On generalized fuzzy weakly interior ideals of ordered semigroups

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Abstract. In this paper, we introduce generalized fuzzy weakly interior ideals of an ordered semigroup. We generalize fuzzy interior ideals of an ordered semigroup. We establish necessary and sufficient condition for a fuzzy set to be a generalized fuzzy weakly interior ideal. We establish three equivalent conditions for an ordered semigroup to be regular. We characterize semisimple ordered semigroups by generalized fuzzy weakly interior ideals and generalized fuzzy right ideals of ordered semigroups. We prove that every generalized fuzzy weakly interior ideal coincides with generalized fuzzy ideal in semisimple ordered semigroups. We establish necessary and sufficient conditions for an ordered semigroup to be right weakly regular in terms of generalized fuzzy weakly interior ideals and generalized fuzzy generalized bi-ideals.

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1. Introduction

The application of fuzzy technology in information processing is important and its significance will certainly increase in importance in the future. Our aim is to promote research and development in fuzzy technology by studying the fuzzy ordered semigroups. The goal is to explain new methodological developments in fuzzy ordered semigroups which is of growing importance in the future. The important concept of a fuzzy set put forth by Zadeh in 1965 [14] has opened up keen insights and applications in a wide range of scientific fields. Since then, many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, groupoids, topology, etc. A theory of fuzzy sets on ordered semigroups has been recently developed. Fuzzy sets in ordered semigroup were


In this paper, fuzzy interior ideal of an ordered semigroup based on [2],[3], which is the generalization of existing \((\varepsilon, \varepsilon \vee q)\)-fuzzy interior ideal of \(S\) is generalized. We introduce \((\lambda, \mu)\)-fuzzy weakly interior ideal. We establish necessary and sufficient condition for a fuzzy set to be a \((\lambda, \mu)\)-fuzzy weakly interior ideal. We establish three equivalent conditions for an ordered semigroup to be regular. We prove that every \((\lambda, \mu)\)-fuzzy weakly interior ideal coincides with \((\lambda, \mu)\)-fuzzy ideal in semisimple ordered semigroups. We characterize semisimple ordered semigroups through \((\lambda, \mu)\)-fuzzy weakly interior ideals and \((\lambda, \mu)\)-fuzzy right ideals of an ordered semigroup. We establish necessary and sufficient conditions for an ordered semigroup to be right weakly regular in terms of \((\lambda, \mu)\)-fuzzy weakly interior ideals and \((\lambda, \mu)\)-fuzzy generalized bi-ideals.

2. Preliminaries

**Definition 2.1.** By an ordered semigroup (po-semigroup), we mean a structure \((S, \leq)\) in which the following conditions are satisfied:

(OS1) \((S, \cdot)\) is a semigroup.

(OS2) \((S, \leq)\) is a poset.

(OS3) \(a \leq b\) implies \(xa \leq xb\) and \(ax \leq bx\) for all \(a, b, x \in S\).

For \(A \subseteq S\), we denote \(\{a\} := \{t \in S| t \leq h\ for\ some\ h \in A\}\). If \(A = \{a\}\), then we write \(\{a\}\) instead of \(\{a\}\). For non-empty subsets \(A, B\) of \(S\), we denote \(AB := \{ab| a \in A, b \in B\}\).

Let \((S, \cdot, \leq)\) be an ordered semigroup. A non-empty subset \(A\) of \(S\) is called a sub-semigroup of \(S\) if \(A^2 \subseteq A\).

**Definition 2.2.** Let \((S, \cdot, \leq)\) be an ordered semigroup. A non-empty subset \(A\) of \(S\) is called a right (left) ideal of \(S\) if

1. For all \(a \in S\), for all \(b \in A\) and \(a \leq b\) implies \(a \in A\).
2. \(AS \subseteq A\) (\(SA \subseteq A\)).

If \(A\) is both a right and left ideal of \(S\), then it is called an ideal of \(S\).
By a fuzzy set $f$ of an ordered semigroup $S$, we mean a mapping $f : S \rightarrow [0, 1]$.

**Definition 2.3.** An ordered semigroup $S$ is called regular if for every $a \in S$, there exists $x \in S$ such that $a \leq axa$.

**Definition 2.4.** Let $f$ be a fuzzy set of an ordered semigroup $S$. The level set of $f$, denoted by $f_t$ on $S$ for $t \in (0, 1]$ is defined as:

$$f_t = \{ x \in S | f(x) \geq t \}$$

**Definition 2.5.** For $a \in S$, we define $A_a = \{ (y, z) \in S \times S | a \leq yz \}$.

**Definition 2.6.** Let $f$ and $g$ be any two fuzzy sets of an ordered semigroup $S$. Then the fuzzy product of $f$ and $g$ denoted by $f \circ g$ is defined as follows:

$$(f \circ g)(a) = \begin{cases} \bigvee_{(y, z) \in A_a} \{ f(y) \wedge g(z) \} & \text{if } A_a \neq \emptyset \\ 0 & \text{if } A_a = \emptyset. \end{cases}$$

For an ordered semigroup $S$, the fuzzy set “1” is defined as follows:

$1 : S \rightarrow [0, 1]$ by $1(x) = 1$ for all $x \in S$.

