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On metrizability and covering properties of $\mathcal{L}$-fuzzy metric spaces

Mami Sharma, Debajit Hazarika

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Abstract. We show that the (classical) topology induced by any complete $\mathcal{L}$-fuzzy metric space (in the sense of Saadati, Razani and Adibi) is metrizable. A relationship between compact and complete $\mathcal{L}$-fuzzy metric in terms of precompactness is provided. Further, we establish a generalized form of the Lebesgue covering lemma for the sequentially compact $\mathcal{L}$-fuzzy metric spaces.

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Corresponding Author: Mami Sharma (mami@tezu.ernet.in)

1. Introduction

Since the inception of fuzzy sets by Zadeh in 1965 [19], various authors have introduced the concept of fuzzy metric spaces in different ways [2, 6, 10, 12, 14]. George and Veeramani [7, 8] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek [14] and defined a Hausdorff topology on this fuzzy metric space. Using the idea of $\mathcal{L}$-fuzzy sets [9], Saadati et al. introduced the notion of $\mathcal{L}$-fuzzy metric spaces [17] as a generalization of fuzzy metric due to George and Veeramani [7] and intuitionistic fuzzy metric due to Park and Saadati [15, 16]. Many authors proved analogues of classical results in metric spaces including the uniform continuity theorem, Ascoli-Arzela theorem, Baire theorem, uniform limit theorem etc. for $\mathcal{L}$-fuzzy metric spaces [4, 18].

2. Preliminaries

Throughout our discussion, we shall assume all lattices $\mathcal{L} = (L, \leq_L)$ to be complete. Let $0_\mathcal{L} = \inf L$ and $1_\mathcal{L} = \sup L$, for a lattice $\mathcal{L}$.

Definition 2.1 ([9]). An $\mathcal{L}$-fuzzy set is defined as a mapping $\mathcal{A} : U \to L$, where U is a non empty set called a universe. For each u in U, $\mathcal{A}(u)$ represents the degree (in L) to which u satisfies $\mathcal{A}$.

Definition 2.2 ([1]). An intuitionistic $\mathcal{L}$-fuzzy set $\mathcal{A}_{\zeta,\eta}$ on U is an object of the form $\mathcal{A}_{\zeta,\eta} = \{(u, \zeta_\mathcal{A}(u), \eta_\mathcal{A}(u)) : u \in U\}$, where the functions $\zeta_\mathcal{A} : U \to L$ and $\eta_\mathcal{A} : U \to L$ are called the degree of membership and degree of nonmembership, respectively, of u in $\mathcal{A}_{\zeta,\eta}$, furthermore the functions $\zeta_\mathcal{A}$ and $\eta_\mathcal{A}$ should satisfy the condition $\zeta_\mathcal{A}(u) \leq_L \mathcal{N}(\eta_\mathcal{A}(u))$, for all $u \in U$, where $\mathcal{N} : L \to L$ is an involutive negation (see definition 2.4 below) in the lattice $\mathcal{L}$.

Classically, a triangular norm T on $([0,1], \leq)$ is a mapping $T : [0,1]^2 \to [0,1]$ which is increasing, commutative, associative and satisfies $T(x,1) = x$, for all $x \in [0,1]$, called the boundary condition. These definitions can be directly extended to any lattice $\mathcal{L}$, irrespective of its completeness.

Definition 2.3. A triangular norm (t-norm) on $\mathcal{L}$ is a mapping $\mathcal{T} : L^2 \to L$ satisfying the following conditions:

(1) $(\forall x \in L)(\mathcal{T}(x, 1_\mathcal{L}) = x)$; (boundary condition)
(2) $(\forall (x,y) \in L^2)(\mathcal{T}(x,y) = \mathcal{T}(y,x))$; (commutativity)
(3) $(\forall (x,y,z) \in L^3)(\mathcal{T}(x, \mathcal{T}(y,z)) = \mathcal{T}(\mathcal{T}(x,y),z))$; (associativity)
(4) $(\forall (x,x',y,y') \in L^4)(x \leq_L x'$ and $y \leq_L y' \Rightarrow \mathcal{T}(x,y) \leq_L \mathcal{T}(x',y'))$; (monotonicity)

A t-norm $\mathcal{T}$ on $\mathcal{L}$ is said to be continuous if for any $x, y \in L$ and any sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ which converges to x and y respectively, we have $\lim_n \mathcal{T}(x_n, y_n) = \mathcal{T}(x,y)$.

For example, $\mathcal{T}(x,y) = \min(x,y)$ and $\mathcal{T}(x,y) = xy$ are two continuous t-norms on $[0,1]$.

