Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 8, No. 5, (November 2014), pp. 687–703
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

@FMI
© Kyung Moon Sa Co.
http://www.kyungmoon.com

Fuzzy soft boundary

Azadeh Zahedi Khameneh, Adem Kiliçman, Abdul Razak Salleh

Received 4 December 2013; Revised 17 April 2014; Accepted 6 May 2014

Abstract. In this paper, we continue study on fuzzy soft topology. Fuzzy soft interior, fuzzy soft closure, and fuzzy soft continuity are considered deeply. The concept of fuzzy soft boundary is introduced and some of its basic properties are also studied.

2010 AMS Classification: 54A05, 54A40, 54B10, 54C05

Keywords: Fuzzy soft set, Fuzzy soft topology, Fuzzy soft boundary, Fuzzy soft continuous, Fuzzy soft product topology.

Corresponding Author: Azadeh Zahedi Khameneh (azadeh503@gmail.com)

1. Introduction

We live in the world of uncertainties where the most of the problems which we deal are vague rather than precise. Even the language used in our daily life situations is usually full of imprecise phrases. During past decades, different mathematical theories such as probability theory, fuzzy set theory [25], and rough set theory [16] were introduced to deal with various types of uncertainties. A wide range of problems can be solved by these methods although they cannot be applied effectively to model imprecise information related to some parameters.

Due to lack of parameterization tools in all previous theories, Molodtsove[14] introduced the concept of soft set in 1999. This concept can be seen as an approximation description of some objects based on some parameters by a given set-valued map. Furthermore, Molodtsove[14] pointed out that theory of soft set is more general than the former theories dealing with uncertainty. He also mentioned several directions for soft set's application such as smoothness of functions, game theory, operations research and so on. Since then, works on soft set theory have progressed rapidly. In 2000, Maji et al.[10] published the first research paper on soft set theory and expanded the theory of soft set fundamentally. They extended some basic concepts from classical set theory to the soft set theory, and then in [11] discussed practical application of soft sets in decision-making problems. In [2], Aktaş and Çağman applied the concept of soft set to introduce the new notion soft group. In [6], Irfan

Ali et al. developed the soft set theory and introduced some new operations for it. Shabir and Naz[22], defined the concept of soft topology and studied some basic topological properties of soft spaces. For more details about soft set theory and soft topology see [7, 4, 21, 5].

But dealing with imprecise and vague information in real life problems encouraged researchers to consider soft set theory in a fuzzy environment. In [12], Maji et al. combined the concepts of fuzzy set and soft set and introduced a new hybrid concept called fuzzy soft set. Later in [18], they applied fuzzy soft set to solve a decision making problem. Kharal and Ahmad[1] also studied some properties of fuzzy soft set and extended some operations in classical set theory to fuzzy soft set theory. Then in [8], they introduced the concept of fuzzy soft map and studied the concept of image and pre-image of a fuzzy soft set. Majumdar and Samanta[13] generalised the concept of fuzzy soft set by attaching the degree of possibility of the relationship between each object and parameter. Topological studies of fuzzy soft sets was begun by Tanya and Kandemir[24]. They applied classical definition of topology to construct a topology over a fuzzy soft set and called this new topological space fuzzy soft topology. Furthermore, they studied some fundamental topological structures such as interior, closure, and base for the fuzzy soft topology. Later Simsekler and Yuksel[23] studied fuzzy soft topological space in the sense of Tanay and Kandemir[24]. But they defined the concept of fuzzy soft topology over a fuzzy soft set with a fixed parameter set and considered some topological concepts such as base, subbase, neighborhood, and Q-neighborhood for fuzzy soft topological spaces. Roy and Samanta[19] remarked a new definition of fuzzy soft topology. They proposed the notion of fuzzy soft topology over an ordinary set by applying fuzzy soft subsets of it where parameter set is supposed fixed everywhere. Then in [20], they continued study on fuzzy soft topology and defined the concept of fuzzy soft point and different neighborhood structures of a fuzzy soft point. The concept of soft quasi-coincidence for fuzzy soft sets was considered by Atmaca and Zorlutuna[3]. They also studied the fundamental topological notions such as interior and closure for a fuzzy soft sets by applying this new concept. Recently, Zahedi et al.[26] introduced the concept of product fuzzy soft topology and studied some of its properties. They also considered the Hausdorff property of finite product of fuzzy soft Hausdorff spaces. Osmanoglu and Tokat [15] introduced fuzzy soft compactness and some basic definitions and theorems by using basic properties of fuzzy soft topology. Li and Cui [9] discussed topologies on intuitionistic fuzzy soft sets.

Although different topological structures of a fuzzy soft space have been studied, the concept of boundary of a fuzzy soft set has received less attention. In ordinary topology the boundary of a set is considered as an exact set. But in real-world experiences, where we mostly deal with qualitative explanation of phenomena, we cannot determine the boundary of a set exactly. In fact, the boundary of a set is mostly a vague area instead of accurate line. So we need to consider the concept of boundary in a fuzzy soft space where we cannot judge exactly which points belong to the boundary zone. To obtain this aim, the present paper is organized as the following. Firstly in section 2, we recall some definitions and properties of fuzzy soft set theory which are used along this work. Then in section 3 the concepts of fuzzy soft interior, fuzzy soft closure, and fuzzy soft continuity are studied in the fuzzy soft topological spaces. Finally in section 4, the concept of fuzzy soft boundary is introduced and some of its properties are studied.

2. Preliminaries

Let X and E are used to show the set of objects and parameters, respectively. Let $A \subseteq E$, and I^X, where $I = [0,1]$, denotes the set of all fuzzy subsets of X.

Definition 2.1 ([14]). A pair (F, A) is called a soft set over X if F is a mapping given by $F : A \to 2^X$ such that for all $a \in A$, $F(a) \subseteq X$.

Definition 2.2 ([12]). A pair (f, A) is called a fuzzy soft set over X if f is a mapping given by $f : A \to I^X$. So $\forall a \in A$, $f(a)$ is a fuzzy subset of X with membership function $f(a) := f_a : X \to [0,1]$.

We denote the fuzzy soft set (f, A) by f_A and abbreviate the terminology "fuzzy soft set" by F.S-set. Moreover, we call the F.S-set f_A as a crisp F.S-set if $\forall a \in A$, the value set of $f(a)$ is a subset of $\{0, 1\}$. The soft set (F, A) can be seen as the F.S-set (χ_F, A) where χ_F refers to the characteristic function of set $F(a)$. In addition, during this paper we use the notation $\mathcal{F}.\mathcal{S}(X, E)$ to show the set of all F.S-sets over X with regard to the parameter set E.

Definition 2.3 ([12, 24]). (Rules of fuzzy soft set) For two F.S-sets f_A and g_B over the common universe X where $A, B \subseteq E$ we have,

i. f_A is a F.S-subset of g_B (or g_B is a F.S-superset of f_A) shown by $f_A \tilde{\leq} g_B$ (or $g_B \tilde{\geq} f_A$) if:
   (a) $A \subseteq B$,
   (b) for all a in A, $f_a(x) \leq g_a(x)$, $\forall x \in X$.
ii. $f_A = g_B$ if $f_A \tilde{\leq} g_B$ and $g_B \tilde{\leq} f_A$.
iii. The complement of F.S-set f_A is denoted by f_A^c and given by the mapping $f^c : A \to I^X$ such that $f_a^c = 1 - f_a$, $\forall a \in A$.
iv. $f_A = \Phi_A$ (null F.S-set with respect to A) if $\forall a \in A$, $f_a(x) = 0$ for all $x \in X$.
v. $f_A = \tilde{X}_A$ (absolute F.S-set with respect to A) if $\forall a \in A$, $f_a(x) = 1$ for all $x \in X$. If $A = E$, the null and absolute fuzzy soft set is denoted by Φ and $\tilde{X}$, respectively.
vi. The union of f_A and g_B, denoted by $f_A \tilde{\vee} g_B$, is a F.S-set over X with membership function $(f \vee g)(c)$ where
$$(f \vee g)_c(x) = \begin{cases} f_c(x) & \text{if } c \in A - B \\ g_c(x) & \text{if } c \in B - A \\ \max\{f_c(x), g_c(x)\} & \text{if } c \in A \cap B \end{cases}$$
for all $x \in X$ and $c \in A \cup B$.
vii. The intersection of f_A and g_B, denoted by $f_A \tilde{\wedge} g_B$, is a F.S-set over X with membership function $(f \wedge g)(c)$ where $(f \wedge g)_c(x) = \min\{f_c(x), g_c(x)\}$ for all $x \in X$ and $e \in A \cap B$.
viii. "f_A AND g_B" is a F.S-set over X and is defined by $h_{A \times B}$ where $A \times B$ is the cartesian product of parameter sets A and B, and h is the map $h : A \times B \to I^X$ such that for all $x \in X$, $a \in A$, and $b \in B$, we have $h_{(a,b)}(x) = \min\{f_a(x), g_b(x)\}$.

ix. "f_A OR g_B" is a F.S-set over X defined by $k_{A\times B}$ where k is the map $k : A\times B \to I^X$ such that for all $x \in X$, $a \in A$, and $b \in B$, we have $k_{(a,b)}(x) = \max\{f_a(x), g_b(x)\}$.

