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Common fixed point theorems in FM-spaces for compatible mappings of type (α) and weakly compatible mappings using implicit relations

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Abstract. In this article, we define a new implicit relation in fuzzy metric spaces. Then, we obtain two common fixed point theorems for compatible mappings of type (α) and weakly compatible in FM-spaces under this implicit relation.

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1. Introduction

In 1965, L. Zadeh [33] introduced the concept of a fuzzy sets. Many researchers have developed the theory of fuzzy sets and its applications and introduced the notion of fuzzy metric spaces (FM-spaces). For example, we can refer to Kramosil and Michalek [20], George and Veeramani [10], Kaleva and Seikkala [19], Ereeg [8], Deng [7], Fang [9] and etc. Recently, many authors, for example ([1], [6], [12], [16], [17], [21], [22], [23], [28]) proved fixed and common fixed point theorems in fuzzy metric spaces. In 1994, Mishra et al. [24] introduced the notion of compatible mappings in FM-spaces. Cho et al. ([3], [4]) introduced the concept of compatible mappings of types (α) and (β) in FM-spaces (compatible mappings of types (α) and (β) introduced by Jungek et al. [18] and Pathak et al. [27] in metric spaces). Also the notion of weakly compatible mappings in fuzzy metric spaces studied by Singh and Jain [32].

2. Preliminaries

Definition 2.1 ([33])**.** Suppose X is a nonempty set. A *fuzzy set* A in X is a function with domain X and values in $[0,1]$.

Definition 2.2 ([29])**.** A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a *continuous t-norm (triangular norm)* if the following conditions hold:
(1) $*$ is associative and commutative;
(2) $a * 1 = a$ for all $a \in [0,1]$;
(3) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$;
(4) $*$ is continuous.

Some typical examples of continuous t-norms are :
$T_M(a,b) = min(a,b)$, $T_P(a,b) = ab$ and $T_L(a,b) = max(a+b-1,0)$.

Definition 2.3. A FM-space in the sense of Kramosil and Michalek [20] is a 3-tuple $(X,M,*)$ where X is an arbitrary (nonempty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0,\infty)$ such that the following properties hold :
(FM-1) $M(x,y,0) = 0 \ \forall x,y \in X$;
(FM-2) $M(x,y,t) = 1 \ \forall t > 0$ iff $x = y$;
(FM-3) $M(x,y,t) = M(y,x,t) \ \forall x,y \in X$ and $t > 0$;
(FM-4) $M(x,y,\cdot) : [0,\infty) \longrightarrow [0,1]$ is left continuous for all $x,y \in X$;
(FM-5) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s) \ \forall x,y,z \in X$, $\forall t,s > 0$.

We refer to these spaces as *KM-fuzzy metric spaces.*

Definition 2.4. A FM-space in the sense of George and Veeramani [10] is a 3-tuple $(X,M,*)$ where X is an arbitrary (nonempty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$ such that the following conditions are satisfied for all $x,y,z \in X$ and $t,s > 0$:
(GV-1) $M(x,y,t) > 0$;
(GV-2) $M(x,y,t) = 1$ if and only if $x = y$;
(GV-3) $M(x,y,t) = M(y,x,t)$;
(GV-4) $M(x,y,\cdot) : (0,\infty) \longrightarrow [0,1]$ is continuous;
(GV-5) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$.

We refer to these spaces as *GV-fuzzy metric spaces.*

We can see some common fixed point theorems in GV-fuzzy metric spaces by Gopal and Imdad in [11].

Suppose $(X,M,*)$ is a fuzzy metric space. For all $t > 0$, the open ball $B(x,r,t)$ with center x in X and radius $0 < r < 1$ is defined by

$$B(x,r,t) = \{y \in X : M(x,y,t) > 1 - r\}.$$

Suppose $(X,M,*)$ is a fuzzy metric space and τ_M is the set of all $A \subset X$ with this property : $x \in A$ if and only if there exists $t > 0$ and $0 < r < 1$ such that $B(x,r,t) \subset A$. Then τ_M is a topology on X (induced by the fuzzy metric M). This topology is Hausdorff and first countable. (See [5]).

Example 2.5 ([10]). Let (X, d) be a metric space. Define $a * b = ab$ (or $a * b = min(a, b)$). For $x, y \in X$ and $t > 0$, put

$$M(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M, *)$ is a GV-fuzzy metric space which is called *the standard fuzzy metric induced* by the metric d.