A t-norm can also be defined recursively as an $(n+1)$-ary operation $(n \in \mathbb{N})$ by $\mathcal{T}^1 = \mathcal{T}$ and
$\mathcal{T}^n(x_1, ..., x_{n+1}) = \mathcal{T}(\mathcal{T}^{n-1}(x_1, ..., x_n), x_{n+1})$ for $n \geq 2$ and $x_i \in L$.

Definition 2.4 ([17]). A negation on $\mathcal{L}$ is any decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_\mathcal{L}) = 1_\mathcal{L}$ and $\mathcal{N}(1_\mathcal{L}) = 0_\mathcal{L}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for $x \in L$, then $\mathcal{N}$ is called an involutive negation.

The negation N_s on $([0,1], \leq)$ defined as, for all $x \in [0,1], N_s(x) = 1 - x$, is the standard negation on $([0,1], \leq)$.

The following definition of an $\mathcal{L}$-fuzzy metric space and its induced topology are from [17].

Definition 2.5. The 3-tuple $(X, \mathcal{M}, \mathcal{T})$ is said to be an $\mathcal{L}$-fuzzy metric space if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{M}$ is an $\mathcal{L}$-fuzzy set on $X^2 \times (0, +\infty)$ satisfying the following conditions for every x, y, z in X and t, s in $(0, +\infty)$:

(a) $\mathcal{M}(x,y,t) >_L 0_{\mathcal{L}}$;
(b) $\mathcal{M}(x,y,t) = 1_{\mathcal{L}}$ for all $t > 0$ if and only if $x = y$;
(c) $\mathcal{M}(x,y,t) = \mathcal{M}(y,x,t)$;
(d) $\mathcal{T}(\mathcal{M}(x,y,t), \mathcal{M}(y,z,s)) \leq_L \mathcal{M}(x,z,t+s)$;
(e) $\mathcal{M}(x,y,.) : (0,+\infty) \to L$ is continuous.

In this case $\mathcal{M}$ is called an $\mathcal{L}$-fuzzy metric. If $\mathcal{M} = \mathcal{M}_{M,N}$ is an intuitionistic $\mathcal{L}$-fuzzy set (see definition 2.2) then the 3-tuple $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is said to be an intuitionistic $\mathcal{L}$-fuzzy metric space.

Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. For $t \in (0,+\infty)$, we define an open ball $B(x,r,t)$ with center $x \in X$ and radius $r \in L \backslash \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, as
$B(x,r,t) = \{y \in X : \mathcal{M}(x,y,t) >_L \mathcal{N}(r)\}$.

We call a subset $A \subseteq X$ to be open if for each $x \in A$, there exist $t > 0$ and $r \in L \backslash \{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $B(x,r,t) \subseteq A$. Let $\tau_{\mathcal{M}}$ denote the family of all open subsets of X. Then $\tau_{\mathcal{M}}$ is a topology (in the classical sense) on X induced by the $\mathcal{L}$ -fuzzy metric $\mathcal{M}$.

Proposition 2.6. *Since $\{B(x, \frac{1}{k}, \frac{1}{k}) : k \in \mathbb{N}\}$ is a countable local base at x, for each $x \in X$, therefore the topology $\tau_{\mathcal{M}}$ is first countable.*

Lemma 2.7 ([3])**.** *Consider the set L^* and operation $\leq_{L^*}$ defined by $L^* = \{(x_1,x_2) : (x_1,x_2) \in [0,1]^2$ and $x_1 + x_2 \leq 1\}$, $(x_1,x_2) \leq_{L^*} (y_1,y_2) \Leftrightarrow x_1 \leq y_1$ and $x_2 \leq y_2$, for every $(x_1,x_2), (y_1,y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice.*

Proposition 2.8. *The pair $([0,1], \leq)$ is a complete lattice where $\leq$ stands for usual comparison of real numbers. Let us denote this pair as $(L', \leq_{L'})$, where L' denotes the set $[0,1]$ and $\leq_{L'}$ stands for the above mentioned comparison on L'.*

Example 2.9 ([16])**.** Let $X = \mathbb{N}$. Define $\mathcal{T}(a,b) = (\max(0, a_1+b_1-1), a_2+b_2-a_2b_2)$ for all $a = (a_1,a_2)$ and $b = (b_1,b_2)$ in L^* and let M and N be fuzzy sets on $X^2 \times (0,+\infty)$ defined as follows:

$$\mathcal{M}_{M,N}(x,y,t) = (M(x,y,t), N(x,y,t)) = \begin{cases} (\frac{x}{y}, \frac{y-x}{y}) & \text{if } x \leq y; \\ (\frac{y}{x}, \frac{x-y}{x}) & \text{othewise.} \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic $\mathcal{L}$-fuzzy metric space.