**Definition 2.7 (\cite{12}).** A fuzzy set $f$ of $S$ is a $(\lambda, \mu)$-fuzzy generalized bi-ideal if

1. $x \leq y$ implies $f(x) \lor \lambda \geq f(y) \land \mu$
2. $f(xyz) \lor \lambda \geq f(x) \land f(z) \land \mu$, for all $x, y, z \in S$.

**Remark 2.8.** 1. If $\lambda = 0, \mu = 1$ in Definition 2.7, then $f$ is a fuzzy generalized bi-ideal of $S$.
2. If $\lambda = 0, \mu = 0.5$ in Definition 2.7, then $f$ is a $(\in, \in \lor q)$-fuzzy generalized bi-ideal of $S$.
3. If $\lambda = 0.5, \mu = 1$ in Definition 2.7, then $f$ is a $(\in, \in \lor q)$-fuzzy generalized bi-ideal of $S$.

**Definition 2.9 (\cite{12}).** A $(\lambda, \mu)$-fuzzy generalized bi-ideal of $S$ is a $(\lambda, \mu)$-fuzzy bi-ideal if $f(xy) \lor \lambda \geq f(x) \land f(y) \land \mu$, for all $x, y \in S$.

**Lemma 2.10 (\cite{9}).** For any two fuzzy sets $f$ and $g$ of $S$,

1. $f \subseteq \lor qg$ if and only if $g(x) \geq f(x) \land 0.5$, for all $x \in S$.
2. $f \supseteq \lor qg$ if and only if $f(x) \lor 0.5 \geq g(x)$, for all $x \in S$.

**Lemma 2.11 (\cite{13}).** Let $A, B$ be any non-empty subsets of an ordered semigroup $S$. Then the following statements are true:

1. $\chi_A \circ \chi_B = \chi_{AB}$.
2. $\chi_A \cap \chi_B = \chi_{A \cap B}$.
3. $\chi_A \cup \chi_B = \chi_{A \cup B}$.

3. **Generalized fuzzy weakly interior ideals**

Let $0 \leq \lambda < \mu \leq 1$. Hereafter $S$ denotes an ordered semigroup unless otherwise specified.
Definition 3.1 ([9]). Let \( f \) be a fuzzy set of an ordered semigroup \( S \). Then \( f \) is said to be a \((\lambda, \mu)\)-fuzzy right (left) ideal of \( S \) if
1. \( x \leq y \) implies \( f(x) \vee \lambda \geq f(y) \wedge \mu \).
2. \( f(xy) \vee \lambda \geq f(x) \wedge f(y) \wedge \mu \), for all \( x, y \in S \).

Definition 3.2 ([11]). Let \( f \) be a fuzzy set of an ordered semigroup \( S \). Then \( f \) is said to be a \((\lambda, \mu)\)-fuzzy ideal of \( S \) if
1. \( x \leq y \) implies \( f(x) \vee \lambda \geq f(y) \wedge \mu \).
2. \( f(xy) \vee \lambda \geq [f(x) \vee f(y)] \wedge \mu \), for all \( x, y \in S \).

Definition 3.3 ([11]). Let \( f \) and \( g \) be two fuzzy sets of \( S \). If \( g(x) \vee \lambda \geq f(x) \wedge \mu \), for all \( x \in S \), then we write \( f \leq_{\mu} g \).

Definition 3.4. Let \((S, \cdot, \leq)\) be an ordered semigroup. A non-empty subset \( I \) of \( S \) is called a weakly interior ideal of \( S \) if
1. \( a \leq b, b \in I \) and \( a \in S \) implies \( a \in I \).
2. \( xay \in I \), for every \( a \in I \) and for every \( x, y \in S \).

Definition 3.5. A weakly interior ideal \( I \) of \( S \) is said to be an interior ideal of \( S \) if \( xy \in I \), for every \( x, y \in I \).

Definition 3.6. A fuzzy set \( f \) of \( S \) is called a fuzzy interior ideal of \( S \) if
1. \( x \leq y \) implies \( f(x) \geq f(y) \).
2. \( f(xy) \geq f(x) \wedge f(y) \).
3. \( f(xy) \geq f(y) \), for all \( x, y, z \in S \).

Theorem 3.7 ([1]). Let \( f \) be a fuzzy set of \( S \). Then \( f \) is an \((\in, \in \vee q)\)-fuzzy interior ideal of \( S \) if and only if
1. \( x \leq y \) implies \( f(x) \geq f(y) \wedge 0.5 \).
2. \( f(xy) \geq f(x) \wedge f(y) \wedge 0.5 \).
3. \( f(xy) \geq f(y) \wedge 0.5 \), for all \( x, y, z \in S \).

Definition 3.8. A fuzzy set \( f \) of \( S \) is called a \((\lambda, \mu)\)-fuzzy weakly interior ideal of \( S \) if
1. \( x \leq y \) implies \( f(x) \vee \lambda \geq f(y) \wedge \mu \).
2. \( f(xy) \vee \lambda \geq f(y) \wedge \mu \), for all \( x, y, z \in S \).