Regarding to vii, the intersection of two F.S-sets f_A and g_B is defined if the parameter sets of them meet each other. To avoid of such a difficulty, in [17] the concept of extended intersection of two F.S-sets has been considered.

Proposition 2.4 ([1]). *If f_A, g_B, and h_C are some F.S-sets over X where A, B, and C are subsets of the parameter set E, then*

(1) $[f_A \tilde{\vee} g_B]\tilde{\wedge} h_C = [f_A \tilde{\wedge} h_C]\tilde{\vee}[g_B \tilde{\wedge} h_C]$,
(2) $[f_A \tilde{\wedge} g_B]\tilde{\vee} h_C = [f_A \tilde{\vee} h_C]\tilde{\wedge}[g_B \tilde{\vee} h_C]$.

Proposition 2.5 ([26]). *If f_E and g_E are two F.S-sets over X, then*

(1) $[f_E \tilde{\vee} g_E]^c = f_E^c \tilde{\wedge} g_E^c$,
(2) $[f_E \tilde{\wedge} g_E]^c = f_E^c \tilde{\vee} g_E^c$.

Definition 2.6 ([8]). Let X_1 and X_2 be universal sets and E_1 and E_2 be corresponding parameter sets. Suppose that f_A is a F.S-set over X_1 and g_B is a F.S-set over X_2 where $A \subseteq E_1$ and $B \subseteq E_2$. If $\Psi_U : X_1 \to X_2$ and $\Psi_P : E_1 \to E_2$ are ordinary functions where U refers to universal set and P refers to parameter set, then

i. The map $\Psi : \mathcal{F}.\mathcal{S}(X_1, E_1) \to \mathcal{F}.\mathcal{S}(X_2, E_2)$ is called a F.S-map from X_1 to X_2 and for any $y \in X_2$ and $e' \in P(E_1) \subseteq E_2$, the image of f_A under Ψ is the F.S-set $\Psi(f_A)$ over X_2 defined as below:

$$[\Psi(f)](e')(y) = \sup_{x \in {\Psi_U}^{-1}(y)} \left[\sup_{e \in {\Psi_P}^{-1}(e') \cap A} f(e) \right](x)$$

if ${\Psi_P}^{-1}(e') \cap A \neq \varnothing$ and ${\Psi_U}^{-1}(y) \neq \varnothing$, otherwise $[\Psi(f)](e')(y) = 0$.

ii. Let $\Psi : \mathcal{F}.\mathcal{S}(X_1, E_1) \to \mathcal{F}.\mathcal{S}(X_2, E_2)$ be a F.S-map from X_1 to X_2. Then the inverse image of F.S-set g_B under Ψ, denoting by $\Psi^{-1}(g_B)$, is a F.S-set over X_1. For all $x \in X_1$ and $e \in E_1$ it is defined as below:

$$[\Psi^{-1}(g)](e)(x) = \begin{cases} g_{\Psi_P(e)}(\Psi_U(x)) & \text{if } \Psi_P(e) \in B \\ 0 & \text{otherwise.} \end{cases}$$

We also show the F.S-map Ψ by the ordered pair (Ψ_P, Ψ_U) of ordinary maps Ψ_U and Ψ_P.

Proposition 2.7 ([8]). *Let $\Psi : \mathcal{F}.\mathcal{S}(X_1, E_1) \to \mathcal{F}.\mathcal{S}(X_2, E_2)$ be a F.S-map as introduced in Definition 2.6. Let $\{f_{iA}\}_{i\in I}$ and $\{g_{iB}\}_{i\in I}$ are two families of F.S-sets in X_1 and X_2, respectively. Then*

(1) $\Psi(\Phi_{E_1}) = \Phi_{E_2}$ *and* $\Psi^{-1}(\Phi_{E_2}) = \Phi_{E_1}$.
(2) $\Psi(\tilde{X}_1) \tilde{\leq} \tilde{X}_2$ *and* $\Psi^{-1}(\tilde{X}_2) = \tilde{X}_1$.
(3) $\Psi[\tilde{\bigvee}_i f_{iA}] = \tilde{\bigvee}_i \Psi(f_{iA})$ *and* $\Psi^{-1}[\tilde{\bigvee}_i g_{iB}] = \tilde{\bigvee}_i \Psi^{-1}(g_{iB})$.
(4) $\Psi[\tilde{\bigwedge}_i f_{iA}] \tilde{\leq} \tilde{\bigwedge}_i \Psi(f_{iA})$ *and* $\Psi^{-1}[\tilde{\bigwedge}_i g_{iB}] = \tilde{\bigwedge}_i \Psi^{-1}(g_{iB})$.
(5) *For each* $i \in I$, $[\Psi(f_{iA})]^c \tilde{\leq} \Psi(f_{iA}^c)$ *and* $[\Psi^{-1}(g_{iB})]^c = \Psi^{-1}(g_{iB}^c)$.

(6) *For any* $i, j \in I$, *if* $f_{iA} \tilde{\leq} f_{jA}$, *then* $\Psi(f_{iA}) \tilde{\leq} \Psi(f_{jA})$ *and if* $g_{iB} \tilde{\leq} g_{jB}$, *then* $\Psi^{-1}(g_{iB}) \tilde{\leq} \Psi^{-1}(g_{jB})$.

Definition 2.8 ([26])**.** Let $f_{E_1} \in \mathcal{F}.\mathcal{S}(X_1, E_1)$ and $g_{E_2} \in \mathcal{F}.\mathcal{S}(X_2, E_2)$ be two F.S-sets over X_1 and X_2, respectively. The "cartesian product" of f_{E_1} and g_{E_2}, denoted by $f_{E_1} \tilde{\otimes} g_{E_2}$, is a F.S-set over $X_1 \times X_2$ defined as below:

$$\begin{aligned} f\tilde{\otimes} g : E_1 \times E_2 &\longrightarrow I^{X_1} \times I^{X_2} \\ (e, e') &\mapsto f(e) \times g(e') \end{aligned}$$

such that $f(e) \times g(e')$ is the fuzzy product of fuzzy sets $f(e)$ and $g(e')$ where

$$\begin{aligned} f(e) \times g(e') : X_1 \times X_2 &\longrightarrow [0,1] \\ (x, y) &\mapsto \min\{f_e(x), g_{e'}(y)\} \end{aligned}$$

Proposition 2.9 ([26])**.** *Let* f_{E_1}, f_{1E_1}, *and* $f_{2E_1} \in \mathcal{F}.\mathcal{S}(X_1, E_1)$; *and* g_{E_2}, g_{1E_2}, *and* $g_{2E_2} \in \mathcal{F}.\mathcal{S}(X_2, E_2)$. *Then*

(1) $\tilde{X}_1 \tilde{\otimes} \tilde{X}_2 = \widetilde{X_1 \times X_2}$.
(2) $f_{E_1} \tilde{\otimes} \Phi_{E_2} = \Phi_{E_1} \tilde{\otimes} g_{E_2} = \Phi_{E_1 \times E_2} = \Phi_{E_1} \tilde{\otimes} \Phi_{E_2}$.
(3) $\forall e \in E_1, \forall e' \in E_2$, $(f \tilde{\otimes} \tilde{X}_2)_{(e,e')}(x, y) = f_e(x)$, $(\tilde{X}_1 \tilde{\otimes} g)_{(e,e')} = g_{e'}(y)$, *where* $x \in X_1$ *and* $y \in X_2$.
(4) $[f_{1E_1} \tilde{\wedge} f_{2E_1}] \tilde{\otimes} [g_{1E_2} \tilde{\wedge} g_{2E_2}] = [f_{1E_1} \tilde{\otimes} g_{1E_2}] \tilde{\wedge} [f_{2E_1} \tilde{\otimes} g_{2E_2}]$.
Specially we have $f_{E_1} \tilde{\otimes} g_{E_2} = [f_{E_1} \tilde{\otimes} \tilde{X}_2] \tilde{\wedge} [\tilde{X}_1 \tilde{\otimes} g_{E_2}]$.