Example 2.10. Let (X,d) be a metric space. Define $\mathcal{T}(a,b) = ab$ for all $a, b \in L'$ and let $\mathcal{M}$ be an $\mathcal{L}$-fuzzy set defined as follows:

$$\mathcal{M}(x,y,t) = \frac{ht^n}{ht^n + md(x,y)}$$

for all $t, h, m, n \in \mathbb{R}^+$. Then $(X, \mathcal{M}, \mathcal{T})$ is an $\mathcal{L}$-fuzzy metric space. If $h = m = n = 1$, then $(X, \mathcal{M}, \mathcal{T})$ is the standard $\mathcal{L}$-fuzzy metric space [7]. Generally, this $\mathcal{L}$-fuzzy metric is denoted by $\mathcal{M}_d$.

Lemma 2.11 ([7, 16])**.** *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then, $\mathcal{M}(x,y,t)$ is nondecreasing with respect to t, for all $x, y \in X$.*

Definition 2.12 ([18]). A sequence (x_n) in an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ is called a Cauchy sequence, if for each $\varepsilon \in L\backslash\{0_{\mathcal{L}}\}$ and $t > 0$, there exist $n_0 \in \mathbb{N}$ such that for all $m \geq n \geq n_0$ $(n \geq m \geq n_0)$

$$\mathcal{M}(x_n, x_m, t) >_L \mathcal{N}(\varepsilon).$$

The sequence (x_n) is said to be convergent to $x \in X$ in the $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ if $\mathcal{M}(x_n, x, t) = \mathcal{M}(x, x_n, t) \to 1_{\mathcal{L}}$ as $n \to \infty$ for every $t > 0$.

An $\mathcal{L}$-fuzzy metric space is said to be complete iff every Cauchy sequence is convergent.

Henceforth, we assume that $\mathcal{T}$ is a continuous t-norm on lattice $\mathcal{L}$ such that for every $\mu \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$, there is a $\lambda \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that
$\mathcal{T}^{n-1}(\mathcal{N}(\lambda), ..., \mathcal{N}(\lambda)) >_L \mathcal{N}(\mu)$.

Remark 2.13. Efe [4] proved that $(X, \tau_{\mathcal{M}})$ is a Hausdorff first countable topological space. Moreover, if (X, d) is a metric space then the topology generated by d coincides with the topology $\tau_{\mathcal{M}_d}$ generated by the induced $\mathcal{L}$-fuzzy metric $\mathcal{M}_d$.

We say that a topological space (X, τ) admits a compatible $\mathcal{L}$-fuzzy metric if there exists an $\mathcal{L}$-fuzzy metric $\mathcal{M}$ such that $\tau = \tau_{\mathcal{M}}$. Thus, by the result of Efe, every metrizable topological space admits a compatible $\mathcal{L}$-fuzzy metric. Conversely, we shall prove that every $\mathcal{L}$-fuzzy metric space is metrizable.

3. Main results

A classical result in the theory of metrizable topological spaces is the Kelley metrization lemma [13], which is stated as follows.

Lemma 3.1. *A T_1 topological space (X, τ) is metrizable if and only if it admits a uniformity with a countable base.*

Lemma 3.2 ([4]). *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then $\tau_{\mathcal{M}}$ is a Hausdorff topology and for each $x \in X$, $\{B(x, \frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ is a neighborhood base at x for the topology $\tau_{\mathcal{M}}$.*

Theorem 3.3. *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then $(X, \tau_{\mathcal{M}})$ is a metrizable topological space.*

Proof. For each $n \in \mathbb{N}$, define $U_n = \{(x, y) \in X \times X : \mathcal{M}(x, y, \frac{1}{n}) >_L \mathcal{N}(\frac{1}{n})\}$. We shall prove that $\{U_n : n \in \mathbb{N}\}$ is a base for a uniformity $\mathcal{U}$ on X whose induced topology coincides with $\tau_{\mathcal{M}}$. We first note that, for each $n \in \mathbb{N}$
$\{(x, x) : x \in X\} \subseteq U_n$, $U_{n+1} \subseteq U_n$ and $U_n = U_n^{-1}$.
On the other hand, for each $n \in \mathbb{N}$, there is, by the continuity of $\mathcal{T}$, an $m \in \mathbb{N}$ such that, $m > 2n$ and $\mathcal{T}(\mathcal{N}(\frac{1}{m}), \mathcal{N}(\frac{1}{m})) >_L \mathcal{N}(\frac{1}{n})$.
Then, $U_m \circ U_m \subseteq U_n$: Let $(x, y), (y, z) \in U_m$. Since $\mathcal{M}(x, y, .)$ is nondecreasing,