Example 2.6. [2] Let $X = I\!R$ and $a * b = T_P(a, b)$ for all $a, b \in [0, 1]$. For all $t > 0$ and $x, y \in X$, define

$$M(x, y, t) = (e^{\frac{|x-y|}{t}})^{-1}.$$

Then $(X, M, *)$ is a GV-fuzzy metric Space.

Example 2.7 ([2]). Let $X = I\!N$ and $a * b = T_P(a, b)$ for all $a, b \in [0, 1]$. For all $t > 0$, define

$$M(x, y, t) = \begin{cases} \frac{x}{y} & \text{if } x \leq y \\ \frac{y}{x} & \text{if } y \leq x. \end{cases}$$

Then $(X, M, *)$ is a GV-fuzzy metric Space.

Example 2.8 ([14]). Let $f : X \longrightarrow (0, \infty)$ be a one to one function, $g : I\!R^+ \longrightarrow [0, \infty)$ be an increasing continuous function and $a * b = T_P(a, b)$ for all $a, b \in [0, 1]$. For fixed $\alpha, \beta > 0$, define M as

$$M(x, y, t) = \left(\frac{(min\{f(x), f(y)\})^\alpha + g(t)}{(max\{f(x), f(y)\})^\alpha + g(t)} \right)^\beta,$$

for all $x, y \in X$ and $t > 0$. Then, $(X, M, *)$ is a FM-space on X.

Example 2.9 ([14]). Let (X, d) be a bounded metric space with $d(x, y) < k$ (for all $x, y \in X$), $g : I\!R^+ \longrightarrow (k, +\infty)$ is an increasing continuous function and $a * b = T_L(a, b)$ for all $a, b \in [0, 1]$. Define a function M as

$$M(x, y, t) = 1 - \frac{d(x, y)}{g(t)},$$

for all $x, y \in X$ and $t > 0$. Then, $(X, M, *)$ is a FM-space on X.

Example 2.10 ([14]). Let $g : I\!R^+ \longrightarrow [0, \infty)$ be a non-decreasing continuous function, $a * b = T_P(a, b)$ for all $a, b \in [0, 1]$ and define function M as

$$M(x, y, t) = e^{(-d(x,y)/g(t))},$$

for all $x, y \in X$ and $t > 0$. Then, $(X, M, *)$ is a FM-space on X.

For more examples of FM-spaces refer to [2].

Lemma 2.11 ([13]). *Let $(X, M, *)$ be a FM-space. Then $M(x, y, t)$ is non-decreasing with respect to t, for all x, y in X.*

Definition 2.12 ([13]). Let $(X, M, *)$ be a (KM- or GV-) fuzzy metric space. A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ if and only if

$$\lim_{n \longrightarrow \infty} M(x_n, x, t) = 1$$

for all $t > 0$.

The sequence $\{x_n\}$ in X is said to be *Cauchy* if

$$\lim_{n,m\longrightarrow\infty} M(x_n, x_m, t) = 1.$$

Or, equivalently, if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in I\!N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for any $n, m \geq n_0$.

The FM-space $(X, M, *)$ is said to be *complete* if every Cauchy sequence in X is convergent.

Lemma 2.13 ([5])**.** *Suppose $(X, M, *)$ is a FM-space. If a sequence $\{x_n\}$ in X satisfies*

$$M(x_n, x_{n+1}, t) \geq M(x_0, x_1, k^n t)$$

for all $k > 1$, $n \in I\!N$, then the sequence $\{x_n\}$ is a Cauchy sequence.

Definition 2.14 ([24])**.** Let f and g be self-maps on a FM-space $(X, M, *)$. Then the mappings f and g are said to be *compatible (asymptotically commuting)* if for each $t > 0$,

$$\lim_{n\longrightarrow\infty} M(fgx_n, gfx_n, t) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\longrightarrow\infty} fx_n = \lim_{n\longrightarrow\infty} gx_n = x$$

for some $x \in X$.
Also we can see definition of noncompatible in [11].

Definition 2.15 ([32])**.** Let f and g be self-maps on a fuzzy metric space $(X, M, *)$. Then the pair (f, g) is said to be *weakly compatible* if f and g commute at their coincidence point, that is, $fx = gx$ implies that $fgx = gfx$.

Also Singh et al. and Pant respectively in [31] , [26] defined semi-compatible and reciprocal continuous and Mishra et al [25] showed that semi-compatible and reciprocal continuous are equivalent.