3. Fuzzy soft topology

The concept of fuzzy soft topology firstly introduced by Tanay and Kandemir[24]. They defined the concept of fuzzy soft topology as a topology over the given fuzzy soft set f_A. So a fuzzy soft topology in the sense of Tanay and Kandemir[24] is the collection τ of F.S-subsets of f_A closed under arbitrary supremum and finite infimum. It also contains Φ_A and $\tilde{X}_A$. But Roy and Samanta[19] redefined the concept of fuzzy soft topology. The most significant reason for such a change is to make sure that the DeMorgan Laws hold in the new definition of fuzzy soft topology. Here we recall the definition of fuzzy soft topology as introduced in [19].

Definition 3.1 ([19])**.** A fuzzy soft topology over X denoted by τ, is a collection of fuzzy soft subsets of X such that:

i. $\tilde{X}$ and $\Phi \in \tau$.
ii. The union of any number of F.S-sets in τ belongs to τ.
iii. The intersection of any two F.S-sets in τ belongs to τ.

The triplet (X, E, τ) is called a fuzzy soft topological space, F.S-topological space in brief, and each element of τ is called a fuzzy soft open set, F.S-open set in brief, in X. The complement of a F.S-open set is called a F.S-closed set. We show the collection of all F.S-closed sets in X by τ^c.

Theorem 3.2. *Let* (X, E, τ) *be a F.S-topological space. Then*

(1) Φ *and* $\tilde{X}$ *are F.S-closed sets.*
(2) *The intersection of any number of F.S-closed sets is a F.S-closed set.*
(3) *The union of any two F.S-closed sets is a F.S-closed set.*

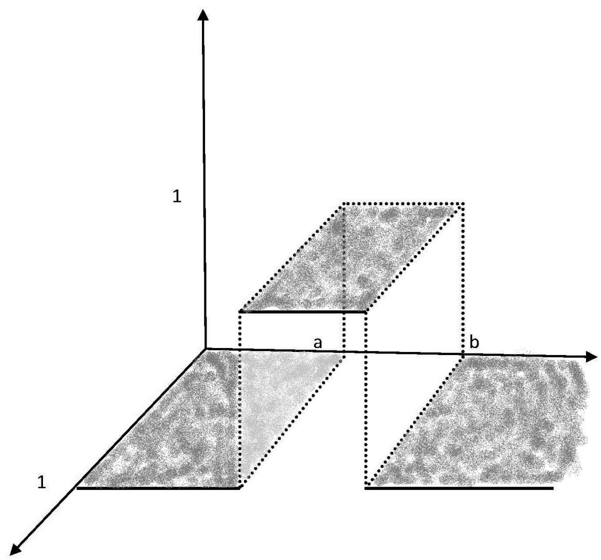

FIGURE 1. F.S-open set U_E in $\tau^*_{F.S}$

Proof. Follows from Proposition 2.5. □

Definition 3.3. Let τ_1 and τ_2 be two F.S-topologies over X. We say that τ_1 is coarser than τ_2 , or τ_2 is finer than τ_1, if $\tau_1 \subset \tau_2$.

Example 3.4 ([26])**.** Let X and E be the universal and parameter set, respectively. Take $\tau = \{\Phi, \tilde{X}\}$ (say trivial F.S-topology over X), then τ is clearly the coarsest one. Take $\tau = \mathcal{F}.\mathcal{S}(X, E)$ (say discrete F.S-topology over X), then τ is the finest one.

Example 3.5 ([26])**.** Let $\mathbb{R}$ be the set of all real numbers with the usual topology τ and $E = [0, 1) \subset \mathbb{R}$. If $U = (a, b) \subset \mathbb{R}$ is an open interval in $\mathbb{R}$, then we define the crisp F.S-set U_E over $\mathbb{R}$ related to the open interval $U \subset \mathbb{R}$ by the mapping

$$\begin{array}{rcl} U : E = [0,1) & \rightarrow & I^{\mathbb{R}} \\ \alpha & \mapsto & U(\alpha) := U_\alpha : \mathbb{R} \rightarrow \{0,1\} \end{array}$$

such that for all $x \in \mathbb{R}$,

$$U_\alpha(x) = \begin{cases} 1 & x \in (a,b) \\ 0 & x \notin (a,b). \end{cases}$$

The family $\{U_E\}$ of such crisp F.S-sets forms a F.S-topology over $\mathbb{R}$ denoted by $\tau^*_{F.S}$. In fact, the F.S-open set U_E in $\tau^*_{F.S}$ is a parameterization extension of open interval $U = (a, b) \subset \mathbb{R}$ (see Fig. 1).

Example 3.6. Consider the real line $\mathbb{R}$ as a topological space with usual topology τ and let $E = [0, 1)$. The collection $\tau_{F.S} = \{f : E \rightarrow I^{\mathbb{R}} : \forall \alpha \in E, (f(\alpha))^{-1}(0, 1] \in \tau\}$ is a F.S-topology over $\mathbb{R}$. Moreover $\tau^*_{F.S} \subset \tau_{FS}$ where $\tau^*_{F.S}$ is the F.S-topology over $\mathbb{R}$ introduced earlier in Example 3.5.

Example 3.7. Let (X,γ) be a fuzzy topological space and $\mu \in \gamma$ be a fuzzy open set in X. The characteristic function of α-cut set of μ is denoted by χ_{μ_α} and is defined by the mapping $\chi_{\mu_\alpha} : X \to \{0,1\}$ such that for all $x \in X$

$$\chi_{\mu_\alpha}(x) = \begin{cases} 1 & \mu(x) > \alpha \\ 0 & \mu(x) \leq \alpha. \end{cases}$$

Now let $E = [0,1)$. We define the crisp F.S-set (χ_μ, E) as below:

$$\begin{aligned} \chi_\mu : E = [0,1) &\to I^X \\ \alpha &\mapsto \chi_\mu(\alpha) := \chi_{\mu_\alpha} : X \to \{0,1\}. \end{aligned}$$

Then the collection $\gamma^* = \{\chi_\mu : [0,1) \to I^X : \mu \in \gamma\}$ is a F.S-topology over X.

Theorem 3.8. *Let (X,E,τ) be a F.S-topological space. For any $e \in E$, the collection $\tau_e = \{(f(e))^{-1}(0,1] : f \in \tau, e \in E\}$ forms a topology over X with respect to e.*

Proof. Follows from Definition 3.1. □

Theorem 3.9. *If $\{\tau_\alpha : \alpha \in I\}$ be a family of F.S-topologies over X, then $\bigcap_{\alpha\in I}\tau_\alpha$ is also a F.S-topology over X.*

Proof. It is straightforward. □

It is clear that $\bigcap_{\alpha\in I}\tau_\alpha$ is the coarsest F.S-topology over X.

Remark 3.10. The $\bigcup_{\alpha\in I}\tau_\alpha$ is not a F.S-topology over X in general case. This is shown in the following example.

Example 3.11. Let $X = \{x_1, x_2\}$ and $E = \{e_1, e_2\}$. Let $\tau = \{\Phi, \tilde{X}, f_{1E}, f_{2E}, f_{3E}\}$ and $\delta = \{\Phi, \tilde{X}, g_{1E}, g_{2E}, g_{3E}, g_{4E}\}$ be two F.S-topologies over X where

$$\begin{aligned} f_{1E} &= \{(e_1, \{\frac{1}{x_1} + \frac{1}{x_2}\}), (e_2, \{\frac{0}{x_1} + \frac{1}{x_2}\})\} \\ f_{2E} &= \{(e_1, \{\frac{1}{x_1} + \frac{0}{x_2}\}), (e_2, \{\frac{1}{x_1} + \frac{1}{x_2}\})\} \\ f_{3E} &= \{(e_1, \{\frac{1}{x_1} + \frac{0}{x_2}\}), (e_2, \{\frac{0}{x_1} + \frac{1}{x_2}\})\} \end{aligned}$$

and

$$\begin{aligned} g_{1E} &= \{(e_1, \{\frac{1}{x_1} + \frac{0.5}{x_2}\}), (e_2, \{\frac{0.5}{x_1} + \frac{0.3}{x_2}\})\} \\ g_{2E} &= \{(e_1, \{\frac{0}{x_1} + \frac{0.8}{x_2}\}), (e_2, \{\frac{1}{x_1} + \frac{1}{x_2}\})\} \\ g_{3E} &= \{(e_1, \{\frac{0}{x_1} + \frac{0.5}{x_2}\}), (e_2, \{\frac{0.5}{x_1} + \frac{0.3}{x_2}\})\} \\ g_{4E} &= \{(e_1, \{\frac{1}{x_1} + \frac{0.8}{x_2}\}), (e_2, \{\frac{1}{x_1} + \frac{1}{x_2}\})\}. \end{aligned}$$

Then $\tau \cup \delta = \{\Phi, \tilde{X}, f_{1E}, f_{2E}, f_{3E}, g_{1E}, g_{2E}, g_{3E}, g_{4E}\}$ is not a F.S-topology over X since $f_{1E}\tilde{\vee}g_{1E} \notin \tau \cup \delta$.