$\mathcal{M}(x,z,\frac{1}{n}) >_L \mathcal{M}(x,z,\frac{2}{m})$. So,

$$\begin{aligned}\mathcal{M}(x,z,\frac{1}{n}) &>_L \mathcal{M}(x,z,\frac{2}{m})\\ &>_L \mathcal{T}(\mathcal{M}(x,y,\frac{1}{m}),\mathcal{M}(y,z,\frac{1}{m}))\\ &>_L \mathcal{T}(\mathcal{N}(\frac{1}{m}),\mathcal{N}(\frac{1}{m})) >_L \mathcal{N}(\frac{1}{n})\end{aligned}$$

Therefore $(x,z) \in U_n$. Thus, $\{U_n : n \in \mathbb{N}\}$ is a countable base for a uniformity $\mathcal{U}$ on X. Since for each $x \in X$ and each $n \in \mathbb{N}$, $U_n(x) = \{y : \mathcal{M}(x,y,\frac{1}{n}) >_L \mathcal{N}(\frac{1}{n})\} = B(x,\frac{1}{n},\frac{1}{n})$, we deduce, from lemma 3.2, that the topology induced by $\mathcal{U}$ coincides with $\tau_{\mathcal{M}}$. Hence, by lemma 3.1 $(X,\tau_{\mathcal{M}})$ is a metrizable topological space. □

Corollary 3.4. *A topological space is metrizable if and only if it admits a compatible $\mathcal{L}$-fuzzy metric.*

Proof. Suppose (X,τ) is a metrizable topological space. Let d be the metric on X compatible with τ. Since the topologies induced by the $\mathcal{L}$-fuzzy metric $\mathcal{M}_d$ and d are the same [7], therefore $\mathcal{M}_d$ is compatible with τ . The converse follows directly from theorem 3.3. □

Corollary 3.5. *Every separable $\mathcal{L}$-fuzzy metric space is second countable.*

Proof. Let $(X,\mathcal{M},\mathcal{T})$ be a separable $\mathcal{L}$-fuzzy metric space. Then by theorem 3.3, $(X,\tau_{\mathcal{M}})$ is a separable metrizable space. So, it is second countable [5]. □

Let us recall that metrizable topological space (X,τ) is completely metrizable if it admits a complete metric [5]. On the other hand, an $\mathcal{L}$-fuzzy metric space $(X,\mathcal{M},\mathcal{T})$ is complete if every Cauchy sequence is convergent [18]. If $(X,\mathcal{M},\mathcal{T})$ is a complete $\mathcal{L}$-fuzzy metric space, then $\mathcal{M}$ is called a complete $\mathcal{L}$-fuzzy metric on X.

Theorem 3.6. *Let $(X,\mathcal{M},\mathcal{T})$ be a complete $\mathcal{L}$-fuzzy metric space. Then $(X,\tau_{\mathcal{M}})$ is completely metrizable.*

Proof. From theorem 3.3, it follows that $\{U_n : n \in \mathbb{N}\}$ is a base for the uniformity $\mathcal{U}$ in X compatible with $\tau_{\mathcal{M}}$ where, $U_n = \{(x,y) : \mathcal{M}(x,y,\frac{1}{n}) >_L \mathcal{N}(\frac{1}{n})\}$ for every $n \in \mathbb{N}$. Then there exists a metric d whose induced uniformity coincides with $\mathcal{U}$. We want to show that d is complete.
Indeed, given a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in (X,d), we shall prove that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X,\mathcal{M},\mathcal{T})$. To this end, fix r,t, with $r \in L\backslash\{0_{\mathcal{L}},1_{\mathcal{L}}\}$ and $t > 0$. We choose a $k \in \mathbb{N}$ such that $\frac{1}{k} < t$ and $\frac{1}{k} <_L r$. Then there exists $n_0 \in \mathbb{N}$ such that $(x_n,x_m) \in U_k$, for every $n,m \geq n_0$. Consequently for every $n,m \geq n_0$,
$\mathcal{M}(x_n,x_m,t) \geq_L \mathcal{M}(x_n,x_m,\frac{1}{k}) >_L \mathcal{N}(\frac{1}{k}) >_L \mathcal{N}(r)$.
Thus, we have shown that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete $\mathcal{L}$-fuzzy metric space $(X,\mathcal{M},\mathcal{T})$. So, it is convergent with respect to $\tau_{\mathcal{M}}$. Hence, d is a complete metric on X. We conclude that $(X,\tau_{\mathcal{M}})$ is completely metrizable. □

Definition 3.7 ([4])**.** An $\mathcal{L}$-fuzzy metric space $(X,\mathcal{M},\mathcal{T})$ is called precompact if for each $r \in L\backslash\{0_{\mathcal{L}},1_{\mathcal{L}}\}$ and $t > 0$, there exists a finite set A of X, such that $X = \cup_{a\in A} B(a,r,t)$. In this case we say that $\mathcal{M}$ is precompact $\mathcal{L}$-fuzzy metric on X.

