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Special hyper BCK-ideals and their fuzzifications

G. Muhiuddin, Abdullah M. Al-roqi

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Abstract. The notions of (weak, strong) special hyper BCK-ideals are introduced, and their relations are investigated. Relations between (weak, strong) special hyper BCK-ideals and (weak, strong) hyper BCK-ideals are also investigated. The fuzzification of (weak, strong) special hyper BCK-ideals is considered, and their relations are discussed. Characterizations of fuzzy (weak) special hyper BCK-ideals are established. Relations between fuzzy (weak, strong) special hyper BCK-ideals and fuzzy (weak, strong) hyper BCK-ideals are investigated.

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Corresponding Author: G. Muhiuddin (chishtygm@gmail.com)

1. Introduction

The hyperstructure theory (called also multialgebras) was introduced in 1934 by Marty [5] at the 8th congress of Scandinavian Mathematicians. Jun et al. [4] applied the hyperstructures to BCK-algebras, and introduced the concept of a hyper BCK-algebra which is a generalization of a BCK-algebra, and investigated several properties. In [2], Jun and Xin considered the fuzzification of a (weak, strong, reflexive) hyper BCK-ideal, and provided their relations.

In this paper, we introduce the notions of (weak, strong) special hyper BCK-ideals, and investigate their relations. We discuss relations between (weak, strong) special hyper BCK-ideals and (weak, strong) hyper BCK-ideals. We consider the fuzzification of (weak, strong) special hyper BCK-ideals and discuss their relations. We provide characterizations of fuzzy (weak) special hyper BCK-ideals, and deal with relations between fuzzy (weak, strong) special hyper BCK-ideals and fuzzy (weak, strong) hyper BCK-ideals.

2. Preliminaries

Let H be a nonempty set endowed with a hyper operation "$\circ$", that is, $\circ$ is a function from $H \times H$ to $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\varnothing\}$. For two subsets A and B of H, denote by $A \circ B$ the set $\cup\{a \circ b \mid a \in A,\, b \in B\}$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$, or $\{x\} \circ \{y\}$.

By a *hyper BCK-algebra* (see [4]) we mean a nonempty set H endowed with a hyper operation "$\circ$" and a constant 0 satisfying the following axioms:

(H1) $(x \circ z) \circ (y \circ z) \ll x \circ y$,
(H2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(H3) $x \circ H \ll \{x\}$,
(H4) $x \ll y$ and $y \ll x$ imply $x = y$,

for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall a \in A$, $\exists b \in B$ such that $a \ll b$.

In a hyper BCK-algebra H, the condition (H3) is equivalent to the condition:

(a1) $x \circ y \ll \{x\}$ for all $x, y \in H$.

In any hyper BCK-algebra H, the following hold (see [4]):

(a2) $x \circ 0 \ll \{x\}$, $0 \circ x \ll \{0\}$ and $0 \circ 0 \ll \{0\}$ for all $x, y \in H$,
(a3) $(A \circ B) \circ C = (A \circ C) \circ B$, $A \circ B \ll A$ and $0 \circ A \ll \{0\}$,
(a4) $0 \circ 0 = \{0\}$,
(a5) $0 \ll x$,
(a6) $x \ll x$,
(a7) $A \ll A$,
(a8) $A \subseteq B$ implies $A \ll B$,
(a9) $0 \circ x = \{0\}$,
(a10) $0 \circ A = \{0\}$,
(a11) $A \ll \{0\}$ implies $A = \{0\}$,
(a12) $A \circ B \ll A$,
(a13) $x \in x \circ 0$,
(a14) $x \circ 0 \ll \{y\}$ implies $x \ll y$,
(a15) $y \ll z$ implies $x \circ z \ll x \circ y$,
(a16) $x \circ y = \{0\}$ implies $(x \circ z) \circ (y \circ z) = \{0\}$ and $x \circ z \ll y \circ z$,
(a17) $A \circ \{0\} = \{0\}$ implies $A = \{0\}$,

for all $x, y, z \in H$ and for all non-empty subsets A, B and C of H.