Definition 3.9. A \((\lambda, \mu)\)-fuzzy weakly interior ideal \( f \) of \( S \) is a \((\lambda, \mu)\)-fuzzy interior ideal if \( f(xy) \vee \lambda \geq f(x) \wedge f(y) \wedge \mu \), for all \( x, y, z \in S \).

Remark 3.10. 1. If \( \lambda = 0, \mu = 1 \) in Definition 3.9, then \( f \) is a fuzzy interior ideal of \( S \).
2. If \( \lambda = 0, \mu = 0.5 \) in Definition 3.9, then \( f \) is an \((\in, \in \vee q)\)-fuzzy interior ideal of \( S \). 
3. If \( \lambda = 0.5, \mu = 1 \) in Definition 3.9, then \( f \) is a \((\in, \in \vee \bar{q})\)-fuzzy interior ideal of \( S \).

Remark 3.11. 1. Every fuzzy weakly interior ideal is a \((\lambda, \mu)\)-fuzzy weakly interior ideal of \( S \).
2. Every \((\lambda, \mu)\)-fuzzy weakly interior ideal need not be a fuzzy weakly interior ideal of \( S \).
3. Every \((\lambda, \mu)\)-fuzzy weakly interior ideal need not be a \((\lambda, \mu)\)-fuzzy interior ideal as shown in the following example.
Example 3.12. Let \((S, \cdot, \leq)\) be an ordered semigroup where \(S = \{a, b, c, d\}\). The order relation “\(\leq\)” is given by
\[
\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b)\}
\]
and the multiplication is given as follows:

\[
\begin{array}{cccc}
\cdot & a & b & c & d \\
\hline
a & a & a & a & a \\
b & a & a & a & a \\
c & a & b & a & a \\
d & a & a & b & b \\
\end{array}
\]

\[
g(x) = \begin{cases} 
0.8 & \text{if } x = a \\
0.5 & \text{if } x = c \\
0.3 & \text{if } x = d \\
0.2 & \text{otherwise}
\end{cases}
\]

Clearly \(g\) is a \((0.3, 0.5)\)-fuzzy weakly interior ideal. But \(g\) is not a \((0.3, 0.5)\)-fuzzy interior ideal, since \(g(c \cdot c) \vee 0.3 = g(b) \vee 0.3 = 0.3 \not\leq g(c) \wedge g(c) \wedge 0.5 = 0.5\), \(g\) is not a \((\in, \in \vee q)\)-fuzzy interior ideal, since \(g(d \cdot d) = g(b) = 0.2 \not\leq g(d) \wedge g(d) \wedge 0.5 = 0.3 \wedge 0.5 = 0.3\) and \(g\) is not a fuzzy interior ideal, since \(g(d \cdot d) = g(b) = 0.2 \not\leq g(d) \wedge g(d) = 0.3\).

Theorem 3.13. For every \(a \in S\), \(\chi_{(a \cup SaS)}\) is a fuzzy weakly interior ideal of \(S\).

Proof. Let \(x \leq y\). If \(y \notin (a \cup SaS)\), then \(\chi_{(a \cup SaS)}(y) = 0\). Therefore \(\forall x \in (a \cup SaS)\) and \(x \leq y\), then \(x \in (a \cup SaS)\). Therefore \(\chi_{(a \cup SaS)}(x) = 1 \geq \chi_{(a \cup SaS)}(y)\). Thus \(x \leq y\) implies \(\chi_{(a \cup SaS)}(x) \geq \chi_{(a \cup SaS)}(y)\). Let \(x, y, z \in S\). If \(y \notin (a \cup SaS)\), then \(\chi_{(a \cup SaS)}(xyz) \geq \chi_{(a \cup SaS)}(y) = 0\). If \(y \in (a \cup SaS)\), then \(xyz \in (SaS)\) implies \(\chi_{(a \cup SaS)}(xyz) = 1 \geq \chi_{(a \cup SaS)}(y)\). Therefore \(\chi_{(a \cup SaS)}\) is a fuzzy weakly interior ideal of \(S\). \(\square\)

Theorem 3.14. For every \(a \in S\), \(\chi_{(a \cup naSa)}\) is a fuzzy generalized bi-ideal of \(S\).

Proof. Proof is similar to the proof of the Theorem 3.13. \(\square\)

Theorem 3.15. A fuzzy set \(f\) is a \((\lambda, \mu)\)-fuzzy weakly interior ideal of \(S\) if and only if the level set \(f_t\) is a weakly interior ideal of \(S\) for all \(t \in (\lambda, \mu]\) whenever non-empty.

Proof. Let \(f\) be a \((\lambda, \mu)\)-fuzzy weakly interior ideal of \(S\). Let \(y \in f_t\), \(t \in (\lambda, \mu]\) and \(x \leq y\). Then \(f(x) \vee \lambda \geq f(y) \wedge \mu \geq t \wedge \mu = t > \lambda\) implies \(f(x) \geq t\). Thus \(x \in f_t\).

