz$. Thus $a \leq xayaz$ implies $a \leq xxayazyaz \leq (x^2ay)(az)(yaz)$. Thus $((x^2ay), (az)(yaz)) \in A_a$ implies $(f \circ g \circ h)(a) \neq 0$. Let f and h be (λ, μ)-fuzzy weakly interior ideals and g be a (λ, μ)-fuzzy right ideal. Now,

$$\begin{aligned}(f \circ g \circ h)(a) &= \bigvee_{(p,q)\in A_a} f(p) \wedge (g \circ h)(q) \\ &\geq f(x^2ay) \wedge (g \circ h)((az)(yaz)) \\ &= f(x^2ay) \wedge [\bigvee_{(s,t)\in A_{((az)(yaz))}} g(s) \wedge h(t)] \\ &\geq f(x^2ay) \wedge (g(az) \wedge h(yaz)).\end{aligned}$$

Therefore,

$$\begin{aligned}(f \circ g \circ h)(a) \vee \lambda &\geq [f(x^2ay) \wedge g(az) \wedge h(yaz)] \vee \lambda \\ &= [f(x^2ay) \vee \lambda] \wedge [g(az) \vee \lambda] \wedge [h(yaz) \vee \lambda] \\ &\geq (f(a) \wedge \mu) \wedge (g(a) \wedge \mu) \wedge (h(a) \wedge \mu) \\ &= (f(a) \wedge g(a) \wedge h(a)) \wedge \mu \\ &= (f \cap g \cap h)(a) \wedge \mu.\end{aligned}$$

Hence $f \cap g \cap h \subseteq_{\mu}^{\lambda} f \circ g \circ h$.

Conversely, by Theorem 3.13, $\chi_{(a \cup SaS]}$ is a $(\lambda, \mu)-$fuzzy weakly interior ideal of S, for all $a \in S$. Thus $\chi_{(a \cup SaS]} \cap \chi_{(a)_r} \cap \chi_{(a \cup SaS]} \subseteq_{\mu}^{\lambda} \chi_{(a \cup SaS]} \circ \chi_{(a)_r} \circ \chi_{(a \cup SaS]}$, for all $a \in S$. For any $a \in S$, $\mu = 1 \wedge \mu = (\chi_{(a \cup SaS]} \cap \chi_{(a)_r} \cap \chi_{(a \cup SaS]})(a) \wedge \mu \leq (\chi_{(a \cup SaS]} \circ \chi_{(a)_r} \circ \chi_{(a \cup SaS]})(a) \vee \lambda$. Therefore $(\chi_{(a \cup SaS]} \circ \chi_{(a)_r} \circ \chi_{(a \cup SaS]})(a) = 1$. Then $a \in ((a \cup SaS] \cdot (a)_r \cdot (a \cup SaS]]$ implies $a \in (((a] \cup (SaS]) \cdot ((a] \cup (aS]) \cdot ((a] \cup (SaS])]$. Thus $a \in (a^3]$ or $a \in (a^2Sa]$ or $a \in (a^2SaS]$ or $a \in (SaSaS]$. Therefore $a \leq a \cdot a \cdot a \leq a \cdot a \cdot a \cdot a \cdot a$ or $a \leq a^2xay = a \cdot a \cdot x \cdot a \cdot y$ or $a \leq a^2xa = a \cdot a \cdot x \cdot a \leq a \cdot a \cdot x \cdot a \cdot a \cdot x \cdot a = a \cdot a \cdot (xa) \cdot a \cdot (xa)$ or $a \leq x \cdot a \cdot y \cdot a \cdot z$. Hence S is semisimple. □

6. Right weakly-regular ordered semigroup

Definition 6.1. An ordered semigroup $(S, \cdot, \leq)$ is called right weakly-regular if for every $a \in S$, there exist $x, y \in S$ such that $a \leq axay$.

Theorem 6.2. *Let S be an ordered semigroup. Then S is right weakly-regular if and only if $f \cap g \subseteq_{\mu}^{\lambda} f \circ g$ for every (λ, μ)-fuzzy generalized bi-ideal f and for every (λ, μ)-fuzzy weakly interior-ideal g of S.*

Proof. Let S be a right weakly-regular ordered semigroup. Then for every $a \in S$, there exist $x, y \in S$ such that $a \leq axay$. Thus $a \leq axay$ implies $a \leq axaxay^2 \leq (axa)(xay^2)$. Thus $((axa), (xay^2)) \in A_a$ implies $(f \circ g)(a) \neq 0$ and

$$\begin{aligned} (f \circ g)(a) &= \bigvee_{(p,q) \in A_a} f(p) \wedge g(q) \\ &\geq f(axa) \wedge g(xay^2). \end{aligned}$$

Therefore,

$$\begin{aligned} (f \circ g)(a) \vee \lambda &\geq [f(axa) \wedge g(xay^2)] \vee \lambda \\ &= [f(axa) \vee \lambda] \wedge [g(xay^2) \vee \lambda] \\ &\geq (f(a) \wedge \mu) \wedge (g(a) \wedge \mu) \\ &= (f(a) \wedge g(a)) \wedge \mu \\ &= (f \cap g)(a) \wedge \mu. \end{aligned}$$

Hence $f \cap g \subseteq_{\mu}^{\lambda} f \circ g$.