An $\mathcal{L}$-fuzzy metric is called compact if $(X,\tau_{\mathcal{M}})$ is a compact topological space.

Lemma 3.8. *An $\mathcal{L}$-fuzzy metric space is precompact if and only if every sequence has a Cauchy subsequence.*

Proof. Let $(X, \mathcal{M}, \mathcal{T})$ be a precompact $\mathcal{L}$-fuzzy metric space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. For each $m \in \mathbb{N}$ there is a finite set A_m of X such that $X = \cup_{a\in A_m} B(a, \dfrac{1}{m}, \dfrac{1}{m})$. Hence for $m = 1$, there exists an $a_1 \in A_1$ and a subsequence $(x_{1(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $x_{1(n)} \in B(a_1, 1, 1)$. Similarly, there exists an $a_2 \in A_2$ and a subsequence $(x_{2(n)})_{n\in\mathbb{N}}$ of $(x_{1(n)})_{n\in\mathbb{N}}$ such that $x_{2(n)} \in B(a_2, \frac{1}{2}, \frac{1}{2})$ for every $n \in \mathbb{N}$. Following this process, for $m \in \mathbb{N}$, $m > 1$, there is an $a_m \in A_m$ and a subsequence $(x_{m(n)})_{n\in\mathbb{N}}$ of $(x_{(m-1)(n)})_{n\in\mathbb{N}}$ such that $x_{m(n)} \in B(a_m, \frac{1}{m}, \frac{1}{m})$ for every $n \in \mathbb{N}$. Now consider the subsequence $(x_{n(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$. Given $r \in L\backslash\{0_\mathcal{L}, 1_\mathcal{L}\}$ and $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that $\mathcal{T}(\mathcal{N}(\frac{1}{n_0}), \mathcal{N}(\frac{1}{n_0})) >_L \mathcal{N}(r)$ and $\frac{2}{n_0} < t$. Then for every $k, m \geq n_0$, we have,

$$\begin{aligned}\mathcal{M}(x_{k(k)}, x_{m(m)}, t) &\geq_L \mathcal{M}(x_{k(k)}, x_{m(m)}, \frac{2}{n_0})\\ &\geq_L \mathcal{T}(\mathcal{M}(x_{k(k)}, a_{n_0}, \frac{1}{n_0}), \mathcal{M}(x_{m(m)}, a_{n_0}, \frac{1}{n_0}))\\ &>_L \mathcal{T}(\mathcal{N}(\frac{1}{n_0}), \mathcal{N}(\frac{1}{n_0})) >_L \mathcal{N}(r)\end{aligned}$$

Hence $(x_{n(n)})_{n\in\mathbb{N}}$ is a Cauchy subsequence of $(x_n)_{n\in\mathbb{N}}$ in $(X, \mathcal{M}, \mathcal{T})$.

Conversely, suppose that $(X, \mathcal{M}, \mathcal{T})$ is non-precompact $\mathcal{L}$-fuzzy metric space. Then, there exists $r \in L\backslash\{0_\mathcal{L}, 1_\mathcal{L}\}$ and $t > 0$ such that for each finite subset A of X, $X \neq \cup_{a\in A} B(a, r, t)$. Fix $x_1 \in X$. There is $x_2 \in X\backslash B(x_1, r, t)$. Moreover there is $x_3 \in X\backslash \cup_{k=1}^{2} B(x_k, r, t)$. In this way, we construct a sequence $(x_n)_{n\in\mathbb{N}}$ of distinct points in X such that $x_{n+1} \notin \cup_{k=1}^{n} B(x_k, r, t)$ for every $n \in \mathbb{N}$. Therefore $(x_n)_{n\in\mathbb{N}}$ has no Cauchy subsequence. This completes the proof. □

Theorem 3.9. *An $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ is separable if and only if $(X, \tau_\mathcal{M})$ admits a precompact $\mathcal{L}$-fuzzy metric.*