Definition 2.16 ([15])**.** A fuzzy metric space $(X, M, *)$ is said to be *compact* if (X, τ_M) is a compact topological space.
The above definition is equivalent to:

Definition 2.17 ([13])**.** A fuzzy metric space $(X, M, *)$ is said to be *compact (sequentially compact)* if every sequence in X has a convergent subsequence.

Definition 2.18 ([3])**.** Let f and g be self-maps on a FM-space $(X, M, *)$. Then the mappings f and g are said to be *compatible of type* (α) if for each $t > 0$,

$$\lim_{n\longrightarrow\infty} M(fgx_n, ggx_n, t) = 1, \lim_{n\longrightarrow\infty} M(gfx_n, ffx_n, t) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\longrightarrow\infty} fx_n = \lim_{n\longrightarrow\infty} gx_n = x$$

for some $x \in X$.

Definition 2.19 ([4]). Let f and g be self-maps on a FM-space $(X, M, *)$. Then the mappings f and g are said to be *compatible of type* (β) if, for each $t > 0$,

$$\lim_{n \longrightarrow \infty} M(ffx_n, ggx_n, t) = 1$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \longrightarrow \infty} fx_n = \lim_{n \longrightarrow \infty} gx_n = x$$

for some $x \in X$.

Remark 2.20 ([29]). If self-maps f and g of a FM-space $(X, M, *)$ are compatible of type (α) or compatible of type (β) then they are weak compatible.

The converse is not true as seen in example below.

Example 2.21 ([29]). Let $X = [0, 2]$ and $(X, M, *)$ be a FM-space. Define $a * b = min(a, b)$, for $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t+d(x,y)}$, for $t > 0$ and $M(x, y, 0) = 0$, for $x, y \in X$. Define self-maps f and g on X as follows:

$$fx = 2 \; if \; 0 \leq x \leq 1 \; and \; gx = 2 \; if \; x = 1;$$

$$fx = \frac{1}{2}x \; if \; 1 < x \leq 2 \; and \; gx = \frac{1}{5}(x + 3) \; otherwise.$$

Taking $x_n = 2 - \frac{1}{2n}$ we have $f(1) = g(1) = 2$ and $f(2) = g(2) = 1$. Also $fg(1) = gf(1) = 1$ and $fg(2) = gf(2) = 2$. Thus (f, g) is weak compatible. Again, $fx_n = 1 - \frac{1}{4n}$ and $gx_n = 1 - \frac{1}{10n}$. Thus, $fx_n \longrightarrow 1$ and $gx_n \longrightarrow 1$. Also

$$\lim_{n \longrightarrow \infty} M(ffx_n, ggx_n, t) = \lim_{n \longrightarrow \infty} M(2, \frac{2}{5} - \frac{1}{50n}, t) = t/(t + \frac{8}{5}) < 1,$$

$\forall t > 0$. Hence f and g are not compatible of type (β).

Now, we give the following example :

Example 2.22. Let $X = [0, 1]$ and $(X, M, *)$ be a FM-space. Define $a * b = min(a, b)$, for $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t+d(x,y)}$, for $t > 0$ and $M(x, y, 0) = 0$, for $x, y \in X$. Define self-maps f and g on X as follows:

$$\forall x \in X; \; fx = gx = 1$$

Taking $x_n = 1 - \frac{1}{n}$ we have,

$$\lim_{n \longrightarrow \infty} fx_n = \lim_{n \longrightarrow \infty} gx_n = 1$$

Also

$$\lim_{n \longrightarrow \infty} M(fgx_n, ggx_n, t) = M(1, 1, t) = 1;$$
$$\lim_{n \longrightarrow \infty} M(gfx_n, ffx_n, t) = M(1, 1, t) = 1$$

$\forall t > 0$. Hence f and g are compatible of type (α). Also

$$\lim_{n \longrightarrow \infty} M(ffx_n, ggx_n, t) = M(1, 1, t) = 1,$$

$\forall t > 0$. Hence f and g are compatible of type (β).

Proposition 2.23 ([3]). *Let $(X, M, *)$ be a fuzzy metric space with this property : $t * t = t$ for all $t \in [0, 1]$. And f, g be continuous mappings from X into itself. Then f and g are compatible if and only if they are compatible of type (α).*

Proposition 2.24 ([4]). *Let $(X, M, *)$ be a fuzzy metric space with this property : $t * t = t$ for all $t \in [0, 1]$. And f, g be continuous mappings from X into itself. Then f and g are compatible if and only if they are compatible of type (β).*

Proposition 2.25 ([4]). *Let $(X, M, *)$ be a fuzzy metric space with this property : $t * t = t$ for all $t \in [0, 1]$. And f, g be continuous mappings from X into itself. Then f and g are compatible of type (α) if and only if they are compatible of type (β).*

Definition 2.26 ([5]). Let f and g be self-maps on a FM-space $(X, M, *)$. Then the pair (f, g) is said to be *compatible of type (I)* if, for each $t > 0$,

$$\lim_{n \longrightarrow \infty} M(fgx_n, x, t) \leq M(gx, x, t)$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \longrightarrow \infty} fx_n = \lim_{n \longrightarrow \infty} gx_n = x$$

for some $x \in X$.

