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Evaluating a fuzzy Henstock double integral using double Simpson's rule

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Abstract. In this paper, double Simpson's rule for the Henstock double integral of a fuzzy number-valued function and the error bound of the method are proposed by using the Hausdorff distance. Also, δ-fine divisions for a Henstock double integral is introduced and numerical examples are presented to show the application and importance of the method.

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1. Introduction

The concept of fuzzy integral was introduced by Sugeno [12]. In order to evaluate a fuzzy integral some numerical methods have been proposed in recent years. Wu in [14, 15], Allahviranloo in [1, 2, 3, 4] and Fariborzi in [9, 10] proposed some numerical methods for computing fuzzy integrals by using quadrature methods and the definition of α-level set. Wu and Gong in [13] proposed the Henstock integral of a fuzzy number-valued function and then developed this work by applying the concept of differentiability of a fuzzy function. Bede and Gal in [6] applied the quadrature rule for evaluating the integral of a fuzzy number valued function. In this paper, we develop this idea for a double fuzzy valued function by applying the double Simpson's rule and introducing the Henstock double integral.

In section 2, we present some basic definitions and properties of fuzzy sets and fuzzy numbers and also some basic theorems which are used in the work.

In section 3, we introduce the double Simpson's rule for computing a fuzzy Henstock double integral (FHDI).

Finally, in order to illustrate an application of the proposed method, in section 4, two double fuzzy integrals are evaluated in order to show the efficiency of the mentioned method.

2. Preliminaries

In this section, some basic definitions of fuzzy sets theory are remined which are used in the rest of this paper.

Definition 2.1 ([8]). A fuzzy number is a function $u : \mathbf{R} \to [0,1]$ satisfying the following properties :

(i) u is normal, i.e. $\exists x_0 \in \mathbf{R}$ with $u(x_0) = 1$,

(ii) u is a convex fuzzy set, i.e.,

$$u(\lambda x + (1-\lambda)y) \geq \min\{u(x), v(y)\} \forall x, y \in \mathbf{R}, \lambda \in [0,1]),$$

(iii) u is upper semi-continuous on $\mathbf{R}$,

(iv) $\overline{\{x \in \mathbf{R} : u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

The set of all fuzzy real numbers is denoted by $\mathbf{R}_F$. For $0 < r \leq 1$, we define $[u]^r = \{x \in \mathbf{R} : u(x) \geq r\}$ and $[u]^0 = \{x \in \mathbf{R} : u(x) > 0\}$ as the $r-$cut and support of a fuzzy number like u respectively. Also, we define $u_r^- = \inf[u]^r$ and $u_r^+ = \sup[u]^r$. A triangular fuzzy number $u = (a, b, c)$ where, $a < b < c$ and $a, b, c \in \mathbf{R}$ is determined such that $u_r^- = a + (b-a)r$ and $u_r^+ = c - (c-b)r$.

For $u, v \in \mathbf{R}_f$ and $\lambda \in \mathbf{R}$, we have the sum $u \oplus v$ and the product $\lambda \odot u$ defined by $[u \oplus v]^r = [u]^r + [v]^r, [\lambda \odot u]^r = \lambda[u]^r \forall r \in [0,1]$, where $[u]^r + [v]^r$ means the usual addition of two interval (as subsets of $\mathbf{R}$) and $\lambda[u]^r$ means the usual product between a scalar and a subset of $\mathbf{R}$.

Definition 2.2 ([5]). The Hausdorff distance between two fuzzy numbers u and v given by $D : \mathbf{R}_F \times \mathbf{R}_F \to \mathbf{R}^+ \cup 0$, is defined as,

$$D(u,v) = \sup_{r\in[0,1]} \max\{|u_r^- - v_r^-|, |u_r^+ - v_r^+|\} = \sup_{r\in[0,1]}\{d_H([u]^r, [v]^r)\}, \tag{2.1}$$

where $[u]_r = [u_r^-, u_r^+], [v]^r = [v_r^-, v_r^+] \subseteq \mathbf{R}$ and d_H is the Hausdorff metric. We define $\| . \| = D(., 0)$.

Theorem 2.3 ([13]). *(i) If we define $\tilde{0} = \chi_{\{0\}}$, then $\tilde{0} \in \mathbf{R}_F$ is neutral element with respect to $\oplus$, i.e. $u \oplus \tilde{0} = \tilde{0} \oplus u = u$ for all $u \in \mathbf{R}_F$.*

(ii) With respect to $\tilde{0}$, none of $u \in \mathbf{R}_F, u \neq \tilde{0}$ has inverse in $\mathbf{R}_F$ (with respect to $\oplus$).