Definition 3.12 ([26]). Let (X, E, τ) be a F.S-topological space. The collection $\mathcal{B}$ of F.S-open subsets of X is called a F.S-base for τ if every F.S-open sets in τ can be written as a union of members of $\mathcal{B}$.

Definition 3.13 ([26]). Let (X_1, E_1, τ_1) and (X_2, E_2, τ_2) be two F.S-topological spaces. The F.S-topology $\tau^{\otimes}$, generated by $\mathcal{B} = \{f_{E_1} \tilde{\otimes} g_{E_2} : f_{E_1} \in \tau_1, g_{E_2} \in \tau_2\}$, is called F.S-product topology over $X_1 \times X_2$ and denoted by $(X, E, \tau^{\otimes})$ where $X = X_1 \times X_2$ and $E = E_1 \times E_2$.

Lemma 3.14. *Let $f_{E_1} \in \mathcal{F.S}(X_1, E_1)$ and $g_{E_2} \in \mathcal{F.S}(X_2, E_2)$. Then we have*

$$[f_{E_1} \tilde{\otimes} g_{E_2}]^c = [f^c_{E_1} \tilde{\otimes} \tilde{X}_2] \tilde{\vee} [\tilde{X}_1 \tilde{\otimes} g^c_{E_2}]$$

Proof. Take $(x, y) \in X_1 \times X_2$ and $(e, e') \in E_1 \times E_2$. By Proposition 2.9 (3), we have

$$\begin{aligned} (f \tilde{\otimes} g)^c_{(e,e')}(x, y) &= 1 - (f \tilde{\otimes} g)_{(e,e')}(x, y) = 1 - \min\{f_e(x), g_{e'}(y)\} \\ &= \max\{1 - f_e(x), 1 - g_{e'}(y)\} = \max\{f^c_e(x), g^c_{e'}(y)\} \\ &= \max\{(f^c \tilde{\otimes} \tilde{X}_2)_{(e,e')}(x, y), (\tilde{X}_1 \tilde{\otimes} g^c)_{(e,e')}(x, y)\} \\ &= [(f^c \tilde{\otimes} \tilde{X}_2) \tilde{\vee} (\tilde{X}_1 \tilde{\otimes} g^c)]_{(e,e')}(x, y) \end{aligned}$$

This means that $[f_{E_1} \tilde{\otimes} g_{E_2}]^c = [f^c_{E_1} \tilde{\otimes} \tilde{X}_2] \tilde{\vee} [\tilde{X}_1 \tilde{\otimes} g^c_{E_2}]$. □

Theorem 3.15. *Let (X, E, τ) be a F.S-topological space. If f_E and g_E are two F.S-closed sets in τ, then $f_E \tilde{\otimes} g_E$ is a F.S-closed set in $\tau^{\otimes}$.*

Proof. Follows from Lemma 3.14. □

3.1. F.S-Closure and F.S-Interior.

Definition 3.16. Let (X, E, τ) be a F.S-topological space.

i. Fuzzy soft closure, F.S-closure in brief, of f_E is denoted by Clf_E and is defined as the intersection of all F.S-closed super sets of f_E. So

$$Clf_E = \tilde{\bigwedge}_{g_E \tilde{\geq} f_E} g_E$$

where g_E's are F.S-closed sets in (X, E, τ).

ii. Fuzzy soft interior, F.S interior in brief, of f_E is denoted by $Intf_E$ and is defined as the union of all F.S-open subsets of f_E. So

$$Intf_E = \tilde{\bigvee}_{h_E \tilde{\leq} f_E} h_E$$

where h_E's are F.S-open sets in (X, E, τ).

Theorem 3.17 ([23]). *Let (X, E, τ) be a F.S-topological space and f_E and g_E be two F.S-sets over X. Then*

(1) $Cl\Phi = \Phi$ and $Cl\tilde{X} = \tilde{X}$.
(2) $f_E \tilde{\leq} Clf_E$ and Clf_E is the smallest F.S-closed set containing the F.S-set f_E.

(3) f_E is a F.S-closed set if and only if $f_E = Clf_E$.
(4) $Cl(Clf_E) = Clf_E$.
(5) if $f_E \tilde{\leq} g_E$, then $Clf_E \tilde{\leq} Clg_E$.
(6) $Cl(f_E \tilde{\vee} g_E) = Clf_E \tilde{\vee} Clg_E$.

(7) $Cl(f_E\tilde{\wedge}g_E)\tilde{\leq}Clf_E\tilde{\wedge}Clg_E$.

Proof. See [23]. □

Theorem 3.18 ([23]). *Let (X, E, τ) be a F.S-topological space and f_E and g_E are two F.S-sets over X. Then*

(1) $Int\Phi = \Phi$ *and* $Int\tilde{X} = \tilde{X}$.
(2) $Intf_E\tilde{\leq}f_E$ *and* $Intf_E$ *is the biggest F.S-open set contained in the F.S-set* f_E.
(3) f_E *is a F.S-open set if and only if* $f_E = Intf_E$.
(4) $Int(Intf_E) = Intf_E$.
(5) *if* $f_E\tilde{\leq}g_E$, *then* $Intf_E\tilde{\leq}Intg_E$.
(6) $Int(f_E\tilde{\vee}g_E)\tilde{\geq}Intf_E\tilde{\vee}Intg_E$.
(7) $Int(f_E\tilde{\wedge}g_E) = Intf_E\tilde{\wedge}Intg_E$.

Proof. See [23]. □

Corollary 3.19. *For any F.S-set f_E in the F.S-topological space (X, E, τ),*
$$Intf_E\tilde{\leq}f_E\tilde{\leq}Clf_E.$$

Proof. Follows from Theorems 3.17 and 3.18. □

Theorem 3.20. *Let (X, E, τ) be a F.S-topological space and $\{f_{iE}\}_{i\in I}$ be a family of F.S-sets over X. Then*

(1) $\tilde{\bigvee}_i Clf_{iE}\tilde{\leq}Cl[\tilde{\bigvee}_i f_{iE}]$.
(2) $\tilde{\bigvee}_i Intf_{iE}\tilde{\leq}Int[\tilde{\bigvee}_i f_{iE}]$.

Proof. Follows from Theorems 3.17 and 3.18. □

Proposition 3.21. *Let (X, E, τ) be a F.S-topological space and f_E be a F.S-set over X. Then*

(1) $Clf^c_E = (Intf_E)^c$.
(2) $Intf^c_E = (Clf_E)^c$.

Proof. (1)

$$\begin{aligned} (Intf_E)^c &= [\tilde{\bigvee} g_E]^c : g_E \in \tau, g_E\tilde{\leq}f_E \\ &= \tilde{\bigwedge} g^c_E : g_E \in \tau, g^c_E\tilde{\geq}f^c_E \\ &= Clf^c_E \end{aligned}$$

(2) It is similar to 1.

□

Theorem 3.22. *If (X, E, τ) is a F.S-topological space and f_E and g_E are two F.S-sets over X. Then*

(1) $Cl(f_E\tilde{\otimes}g_E)\tilde{\leq}Clf_E\tilde{\otimes}Clg_E$.
(2) $Int(f_E\tilde{\otimes}g_E)\tilde{\geq}Intf_E\tilde{\otimes}Intg_E$.

Proof. (1) Since $f_E\tilde{\leq}Clf_E$ and $g_E\tilde{\leq}Clg_E$, then it is easily to see that
$$f_E\tilde{\otimes}g_E\tilde{\leq}Clf_E\tilde{\otimes}Clg_E.$$

$Cl(f_E\tilde{\otimes}g_E)\tilde{\leq}Clf_E\tilde{\otimes}Clg_E$ follows from Theorems 3.15 and 3.17.