Proof. Suppose $(X, \mathcal{M}, \mathcal{T})$ is a separable $\mathcal{L}$-fuzzy metric space. Then by theorem 3.3 and corollary 3.5, $(X, \tau_\mathcal{M})$ is a separable metrizable space. Therefore, $\tau_\mathcal{M}$ admits a compatible precompact metric d [5]. We want to show that, then, the standard $\mathcal{L}$-fuzzy metric $\mathcal{M}_d$ induced by d is precompact. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. By the precompactness of d, $(x_n)_{n\in\mathbb{N}}$ has a Cauchy subsequence $(x_{k(n)})_{n\in\mathbb{N}}$ in (X, d). Fix $r \in L\backslash\{0_\mathcal{L}, 1_\mathcal{L}\}$ and $t > 0$ and choose ε such that $\frac{t}{t+\varepsilon} >_L \mathcal{N}(r)$. Then, there exists $n_0 \in \mathbb{N}$ such that $d(x_{k(n)}, x_{k(m)}) < \varepsilon$, for every $n, m \geq n_0$. Therefore,
$\mathcal{M}_d(x_{k(n)}, x_{k(m)}, t) >_L \frac{t}{t+\varepsilon} >_L \mathcal{N}(r)$, for every $n, m \geq n_0$.
So, $(x_{k(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in the $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}_d, \mathcal{T})$. By lemma 3.8, $(X, \mathcal{M}_d, \mathcal{T})$ is precompact.
Conversely, suppose that $(X, \tau_\mathcal{M})$ admits a compatible precompact $\mathcal{L}$-fuzzy metric $\mathcal{D}$. Then for each $n \in \mathbb{N}$, there exists a finite subset A_n of X such that $X = \cup_{a\in A_n} B(a, \frac{1}{n}, \frac{1}{n})$. Put $A = \cup_{n=1}^{\infty} A_n$. Then A is countable. We shall show that, A is dense in X. Let, $x \in X$ and $B(x, \frac{1}{m}, \frac{1}{m})$ be a basic neighborhood of x. Then, there exists $a \in A_m$ such that $x \in B(a, \frac{1}{m}, \frac{1}{m})$. Thus, A is dense in X. We conclude that $(X, \tau_\mathcal{M})$ is separable. □

Remark 3.10. The question whether the admitted $\mathcal{L}$-fuzzy metric has any relation with the original $\mathcal{L}$-fuzzy metric remains an open question.

Lemma 3.11. *Let $(X, \mathcal{M}, \mathcal{T})$ be a $\mathcal{L}$-fuzzy metric space. If a Cauchy sequence clusters to a point $x \in X$, then the sequence converges to x.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(X, \mathcal{M}, \mathcal{T})$ having a cluster point $x \in X$. Then there is a subsequence $(x_{k(n)})_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ that converges to x with respect to $\tau_{\mathcal{M}}$. Thus, given r, with $r \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $\mathcal{M}(x, x_{k(n)}, \frac{t}{2}) >_L \mathcal{N}(s)$, where $s \in L\backslash\{0_{\mathcal{L}}\}$ satisfies $\mathcal{T}(\mathcal{N}(s), \mathcal{N}(s)) >_L \mathcal{N}(r)$. On the other hand, there is $n_1 \geq k(n_0)$ such that for each $n, m \geq n_1$, $\mathcal{M}(x_n, x_m, \frac{t}{2}) >_L \mathcal{N}(s)$. Therefore, for each $n \geq n_1$, we have

$$\begin{aligned}\mathcal{M}(x_n, x, t) &\geq_L \mathcal{T}(\mathcal{M}(x_n, x_{k(n)}, \frac{t}{2}), \mathcal{M}(x_{k(n)}, x, \frac{t}{2}))\\ &\geq_L \mathcal{T}(\mathcal{N}(s), \mathcal{N}(s)) >_L \mathcal{N}(r).\end{aligned}$$

Hence, we conclude that Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ converges to x. □

Theorem 3.12. *An $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ is compact if and only if it is precompact and complete.*

Proof. Suppose $(X, \mathcal{M}, \mathcal{T})$ is a compact $\mathcal{L}$-fuzzy metric space. Then for each $r \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and $t > 0$, the open cover $\{B(x, r, t) : x \in X\}$ of X has a finite subcover. Thus X is precompact. On the other hand, every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in $(X, \mathcal{M}, \mathcal{T})$ has a cluster point $y \in X$. By lemma 3.11 $(x_n)_{n\in\mathbb{N}}$ converges to y. So, $(X, \mathcal{M}, \mathcal{T})$ is complete.

Conversely, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the precompact and $\mathcal{L}$-complete fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$. Then, by lemma 3.8 and completeness of $(X, \mathcal{M}, \mathcal{T})$, we get that $(x_n)_{n\in\mathbb{N}}$ has a cluster point. Since by theorem 3.3, $(X, \tau_{\mathcal{M}})$ is metrizable and every sequentially compact metrizable space is compact, so $(X, \mathcal{M}, \mathcal{T})$ is compact. □

Definition 3.13. Let A be a subset of an $\mathcal{L}$-fuzzy metric space. Then A is precompact if for each $r \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and $t > 0$, there exists a finite subset S of X such that $A \subseteq \cup_{a\in S} B(a, r, t)$.

Definition 3.14 ([11])**.** Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space and $A \subseteq X$. The $\mathcal{L}$-fuzzy diameter of a set A is defined as:

$\delta_A = \sup_{t>0} \inf_{x,y\in A} \sup_{\varepsilon<t} \mathcal{M}(x, y, \varepsilon)$.