Let \(y \in f_t\) and \(x, z \in S\). Then \(f(xyz) \vee \lambda \geq f(y) \wedge \mu \geq t \wedge \mu = t > \lambda\). Then \(f(xyz) \geq t\) implies \(xyz \in f_t\). Therefore \(f_t\) is a weakly interior ideal of \(S\) for all \(t \in (\lambda, \mu]\) whenever non-empty.

Conversely, let \(x \leq y\). Take \(f(y) \wedge \mu = t\). If \(t > \lambda\), then \(f_t\) is an interior ideal of \(S\). Now, \(x \leq y\) and \(y \in f_t\) imply \(x \in f_t\). Thus \(f(x) \vee \lambda \geq t \vee \lambda = t = f(y) \wedge \mu\). If \(t \leq \lambda\), then \(f(y) \wedge \mu = t \leq \lambda \leq f(x) \vee \lambda\). Therefore \(x \leq y\) implies \(f(x) \vee \lambda \geq f(y) \wedge \mu\). If there exist \(x, y, z \in S\) such that \(f(xyz) \vee \lambda < f(y) \wedge \mu\), then choose a \(t \in (\lambda, \mu]\) such that \(f(xyz) \vee \lambda < t < f(y) \wedge \mu\). Then \(y \in f_t\) and \(t \in (\lambda, \mu]\) but \(xyz \notin f_t\), which is a contradiction. Therefore \(f\) is a \((\lambda, \mu)\)-fuzzy weakly interior ideal of \(S\). \(\square\)
Corollary 3.16. A fuzzy set \( f \) is a \((\lambda, \mu)\)-fuzzy interior ideal of \( S \) if and only if the level set \( f_t \) is an interior ideal in \( S \) for all \( t \in (\lambda, \mu) \) whenever non-empty.

Proof. Let \( f \) be a \((\lambda, \mu)\)-fuzzy interior ideal of \( S \). By Theorem 3.15, \( f_t \) is a weakly interior ideal in \( S \) for all \( t \in (\lambda, \mu) \), whenever non-empty. Now, \( x, y \in f_t \) for \( t \in (\lambda, \mu) \) implies \( f(x) \land f(y) \geq t \). Thus \( f(xy) \lor \lambda \geq f(x) \land f(y) \land \mu \geq t \land \mu = t > \lambda \). Therefore \( xy \in f_t \). Hence \( f_t \) is an interior ideal in \( S \).

On the other hand, by Theorem 3.15, \( f \) is a \((\lambda, \mu)\)-fuzzy weakly interior ideal of \( S \). If there exist \( x, y \in S \) such that \( f(xy) \lor \lambda < f(x) \land f(y) \land \mu \), then choose a \( t \in (\lambda, \mu) \) such that \( f(xy) \lor \lambda < t < f(x) \land f(y) \land \mu \). Then \( x, y \in f_t \) and \( t \in (\lambda, \mu) \) but \( xy \notin f_t \), which is a contradiction. Therefore \( f \) is a \((\lambda, \mu)\)-fuzzy interior ideal of \( S \). \( \square \)

Corollary 3.17. A fuzzy set \( f \) is a fuzzy interior ideal of \( S \) if and only if the level set \( f_t \) is an interior ideal in \( S \) for all \( t \in [0,1] \) whenever non-empty.

Proof. By taking \( \lambda = 0, \mu = 1 \) in Corollary 3.16, then we get the result. \( \square \)

Corollary 3.18. Let \( f \) be fuzzy set of \( S \). Then \( f \) is a \((\lambda, \mu)\)-fuzzy interior ideal of \( S \) if and only if the level set \( f_t \) is an interior ideal of \( S \) for all \( t \in (0,0.5] \) whenever nonempty.

Proof. By taking \( \lambda = 0, \mu = 0.5 \) in Corollary 3.16, then we get the result. \( \square \)

Now, \((\lambda, \mu)\)-fuzzy weakly interior ideal is a generalization of fuzzy interior ideal, \((\in, \lor \in \lor q)\)-fuzzy weakly interior ideal and \((\in, \lor q)\)-fuzzy weakly interior ideal of \( S \) as shown in the following example.

Example 3.19. Let \((S, \cdot, \leq)\) be an ordered semigroup where \( S = \{a, b, c, d, e\} \). The order relation “\( \leq \)” is given by

\[
\begin{align*}
\leq := \{(a, a), (a, c), (a, d), (a, e), (b, b), (b, d), (b, e), (c, c), (c, e), (d, d), (d, e), (e, e)\}
\end{align*}
\]

and the multiplication is given as follows:

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\[
g(x) = \begin{cases}
0.8 & \text{if } x \in \{b, d\} \\
0.7 & \text{if } x = a \\
0.3 & \text{if } x = c \\
0.2 & \text{if } x = e 
\end{cases}
\]

\[
g_t = \begin{cases}
\emptyset & \text{if } t \in (0.8, 1] \\
\{b, d\} & \text{if } t \in (0.7, 0.8] \\
\{a, b, d\} & \text{if } t \in (0.3, 0.7] \\
\{a, b, d, c\} & \text{if } t \in (0.2, 0.3] \\
S & \text{if } t \in (0, 0.2]
\end{cases}
\]

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Clearly $g$ is a $(0.3,0.7)$-fuzzy weakly interior ideal. But $g$ is not a fuzzy weakly interior ideal since $a \leq d$ implies $g(a) = 0.7 \nleq g(d) = 0.8$ and $g$ is not a $(\epsilon, \epsilon \lor q)$-fuzzy weakly interior ideal since $g(c \cdot c \cdot e) = g(e) = 0.2 \nleq g(c) \land 0.5 = 0.3 \land 0.5 = 0.3$.