(iii) For any $a, b \in \mathbf{R}$ with $a, b \geq 0$ or $a, b \leq 0$, and any $u \in \mathbf{R}_F$, we have $(a + b) \odot u = a \odot u \oplus b \odot u$.
For general $a, b \in \mathbf{R}$, the above property does not hold.

(iv) For any $\lambda \in \mathbf{R}$ and any $u, v \in \mathbf{R}_F$, we have $\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot v$.

(v) For any $\lambda, \mu \in \mathbf{R}$ and any $u \in \mathbf{R}_F$, we have $\lambda \odot (\mu \odot u) = (\lambda \odot \mu) \odot u$.

(vi) $\| \, . \, \|_F$ *has the properties of a usual norm on* $\mathbf{R}_F$*, i.e.* $\| u \|_F = 0$ *iff* $u = \tilde{0}$, $\| \lambda \odot u \|_F = |\lambda|. \| u \|_F$ *and* $\| u \oplus v \|_F \leq \| u \|_F + \| v \|_F$.

(vii) $\| u \|_F \leq D(u, v)$ *and* $D(u, v) \leq \| u \|_F \oplus \| v \|_F$ *for any* $u, v \in \mathbf{R}_F$.

Theorem 2.4 ([5]). *(i)* $(\mathbf{R}_F, D)$ *is a complete metric space,*

(ii) $D(u \oplus v, v \oplus w) = D(u, w) \forall u, v, w \in \mathbf{R}_F$,

(iii) $D(k \odot u, k \odot v) = |k| D(u, v) \forall u, v \in \mathbf{R}_F, \forall k \in \mathbf{R}$,

(iv) $D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e) \forall u, v, w, e \in \mathbf{R}_F$.

Wu and Gong in [13] introduced the concept of the Henstock integral for a fuzzy number-valued function. We introduce this definition for a two-dimensional fuzzy number-valued function.
Let $f : [a, b] \times [c, d] \to \mathbf{R}_F$ and $\triangle_m : a = x_0 < x_1 < ... < x_m = b$ and $\triangle_n : c = y_0 < y_1 < ... < y_n = d$ be the partitions of the intervals $[a, b]$ and $[c, d]$ respectively.
We consider the points $\xi_i \in [x_{i-1}, x_i], i = 1, ..., m$ and $\eta_j \in [y_{j-1}, y_j], j = 1, ..., n$ and $\delta : [a, b] \times [c, d] \to \mathbf{R}^+$. The division $P = \{([x_{i-1}, x_i]; \xi_i); i = 1, ..., m\}$ and $Q = \{([y_{j-1}, y_j]; \eta_j); j = 1, ..., n\}$ denoted shortly by $P = (\triangle_m, \xi)$ and $Q = (\triangle_n, \eta)$ are said to be δ-fine if $[x_{i-1}, x_i] \subseteq (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $[y_{j-1}, y_j] \subseteq (\eta_j - \delta(\eta_j), \eta_j + \delta(\eta_j))$.

Definition 2.5. The function f is called Henstock double integrable to $I \in \mathbf{R}_f$ if for every $\epsilon > 0$ there is a function $\delta : [a, b] \times [c, d] \to \mathbf{R}^+$ such that for any δ-fine divisions P and Q we have $D(\sum_{i=1}^{m} \sum_{j=1}^{n} (x_i - x_{i-1})(y_j - y_{j-1}) \bigodot f(\xi_i, \eta_j), I) < \epsilon$. Then I is called the fuzzy Henstock double integral of f and it is denoted by (FHDI) $\int_a^b \int_c^d f(x, y) dx dy$.

Lemma 2.6. *(i) If* f *and* g *are Henstock double integrable mappings and if* $D(f(x, y), g(x, y))$ *is Lebesgue integrable, then*

$$
\begin{aligned}
&D((FHDI) \int_a^b \int_c^d f(x, y) dy dx, (FHDI) \int_a^b \int_c^d g(x, y) dy dx) \\
&\leq (L) \int_a^b \int_c^d D(f(x, y), g(x, y)) dy dx.
\end{aligned} \tag{2.2}
$$

(ii) Let $f : [a, b] \times [c, d] \to \mathbf{R}_F$ *be a Henstock double integrable bounded mapping. Then, for any fixed* $(u, v) \in [a, b] \times [c, d]$*, the function* $\varphi_{(u,v)} : [a, b] \times [c, d] \to \mathbf{R}^+$ *defined by* $\varphi_{(u,v)}(x, y) = D(f(u, v), f(x, y))$ *is Lebesgue integrable on* $[a, b] \times [c, d]$.