Definition 2.27 ([5]). Let f and g be self-maps on a FM-space $(X, M, *)$. Then the pair (f, g) is said to be *compatible of type (II)* if and only if (g, f) is Compatible of type (I).

3. Implicit relation

Suppose Φ denote the set of all functions $E : [0, 1]^4 \longrightarrow I\!R$ such that one of the following conditions is true for all $u, v \in [0, 1)$:

(A_1) $E(v, u, u, v) > 0$ or $E(v, u, v, u) > 0$ implies $u < v$ and $E(v, 1, 1, v) \leq 0$, $E(v, v, 1, 1) \leq 0$ and $E(v, 1, v, 1) \leq 0$.
(A_2) $E(v, u, u, v) < 0$ or $E(v, u, v, u) < 0$ implies $u < v$ and $E(v, 1, 1, v) \geq 0$, $E(v, v, 1, 1) \geq 0$ and $E(v, 1, v, 1) \geq 0$.
(A_3) $E(v, u, u, v) \geq 0$ or $E(v, u, v, u) \geq 0$ implies $u < v$ and $E(v, 1, 1, v) < 0$, $E(v, v, 1, 1) < 0$ and $E(v, 1, v, 1) < 0$.
(A_4) $E(v, u, u, v) \leq 0$ or $E(v, u, v, u) \leq 0$ implies $u < v$ and $E(v, 1, 1, v) > 0$, $E(v, v, 1, 1) > 0$ and $E(v, 1, v, 1) > 0$.

Example 3.1.

$$E_1(t_1, t_2, t_3, t_4) = t_1^3 - t_2 t_3 t_4,$$

$$E_2(t_1, t_2, t_3, t_4) = t_2 t_3 t_4 - t_1^3,$$

$$E_3(t_1, t_2, t_3, t_4) = t_1 - min\{t_2, t_3, t_4\}.$$

It is easy to see that $E_k \in \Phi$ for $k = 1, 2, 3$.

4. Main results

Theorem 4.1. *Let A, B, S and T be self-maps of a complete GV-fuzzy metric space $(X, M, *)$ such that:*
(a) $T(X) \subseteq A(X)$, $S(X) \subseteq B(X)$,
*(b) $E(M(Sx, Ty, kt), M(Ax, By, (1-k)t), M(Ax, Sx, (1-k)t),$
$M(By, Ty, kt)) > 0$;*
for any $x, y \in X$, $t > 0$, $k \in (0, 1)$, $E \in \Phi$ and E satisfies (A_1) [if $E(t_1, t_2, t_3, t_4) < 0, \geq 0$ and ≤ 0 respectively E satisfies A_2, A_3, A_4],
(c) The mappings A, B, S and T are continuous,
(d) The pairs (A, S) and (B, T) are compatible of type (α).
Then A, B, S and T have a unique common fixed point as z in X. Also z is the unique common fixed point of A, S, B and T

Proof. We prove theorem for the case (A_1). The other cases are similar. Let x_0 be an arbitrary element in X. Since $S(X) \subseteq B(X)$ and $T(X) \subseteq A(X)$, there exist $x_1, x_2 \in X$ such that $Sx_0 = Bx_1$, $Tx_1 = Ax_2$. Inductively, we can make the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Sx_{2n} = Bx_{2n+1}, y_{2n+1} = Tx_{2n+1} = Ax_{2n+2} \tag{4.1}$$