4. Regular ordered semigroup

**Theorem 4.1.** If $S$ is a regular ordered semigroup, then every $(\lambda, \mu)$-fuzzy weakly interior ideal is a $(\lambda, \mu)$-fuzzy interior ideal of $S$.

**Proof.** Let $f$ be a $(\lambda, \mu)$-fuzzy weakly interior ideal of regular ordered semigroup $S$. Let $x, y \in S$. Then for every $y \in S$, there exists $z \in S$ such that $y \leq yzy$. Thus $y \leq yzy$ implies $xy \leq xzy$. Now,

$$f(xy) \lor \lambda = (f(xy) \lor \lambda) \lor \lambda$$

$$\geq (f(xyzy)) \lor \lambda \lor \lambda$$

$$= (f(xyzy)) \lor \lambda \lor \lambda$$

$$\geq (f(y)) \lor \lambda \lor \lambda$$

$$= f(y) \lor \lambda \lor \lambda$$

$$\geq f(x) \lor f(y) \lor \lambda.$$

Hence $f$ is a $(\lambda, \mu)$-fuzzy interior ideal of $S$. \qed

**Theorem 4.2.** If $S$ is a regular ordered semigroup, then every $(\lambda, \mu)$-fuzzy weakly interior ideal is a $(\lambda, \mu)$-fuzzy ideal of $S$.

**Proof.** Let $f$ be a $(\lambda, \mu)$-fuzzy weakly interior ideal of a regular ordered semigroup $S$. Let $x, y \in S$. Then for every $y \in S$, there exists $z \in S$ such that $y \leq yzy$. Thus $y \leq yzy$ implies $xy \leq xzy$. Now,

$$f(xy) \lor \lambda = (f(xy) \lor \lambda) \lor \lambda$$

$$\geq (f(xyzy)) \lor \lambda \lor \lambda$$

$$= (f(xyzy)) \lor \lambda \lor \lambda$$

$$\geq (f(y)) \lor \lambda \lor \lambda$$

$$= f(y) \lor \lambda \lor \lambda$$

Also, for every $x \in S$, there exists $a \in S$ such that $x \leq xax$. Thus $x \leq xax$ implies $xy \leq xaxy$. Now,

$$f(xy) \lor \lambda = (f(xy) \lor \lambda) \lor \lambda$$

$$\geq (f((xa)xy)) \lor \lambda \lor \lambda$$

$$= (f((xa)xy)) \lor \lambda \lor \lambda$$

$$\geq (f(x)) \lor \lambda \lor \lambda$$

$$= f(x) \lor \lambda \lor \lambda.$$

Therefore $f(xy) \lor \lambda \geq (f(x)) \lor (f(y)) \lor \lambda = (f(x) \lor f(y)) \lor \lambda$. Hence $f$ is a $(\lambda, \mu)$-fuzzy ideal of $S$. \qed

**Lemma 4.3.** Every $(\lambda, \mu)$-fuzzy right (left) ideal is a $(\lambda, \mu)$-fuzzy bi-ideal of $S$. 809
Proof. Let $f$ and $g$ be $(\lambda, \mu)$-fuzzy right and left ideal of $S$ respectively. Let $x, y, z \in S$. Then

$$f(xyz) \lor \lambda \geq f(x) \land \mu$$

$$\geq f(x) \land f(z) \land \mu$$

$$g((xy)z) \lor \lambda \geq g(z) \land \mu$$

$$\geq g(x) \land g(z) \land \mu.$$ 

Therefore $f(xyz) \lor \lambda \geq f(x) \land f(z) \land \mu$ and $g(xyz) \lor \lambda \geq g(x) \land g(z) \land \mu$. Hence $f$ and $g$ are $(\lambda, \mu)$-fuzzy bi-ideals of $S$. \hfill \Box