Proof. (ii) If f is Henstock integrable and bounded on $[a, b] \times [c, d]$, then it follows that $f_-^r(x, y), f_+^r(x, y)$ (as real functions of $(x, y) \in [a, b] \times [c, d]$) are Henstock double integrable (and uniformly bounded) with respect to $r \in [0, 1]$. Therefore, $f_r^-(x, y)$ and $f_r^+(x, y)$ are Lebesgue measurable (as functions of (x, y))and uniformly bounded with respect to $r \in [0, 1]$, [13] Furthermore,

$$
\begin{aligned}
\varphi(x, y) &= D(f(x_1, y_1), f(x_2, y_2)) \\
&= \sup_{r \in [0,1]} \max\{|f_-^r(x_1, y_1) - f_-^r(x_2, y_2)|, |f_+^r(x_1, y_1) - f_+^r(x_2, y_2)|\} \\
&= \sup_{r_n \in [0,1]} \max\{|f_-^{r_n}(x_1, y_1) - f_-^{r_n}(x_2, y_2)|, |f_+^{r_n}(x_1, y_1) - f_+^{r_n}(x_2, y_2)|\},
\end{aligned} \tag{2.3}
$$

where $r_n, n \in \mathbf{N}$, represent all the rational numbers in $[0,1]$. By Lebesgue's theorem of dominated convergence, it follows that $\varphi(x,y)$ is Lebesgue integrable on $[a,b] \times [c,d]$ and this ends the proof. □

Definition 2.7. Let $f : [a,b] \times [c,d] \to \mathbf{R}_F$ be a bounded mapping. Then the function $\omega_{([a,b]\times[c,d])}(f,.) : \mathbf{R}^+ \cup 0 \to \mathbf{R}^+$

$$\omega_{([a,b]\times[c,d])}(f,\delta_1,\delta_2) = \sup\{D(f(x_1,y_1), f(x_2,y_2)); \quad (x_1,y_1),(x_2,y_2) \in [a,b]\times[c,d], |x_1-y_1| \le \delta_1, |x_2-y_2| \le \delta_2\} \tag{2.4}$$

is called the modulus of oscillation of f on $[a,b] \times [c,d]$.
If $f : [a,b] \times [c,d] \to \mathbf{R}_F$ is continuous on $[a,b] \times [c,d]$, then $\omega_{([a,b]\times[c,d])}(f,\delta_1,\delta_2)$ is called uniform modulus of continuity of f.

From definition 2.7, the following theorem can be proved.

Theorem 2.8. *The following statements, concerning the modulus of oscillation, are true:*

(a) $D(f(x_1,y_1), f(x_2,y_2)) \le \omega_{[a,b]\times[c,d]}(f, |x_1-y_1|, |x_2-y_2|)$ *for any* (x_1,y_1), $(x_2,y_2) \in [a,b]\times[c,d]$,

(b) $\omega_{[a,b]\times[c,d]}(f,\delta_1,\delta_2)$ *is a non-decreasing mapping in* δ_1, δ_2,

(c) $\omega_{[a,b]\times[c,d]}(f,0,0) = 0$,

(d) $\omega_{[a,b]\times[c,d]}(f,n\delta_1,m\delta_2) \le nm\omega_{[a,b]\times[c,d]}(f,\delta_1,\delta_2)$ *for any* $\delta_1,\delta_2 \ge 0$ *and* $n, m \in \mathbf{N}$,

(e) $\omega_{[a,b]\times[c,d]}(f,\lambda_1\delta_1,\lambda_2\delta_2) \le (\lambda_1+1)(\lambda_2+1)\omega_{[a,b]\times[c,d]}(f,\delta_1,\delta_2)$ *for any* δ_1, δ_2, λ_1, $\lambda_2 \ge 0$.

(f) If $[e,f]\times[g,h] \subseteq [a,b]\times[c,d]$, *then* $\omega_{[e,f]\times[g,h]}(f,\delta_1,\delta_2) \le \omega_{[a,b]\times[c,d]}(f,\delta_1,\delta_2)$.