For each n=0,1,2,...
If we set $d_m(t) = M(y_m, y_{m+1}, t)$ for $t > 0$, then we prove that $\{y_n\}$ is a Cauchy sequence.
Putting $x = x_{2n}$, $y = x_{2n+1}$, $k \in (0, \frac{1}{2})$ in (b), we have
$0 < E(M(Sx_{2n}, Tx_{2n+1}, kt), M(Ax_{2n}, Bx_{2n+1}, (1-k)t), M(Ax_{2n}, Sx_{2n}, (1-k)t)$
$, M(Bx_{2n+1}, Tx_{2n+1}, kt))$
$= E(M(y_{2n}, y_{2n+1}, kt), M(y_{2n-1}, y_{2n}, (1-k)t), M(y_{2n-1}, y_{2n}, (1-k)t),$
$M(y_{2n}, y_{2n+1}, kt))$
$= E(d_{2n}(kt), d_{2n-1}((1-k)t), d_{2n-1}((1-k)t), d_{2n}(kt))$
From (A_1), we have

$$d_{2n-1}((1-k)t) < d_{2n}(kt).$$

If we set $q = 1 - k$, then we have

$$d_{2n-1}(qt) < d_{2n}(kt). \tag{4.2}$$

Putting $x = x_{2n+1}$, $y = x_{2n+2}$, $k \in (0, \frac{1}{2})$ in (b), we have
$0 < E(M(Sx_{2n+1}, Tx_{2n+2}, kt), M(Ax_{2n+1}, Bx_{2n+2}, qt), M(Ax_{2n+1}, Sx_{2n+1}, qt)$
$, M(Bx_{2n+2}, Tx_{2n+2}, kt))$
$= E(M(y_{2n+1}, y_{2n+2}, kt), M(y_{2n}, y_{2n+1}, qt), M(y_{2n}, y_{2n+1}, qt), M(y_{2n+1}, y_{2n+2}, kt))$
$= E(d_{2n+1}(kt), d_{2n}(qt), d_{2n}(qt), d_{2n+1}(kt))$
From (A_1), we have

$$d_{2n}(qt) < d_{2n+1}(kt). \tag{4.3}$$

Then, from (4.2) and (4.3) for each $n \in I\!N$, we have
$d_n(kt) > d_{n-1}(qt)$.

Consequently
$M(y_n, y_{n+1}, kt) > M(y_{n-1}, y_n, qt)$.
That is
$M(y_n, y_{n+1}, t) > M(y_{n-1}, y_n, \frac{q}{k}t) > ... > M(y_0, y_1, (\frac{q}{k})^n t)$.
Putting $k_1 = \frac{q}{k}$ in the above inequality, we have
$M(y_n, y_{n+1}, t) > M(y_0, y_1, k_1^n t)$ $(k_1 > 1)$.
Hence, by Lemma 2.13, $\{y_n\}$ is a Cauchy sequence. Completeness of X, follows that $\{y_n\}$ converges to a point z in X. Hence we have
$\lim_{n\longrightarrow\infty} y_{2n} = \lim_{n\longrightarrow\infty} Sx_{2n} = \lim_{n\longrightarrow\infty} Bx_{2n+1} = \lim_{n\longrightarrow\infty} y_{2n+1}$
$= \lim_{n\longrightarrow\infty} Tx_{2n+1} = \lim_{n\longrightarrow\infty} Ax_{2n+2} = z$.
Suppose that $M(Sz, Tz, \frac{t}{2}) \neq 1$. Putting $x = Ax_{2n}$, $y = Bx_{2n+1}$, $k = \frac{1}{2}$ in (b), we have

$$0 < E(M(SAx_{2n}, TBx_{2n+1}, \frac{t}{2}), M(AAx_{2n}, BBx_{2n+1}, \frac{t}{2}), M(AAx_{2n}, SAx_{2n}, \frac{t}{2}), M(BBx_{2n+1}, TBx_{2n+1}, \frac{t}{2})). \tag{4.4}$$

$SAx_{2n}, \frac{t}{2}), M(BBx_{2n+1}, TBx_{2n+1}, \frac{t}{2}))$.
Since (A, S) is compatible of type (α), then we have

$$\lim_{n\longrightarrow\infty} M(SAx_{2n}, AAx_{2n}, t) = 1.$$

Hence

$$\lim_{n\longrightarrow\infty} AAx_{2n} = \lim_{n\longrightarrow\infty} SAx_{2n}.$$

Now, since A and S are continuous, then we have

$$Az = \lim_{n\longrightarrow\infty} AAx_{2n} = \lim_{n\longrightarrow\infty} SAx_{2n} = Sz. \tag{4.5}$$