**Theorem 4.4.** Let $S$ be an ordered semigroup. Then the following statements are equivalent.

1. $S$ is regular.
2. $f \cap g \cap h \subseteq f \circ g \circ h$ for every $(\lambda, \mu)$-fuzzy weakly interior ideal $g$ and for every $(\lambda, \mu)$-fuzzy generalized bi-ideals $f$ and $h$ of $S$.
3. $f \cap g \cap h \subseteq f \circ g \circ h$ for every $(\lambda, \mu)$-fuzzy right ideal $f$, for every $(\lambda, \mu)$-fuzzy weakly interior ideal $g$ and for every $(\lambda, \mu)$-fuzzy generalized bi-ideal $h$ of $S$.
4. $f \cap g \cap h \subseteq f \circ g \circ h$ for every $(\lambda, \mu)$-fuzzy right ideal $f$, for every $(\lambda, \mu)$-fuzzy weakly interior ideal $g$ and for every $(\lambda, \mu)$-fuzzy left ideal $h$ of $S$.
5. $\chi_{(a)} \cap \chi_{(a) \cap (a, a)} \cap \chi_{(a) \land (a, a)} \subseteq \chi_{(a) \lor (a, a)} \lor \chi_{(a) \land (a, a)} \lor \chi_{(a) \land (a, a)}$, for all $a \in S$, where $(a)_r$ is the right ideal generated by $a$ and $(a)_l$ is the left ideal generated by $a$.

**Proof.** (1) $\Rightarrow$ (2) Let $S$ be a regular ordered semigroup. Then for every $a \in S$, there exists $x \in S$ such that $a \leq axa$. Then $a \leq axa \leq (axa)xa = axaxa \leq (axa)(xax)(axa)$. Let $f$ and $h$ be $(\lambda, \mu)$-fuzzy weakly bi-ideals and $g$ be a $(\lambda, \mu)$-fuzzy weakly interior ideal of $S$. Thus $(axa, (xax)(axa)) \in A_a$ implies $(f \circ g \circ h)(a) \neq 0$ and

$$(f \circ g \circ h)(a) = \bigvee_{(y, z) \in A_a} f(y) \land (g \circ h)(z)$$

$$\geq f(axa) \land (g \circ h)((xax)(axa))$$

$$= f(axa) \land \bigvee_{(s, t) \in A_{(xax)(axa)}} g(s) \land h(t)$$

$$\geq f(axa) \land (g(xax) \land h(axa)).$$

Therefore,

$$(f \circ g \circ h)(a) \lor \lambda \geq [f(axa) \land g(xax) \land h(axa)] \lor \lambda$$

$$= [f(axa) \lor \lambda] \land [g(xax) \lor \lambda] \land [h(axa) \lor \lambda]$$

$$\geq (f(a) \land \mu) \land (g(a) \land \mu) \land (h(a) \land \mu)$$

$$= (f(a) \land g(a) \land h(a)) \land \mu$$

$$= (f \cap g \cap h)(a) \land \mu.$$ 

Hence $f \cap g \cap h \subseteq f \circ g \circ h$.

(2) $\Rightarrow$ (3) Let $f$ be a $(\lambda, \mu)$-fuzzy right ideal of $S$. Then by Lemma 4.3, $f$ is a $(\lambda, \mu)$-fuzzy bi-ideal. Therefore by (2), $f \cap g \cap h \subseteq f \circ g \circ h$. 

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(3) ⇒ (4) Let \( h \) be a \((\lambda, \mu)\)-fuzzy left ideal of \( S \). Then by Lemma 4.3, \( h \) is a \((\lambda, \mu)\)-fuzzy bi-ideal. Therefore by (3), \( f \cap g \cap h \leq \mu \ f \circ g \circ h \).

(4) ⇒ (5) By Theorem 3.13, \( \chi_{(a \cup S \cap S)} \) is a \((\lambda, \mu)\)-fuzzy weakly interior ideal of \( S \), for all \( a \in S \). Thus \( \chi_{(a \cup S \cap S)} \cap \chi_{(a \cup S \cap S)} \leq \mu \ \chi_{(a \cup S \cap S)} \circ \chi_{(a \cup S \cap S)} \), for all \( a \in S \).

(5) ⇒ (1) For any \( a \in S \), \( \mu = 1 \land \mu = (\chi_{(a \cup S \cap S)} \cap \chi_{(a \cup S \cap S)}) \land \mu \leq (\chi_{(a \cup S \cap S)} \circ \chi_{(a \cup S \cap S)}) \lor \lambda \). Therefore \( \chi_{(a \cup S \cap S)} \circ \chi_{(a \cup S \cap S)} \mu = 1 \). Then \( a \in ((a \cup S \cap S \cap S) \cap (a S \cap S)) \) implies \( a \in ((a \cup S \cap S \cap S) \cap (a S \cap S)) \cdot ((a \cup S \cap S) \cap (a S \cap S)) \). Thus \( a \in (a^3) \) or \( a \in (a S \cap S) \). Therefore \( a \leq a \cdot a \cdot a \) or \( a \leq axa \). Hence \( S \) is regular. \( \square \)

5. Semisimple ordered semigroup

Definition 5.1 ([1]). An ordered semigroup \((S, \cdot, \leq)\) is called semisimple if for every \( a \in S \), there exist \( x, y, z \in S \) such that \( a \leq x y a z \).

Theorem 5.2. Let \( S \) be a semisimple ordered semigroup. Then every \((\lambda, \mu)\)-fuzzy weakly interior ideal is a \((\lambda, \mu)\)-fuzzy ideal of \( S \).