Proof. (f) According to the hypothesis,

$$\begin{aligned} &\sup\{D(f(x_1,y_1), f(x_2,y_2)); (x_1,y_1),(x_2,y_2) \in [e,f]\times[g,h], \\ &\qquad\qquad |x_1-y_1| \le \delta_1, |x_2-y_2| \le \delta_2\} \\ &\le \sup\{D(f(x_1,y_1), f(x_2,y_2)); (x_1,y_1),(x_2,y_2) \in [a,b]\times[c,d], \\ &\qquad\qquad |x_1-y_1| \le \delta_1, |x_2-y_2| \le \delta_2\} \end{aligned}$$

so the relation is proved.
The other statements can be proved similarly. □

Definition 2.9. . A function $f : [a,b]\times[c,d] \to \mathbf{R}_F$ is said to be (L_1, L_2) Lipschitz if for any $(x_1,y_1),(x_2,y_2) \in [a,b]\times[c,d]$,

$$D(f(x_1,y_1), f(x_2,y_2)) \le L_1|x_1-x_2| + L_2|y_1-y_2| \tag{2.5}$$

3. Double Simpson's rule for the fuzzy Henstock double integrals

In order to introduce double Simpson's rule for evaluating FHDI, at first, we prove the following theorem.

Theorem 3.1. *Let* $f : [a,b]\times[c,d] \to \mathbf{R}_F$ *be a Henstock integrable, bounded mapping. Then, for any division* $a = x_0 < x_1 < ... < x_m = b$, $c = y_0 < y_1 < ... < y_n = d$

and any points $\xi_i \in [x_{i-1}, x_i]$, $\eta_j \in [y_{j-1}, y_j]$ *we have*

$$D((FHDI)\int_a^b \int_c^d f(x,y)dxdy, \sum_{i=1}^m \sum_{j=1}^n (x_i - x_{i-1})(y_j - y_{j-1}) \odot f(\xi_i, \eta_j))$$
$$\leq \sum_{i=1}^m \sum_{j=1}^n (x_i - x_{i-1})(y_j - y_{j-1}) \omega_{[x_{i-1},x_i]\times[y_{j-1},y_j]}(f, (x_i - x_{i-1}), (y_j - y_{j-1})).$$

Proof. Since that the Henstock integral is additive related to interval [11], hence,

$$D((FHDI)\int_a^b \int_c^d f(x,y)dxdy, \sum_{i=1}^m \sum_{j=1}^n (x_i - x_{i-1})(y_j - y_{j-1}) \odot f(\xi_i, \eta_j))$$
$$= D(\sum_{i=1}^m \sum_{j=1}^n (FHDI)\int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} f(x,y)dxdy,$$
$$\sum_{i=1}^m \sum_{j=1}^n (x_i - x_{i-1})(y_j - y_{j-1}) \odot f(\xi_i, \eta_j).$$

Since it is obvious that $(FHDI)\int_a^b \int_c^d kdxdy = (c-a)(d-b) \bigodot k$ for any fuzzy constant $k \in \mathbf{R}_F$, we obtain

$$D((FHDI)\int_a^b \int_c^d f(x,y)dxdy, \sum_{i=1}^m \sum_{j=1}^n (x_i - x_{i-1})(y_j - y_{j-1}) \odot f(\xi_i, \eta_j)$$
$$= D(\sum_{i=1}^m \sum_{j=1}^n (FHDI)\int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} f(x,y)dxdy,$$
$$\sum_{i=1}^m \sum_{j=1}^n (FHDI)\int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} f(\xi_i, \eta_j)dxdy).$$

By property (iv) of theorem 2.4, we have

$$D(\sum_{i=1}^m \sum_{j=1}^n (FHDI)\int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} f(x,y)dxdy,$$
$$\sum_{i=1}^m \sum_{j=1}^n (FHDI)\int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} f(\xi_i, \eta_j)dxdy)$$
$$\leq \sum_{i=1}^m \sum_{j=1}^n D((FHDI)\int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} f(x,y)dxdy, (FHDI)\int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} f(\xi_i, \eta_j)dxdy).$$

Since the functions $D(f(x,y), f(\xi_i, \eta_j))$ are Lebesgue integrable for $i = 1, ..., m$ and $j = 1, ..., n$, from lemma 2.6 we obtain

$$D((FHDI)\int_a^b \int_c^d f(x,y)dxdy, \sum_{i=1}^m \sum_{j=1}^n (x_i - x_{i-1})(y_j - y_{j-1}) \odot f(\xi_i, \eta_j)$$
$$\leq \sum_{i=1}^m \sum_{j=1}^n (L)\int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} D(f(x,y), f(\xi_i, \eta_j))dxdy.$$