(2) By Proposition 3.21, Lemma 3.14, and Theorem 3.17 we have

$$\begin{aligned}[Int(f_E\tilde{\otimes}g_E)]^c &= Cl[(f_E\tilde{\otimes}g_E)^c] = Cl[(f_E^c\tilde{\otimes}\tilde{X})\tilde{\vee}(\tilde{X}\tilde{\otimes}g_E^c)]\\ &= Cl(f_E^c\tilde{\otimes}\tilde{X})\tilde{\vee}Cl(\tilde{X}\tilde{\otimes}g_E^c)\\ &\tilde{\leq} [Clf_E^c\tilde{\otimes}Cl\tilde{X}]\tilde{\vee}[Cl\tilde{X}\tilde{\otimes}Clg_E^c] = [Clf_E^c\tilde{\otimes}\tilde{X}]\tilde{\vee}[\tilde{X}\tilde{\otimes}Clg_E^c]\\ &= [(Intf_E)^c\tilde{\otimes}\tilde{X}]\tilde{\vee}[\tilde{X}\tilde{\otimes}(Intg_E)^c]\\ &= (Intf_E\tilde{\otimes}Intg_E)^c\end{aligned}$$

So $[Int(f_E\tilde{\otimes}g_E)]^c\tilde{\leq}(Intf_E\tilde{\otimes}Intg_E)^c$. This implies that

$$Int(f_E\tilde{\otimes}g_E)\tilde{\geq}Intf_E\tilde{\otimes}Intg_E.$$

□

3.2. F.S-Continuity.

Definition 3.23 ([26]). Let $\Psi : (X_1, E_1, \tau_1) \to (X_2, E_2, \tau_2)$ be a F.S-map as introduced in Definition 2.6. Ψ is called

i. A F.S-continuous map if and only if $\forall g_{E_2} \in \tau_2$, $\Psi^{-1}(g_{E_2}) \in \tau_1$.
ii. A F.S-open map if and only if $\forall f_{E_1} \in \tau_1$, $\Psi(f_{E_1}) \in \tau_2$.

Theorem 3.24. *The F.S-map $\Psi : (X_1, E_1, \tau_1) \to (X_2, E_2, \tau_2)$ is F.S-continuous if and only if $\forall g_{E_2} \in \tau_2^c$, $\Psi^{-1}(g_{E_2}) \in \tau_1^c$.*

Proof. Follows from Proposition 2.7 (5). □

Theorem 3.25. *If $\Psi : (X_1, E_1, \tau_1) \to (X_2, E_2, \tau_2)$ is a F.S-continuous map, then $\forall g_{E_2} \in \mathcal{F}.\mathcal{S}(X_2, E_2)$, $Cl[\Psi^{-1}(g_{E_2})]\tilde{\leq}\Psi^{-1}(Clg_{E_2})$.*

Proof. Follows from Proposition 2.7 and Theorems 3.24 and 3.17. □

Definition 3.26. Let $\Psi_1 : \mathcal{F}.\mathcal{S}(X_1, E_1) \to \mathcal{F}.\mathcal{S}(X_2, E_2)$ and $\Psi_2 : \mathcal{F}.\mathcal{S}(X_2, E_2) \to \mathcal{F}.\mathcal{S}(X_3, E_3)$ be two F.S-maps. The mapping

$$\Psi = \Psi_2 o \Psi_1 : \mathcal{F}.\mathcal{S}(X_1, E_1) \to \mathcal{F}.\mathcal{S}(X_3, E_3)$$

is called the F.S-composition map such that for any $f_{E_1} \in \mathcal{F}.\mathcal{S}(X_1, E_1)$ it is defined as

$$[(\Psi_2 o \Psi_1)(f)](\alpha)(z) = \sup_{x\in(\Psi_{2U} o \Psi_{1U})^{-1}(z)} \left[\sup_{\beta\in(\Psi_{2P} o \Psi_{1P})^{-1}(\alpha)} f(\beta)\right](x)$$

where $z \in X_3$, $\alpha \in E_3$; and $\Psi_{1P} : E_1 \to E_2$, $\Psi_{2P} : E_2 \to E_3$, $\Psi_{1U} : X_1 \to X_2$, and $\Psi_{2U} : X_2 \to X_3$ are ordinary maps. Moreover For $x \in X_1$ and $\beta \in E_1$ we have

$$[(\Psi_2 o \Psi_1)^{-1}(g)](\beta)(x) = g[(\Psi_{2P} o \Psi_{1P})(\beta)][(\Psi_{2U} o \Psi_{1U})(x)]$$

where $g_{E_3} \in \mathcal{F}.\mathcal{S}(X_3, E_3)$.

Lemma 3.27. *If $\Psi_1 : \mathcal{F}.\mathcal{S}(X_1, E_1) \to \mathcal{F}.\mathcal{S}(X_2, E_2)$ and $\Psi_2 : \mathcal{F}.\mathcal{S}(X_2, E_2) \to \mathcal{F}.\mathcal{S}(X_3, E_3)$ are two F.S-maps, then $(\Psi_2 o \Psi_1)^{-1} = \Psi_1^{-1} o \Psi_2^{-1}$.*

Proof. Take $f_{E_3} \in \mathcal{F}.\mathcal{S}(X_3, E_3)$. Then for $x \in X_1$ and $e \in E_1$, we have

$$\begin{aligned}
[(\Psi_2 o \Psi_1)^{-1}(f)](e)(x) &= f[(\Psi_{2P} o \Psi_{1P})(e)][(\Psi_{2U} o \Psi_{1U})(x)] \\
&= f_{\Psi_{2P}(\Psi_{1P}(e))}[\Psi_{2U}(\Psi_{1U}(x))] \\
&= [\Psi_2^{-1}(f)](\Psi_{1P}(e))(\Psi_{1U}(x))] \\
&= \Psi_1^{-1}[\Psi_2^{-1}(f)](e)(x) \\
&= [(\Psi_1^{-1} o \Psi_2^{-1})(f)](e)(x)
\end{aligned}$$

Thus $(\Psi_2 o \Psi_1)^{-1} = \Psi_1^{-1} o \Psi_2^{-1}$. □

Theorem 3.28. *If $\Psi_1 : (X_1, E_1, \tau_1) \to (X_2, E_2, \tau_2)$ and $\Psi_2 : (X_2, E_2, \tau_2) \to (X_3, E_3, \tau_3)$ are two F.S-continuous maps, then $\Psi_2 o \Psi_1 : (X_1, E_1, \tau_1) \to (X_3, E_3, \tau_3)$ is a F.S-continuous map.*

Proof. Follows from Lemma 3.27. □

Lemma 3.29. *Let for i=1,2, $\Psi_i : \mathcal{F}.\mathcal{S}(X, E) \to \mathcal{F}.\mathcal{S}(Y_i, V_i)$, $\Psi_{iP} : E \to V_i$, and $\Psi_{iU} : X \to Y_i$ be some F.S-maps and ordinary maps, respectively as introduced in Definition 2.6. If $\Psi_1 \tilde{\otimes} \Psi_2 : \mathcal{F}.\mathcal{S}(X, E) \to \mathcal{F}.\mathcal{S}(Y_1, V_1) \tilde{\otimes} \mathcal{F}.\mathcal{S}(Y_2, V_2)$ is a F.S-map defined by*

$$(\Psi_1 \tilde{\otimes} \Psi_2)(f_E) = \Psi_1(f_E) \tilde{\otimes} \Psi_2(f_E)$$

where $f_E \in \mathcal{F}.\mathcal{S}(X, E)$, then for $g_{1V_1} \in \mathcal{F}.\mathcal{S}(Y_1, V_1)$ and $g_{2V_2} \in \mathcal{F}.\mathcal{S}(Y_2, V_2)$,

$$(\Psi_1 \tilde{\otimes} \Psi_2)^{-1}(g_{1V_1} \tilde{\otimes} g_{2V_2}) = \Psi_1^{-1}(g_{1V_1}) \tilde{\wedge} \Psi_2^{-1}(g_{2V_2})$$

Proof. Consider the two following ordinary maps

$$\begin{aligned}
\Psi_{1P} \times \Psi_{2P} : E &\to V_1 \times V_2 \\
e &\mapsto (\Psi_{1P}(e), \Psi_{2P}(e))
\end{aligned}$$

and

$$\begin{aligned}
\Psi_{1U} \times \Psi_{2U} : X &\to Y_1 \times Y_2 \\
x &\mapsto (\Psi_{1U}(x), \Psi_{2U}(x)).
\end{aligned}$$

Take $g_{1V_1} \in \mathcal{F}.\mathcal{S}(Y_1, V_1)$ and $g_{2V_2} \in \mathcal{F}.\mathcal{S}(Y_2, V_2)$. For given $e \in E$ and $x \in X$, we have

$$\begin{aligned}
[(\Psi_1 \tilde{\otimes} \Psi_2)^{-1}(g_1 \tilde{\otimes} g_2)](e)(x) &= (g_1 \tilde{\otimes} g_2)((\Psi_{1P} \times \Psi_{2P})(e))((\Psi_{1U} \times \Psi_{2U})(x)) \\
&= (g_1 \tilde{\otimes} g_2)(\Psi_{1P}(e), \Psi_{2P}(e))(\Psi_{1U}(x), \Psi_{2U}(x)) \\
&= \min\{g_1(\Psi_{1P}(e))(\Psi_{1U}(x)), g_2(\Psi_{2P}(e))(\Psi_{2U}(x))\} \\
&= \min\{\Psi_1^{-1}(g_1)(e)(x), \Psi_2^{-1}(g_2)(e)(x)\} \\
&= [\Psi_1^{-1}(g_1) \tilde{\wedge} \Psi_2^{-1}(g_2)](e)(x)
\end{aligned}$$