A non-empty subset A of a hyper BCK-algebra H is called a *hyper BCK-ideal* of H (see [4]) if it satisfies

$$0 \in A, \tag{2.1}$$

$$(\forall x, y \in H)\,(x \circ y \ll A,\ y \in A \ \Rightarrow\ x \in A)\,. \tag{2.2}$$

A non-empty subset A of a hyper BCK-algebra H is called a *strong hyper BCK-ideal* of H (see [3]) if it satisfies (2.1) and

$$(\forall x, y \in H)\,((x \circ y) \cap A \neq \varnothing,\ y \in A \ \Rightarrow\ x \in A)\,. \tag{2.3}$$

A hyper BCK-ideal A is said to be *reflexive* (see [3]) if $x \circ x \subseteq A$ for all $x \in H$.

A non-empty subset A of a hyper BCK-algebra H is called a *weak hyper BCK-ideal* of H (see [4]) if it satisfies (2.1) and

$$(\forall x, y \in H)\,(x \circ y \subseteq A,\ y \in A \ \Rightarrow\ x \in A)\,. \tag{2.4}$$

Definition 2.1 ([2])**.** A fuzzy set μ in H is called a *fuzzy hyper BCK-ideal* of H if

$$(\forall x, y \in H)\,(x \ll y \ \Rightarrow\ \mu(x) \geq \mu(y))\,, \tag{2.5}$$

$$(\forall x, y \in H)\left(\mu(x) \geq \min\left\{\inf_{a \in x \circ y} \mu(a), \mu(y)\right\}\right). \tag{2.6}$$

Definition 2.2 ([2])**.** A fuzzy set μ in H is called a *fuzzy strong hyper BCK-ideal* of H if

$$(\forall x, y \in H)\left(\inf_{a \in x \circ x} \mu(a) \geq \mu(x) \geq \min\left\{\sup_{b \in x \circ y} \mu(b), \mu(y)\right\}\right). \tag{2.7}$$

Definition 2.3 ([2])**.** A fuzzy set μ in H is called a *fuzzy weak hyper BCK-ideal* of H if

$$(\forall x, y \in H)\left(\mu(0) \geq \mu(x) \geq \min\left\{\inf_{a \in x \circ y} \mu(a), \mu(y)\right\}\right). \tag{2.8}$$

3. Special hyper BCK-ideals

In what follows, let H denote a hyper BCK-algebra unless otherwise specified.

Every hyper BCK-algebra H satisfies the identity:

$$(\forall x, y \in H)\,(((x \circ 0) \circ 0) \circ y = x \circ y)\,. \tag{3.1}$$

Now, we consider more general form $((x \circ y) \circ y) \circ z$ than the given form (3.1). For any subset A of H, we take three conditions:

$$(\forall x, y \in H)\,(\forall z \in A)\,((((x \circ y) \circ y) \circ z) \cap A \neq \varnothing)\,, \tag{3.2}$$

$$(\forall x, y \in H)\,(\forall z \in A)\,(((x \circ y) \circ y) \circ z \subseteq A)\,, \tag{3.3}$$

$$(\forall x, y \in H)\,(\forall z \in A)\,(((x \circ y) \circ y) \circ z \ll A)\,. \tag{3.4}$$

Definition 3.1. Let A be a subset of H containing 0. Then A is called a *special hyper BCK-ideal* of H if $x \circ (y \circ (y \circ x)) \subseteq A$ for all $x, y \in H$ for which the condition (3.4) is valid.

Example 3.2. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" which is given by Table 1.

Table 1. Cayley table for the hyper operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

Then the set $A = \{0, a\}$ is a special hyper BCK-ideal of H by the calculations in Tables 2 and 3.