From property (a) of theorem 2.8 applied to each of the above integrals we have

$$
\begin{aligned}
&D((FHDI)\int_a^b\int_c^d f(x,y)dxdy, \sum_{i=1}^{m}\sum_{j=1}^{n}(x_i-x_{i-1})(y_j-y_{j-1})\odot f(\xi_i,\eta_j)\\
&\leq \sum_{i=1}^{m}\sum_{j=1}^{n}(L)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j}\omega_{[x_{i-1},x_i]\times[y_{j-1},y_j]}(f,x_i-x_{i-1})(y_j-y_{j-1}))dxdy\\
&=\sum_{i=1}^{m}\sum_{j=1}^{n}(x_i-x_{i-1})(y_j-y_j)\omega_{[x_{i-1},x_i]\times[y_{j-1},y_j]}(f,x_i-x_{i-1})(y_j-y_{j-1}),
\end{aligned}
$$

which completes the proof. □

Corollary 3.2. *Let $f:[a,b]\times[c,d]\to\mathbf{R}_F$ be a Henstock double integrable, bounded mapping. Then*

$$
\begin{aligned}
&D((FHDI)\int_a^b\int_c^d f(x,y)dxdy, \sum_{i=1}^{3}\sum_{j=1}^{3}(x_i-x_{i-1})(y_j-y_{j-1})\odot f(\xi_i,\eta_j))\\
&\leq (\alpha-a)(\gamma-c)\omega_{[a,\alpha]\times[c,\gamma]}(f,(\alpha-a),(\gamma-c))\\
&\qquad + (\beta-a)(\gamma-c)\omega_{[\alpha,\beta]\times[c,\gamma]}(f,(\beta-\alpha),(\gamma-c))\\
&\qquad + (b-\beta)(\gamma-c)\omega_{[\beta,b]\times[c,\gamma]}(f,(b-\beta),(\gamma-c))\\
&\qquad + (\alpha-a)(\delta-\gamma)\omega_{[a,\alpha]\times[\gamma,\delta]}(f,(\alpha-a),(\delta-\gamma))\\
&\qquad + (\beta-\alpha)(\delta-\gamma)\omega_{[\alpha,\beta]\times[\gamma,\delta]}(f,(\beta-\alpha),(\delta-\gamma))\\
&\qquad + (b-\beta)(\delta-\gamma)\omega_{[\beta,b]\times[\gamma,\delta]}(f,(b-\beta),(\delta-\gamma))\\
&\qquad + (\alpha-a)(d-\delta)\omega_{[a,\alpha]\times[\delta,d]}(f,(\alpha-a),(d-\delta))\\
&\qquad + (\beta-\alpha)(d-\delta)\omega_{[\alpha,\beta]\times[\delta,d]}(f,(\beta-\alpha),(d-\delta))\\
&\qquad + (b-\beta)(d-\delta)\omega_{[\beta,b]\times[\delta,d]}(f,(b-\beta),(d-\delta))
\end{aligned}
$$

for any $\alpha,\beta\in[a,b]$ and $\gamma,\delta\in[c,d]$, $(u,u')\in[a,\alpha]\times[c,\gamma]$ and $(v,v')\in[\alpha,\beta]\times[\gamma,\delta]$ and $(w,w')\in[\beta,b]\times[\delta,d]$ where $\xi_1=u,\xi_2=v,\xi_3=w$ and $\eta_1=u',\eta_2=v',\eta_3=w'$.

Proof. It is obvious that for $m=3$ and $n=3$ in theorem 3.1 the inequality stated above is obtained. □

The following corollary is the fuzzy variant of the classical double Simpson's rule mentioned in [7] with a new error bound.

Corollary 3.3. *Let $f : [a,b] \times [c,d] \to \mathbf{R}_F$ be a Henstock double integrable, bounded mapping. Then*

$$
\begin{aligned}
D((FHDI)&\int_a^b \int_c^d f(x,y)dxdy, \frac{(b-a)(d-c)}{36} \\
&\odot \{[f(a,c) \bigoplus 4 \bigodot f(\frac{a+b}{2}, c) \bigoplus f(b,c)] \\
&\bigoplus 4 \bigodot [f(a, \frac{c+d}{2}) \bigoplus 4 \bigodot f(\frac{a+b}{2}, \frac{c+d}{2}) \bigoplus f(b, \frac{c+d}{2})] \\
&\bigoplus [f(a,d) + 4 \bigodot f(\frac{a+b}{2}), d) \bigoplus f(b,d)]\} \\
\leq 9(b-a)&(d-c)\omega_{[a,b]\times[c,d]}(f, \frac{(b-a)}{6}, \frac{(c-d)}{6}).
\end{aligned}
$$