If $\delta_A = 1_{\mathcal{L}}$,then we say that the set A is $\mathcal{L}F$-strongly bounded.

Lemma 3.15 ([11])**.** *The set $A \subseteq X$ is $\mathcal{L}F$-strongly bounded if an only if for arbitrary negation $\mathcal{N}(r)$ with $r \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ there exists $t > 0$ such that $\mathcal{M}(x, y, t) >_L \mathcal{N}(r)$, for all $x, y \in A$.*

Theorem 3.16. *Every precompact subset A of an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ is $\mathcal{L}F$-strongly bounded.*

Proof. Let A be a precompact subset of X. Fix $t > 0$ and $r \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$. Since A is precompact, there exists a finite subset S of X such that $A \subseteq \cup_{a\in S} B(a, r, t)$.

Let $x, y \in A$. Then $x \in B(x_i, r, t)$ and $y \in B(x_j, r, t)$ for some i, j. Thus, we have $\mathcal{M}(x, x_i, t) >_L \mathcal{N}(r)$ and $\mathcal{M}(y, x_j, t) >_L \mathcal{N}(r)$. Now, let

$$\alpha = \{\min \mathcal{M}(x_i, x_j, t) : 1 \leq i, j \leq n\}.$$

Then $\alpha \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and there exists $s \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $\mathcal{T}^2(\mathcal{N}(r), \mathcal{N}(r), \alpha) >_L \mathcal{N}(s)$. Therefore,

$$\begin{aligned}\mathcal{M}(x, y, 3t) &\geq_L \mathcal{T}^2(\mathcal{M}(x, x_i, t), \mathcal{M}(x_i, x_j, t), \mathcal{M}(x_j, y, t)) \\ &\geq_L \mathcal{T}^2(\mathcal{N}(r), \mathcal{N}(r), \alpha) >_L \mathcal{N}(s)\end{aligned}$$

for all $x, y \in A$. Hence A is $\mathcal{L}F$-strongly bounded. □

Remark 3.17. Every sequentially compact $\mathcal{L}$-fuzzy metric space is precompact.

Definition 3.18. An element $\varepsilon \in L\backslash\{0_{\mathcal{L}}\}$ is called a covering factor for an open cover $\mathcal{G} = \{G_i\}_{i \in \bigwedge}$ of an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ if for every set A in X with fuzzy diameter $\delta_A >_L \mathcal{N}(\varepsilon)$ is contained in any G_i in $\mathcal{G}$.

Lemma 3.19. *Let $B(x, r, t)$ be an open ball of an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ with $r \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and $t > 0$. Let A be a subset of X such that $\delta_A >_L \mathcal{N}(s)$, where $s \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ satisfying $\mathcal{T}(\mathcal{N}(s), \mathcal{N}(s)) >_L \mathcal{N}(r)$. If A intersects $B(x, s, \frac{t}{2})$, then $A \subseteq B(x, r, t)$.*

Lemma 3.20. *Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. If $t > 0$ and $r, s \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $\mathcal{T}(\mathcal{N}(s), \mathcal{N}(s)) >_L \mathcal{N}(r)$, then $B(x, s, \frac{t}{2}) \subset B(x, r, t)$.*

The proofs of the above two results are straight forward, so we omit them. The following result is a generalized form of Lebesgue's covering lemma.

Theorem 3.21. *In a sequentially compact $\mathcal{L}$-fuzzy metric space with involutive negation, every open cover has a covering factor.*

Proof. Let $(X, \mathcal{M}, \mathcal{T})$ be a sequentially compact $\mathcal{L}$-fuzzy metric space with an involutive negation $\mathcal{N}$ and $\mathcal{G} = \{G_\alpha : \alpha \in \bigwedge\}$ be an open cover of X. We assume that there exists sets in X which are not contained in any G_α; otherwise any $\varepsilon \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ will work as covering factor and the result is established. Let us call these sets as "big sets". Let δ' be the infimum of negation of fuzzy diameter of these big sets. There are three possibilities:
If $\delta' = 1_{\mathcal{L}}$; it follows that every set $A \subset X$ with $\mathcal{N}(\delta_A) <_L 1_{\mathcal{L}}$ i.e. $\delta_A >_L 0_{\mathcal{L}}$ is a subset of G_α for some $\alpha \in \bigwedge$. Hence any element δ is the covering factor.
If $\delta' = 0_{\mathcal{L}}$; we arrive at a contradiction:
Assume $\delta' = 0_{\mathcal{L}}$ then, by definition of δ', for a given $n \in \mathbb{N}$ there exists a big set B_n such that $\mathcal{N}(\delta_{B_n}) <_L \frac{1}{n}$ which gives $\delta_{B_n} >_L \mathcal{N}(\frac{1}{n})$. Since, a big set must contain at least two elements, therefore, we get $1_{\mathcal{L}} >_L \delta_{B_n} >_L \mathcal{N}(\frac{1}{n})$. Construct a sequence $(x_n)_{n \in \mathbb{N}}$ choosing $x_n \in B_n$ for $n \in \mathbb{N}$. Since X is sequentially compact, the sequence $(x_n)n \in \mathbb{N}$ has a convergent subsequence converging to some $x \in X$. As $X = \cup_{\alpha \in \bigwedge} G_\alpha$, so there is a $\beta \in \bigwedge$ such that $x \in G_\beta$. As G_β is open in X, therefore, there exists $r \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ and $t > 0$ such that $B(x, r, t) \subseteq G_\beta$. Choose $s \in L\backslash\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ such that $\mathcal{T}(\mathcal{N}(s), \mathcal{N}(s)) >_L \mathcal{N}(r)$. Then, lemma 3.20 gives $B(x, s, \frac{t}{2}) \subset B(x, r, t)$. Consider the open ball $B(x, s, \frac{t}{2})$ centered at x. Since a