Proof. Let \( f \) be a \((\lambda, \mu)\)-fuzzy weakly interior ideal of \( S \) and \( S \) be semisimple ordered semigroup. Then for every \( x \in S \), there exist \( a, b, c \in S \) such that \( x \leq a x b x c \). Thus \( x \leq a x b x c \) implies \( x y \leq a x b x c y \). Now,

\[
\begin{align*}
 f(xy) \land \lambda &= (f(xy) \lor \lambda) \land \lambda \\
 &\geq (f(axb \cdot cxy) \land \mu) \lor \lambda \\
 &= (f(axb \cdot cxy) \land \mu) \\
 &\geq (f(x) \land \mu) \land \mu \\
 &= f(x) \land \mu.
\end{align*}
\]

Also, for every \( y \in S \), there exist \( d, e, g \in S \) such that \( y \leq d y e y g \). Thus \( y \leq d y e y g \) implies \( x y \leq x d y e y g \). Now,

\[
\begin{align*}
 f(xy) \land \lambda &= (f(xy) \lor \lambda) \land \lambda \\
 &\geq (f(xd) \cdot eyg \land \mu) \lor \lambda \\
 &= (f(xd) \cdot eyg \land \mu) \\
 &\geq (f(y) \land \mu) \land \mu \\
 &= f(y) \land \mu.
\end{align*}
\]

Therefore \( f(xy) \land \lambda \geq (f(x) \land \mu) \lor (f(y) \land \mu) = (f(x) \lor f(y)) \land \mu \). Hence \( f \) is a \((\lambda, \mu)\)-fuzzy ideal of \( S \). \( \square \)

Corollary 5.3. Let \( S \) be a semisimple ordered semigroup. Then every \((\lambda, \mu)\)-fuzzy weakly interior ideal is a \((\lambda, \mu)\)-fuzzy ideal of \( S \).

Proof. By Theorem 5.2, \( f \) is a \((\lambda, \mu)\)-fuzzy ideal of \( S \). Therefore \( f \) is a \((\lambda, \mu)\)-fuzzy interior ideal of \( S \). \( \square \)

Theorem 5.4. Let \( S \) be an ordered semigroup. Then \( S \) is semisimple if and only if \( f \land g \leq \mu \ f \lor g \) for every \((\lambda, \mu)\)-fuzzy weakly interior ideals \( f \) and \( g \) of \( S \).
Proof. Let $S$ be a semisimple ordered semigroup. Then for every $a \in S$, there exist $x, y, z \in S$ such that $a \leq xayz$. Thus $a \leq xayz$ implies $a \leq xayz \leq (xay)(xayazz)$. Thus $((xay), (xa(yaz^2))) \in A_a$ implies $(f \circ g)(a) \neq 0$ and

$$
(f \circ g)(a) = \bigvee_{(p,q) \in A_a} f(p) \land g(q)
$$

Therefore,

$$
(f \circ g)(a) \lor \lambda \geq [f(xay) \land g(xa(yaz^2))] \lor \lambda
$$

On the other hand, by Theorem 3.13, $\chi_{(a \cup S_S)}$ is a $(\lambda, \mu)$–fuzzy weakly interior ideal of $S$, for all $a \in S$. Thus $\chi_{(a \cup S_S)} \cap \chi_{(a \cup S_S)} = \chi_{(a \cup S_S)} \cap \chi_{(a \cup S_S)}$, for all $a \in S$. For any $a \in S$, $\mu = 1 \land \mu = (\chi_{(a \cup S_S)} \cap \chi_{(a \cup S_S)})(a) \land \mu \leq (\chi_{(a \cup S_S)} \cap \chi_{(a \cup S_S)})(a) \lor \lambda$. Therefore $(\chi_{(a \cup S_S)} \cap \chi_{(a \cup S_S)})(a) = 1$. Thus $a \in (a \cup S_S \cdot (a \cup S_S))$ implies $a \in (a^2)$ or $a \in (aSaS)$ or $a \in (S_S)$ or $a \in (S_S)$. Therefore $a \leq a \cdot a \leq a \cdot a \cdot a$ or $a \leq xayz \leq axaxaya = xa(ya(y^2))$ or $a \leq xayz \leq xayz = xa(yz)$. Hence $S$ is semisimple.

Theorem 5.5. Let $S$ be an ordered semigroup. Then $S$ is semisimple if and only if $f \land g \land h \leq \mu f \circ g \circ h$ for every $(\lambda, \mu)$-fuzzy right ideal $g$ and for every $(\lambda, \mu)$-fuzzy weakly interior ideals $f$ and $h$ of $S$.

Proof. Let $S$ be a semisimple ordered semigroup. Then for every $a \in S$, there exist $x, y, z \in S$ such that $a \leq xayz$. Thus $a \leq xayz$ implies $a \leq xayz \leq (x^2ay)(az)(yaz)$. Thus $((x^2ay), (az)(yaz)) \in A_a$ implies $(f \circ g \circ h)(a) \neq 0$. Let $f$ and $h$ be $(\lambda, \mu)$-fuzzy weakly interior ideals and $g$ be a $(\lambda, \mu)$-fuzzy right ideal. Now,

$$
(f \circ g \circ h)(a) = \bigvee_{(p,q) \in A_a} f(p) \land (g \circ h)(q)
$$

Therefore,

$$
(f \circ g \circ h)(a) \lor \lambda \geq [f(x^2ay) \land (g \circ h)(az)(yaz)] \lor \lambda
$$

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Hence \( f \cap g \cap h \subseteq \lambda \ f \circ g \circ h \).