Proof. The inequality follows from the corresponding assertion of the previous corollary by taking $\alpha = (5a+b)/6$,$\beta = (a+5b)/6$,$\gamma = (5c+d)/6$,$\delta = (c+5d)/6$,$u = a$,$v = (a+b)/2$,$u' = c$ and $v' = (d+c)/2$ and $w = b$,$w' = d$. □

Theorem 3.4. *Let $f : [a,b] \times [c,d] \to \mathbf{R}_F$ be a Lipschitz mapping with constants L_1 and L_2. Then, for any division $\triangle_m : a = x_0 < x_1 < ... < x_m = b$ and $\triangle_n : c = y_0 < y_1 < ... < y_n = d$ and any points $\xi_i \in [x_{i-1}, x_i], i = 1, ..., m$ and $\eta_j \in [y_{j-1}, y_j], j = 1, ..., n$, we have*

$$
\begin{aligned}
&D((FHDI)\int_a^b \int_c^d f(x,y)dxdy, \sum_{i=1}^{m}\sum_{j=1}^{n}(x_i - x_{i-1})(y_j - y_{j-1}) \odot f(\xi_i, \eta_j)) \\
&\leq \sum_{i=1}^{m}\sum_{j=1}^{n}(L_1(y_j - y_{j-1})(x_i - x_{i-1}) + L_2(x_i - x_{i-1})(y_j - y_{j-1})).
\end{aligned}
\tag{3.1}
$$

Proof. Similar to the proof of theorem 3.1 we have

$$
\begin{aligned}
D((FHDI)&\int_a^b \int_c^d f(x,y)dxdy, \sum_{i=1}^{m}\sum_{j=1}^{n}(x_i - x_{i-1})(y_j - y_{j-1}) \odot f(\xi_i, \eta_j)) \\
\leq \sum_{i=1}^{m}&\sum_{j=1}^{n}(L)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} D(f(x,y), f(\xi_i, \eta_j))dydx.
\end{aligned}
$$

By the definition of a Lipschitz mapping, we have

$$
\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1}^{n}(L)\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} D(f(x,y), f(\xi_i, \eta_j))dydx \\
&\leq \sum_{i=1}^{m}\sum_{j=1}^{n} L_1 \int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} |x - \xi_i|dydx + L_2 \int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} |y - \eta_j|dydx.
\end{aligned}
$$

By direct computation it follows that

$$\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1}^{n} L_1 \int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} |x-\xi_i|dydx + L_2 \int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} |y-\eta_j|dydx \\
&= \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}(L_1(y_j-y_{j-1})[(x_{i-1}-\xi_i)^2+(x_i-\xi_i)^2] \\
&\qquad\qquad + L_2(x_i-x_{i-1})[(y_{j-1}-\eta_j)^2+(y_j-\eta_j)^2]) \\
&\le \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}(L_1(y_j-y_{j-1})(x_i-x_{i-1})^2+L_2(x_i-x_{i-1})(y_j-y_{j-1})^2).
\end{aligned}$$

□

Remark 3.5. If $x_i - x_{i-1} = h$ and $y_j - y_{j-1} = k$ then,

$$\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1}^{n}(L_1(y_j-y_{j-1})(x_i-x_{i-1})^2+L_2(x_i-x_{i-1})(y_j-y_{j-1})^2) \\
&= \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}(L_1kh^2+L_2k^2h),
\end{aligned}$$

where $mh = b-a$ and $nk = d-c$. Therefore, we have

$$\begin{aligned}
&D((FHDI)\int_a^b\int_c^d f(x,y)dxdy, \sum_{i=1}^{m}\sum_{j=1}^{n}(x_i-x_{i-1})(y_j-y_{j-1})\odot f(\xi_i,\eta_j)) \\
&\le U(h,k) = \frac{(b-a)(d-c)}{2}(L_1h+L_2k).
\end{aligned} \tag{3.2}$$

4. Numerical examples

Example 4.1. Let us consider $f:[1,3]\times[2,4]\to \mathbf{R}_F$, $f(x,y) = (\tilde{x}\otimes\tilde{x})\oplus(\tilde{2}\otimes\tilde{y})\oplus\tilde{1}$, with $\tilde{x} = (x-1,x,x+1)$, $\tilde{1} = (0,1,2)$ and $\tilde{2} = (1,2,3)$, where (a_1,a_2,a_3) denotes a triangular fuzzy number having the membership function

$$\mu(x) = \begin{cases} \frac{x-a_1}{a_2-a_1} & if\ a_1 \le x \le a_2, \\ \frac{a_3-x}{a_3-a_2} & if\ a_2 \le x \le a_3, \\ 0 & otherwise. \end{cases} \tag{4.1}$$