Similarly, we have

$$Bz = Tz. \tag{4.6}$$

Letting $n \to \infty$, in (4.4) and (4.5) we have
$0 < E(M(Sz, Tz, \frac{t}{2}), M(Az, Bz, \frac{t}{2}), 1, 1) = E(M(Sz, Tz, \frac{t}{2}), M(Sz, Tz, \frac{t}{2}), 1, 1)$
which is a contradiction with (A_1). Hence, $M(Sz, Tz, \frac{t}{2}) = 1$ and consequently we have $Sz = Tz$.
So, $Az = Bz = Sz = Tz$.
Suppose that $M(Sz, z, \frac{t}{2}) \neq 1$. Putting $x = Ax_{2n}$, $y = x_{2n+1}$, $k = \frac{1}{2}$ in (b), we have
$0 < E(M(SAx_{2n}, Tx_{2n+1}, \frac{t}{2}), M(AAx_{2n}, Bx_{2n+1}, \frac{t}{2}), M(AAx_{2n}, SAx_{2n}, \frac{t}{2}),$
$M(Bx_{2n+1}, Tx_{2n+1}, \frac{t}{2}))$.
In the above inequality, letting $n \to \infty$, we have
$0 < E(M(Sz, z, \frac{t}{2}), M(Az, z, \frac{t}{2}), 1, 1) = E(M(Sz, z, \frac{t}{2}), M(Sz, z, \frac{t}{2}), 1, 1)$
which is a contradiction with (A_1). Hence $M(Sz, z, \frac{t}{2}) = 1$ and consequently we have $Sz = z$.
So, we have $Az = Bz = Sz = Tz = z$.

Now, suppose that $z^{'}$ is another common fixed point of A, B, S and T. Hence $M(z, z^{'}, \frac{t}{2}) \neq 1$. Putting $x = z$, $y = z^{'}$, $k = \frac{1}{2}$ in (b), we have
$0 < E(M(Sz, Tz^{'}, \frac{t}{2}), M(Az, Bz^{'}, \frac{t}{2}), M(Az, Sz, \frac{t}{2}), M(Bz^{'}, Tz^{'}, \frac{t}{2}))$
$= E(M(z, z^{'}, \frac{t}{2}), M(z, z^{'}, \frac{t}{2}), M(z, z, \frac{t}{2}), M(z^{'}, z^{'}, \frac{t}{2}))$.
$= E(M(z, z^{'}, \frac{t}{2}), M(z, z^{'}, \frac{t}{2}), 1, 1)$.
which is a contradiction with (A_1). Hence $M(z, z^{'}, \frac{t}{2}) = 1$ and consequently $z = z^{'}$, i.e. z is a common fixed point of A, B, S and T. Now, suppose that r is another common fixed point of A and S. Hence $M(r, z, \frac{t}{2}) \neq 1$. Putting $x = r$, $y = z$, $k = \frac{1}{2}$ in (b), we have
$0 < E(M(Sr, Tz, \frac{t}{2}), M(Ar, Bz, \frac{1}{2}), M(Ar, Sr, \frac{t}{2}), M(Bz, Tz, \frac{t}{2}))$
$= E(M(r, z, \frac{t}{2}), M(r, z, \frac{t}{2}), M(r, r, \frac{t}{2}), M(z, z, \frac{t}{2}))$.
$= E(M(r, z, \frac{t}{2}), M(r, z, \frac{t}{2}), 1, 1)$.
which is a contradiction with (A_1). Hence, we have $M(r, z, \frac{t}{2}) = 1$ and consequently, we have $r = z$.
Therefore, z is the unique common fixed point of A and S. Similarly we can show that z is the unique common fixed point of B and T . □

Theorem 4.2. *Let A, B, S and T be self-maps of a compact GV-fuzzy metric spaces $(X, M, *)$ such that:*
(a) $T(X) \subseteq A(X)$, $S(X) \subseteq B(X)$,
(b) $E(M(Sx, Ty, t), M(Ax, By, t), M(Ax, Sx, t), M(By, Ty, t)) > 0$;
For all $x, y \in X$ such that one of the relations $Ax \neq By$, $Ax \neq Sx$ and $By \neq Ty$ holds and for all $t > 0$, $E \in \Phi$ and E satisfies (A_1)
[if $E(t_1, t_2, t_3, t_4) < 0, \geq 0$ and ≤ 0 respectively E satisfies A_2, A_3, A_4],
(c) The pairs (A, S) and (B, T) are weakly compatible,
(d) Either A and S are continuous or B and T are continuous.
Then A, B, S and T have a unique common fixed point as z in X. Also z is the unique common fixed point of A and S and of B and T.