Conversely, by Theorem 3.13, \( \chi_{(a \cup SaS)} \) is a \((\lambda, \mu)\)-fuzzy weakly interior ideal of \( S \), for all \( a \in S \). Thus \( \chi_{(a \cup SaS)} \cap \chi_{(a \cup SaS)} \subseteq \lambda \chi_{(a \cup SaS)} \circ \chi_{(a \cup SaS)} \), for all \( a \in S \). For any \( a \in S \), \( \mu = 1 \land \mu = (\chi_{(a \cup SaS)} \cap \chi_{(a \cup SaS)})(a) \wedge \mu \leq (\chi_{(a \cup SaS)} \circ \chi_{(a \cup SaS)})(a) \vee \lambda \). Therefore \( \chi_{(a \cup SaS)} \circ \chi_{(a \cup SaS)}(a) = 1 \). Then \( a \in \{(a \cup SaS) \cap \chi_{(a \cup SaS)} \} \) implies \( a \in \{(a) \cup (a \cup SaS) \} \). Thus \( a \in \{a\} \) or \( a \in \{a^2SaS\} \) or \( a \in \{SaS\} \). Therefore \( a \leq a \cdot a \cdot a \leq a \cdot a \cdot x \cdot a \cdot y \) or \( a \leq a^2 x a = a \cdot a \cdot x \cdot a \cdot x \cdot a = a \cdot a \cdot (xa) \cdot a \cdot (xa) \) or \( a \leq x \cdot a \cdot y \cdot a \cdot z \). Hence \( S \) is semisimple. \( \Box \)

6. RIGHT WEAKLY-REGULAR ORDERED SEMIGROUP

**Definition 6.1.** An ordered semigroup \((S, \cdot, \leq)\) is called right weakly-regular if for every \( a \in S \), there exist \( x, y \in S \) such that \( a \leq axay \).

**Theorem 6.2.** Let \( S \) be an ordered semigroup. Then \( S \) is right weakly-regular if and only if \( f \cap g \subseteq \lambda f \circ g \) for every \((\lambda, \mu)\)-fuzzy generalized bi-ideal \( f \) and for every \((\lambda, \mu)\)-fuzzy weakly interior-ideal \( g \) of \( S \).

**Proof.** Let \( S \) be a right weakly-regular ordered semigroup. Then for every \( a \in S \), there exist \( x, y \in S \) such that \( a \leq axay \). Thus \( a \leq axay \) implies \( a \leq axaxay^2 \leq (axa)(xy^2) \). Thus \((axa)(xy^2) \in A_a \) implies \((f \circ g)(a) \neq 0 \) and

\[
(f \circ g)(a) = \bigvee_{(p,q) \in A_a} f(p) \wedge g(q) \geq f(axa) \wedge g(xy^2).
\]

Therefore,

\[
(f \circ g)(a) \vee \lambda \geq [f(axa) \wedge g(xy^2)] \vee \lambda = [f(axa) \vee \lambda] \wedge [g(xy^2) \vee \lambda] \geq (f(a) \wedge \mu) \wedge (g(a) \wedge \mu) = (f(a) \wedge g(a)) \wedge \mu = (f \cap g)(a) \wedge \mu.
\]

Hence \( f \cap g \subseteq \lambda f \circ g \).

Conversely, let \( f \cap g \subseteq \lambda f \circ g \). By Theorems 3.13 and 3.14, \( \chi_{(a \cup SaS)} \) is a \((\lambda, \mu)\)-fuzzy weakly interior ideal of \( S \) and \( \chi_{a \cup SaS} \) is a \((\lambda, \mu)\)-fuzzy generalized bi-ideal of \( S \) for all \( a \in S \). Thus \( \chi_{(a \cup SaS) \cap (a \cup SaS)} \subseteq \lambda \chi_{(a \cup SaS)} \circ \chi_{(a \cup SaS)} \). For any \( a \in S \), \( \mu = 1 \land \mu = (\chi_{(a \cup SaS) \cap (a \cup SaS)})(a) \wedge \mu \leq (\chi_{(a \cup SaS)} \circ \chi_{(a \cup SaS)})(a) \vee \lambda \). Therefore \( \chi_{(a \cup SaS)} \circ \chi_{(a \cup SaS)}(a) = 1 \). Hence \( a \in \{(a \cup SaS) \cap (a \cup SaS) \} \). Thus \( a \in \{a^2\} \) or \( a \in \{SaS\} \). Therefore \( a \leq a \cdot a \leq a \cdot a \cdot a \cdot a \) or \( a \leq axay \). Hence \( S \) is right weakly-regular. \( \Box \)

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