Thus $(\Psi_1 \tilde{\otimes} \Psi_2)^{-1}(g_{1V_1} \tilde{\otimes} g_{2V_2}) = \Psi_1^{-1}(g_{1V_1}) \tilde{\wedge} \Psi_2^{-1}(g_{2V_2})$. □

Theorem 3.30. *Let $\Psi_i : (X, E, \tau) \to (Y_i, V_i, \tau_i)$ be two F.S-maps (for $i = 1, 2$). The F.S-map $\Psi_1 \tilde{\otimes} \Psi_2 : (X, E, \tau) \to (Y_1 \times Y_2, V_1 \times V_2, \tau^{\otimes})$ is continuous if and only if Ψ_1 and Ψ_2 are F.S-continuous.*

Proof. $\Rightarrow$**:** Let Ψ_1 and Ψ_2 be two F.S-continuous maps. Let $g_{1V_1}\tilde{\otimes}g_{2V_2}$ be a F.S-open set in F.S-topological space $(Y_1 \times Y_2, V_1 \times V_2, \tau^{\otimes})$ where $\tau^{\otimes}$ is the F.S-product topology over $Y_1 \times Y_2$ introduced in [26]. By applying Lemma 3.29 we have $(\Psi_1\tilde{\otimes}\Psi_2)^{-1}(g_{1V_1}\tilde{\otimes}g_{2V_2}) = \Psi_1^{-1}(g_{1V_1})\tilde{\wedge}\Psi_2^{-1}(g_{2V_2}) \in \tau$. This means that Ψ is a F.S-continuous.

$\Leftarrow$**:** Take $g_{1V_1} \in \tau_1$. Then by Lemma 3.29, for $e \in V$ and $x \in Y$ we have

$$\begin{aligned}(\Psi_1\tilde{\otimes}\Psi_2)^{-1}(g_1\tilde{\otimes}\tilde{Y}_2)(e)(x) &= [\Psi_1^{-1}(g_1)\tilde{\wedge}\Psi_2^{-1}(\tilde{Y}_2)](e)(x)\\ &= [\Psi_1^{-1}(g_1)\tilde{\wedge}\tilde{X}](e)(x)\\ &= \Psi_1^{-1}(g_1)(e)(x)\end{aligned}$$

This means that $\Psi_1^{-1}(g_{1V_1}) = (\Psi_1\tilde{\otimes}\Psi_2)^{-1}(g_{1V_1}\tilde{\otimes}\tilde{Y}_2) \in \tau$. Similarly if $g_{2V_2} \in \tau_2$, then $\Psi_2^{-1}(g_{2V_2}) = (\Psi_1\tilde{\otimes}\Psi_2)^{-1}(\tilde{Y}_1\tilde{\otimes}\tilde{g}_{2V_2}) \in \tau$. This completes the proof. □

Lemma 3.31. *Let for i=1,2, $\Psi_i : \mathcal{F}.\mathcal{S}(X_i, E_i) \to \mathcal{F}.\mathcal{S}(Y_i, V_i)$ be two F.S-maps and $\Psi_{iP} : E_i \to V_i$ and $\Psi_{iU} : X_i \to Y_i$ be some ordinary maps as introduced in Definition 2.6. If $\Psi_1\tilde{\otimes}\Psi_2 : \mathcal{F}.\mathcal{S}(X_1, E_1)\tilde{\otimes}\mathcal{F}.\mathcal{S}(X_2, E_2) \to \mathcal{F}.\mathcal{S}(Y_1, V_1)\tilde{\otimes}\mathcal{F}.\mathcal{S}(Y_2, V_2)$ is a F.S-map defined by*

$$(\Psi_1\tilde{\otimes}\Psi_2)(f_{1E_1}\tilde{\otimes}f_{2E_2}) = \Psi_1(f_{1E_1})\tilde{\otimes}\Psi_2(f_{2E_2})$$

where $f_{1E_1} \in \mathcal{F}.\mathcal{S}(X_1, E_1)$ and $f_{2E_2} \in \mathcal{F}.\mathcal{S}(X_2, E_2)$, then

$$(\Psi_1\tilde{\otimes}\Psi_2)^{-1}(g_{1V_1}\tilde{\otimes}g_{2V_2}) = \Psi_1^{-1}(g_{1V_1})\tilde{\otimes}\Psi_2^{-1}(g_{2V_2})$$

Proof. It is similar to Lemma 3.29. □

Theorem 3.32. *Let $\Psi_i : \mathcal{F}.\mathcal{S}(X_i, E_i, \tau_i) \to \mathcal{F}.\mathcal{S}(Y_i, V_i, \gamma_i)$ where $i = 1, 2$ be two F.S-maps. The F.S-map $\Psi_1\tilde{\otimes}\Psi_2 : (X_1 \times X_2, E_1 \times E_2, \tau^{\otimes}) \to (Y_1 \times Y_2, V_1 \times V_2, \gamma^{\otimes})$ is continuous if and only if Ψ_1 and Ψ_2 are F.S-continuous.*

Proof. Follows from Lemma 3.31. □

4. Fuzzy soft Boubdary

Definition 4.1. Let (X, E, τ) be a F.S-topological space and f_E be a F.S-set over X. We define the boundary of f_E, denoted by Bdf_E, as below

$$Bdf_E = Clf_E\tilde{\wedge}Clf_E^c$$

Proposition 4.2. *If (X, E, τ) is a F.S-topological space, then*

(1) Bdf_E is a F.S-closed set.
(2) $Bdf_E\tilde{\leq}Clf_E$.
(3) $Bd\Phi = \Phi$ and $Bd\tilde{X} = \Phi$.

Proof. Follows from Definition 4.1. □

Proposition 4.3. *If (X, E, τ) is a F.S-topological space, then*

(1) $Bdf_E = Bdf_E^c$.
(2) $[Bdf_E]^c = Intf_E^c\tilde{\vee}Intf_E$.
(3) $Bdf_E\tilde{\vee}f_E\tilde{\leq}Clf_E$.

(4) $Bdf_E\tilde{\leq}f_E$ *if* f_E *be a F.S-closed set, and so* $BdClf_E\tilde{\leq}Clf_E$ *for all F.S-set* f_E.
(5) $Bdf_E\tilde{\leq}f_E^c$ *if* f_E *be a F.S-open set, and so* $BdIntf_E\tilde{\leq}Clf_E^c$ *for all F.S-set* f_E.
(6) $BdBdf_E\tilde{\leq}Bdf_E$.
(7) $BdIntf_E\tilde{\leq}Bdf_E$.
(8) $BdClf_E\tilde{\leq}Bdf_E$.

Proof. (1) It follows from Definition 4.1.
(2) It follows from Definition 4.1, Proposition 2.5, and Proposition 3.21.
(3) It follows from Definition 4.1.
(4) If f_E be a F.S-closed set, then Theorem 3.17 (3), implies that $Clf_E = f_E$. So we have $Bdf_E = Clf_E\tilde{\wedge}Clf_E^c = f_E\tilde{\wedge}Clf_E^c$ which implies that $Bdf_E\tilde{\leq}f_E$. Moreover if $f_E \in \mathcal{F}.\mathcal{S}(X,E)$, then we have

$$BdClf_E = ClClf_E\tilde{\wedge}Cl(Clf_E)^c = Clf_E\tilde{\wedge}Cl(Clf_E)^c$$

So $BdClf_E\tilde{\leq}Clf_E$.
(5) It is similar to 4.
(6) It is clear since Bdf_E is a F.S-closed set.
(7)

$$\begin{aligned} BdIntf_E &= Cl(Intf_E)\tilde{\wedge}Cl(Intf_E)^c = ClIntf_E\tilde{\wedge}ClClf_E^c = ClIntf_E\tilde{\wedge}Clf_E^c \\ &\tilde{\leq} Clf_E\tilde{\wedge}Clf_E^c = Bdf_E \end{aligned}$$

Thus $BdIntf_E\tilde{\leq}Bdf_E$.
(8)

$$\begin{aligned} BdClf_E &= Cl(Clf_E)\tilde{\wedge}Cl(Clf_E)^c = ClClf_E\tilde{\wedge}ClIntf_E^c = Clf_E\tilde{\wedge}ClIntf_E^c \\ &\tilde{\leq} Clf_E\tilde{\wedge}Clf_E^c = Bdf_E \end{aligned}$$

So $BdClf_E\tilde{\leq}Bdf_E$.