We compute the integral (FHDI) $\int_1^3\int_2^4 f(x,y)dydx$ numerically. We obtain

$$[f(x,y)]_-^r = 2r^2 + r(2x+y-1) + (y-2x+x^2)$$

and $[f(x,y)]^r_+ = 2r^2 - r(2x+y+7) + (x^2+2x+3y+6)$. We observe that

$$\begin{aligned}
&D(f(x_1,y_1), f(x_2,y_2))\\
&= \sup_{r\in[0,1]} \max\{|f^r_-(x_1,y_1) - f^r_-(x_2,y_2)|, |f^r_+(x_1,y_1) - f^r_+(x_2,y_2)|\}\\
&= \sup_{r\in[0,1]} \max\{r|2(x_1-x_2)+(y_1-y_2)| + |2(x_2-x_1)+(y_1-y_2)+x_1^2-x_2^2|,\\
&\qquad\qquad r|2(x_2-x_1)+(y_1-y_2)| + |2(x_1-x_2+3(y_1-y_2)+x_1^2-x_2^2)\}\\
&\leq \sup_{r\in[0,1]} \max\{|x_1-x_2|(2r+2+|x_1+x_2|, 2r+2+|x_1+x_2|\}\\
&\qquad + \sup_{r\in[0,1]} \max\{|y_1-y_2|(r+1, r+3)\} \leq 10|x_1-x_2| + 4|y_1-y_2|
\end{aligned}$$

i.e., f is a Lipschitz mapping with constants $L_1 = 10$ and $L_2 = 4$. For $r = 1$ we have $\underline{I}^1 = \bar{I}^1 = \int_1^3\int_2^4 (x^2+2y+1)dydx = 45.3333333$. Table 1 shows the results for different r and $m = 100$, $n = 40$. In the table, the notations $\underline{I}^{m,n}_r$ and $\overline{I}^{m,n}_r$ are the approximate values of r-cut for (FHDI) $\int_1^3\int_2^4 f(x,y)dydx$ obtained by the double Simpson's rule with $h = \frac{b-a}{m}$ and $k = \frac{d-c}{n}$ [7].

In this case, from (3.2) we have $U(h,k) = 0.8$.

r	$\underline{I}^{m,n}_r$	$\overline{I}^{m,n}_r$
1	45.3333333	45.3333333
0.9	41.4133333	49.4133333
0.8	37.6533333	53.6533333
0.7	34.0533333	58.0533333
0.6	30.6133333	62.6133333
0.5	27.3333333	67.3333333
0.4	24.2133333	72.2133333
0.3	21.2533333	77.2533333
0.2	18.4533333	82.4533333
0.1	15.8133333	87.8133333
0.0	13.3333333	93.3333333

Table 1. The results of example 4.1

Example 4.2. Consider $f : [1,2]\times[2,4] \to \mathbf{R}_F$, $f(x,y) = (\tilde{x}\otimes\tilde{x}\otimes\tilde{x}\otimes\tilde{x})\oplus(\tilde{y}\otimes\tilde{y})\oplus\tilde{2}$, with $\tilde{x} = (x-1, x, x+1)$, $\tilde{y} = (y-1, y, y+1)$ and $\tilde{2} = (1,2,3)$, where (a_1,a_2,a_3) denotes a triangular fuzzy number having the membership function mentioned in (4.1). We compute numerically the integral (FHDI) $\int_1^2\int_2^4 f(x,y)dxdy$. In this case,

$$\begin{aligned}
[f(x,y)]^r_- &= x^4 + (4r-4)x^3 + (6r^2-12r+6)x^2 + (4r^3-12r^2+12r-4)x\\
&\quad + (2r-2)y + y^2 + r^4 - 4r^3 + 7r^2 - 5r + 3
\end{aligned}$$

and

$$\begin{aligned}
[f(x,y)]^r_+ &= x^4 + (-4r+4)x^3 + (6r^2-12r+6)x^2 + (-4r^3\\
&\quad + 12r^2 - 12r + 4)x + (-2r+2)y + y^2 + r^4 - 4r^3 + 7r^2 - 7r + 5.
\end{aligned}$$