Proof. We only prove the case (A_1).The other cases are similar. At first, we suppose that A and S are continuous. For all $t > 0$, let

$$m = sup\{M(Ax, Sx, t) : x \in X\}. \tag{4.7}$$

Since A and S on a compact fuzzy metric space are continuous, there exists u in X such that $m = M(Au, Su, t)$.
Since $S(X) \subseteq B(X)$, there exists $v \in X$ such that

$$Su = Bv. \tag{4.8}$$

Since $T(X) \subseteq A(X)$, there exists $w \in X$ such that

$$Tv = Aw. \tag{4.9}$$

Let A, S, B and T have not any coincidence point in X. Then
$m = M(Au, Su, t) \neq 1$, $M(Bv, Tv, t) \neq 1$ and $M(Aw, Sw, t) \neq 1$.
Hence, putting $x = u$ and $y = v$ in (b), we have

$0 < E(M(Su,Tv,t), M(Au,Bv,t), M(Au,Su,t), M(Bv,Tv,t))$
$\quad = E(M(Bv,Tv,t), M(Au,Su,t), M(Au,Su,t), M(Bv,Tv,t))$
$\quad = E(M(Bv,Tv,t), m, m, M(Bv,Tv,t))$.
So, from (A_1), we have

$$m < M(Bv,Tv,t). \tag{4.10}$$

Putting $x = w$, $y = v$ in (b), we have
$0 < E(M(Sw,Tv,t), M(Aw,Bv,t), M(Aw,Sw,t), M(Bv,Tv,t))$
$\quad = E(M(Sw,Aw,t), M(Tv,Bv,t), M(Aw,Sw,t), M(Bv,Tv,t))$
So, from (A_1), we have

$$M(Bv,Tv,t) < M(Aw,Sw,t). \tag{4.11}$$

Now, from (4.7), (4.10) and (4.11) we have

$$m \geq M(Aw,Sw,t) > M(Bv,Tv,t) > m,$$

which is a contradiction. Hence, either A and S or B and T have a coincidence point in X. That is, there exists $a \in X$ such that $Aa = Sa$ or $Ba = Ta$.
Case (1): Suppose that $Aa = Sa$. Since $S(X) \subseteq B(X)$, there exists $b \in X$ such that $Sa = Bb$. Let $M(Bb,Tb,t) \neq 1$. Then, putting $x = a$, $y = b$ in (b) we have
$0 < E(M(Sa,Tb,t), M(Aa,Bb,t), M(Aa,Sa,t), M(Bb,Tb,t))$
$\quad = E(M(Bb,Tb,t), M(Aa,Sa,t), M(Aa,Sa,t), M(Bb,Tb,t))$
$\quad = E(M(Bb,Tb,t), 1, 1, M(Bb,Tb,t))$,
which is a contradiction with (A_1). Hence, we have $M(Bb,Tb,t) = 1$. So, $Bb = Tb$. Thus

$$Aa = Sa = Bb = Tb = z. \tag{4.12}$$

Now, since the pair (A,S) is weakly compatible we have

$$Az = ASa = SAa = Sz. \tag{4.13}$$

Suppose that $M(Sz,z,t) \neq 1$. Putting $x = z$, $y = b$ in (b), we have
$0 < E(M(Sz,Tb,t), M(Az,Bb,t), M(Az,Sz,t), M(Bb,Tb,t))$
$\quad = E(M(Sz,z,t), M(Sz,z,t), 1, 1)$,
which is a contradiction with (A_1). Hence $M(Sz,z,t) = 1$ and consequently we have $Sz = z$. Thus

$$Az = Sz = z. \tag{4.14}$$

Since the pair (B,T) is weakly compatible we have

$$Bz = BTb = TBb = Tz. \tag{4.15}$$

Suppose that $M(z,Tz,t) \neq 1$. Putting $x = a$, $y = z$ in (b), we have
$0 < E(M(Sa,Tz,t), M(Aa,Bz,t), M(Aa,Sa,t), M(Bz,Tz,t))$
$\quad = E(M(z,Tz,t), M(z,Tz,t), 1, 1)$,

which is a contradiction with (A_1). Hence $M(z, Tz, t) = 1$ and consequently we have $Tz = z$. Thus

$$Bz = Tz = z. \tag{4.16}$$

Hence, from (4.14) and (4.16) we have
$Az = Sz = Bz = Tz = z$.
That is z is a common fixed point of A, B, S and T.
Case (2): Suppose that $Ba = Ta$. Since $T(X) \subseteq A(X)$, there exists $b \in X$ such that $Ta = Ab$. Suppose that $M(Ab, Sb, t) \neq 1$. Then, putting $x = b$, $y = a$ in (b) we have
$0 < E(M(Sb, Ta, t), M(Ab, Ba, t), M(Ab, Sb, t), M(Ba, Ta, t))$
$\quad = E(M(Sb, Ab, t), 1, M(Ab, Sb, t), 1)$,
which is a contradiction with (A_1). Hence, we have $M(Ab, Sb, t) = 1$. Consequently $Ab = Sb$. Thus