Table 2. True (T) or False (F) of $((x \circ y) \circ y) \circ z \ll A$

x	y	z	$x \circ y$	$(x \circ y) \circ y$	$((x \circ y) \circ y) \circ z$	$((x \circ y) \circ y) \circ z \ll A$
0	0	0	$\{0\}$	$\{0\}$	$\{0\}$	T
0	0	a	$\{0\}$	$\{0\}$	$\{0\}$	T
0	a	0	$\{0\}$	$\{0\}$	$\{0\}$	T
0	a	a	$\{0\}$	$\{0\}$	$\{0\}$	T
0	b	0	$\{0\}$	$\{0\}$	$\{0\}$	T
0	b	a	$\{0\}$	$\{0\}$	$\{0\}$	T
a	0	0	$\{a\}$	$\{a\}$	$\{a\}$	T
a	0	a	$\{a\}$	$\{a\}$	$\{0\}$	T
a	a	0	$\{0\}$	$\{0\}$	$\{0\}$	T
a	a	a	$\{0\}$	$\{0\}$	$\{0\}$	T
a	b	0	$\{0\}$	$\{0\}$	$\{0\}$	T
a	b	a	$\{0\}$	$\{0\}$	$\{0\}$	T
b	0	0	$\{b\}$	$\{b\}$	$\{b\}$	F
b	0	a	$\{b\}$	$\{b\}$	$\{a, b\}$	F
b	a	0	$\{a, b\}$	$\{0, a, b\}$	$\{0, a, b\}$	F
b	a	a	$\{a, b\}$	$\{0, a, b\}$	$\{0, a, b\}$	F
b	b	0	$\{0, a, b\}$	$\{0, a, b\}$	$\{0, a, b\}$	F
b	b	a	$\{0, a, b\}$	$\{0, a, b\}$	$\{0, a, b\}$	F

Table 3. True (T) or False (F) of $x \circ (y \circ (y \circ x)) \subseteq A$

x	y	$y \circ x$	$y \circ (y \circ x)$	$x \circ (y \circ (y \circ x))$	$x \circ (y \circ (y \circ x)) \subseteq A$
0	0	$\{0\}$	$\{0\}$	$\{0\}$	T
0	a	$\{a\}$	$\{0\}$	$\{0\}$	T
0	b	$\{b\}$	$\{0, a, b\}$	$\{0\}$	T
a	0	$\{0\}$	$\{0\}$	$\{a\}$	T
a	a	$\{0\}$	$\{a\}$	$\{0\}$	T
a	b	$\{a, b\}$	$\{0, a, b\}$	$\{0, a\}$	T
b	0	$\{0\}$	$\{0\}$	$\{b\}$	F
b	a	$\{0\}$	$\{a\}$	$\{a, b\}$	F
b	b	$\{0, a, b\}$	$\{0, a, b\}$	$\{0, a, b\}$	F

In (3.4), if we take $y = 0$ then $x \circ z = ((x \circ 0) \circ 0) \circ z \ll A$ for all $x \in H$ and $z \in A$. Hence every special hyper BCK-ideal is a hyper BCK-ideal.

The following example shows that any hyper BCK-ideal may not be a special hyper BCK-ideal.

Example 3.3. Consider a hyper BCK-algebra $H = \{0, a, b, c\}$ with the hyper operation "$\circ$" which is given by Table 4.

Table 4. Cayley table for the hyper operation "$\circ$"

$\circ$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,a\}$	$\{0\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{0,a\}$

We know that the set $A = \{0, a\}$ is a hyper BCK-ideal of H. But it is not a special hyper BCK-ideal of H since if we take $x = b$, $y = c$ and $z = a$, then the condition (3.4) is valid, but $b \circ (c \circ (c \circ b)) = \{b\} \nsubseteq A$.

Definition 3.4. Let A be a subset of H containing 0. Then A is called a *weak-special hyper BCK-ideal* of H if $x \circ (y \circ (y \circ x)) \subseteq A$ for all $x, y \in H$ for which the condition (3.3) is valid.

Example 3.5. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" which is given by Table 5.

Table 5. Cayley table for the hyper operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$

Then the set $A = \{0, b\}$ is a weak-special hyper BCK-ideal of H.

Theorem 3.6. *Every weak-special hyper BCK-ideal is a weak hyper BCK-ideal.*

Proof. The proof is straightforward. □

The set $A = \{0, a\}$ in Example 3.3 is a weak hyper BCK-ideal of H. If we take $x = b$, $y = c$ and $z = a$, then the condition (3.3) is valid. But we have $b \circ (c \circ (c \circ b)) = \{b\} \nsubseteq A$. Therefore A is not a weak-special hyper BCK-ideal of H. This shows that the converse of Theorem 3.6 is not true in general.

Definition 3.7. Let A be a subset of H containing 0. Then A is called a *strong-special hyper BCK-ideal* of H if $x \circ (y \circ (y \circ x)) \subseteq A$ for all $x, y \in H$ for which the condition (3.2) is valid.