□

Example 4.4. Let $X = \mathbb{R}$, $E = [0,1)$, and $\tau^*_{F.S}$ be the F.S-topology over $\mathbb{R}$ as introduced in Example 3.5. We define the crisp F.S-set V_E over $\mathbb{R}$ related to the half-open interval $[a,b) \subset \mathbb{R}$ by the mapping $V : E = [0,1) \to I^{\mathbb{R}}$ as below

$$V_\alpha(x) = \begin{cases} 1 & x \in [a,b) \\ 0 & x \notin [a,b) \end{cases}$$

where $\alpha \in E$ and $x \in \mathbb{R}$. By applying Definition 3.16, the closure and the interior of the crisp F.S-set V_E, denoted by ClV_E and $IntV_E$ respectively, are defined by the mappings $ClV : E = [0,1) \to I^{\mathbb{R}}$ and $IntV : E = [0,1) \to I^{\mathbb{R}}$ such that for any $x \in \mathbb{R}$ and $\forall\alpha \in E$ we have

$$(ClV)_\alpha(x) = \begin{cases} 1 & x \in [a,b] \\ 0 & x \notin [a,b], \end{cases}$$

and

$$(IntV)_\alpha(x) = \begin{cases} 1 & x \in (a,b) \\ 0 & x \notin (a,b). \end{cases}$$

Moreover by Definition 4.1, the boundary of crisp F.S-set V_E is defined by the mapping $BdV : E = [0, 1) \to I^{\mathbb{R}}$ as below

$$(BdV)_\alpha(x) = \begin{cases} 1 & x \in \{a, b\} \\ 0 & x \notin \{a, b\} \end{cases}$$

where $x \in \mathbb{R}$ and $\alpha \in E$.

Remark 4.5. In general topology, it is well-known that $BdA = \varnothing$ if and only if A is an open and a closed set, both in X. But in F.S-topology, it may not hold in general. This is shown by the following example.

Example 4.6. Take the set of real numbers $\mathbb{R}$ with usual topology τ. Consider the F.S-topological space $(\mathbb{R}, [0, 1), \tau_{F.S})$ as introduced earlier in Example 3.6. Let $a, b, c \in \mathbb{R}$ such that $a < b < c$ and let f_E be a F.S-set over $\mathbb{R}$ such that for any $\alpha \in E$, the mapping $f_\alpha : \mathbb{R} \to [0, 1]$ is defined as below

$$f_\alpha(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{c-a} & a < x \leq c \\ \frac{x-b}{c-b} & c < x < b \\ 0 & x \geq b \end{cases}$$

where $x \in \mathbb{R}$. It is clear that the complement of f_E, denoted by f^c_E, given by the mapping $f^c : E = [0, 1) \to I^{\mathbb{R}}$ such that $\forall x \in \mathbb{R}$ and $\forall \alpha \in E$ is defined as below

$$f^c_\alpha(x) = \begin{cases} 1 & x \leq a \\ \frac{c-x}{c-a} & a < x \leq c \\ \frac{c-x}{c-b} & c < x < b \\ 1 & x \geq b. \end{cases}$$

Regarding to Example 3.6, f_E is a F.S-open and a F.S-closed set both, in F.S-topological space $(\mathbb{R}, [0, 1), \tau_{F.S})$ since $\forall \alpha \in E$, ${f_\alpha}^{-1}(0, 1] = (a, b) \in \tau$ and ${f^c_\alpha}^{-1}(0, 1]$ $= (-\infty, c) \cup (c, +\infty) \in \tau$. So $f_E = Clf_E$ and $f^c_E = Clf^c_E$. This implies that $Bdf_E = f_E \tilde{\wedge} f^c_E$. Thus $\forall x \in \mathbb{R}$ and $\forall \alpha \in E$,

$$(Bdf)_\alpha(x) = \min\{f_\alpha(x), f^c_\alpha(x)\} = \begin{cases} 0 & x \leq a \ , \ x \geq b \\ f_\alpha(x) & a < x \leq \frac{a+c}{2} \ , \ \frac{c+b}{2} \leq x < b \\ f^c_\alpha(x) & \frac{a+c}{2} < x < \frac{c+b}{2} \end{cases}$$

means that $Bdf_E \neq \Phi$.

Theorem 4.7. *If (X, E, τ) is a F.S-topological space, then*
(1) $Bd[f_E \tilde{\vee} g_E] \tilde{\leq} Bdf_E \tilde{\vee} Bdg_E$.
(2) $Bd[f_E \tilde{\wedge} g_E] \tilde{\leq} Bdf_E \tilde{\vee} Bdg_E$.

Proof. (1)

$$\begin{aligned} Bd[f_E \tilde{\vee} g_E] &= Cl[f_E \tilde{\vee} g_E] \tilde{\wedge} Cl[f_E \tilde{\vee} g_E]^c \\ &= Cl[f_E \tilde{\vee} g_E] \tilde{\wedge} Cl[f^c_E \tilde{\wedge} g^c_E] \\ &\tilde{\leq} [Clf_E \tilde{\vee} Clg_E] \tilde{\wedge} [Clf^c_E \tilde{\wedge} Clg^c_E] \\ &= [Clf_E \tilde{\wedge} (Clf^c_E \tilde{\wedge} Clg^c_E)] \tilde{\vee} [Clg_E \tilde{\wedge} (Clf^c_E \tilde{\wedge} Clg^c_E)] \\ &= [Bdf_E \tilde{\wedge} Clg^c_E] \tilde{\vee} [Bdg_E \tilde{\wedge} Clf^c_E] \\ &\tilde{\leq} Bdf_E \tilde{\vee} Bdg_E \end{aligned}$$

(2)

$$\begin{array}{rcl} Bd[f_E\tilde{\wedge}g_E] & = & Cl[f_E\tilde{\wedge}g_E]\tilde{\wedge}Cl[f_E\tilde{\wedge}g_E]^c \\ & = & Cl[f_E\tilde{\wedge}g_E]\tilde{\wedge}Cl[f_E^c\tilde{\vee}g_E^c] \\ & \tilde{\leq} & [Clf_E\tilde{\wedge}Clg_E]\tilde{\wedge}[Clf_E^c\tilde{\vee}Clg_E^c] \\ & = & [(Clf_E\tilde{\wedge}Clg_E)\tilde{\wedge}Clf_E^c]\tilde{\vee}[(Clf_E\tilde{\wedge}Clg_E)\tilde{\wedge}Clg_E^c] \\ & = & [Bdf_E\tilde{\wedge}Clg_E]\tilde{\vee}[Bdg_E\tilde{\wedge}Clf_E] \\ & \tilde{\leq} & Bdf_E\tilde{\vee}Bdg_E \end{array}$$

□

Theorem 4.8. *Let (X, E, τ) be a F.S-topological space and f_E be a F.S-set over X. If $(Bdf)_e(x) = 0$, then $f_e(x) = 0$ or $f_e(x) = 1$.*

Proof. Let $x \in X$ such that for some $e \in E$ we have $(Bdf)_e(x) = 0$, means $\min\{(Clf)_e(x), (Clf^c)_e(x)\} = 0$. Then $(Clf)_e(x) = 0$ or $(Clf^c)_e(x) = 0$ which implies that $f_e(x) = 0$ or $f_e^c(x) = 0$. Thus $f_e(x) = 0$ or $f_e(x) = 1$. □

Corollary 4.9. *In the F.S-topological space (X, E, τ), the F.S-set f_E is a crisp F.S-set if $Bdf_E = \Phi$.*

Proof. Let $Bdf_E = \Phi$. Theorem 4.8 implies that $f_e(x) = 0$ or $f_e(x) = 1$, $\forall x \in X$ and for all $e \in E$. This means that the set value of mapping $f(e)$ is a subset of $\{0, 1\}$ i.e., f_E is a crisp F.S-set. □

Theorem 4.10. *If $\Psi : (X_1, E_1, \tau_1) \to (X_2, E_2, \tau_2)$ is a F.S-continuous map, then $\forall g_{E_2} \in \mathcal{F}.\mathcal{S}(X_2, E_2)$,*

$$Bd[\Psi^{-1}(g_{E_2})]\tilde{\leq}\Psi^{-1}[Bdg_{E_2}]$$

Proof. If g_{E_2} is a F.S-set over X_2, then by Proposition 2.7 and Theorem 3.25 we have

$$\begin{array}{rcl} Bd[\Psi^{-1}(g_{E_2})] & = & Cl[\Psi^{-1}(g_{E_2})]\tilde{\wedge}Cl[\Psi^{-1}(g_{E_2})]^c \\ & = & Cl[\Psi^{-1}(g_{E_2})]\tilde{\wedge}Cl[\Psi^{-1}(g_{E_2}^c)] \\ & \tilde{\leq} & \Psi^{-1}[Clg_{E_2}]\tilde{\wedge}\Psi^{-1}[Clg_{E_2}^c] \\ & = & \Psi^{-1}[Clg_{E_2}\tilde{\wedge}Clg_{E_2}^c] \\ & = & \Psi^{-1}[Bdg_{E_2}] \end{array}$$

Thus $Bd[\Psi^{-1}(g_{E_2})]\tilde{\leq}\Psi^{-1}[Bdg_{E_2}]$. □

Theorem 4.11. *If (X, E, τ) is a F.S-topological space, then*

(1) $Bd[f_E\tilde{\otimes}g_E]\tilde{\leq}Clf_E\tilde{\otimes}Clg_E$.
(2) $Bd[f_E\tilde{\otimes}g_E]\tilde{\leq}[Bdf_E\tilde{\otimes}Clg_E]\tilde{\vee}[Clf_E\tilde{\otimes}Bdg_E]$.