$$Sb = Ab = Ta = Ba = z. \tag{4.17}$$

Now, since the pair (A, S) is weakly compatible we have

$$Az = ASb = SAb = Sz. \tag{4.18}$$

Suppose that $M(Sz, z, t) \neq 1$. Putting $x = z$, $y = a$ in (b), we have
$0 < E(M(Sz, Ta, t), M(Az, Ba, t), M(Az, Sz, t), M(Ba, Ta, t))$
$\quad = E(M(Sz, z, t), M(Sz, z, t), 1, 1)$,
which is a contradiction with (A_1). Hence $M(Sz, z, t) = 1$ and consequently we have $Sz = z$. Thus

$$Az = Sz = z. \tag{4.19}$$

Since the pair (B, T) is weakly compatible we have

$$Bz = BTa = TBa = Tz. \tag{4.20}$$

Suppose that $M(z, Tz, t) \neq 1$. Putting $x = b$, $y = z$ in (b), we have
$0 < E(M(Sb, Tz, t), M(Ab, Bz, t), M(Ab, Sb, t), M(Bz, Tz, t))$
$\quad = E(M(z, Tz, t), M(z, Tz, t), 1, 1)$,
which is a contradiction with (A_1). Hence $M(z, Tz, t) = 1$ and consequently we have $Tz = z$. Thus

$$Bz = Tz = z. \tag{4.21}$$

Hence, from (4.19) and (4.21) we have
$Az = Sz = Bz = Tz = z$.
That is z is a common fixed point of A, B, S and T.
Uniqueness: Now suppose that $z^{'}$ is another common fixed point of A, B, S and T. Hence $M(z, z^{'}, t) \neq 1$. Putting $x = z$, $y = z^{'}$ in (b), we have
$0 < E(M(Sz, Tz^{'}, t), M(Az, Bz^{'}, t), M(Az, Sz, t), M(Bz^{'}, Tz^{'}, t)$

$= E(M(z,z',t), M(z,z',t), M(z,z,t), M(z',z',t)).$
$= E(M(z,z',t), M(z,z',t), 1, 1),$
which is a contradiction with (A_1). Hence $M(z,z',t) = 1$ and consequently $z = z'$. That is z is the unique common fixed point of A, B, S and T.
Now, suppose that r is another common fixed point of A and S. Thus $M(r,z,t) \neq 1$. Putting $x = r$, $y = z$ in (b), we have
$0 < E(M(Sr,Tz,t), M(Ar,Bz,t), M(Ar,Sr,t), M(Bz,Tz,t)$
$= E(M(r,z,t), M(r,z,t), M(r,r,t), M(z,z,t)).$
$= E(M(r,z,t), M(r,z,t), 1, 1),$
which is a contradiction with (A_1). Hence we have $M(r,z,t) = 1$ and consequently $r = z$.
Hence z is the unique common fixed point of A and S. Similarly we can show that z is the unique common fixed point of B and T, and the theorem is true when B and T are continuous. □

5. Examples

Example 5.1. Let $X = \mathbb{R}$. For all $a, b \in [0,1]$, define $a * b = T_P(a,b)$. For any $t > 0$, define

$$M(x,y,t) = \frac{t}{t + |x-y|}$$

for $x, y \in X$. Define $Sx = Tx = 1$, $Ax = x^2$ and $Bx = x^3$ for all $x \in X$. In Theorem 3.1, put

$$E(t_1,t_2,t_3,t_4) = t_1^3 - t_2t_3t_4.$$

We can show that 1 is the unique common fixed point of A, B, S and T.

Example 5.2. Let $X = [0,1]$. For all $a, b \in [0,1]$, define $a * b = T_P(a,b)$ (or $a * b = T_M(a,b)$). For all x, y in X and $t > 0$, define

$$M(x,y,t) = \frac{t}{t + |x-y|}.$$

Define $Sx = Tx = 1$, $Ax = \frac{3+x}{4}$ and $Bx = \frac{x+9}{10}$ for all $x \in X$. In Theorem 3.1, put

$$E(t_1,t_2,t_3,t_4) = t_1 - min\{t_2,t_3,t_4\}.$$

We can show that 1 is the unique common fixed point of A, B, S and T.

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