Conversely, let $f \cap g \subseteq_{\mu}^{\lambda} f \circ g$. By Theorems 3.13 and 3.14, $\chi_{(a \cup SaS]}$ is a $(\lambda, \mu)-$fuzzy weakly interior ideal of S and $\chi_{(a \cup aSa]}$ is a $(\lambda, \mu)-$fuzzy generalized bi-ideal of S for all $a \in S$. Thus $\chi_{(a \cup aSa]} \cap \chi_{(a \cup SaS]} \subseteq_{\mu}^{\lambda} \chi_{(a \cup aSa]} \circ \chi_{(a \cup SaS]}$. For any $a \in S$, $\mu = 1 \wedge \mu = (\chi_{(a \cup aSa]} \cap \chi_{(a \cup SaS]})(a) \wedge \mu \leq (\chi_{(a \cup aSa]} \circ \chi_{(a \cup SaS]})(a) \vee \lambda$. Therefore $(\chi_{(a \cup aSa]} \circ \chi_{(a \cup SaS]})(a) = 1$. Hence $a \in ((a \cup aSa] \cdot (a \cup SaS]]$. Thus $a \in (a^2]$ or $a \in (aSaS]$. Therefore $a \leq a \cdot a \leq a \cdot a \cdot a \cdot a$ or $a \leq axay$. Hence S is right weakly-regular. □

References

[1] A. Khan, Y. B. Jun and Muhammad Zaheer Abbas, Characterization of ordered semigroups in terms of $(\in, \in \vee q)$-fuzzy interior-ideals, Neural Computing and Application 21(2) (2012) 433–440.

[2] P. Dheena and G. Mohanraj, (λ, μ)-fuzzy ideals in semirings, Advances in Fuzzy Mathematics 6(2) (2011) 183–192.
[3] P. Dheena and G. Mohanraj, On (λ, μ)-fuzzy prime ideals of semirings semigroup, J. Fuzzy Math. 20(4) (2012) 889–892.
[4] Y. B. Jun, A. Khan and M. Shabir., Ordered semigroups characterized by their $(\in, \in \vee q)$-fuzzy bi-ideals, Bull. Malays. Math. Sci. Soc. 32(3) (2009) no.2, 391-408.
[5] Y. B. Jun and S. Z. Song, Generalized fuzzy interior-ideals in semigroups, Inform. Sci. 176 (2006) 3079–3093.
[6] O. Kazanchi and S. Yamak, Generalized fuzzy bi-ideals of semigroups, Soft Computing 12(11) (2008) 1119–1124.
[7] N. Kehayopulu and M. Tsingelis, Fuzzy sets in ordered groupoids, Semigroup Forum 65(1) (2002) 128–132.
[8] G. Mohanraj, D. Krishnaswamy and R. Hema, On $(\in, \in \vee q)$-anti fuzzy bi-ideals of ordered semigroup, Journal of Hyperstructures 1(2) (2012) 31–45.
[9] G. Mohanraj, D. Krishnaswamy and R. Hema, On generalized redefined fuzzy prime ideals of ordered semigroup, Ann. Fuzzy Math. Inform. 1(6) (2013) 171–179.
[10] G. Mohanraj, D. Krishnasamy and R. Hema, On fuzzy m-systems and n-systems of ordered semigroups, Ann. Fuzzy Math. Inform. 7(1) (2014) 173–179.
[11] G. Mohanraj, D. Krishnaswamy and R. Hema, Characterization of ordered semigroup by the generalized fuzzy ideals, (submitted).
[12] G. Mohanraj, D. Krishnaswamy and R. Hema, Characterization of ordered semigroup by the generalized fuzzy generalized bi-ideals, (submitted).
[13] Xiang-Yun Xie and Jian Tang, Fuzzy radicals and prime fuzzy ideals of ordered semi groups, Inform. Sci. 178 (2008) 4357–4374.
[14] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

G. Mohanraj (gmohanraaj@gmail.com)
Department of Mathematics, Annamalai University, Annamalainagar -608 002, India

D. Krishnaswamy (krishna_swamy2004@yahoo.co.in)
Department of Mathematics, Annamalai University, Annamalainagar -608 002, India

R. Hema (hemadu75@gmail.com)
Department of Mathematics, Annamalai University, Annamalainagar - 608 002, India