We observe that

$$
\begin{aligned}
D(f(x_1,y_1),f(x_2,y_2)) &= \sup_{r\in[0,1]} \max\{|f_-^r(x_1,y_1)-f_-^r(x_2,y_2)|,\\
&\qquad |f_+^r(x_1,y_1)-f_+^r(x_2,y_2)|\}\\
= \sup_{r\in[0,1]} \max\{&|(x_1-x_2)(x_1+x_2)(x_1^2+x_2^2)+(x_1-x_2)(x_1^2+x_1x_2+x_2^2)(4r-4)\\
&+(x_1-x_2)(x_1+x_2)(6r^2-12r+6)+(x_1-x_2)(4r^3-12r^2+12r-4)\\
&+(y_1-y_2)(y_1+y_2)+(y_1-y_2)(2r-2)|,|(x_1-x_2)(x_1+x_2)(x_1^2+x_2^2)\\
&+(x_1-x_2)(x_1^2+x_1x_2+x_2^2)(-4r+4)+(x_1-x_2)(x_1+x_2)(6r^2-12r+6)\\
&+(x_1-x_2)(-4r^3+12r^2-12r+4)+(y_1-y_2)(y_1+y_2)+(y_1-y_2)(-2r+2)|\}\\
\le \sup_{r\in[0,1]} \max\{&|(x_1-x_2)|(|(x_1+x_2)(x_1^2+x_2^2)+(x_1^2+x_1x_2+x_2^2)(4r-4)\\
&+(x_1+x_2)(6r^2-12r+6)+(4r^3-12r^2+12r-4)|,\\
&|(x_1+x_2)(x_1^2+x_2^2)+(x_1^2+x_1x_2+x_2^2)(-4r+4)+(x_1+x_2)(6r^2-12r+6)\\
&+(-4r^3+12r^2-12r+4)|\}\\
&+\sup_{r\in[0,1]} \max\{|y_1-y_2|(|(y_1+y_2)+(2r-2)|,|(y_1+y_2)+(-2r+2)|)\}\\
\le 256|x_1-x_2|&+12|y_1-y_2|
\end{aligned}
$$

i.e., f is a Lipschitz mapping with constants $L_1=256$ and $L_2=12$. For $r=1$ we have $\underline{I}^1=\bar{I}^1=\int_1^2\int_2^4(x^4+y^2+2)dydx=35.0666667$. Tables 2 and 3 show the results for different r.

In the tables, the notations $\underline{I}_r^{m,n}$ and $\overline{I}_r^{m,n}$ are the approximate values of r-cut for (FHDI) $\int_1^3\int_2^4 f(x,y)dydx$ obtained by the double Simpson's rule with $h=\frac{b-a}{m}$ and $k=\frac{d-c}{n}$. In table 2 we consider $m=200$, $n=80$ and then $U(h,k)=1.58$ and in table 3 we choose $m=300$, $n=180$ and hence $U(h,k)=0.9867$ which are computed using (3.2).

r	$\underline{I}_r^{m,n}$	$\overline{I}_r^{m,n}$
1	35.06666667317709	35.06666667317709
0.9	30.95486667317708	39.77886667317709
0.8	27.37386667317709	45.16586667317707
0.7	24.25886667317708	51.30686667317710
0.6	21.54986667317708	58.28586667317709
0.5	19.19166667317709	66.19166667317708
0.4	17.13386667317709	75.11786667317711
0.3	15.33086667317709	85.16286667317711
0.2	13.74186667317708	96.42986667317710
0.1	12.33086667317708	109.0268666731771
0.0	11.06666667317709	123.0666666731771

Table 2. The results of example 4.2

r	$\underline{I}_r^{m,n}$	$\overline{I}_r^{m,n}$
1	35.06666666692070	35.06666666692070
0.9	30.95486666692069	39.77886666692069
0.8	27.37386666692069	45.16586666692069
0.7	24.25886666692070	51.30686666692071
0.6	21.54986666692070	58.28586666692072
0.5	19.19166666692070	66.19166666692071
0.4	17.13386666692070	75.11786666692071
0.3	15.33086666692070	85.16286666692071
0.2	13.74186666692070	96.42986666692072
0.1	12.33086666692069	109.0268666669207
0.0	11.06666666692070	123.0666666669207

Table 3. The results of example 4.2

5. Conclusions

In this work, the fuzzy Henstock double integral was introduced and evaluated by applying double Simpson's rule. In this case, a theorem was proved to show the upper boundedness of the distance between the exact and approximate values. The examples illustrated the accuracy of the double Simpson's rule and the efficiency of the proposed method. In the sequel, one can use the Monte Carlo method [15] for the fuzzy Henstock double integral and compare the results of the methods with each other.

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