subsequence of $(x_n)_{n\in\mathbb{N}}$ converges to x, so $x_n \in B(x, s, \frac{t}{2})$ for infinitely many n. Let N be one of those values of n such that $\frac{1}{N} <_L s$ which implies that $\mathcal{N}(\frac{1}{N}) >_L \mathcal{N}(s)$ and $x_N \in B(x, s, \frac{t}{2})$. But $x_N \in B_N$ and by definition of B_N, $\delta_{B_N} >_L \mathcal{N}(\frac{1}{N})$. This gives $\delta_{B_N} >_L \mathcal{N}(s)$. By lemma 3.19 $B_N \subseteq B(x, r, t) \subseteq G_\beta$, a contradiction to the assumption. Thus $\delta' \neq 0_{\mathcal{L}}$.
The only possibility left out is $0_{\mathcal{L}} <_L \delta' <_L 1_{\mathcal{L}}$. The element $\delta = \mathcal{N}(\delta')$ will be the required covering factor. This completes the proof. □

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References

[1] A. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.
[2] Z. K. Deng, Fuzzy pseudo-metric spaces, J. Math. Anal. Appl. 86 (1982) 74–95.
[3] G. Deschrijver and E. E. Kerre, On the relationship between some extension of fuzzy set theory, Fuzzy Sets and Systems 133 (2003) 227–235.
[4] H. Efe, Some results in $\mathcal{L}$-fuzzy metric space, Carpathian J. Math. 24 (2008) 37–44.
[5] R. Engelking, General Topology, PWN-Polish Sci. Publ., Warasaw, 1977.
[6] M. A. Erceg, Metric spaces in fuzzy set theory, J. Math. Anal. Appl 69 (1979) 205–230.
[7] A. George and P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets and Systems 64 (1994) 395–399.
[8] A. George and P. Veeramani, On some results of analysis for fuzzy metric spaces, Fuzzy Sets and Systems 90 (1997) 365–368.
[9] J. Goguen, $\mathcal{L}$-fuzzy sets, J. Math. Anal. Appl 18 (1967) 145–174.
[10] M. Grabiec, Fixed points in fuzzy metric spaces, Fuzzy Sets and Systems 27 (1988) 385–389.
[11] S. N. Ješić and N. A. Babačev, Common fixed point theorems in intuitionistic fuzzy metric spaces and $\mathcal{L}$-fuzzy metric spaces with nonlinear contractive condition, Chaos Solitons Fractals 37(3) (2008) 675–687.
[12] O. Kaleva and S. Seikkala, On fuzzy metric spaces, Fuzzy Sets and Systems 27 (1984) 225–229.
[13] J. K. Kelley, General Topology, Van Nostrand, 1995.
[14] O. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetika (Prague) 11(5) (1975) 336–344.
[15] J. Park, Intuitionistic fuzzy metric spaces, Chaos Solitons Fractals 22(5) (2004) 1039–1046.
[16] R. Saadati and J. H. Park, On the intuitionistic fuzzy topological spaces, Chaos Solitons Fractals 27(2) (2006) 331–344.
[17] R. Saadati, A. Razani and H. Adibi, A common fixed point theorem in $\mathcal{L}$-fuzzy metric spaces, Chaos Solitons Fractals 33 (2007) 358–363.
[18] R. Saadati, On the $\mathcal{L}$-fuzzy topological spaces, Chaos Solitons Fractals 37 (2008) 1419–1426.
[19] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

Mami Sharma (mami@tezu.ernet.in)
Department of Mathematical Sciences, School of Sciences, Tezpur University, Assam, India

Debajit Hazarika (debajit@tezu.ernet.in)
Department of Mathematical Sciences, School of Sciences, Tezpur University, Assam, India