Proof. (1) By applying Proposition 4.2 we have $Bd[f_E\tilde{\otimes}g_E]\tilde{\leq}Cl[f_E\tilde{\otimes}g_E]$. So $Bd[f_E\tilde{\otimes}g_E]\tilde{\leq}Clf_E\tilde{\otimes}Clg_E$ follows from Theorem 3.22 (1).

(2) By Lemma 3.14, Theorem 3.17, Theorem 3.22 (1), Proposition 2.4 (2), and Proposition 2.9 (4) we have

$$\begin{aligned}
Bd[f_E\tilde{\otimes}g_E] &= Cl[f_E\tilde{\otimes}g_E]\tilde{\wedge}Cl[f_E\tilde{\otimes}g_E]^c \\
&= Cl[f_E\tilde{\otimes}g_E]\tilde{\wedge}Cl[(f_E^c\tilde{\otimes}\tilde{X})\tilde{\vee}(\tilde{X}\tilde{\otimes}g_E^c)] \\
&\tilde{\leq} [Clf_E\tilde{\otimes}Clg_E]\tilde{\wedge}[Cl(f_E^c\tilde{\otimes}\tilde{X})\tilde{\vee}Cl(\tilde{X}\tilde{\otimes}g_E^c)] \\
&\tilde{\leq} [Clf_E\tilde{\otimes}Clg_E]\tilde{\wedge}[(Clf_E^c\tilde{\otimes}\tilde{X})\tilde{\vee}(\tilde{X}\tilde{\otimes}Clg_E^c)] \\
&= [(Clf_E\tilde{\otimes}Clg_E)\tilde{\wedge}(Clf_E^c\tilde{\otimes}\tilde{X})]\tilde{\vee}[(Clf_E\tilde{\wedge}Clg_E)\tilde{\wedge}(\tilde{X}\tilde{\otimes}Clg_E^c)] \\
&= [(Clf_E\tilde{\wedge}Clf_E^c)\tilde{\otimes}(Clg_E\tilde{\wedge}\tilde{X})]\tilde{\vee}[(Clf_E\tilde{\wedge}\tilde{X})\tilde{\otimes}(Clg_E\tilde{\wedge}Clg_E^c)] \\
&= [Bdf_E\tilde{\otimes}Clg_E]\tilde{\vee}[Clf_E\tilde{\otimes}Bdg_E]
\end{aligned}$$

Hence $Bd[f_E\tilde{\otimes}g_E]\tilde{\leq}[Bdf_E\tilde{\otimes}Clg_E]\tilde{\vee}[Clf_E\tilde{\otimes}Bdg_E]$. □

5. Conclusions

The aim of this work is to introduce and to study the concept of fuzzy soft boundary. We define fuzzy soft boundary as an parameterization extension of the concept of boundary in the classical sense and then consider some properties of it. The fuzzy soft topology is also considered and closure, interior, and continuity are studied in a fuzzy soft topological space.

Acknowledgements. This work was partially supported by the Institute for Mathematical Research, Universiti Putra Malaysia under the Grant number 5527179.

References

[1] B. Ahmad and A. Kharal, On fuzzy soft sets, Adv. Fuzzy Syst. 2009, Art. ID 586507, 6 pp.
[2] H. Aktaş and N. Çağman, Soft sets and soft groups, Inform. Sci. 177 (2007) 2726–3332.
[3] S. Atmaca and I. Zorlutuna, On fuzzy soft topological spaces, Ann. Fuzzy Math. Inform. 5(2) (2013) 377–386.
[4] N. Çağman, S. Karataş and S. Enginoğlu, Soft topology, Comput. Math. Appl. 62 (2011) 351–358.
[5] H. Hazra, P. Majumdar and S. K. Samanta, Soft topology, Fuzzy Inf. Eng. 4(1) (2012) 105–115.
[6] M. Irfan Ali, F. Feng, X. Liu and W.K. Min, On some new operations in soft set theory, Comput. Math. Appl. 57(9) (2009) 1547–1553.
[7] M. Irfan Ali, M. Shabir and Munzza Naz, Algebraic structures of soft sets associated with new operations, Comput. Math. Appl. 61(9) (2011) 2647–2654.
[8] A. Kharal and B. Ahmad, Mappings on fuzzy soft classes, Adv. Fuzzy Syst. 2009, Art. ID 407890, 6 pp.
[9] Z. Li and R. Cui, On the topological structure of intuitionistic fuzzy soft sets, Ann. Fuzzy Math. Inform. 5(1) (2013) 229–239.
[10] P. K. Maji, R. Biswas and A. R. Roy, soft set theory, Comput. Math. Appl. 45 (2003) 555–562.
[11] P. K. Maji, R. Biswas and A. R. Roy, An application of soft sets in a decision making problem, Comput. Math. Appl. 44 (2002) 1077–1083.
[12] P. K. Maji, R. Biswas and A. R. Roy, Fuzzy soft set, J. Fuzzy Math. 9(3) (2001) 589–602.
[13] P. Majumdar and S. K. Samanta, Generalized fuzzy soft sets, Comput. Math. Appl. 59(4) (2010) 1425–1432.
[14] D. Molodtsov, Soft set theory first-results, Comput. Math. Appl. 37 (1999) 19–31.
[15] I. Osmanoglu and D. Tokat, Compact fuzzy soft spaces, Ann. Fuzzy Math. Inform. 7(1) (2014) 45–51.

[16] Z. Pawlak, Rough sets, Internat. J. Comput. Inform. Sci. 11(5) (1982) 341–356.
[17] A. Rehman, S. Abdullah, M. Aslam and M. S. Kamran, A study on fuzzy soft set and its operations, Ann. Fuzzy Math. Inform. 6(2) (2013) 339–362.
[18] A. R. Roy and P. K. Maji, A fuzzy soft set theoretic approach to decision making problems, J. Comput. Appl. Math. 203 (2007) 412–418.
[19] S. Roy and T. K. Samanta, A note on fuzzy soft topological spaces, Ann. Fuzzy Math. Inform. 3(2) (2012) 305–311.
[20] S. Roy and T. K. Samanta, An introduction to open and closed sets on fuzzy soft topological spaces, Ann. Fuzzy Math. Inform. 6(2) (2013) 425–431.
[21] A. Sezgin and A. O. Atagun, On operations of soft sets, Comput. Math. Appl. 61 (2011) 1457–1467.
[22] M. Shabir and M. Naz, On soft topological spaces, Comput. Math. Appl. 61 (2011) 1786–1799.
[23] T. Simsekler and S. Yuksel, Fuzzy soft topological spaces, Ann. Fuzzy Math. Inform. 5(1) (2013) 87–96.
[24] B. Tanay and M. B. Kandemir, Topological structure of fuzzy soft sets, Comput. Math. Appl. 61 (2011) 2952–2957.
[25] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[26] A. Zahedi Khameneh, A. Kılıçman and A. R. Salleh, Fuzzy soft product topology, Ann. Fuzzy Math. Inform. 7(6) (2014) 935–947.

Azadeh Zahedi Khameneh (azadeh503@gmail.com)
Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor D.E., Malaysia

Adem Kılıçman (akilic@upm.edu.my)
Institute for Mathematical Research and Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor D.E., Malaysia

Abdul Razak Salleh (aras@ukm.my)
School of Mathematical Sciences, Universiti Kebangsaan Malaysia, 43600 UKM Bangi, Selangor D.E., Malaysia