Example 3.8. Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" which is given by Table 6.
Then the set $A = \{0, a\}$ is a strong-special hyper BCK-ideal of H by the calculations in Tables 7 and 8. By the similar calculation, we know that the set $B = \{0, b\}$ is a strong-special hyper BCK-ideal of H.

Theorem 3.9. *Every strong-special hyper BCK-ideal is a strong hyper BCK-ideal.*

Proof. The proof is straightforward. □

Table 6. Cayley table for the hyper operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

Table 7. True (T) or False (F) of $((x \circ y) \circ y) \circ z \cap A \neq \varnothing$

x	y	z	$x \circ y$	$(x \circ y) \circ y$	$((x \circ y) \circ y) \circ z$	$((x \circ y) \circ y) \circ z \cap A \neq \varnothing$
0	0	0	$\{0\}$	$\{0\}$	$\{0\}$	T
0	0	a	$\{0\}$	$\{0\}$	$\{0\}$	T
0	a	0	$\{0\}$	$\{0\}$	$\{0\}$	T
0	a	a	$\{0\}$	$\{0\}$	$\{0\}$	T
0	b	0	$\{0\}$	$\{0\}$	$\{0\}$	T
0	b	a	$\{0\}$	$\{0\}$	$\{0\}$	T
a	0	0	$\{a\}$	$\{a\}$	$\{a\}$	T
a	0	a	$\{a\}$	$\{a\}$	$\{0\}$	T
a	a	0	$\{0\}$	$\{0\}$	$\{0\}$	T
a	a	a	$\{0\}$	$\{0\}$	$\{0\}$	T
a	b	0	$\{a\}$	$\{0\}$	$\{0\}$	T
a	b	a	$\{a\}$	$\{0\}$	$\{0\}$	T
b	0	0	$\{b\}$	$\{b\}$	$\{b\}$	F
b	0	a	$\{b\}$	$\{b\}$	$\{b\}$	F
b	a	0	$\{b\}$	$\{b\}$	$\{b\}$	F
b	a	a	$\{b\}$	$\{b\}$	$\{b\}$	F
b	b	0	$\{0, b\}$	$\{0, b\}$	$\{b\}$	F
b	b	a	$\{0, b\}$	$\{0, b\}$	$\{b\}$	F

Table 8. True (T) or False (F) of $x \circ (y \circ (y \circ x)) \subseteq A$

x	y	$y \circ x$	$y \circ (y \circ x)$	$x \circ (y \circ (y \circ x))$	$x \circ (y \circ (y \circ x)) \subseteq A$
0	0	$\{0\}$	$\{0\}$	$\{0\}$	T
0	a	$\{a\}$	$\{0\}$	$\{0\}$	T
0	b	$\{b\}$	$\{0, b\}$	$\{0\}$	T
a	0	$\{0\}$	$\{0\}$	$\{a\}$	T
a	a	$\{0\}$	$\{a\}$	$\{0\}$	T
a	b	$\{b\}$	$\{0, b\}$	$\{a\}$	T
b	0	$\{0\}$	$\{0\}$	$\{b\}$	F
b	a	$\{a\}$	$\{0\}$	$\{b\}$	F
b	b	$\{0, b\}$	$\{0, b\}$	$\{0, b\}$	F

Since every strong hyper BCK-ideal is a (weak) hyper BCK-ideal (see [3, Theorem 3.8]), we have the following corollary.

Corollary 3.10. *Every strong-special hyper BCK-ideal is a (weak) hyper BCK-ideal.*

The set $A = \{0, a\}$ in Example 3.3 is a strong hyper BCK-ideal of H. If we take $x = b$, $y = c$ and $z = a$, then the condition (3.2) is valid. But we have $b \circ (c \circ (c \circ b)) = \{b\} \nsubseteq A$. Therefore A is not a strong-special hyper BCK-ideal of H. This shows that the converse of Theorem 3.9 is not true in general.

Using (a8), we know that every special hyper BCK-ideal is a weak-special hyper BCK-ideal, but the converse is not true. In fact, the weak-special hyper BCK-ideal $A = \{0, b\}$ in Example 3.5 is not a special hyper BCK-ideal of H since the condition (3.4) is valid for $x = a$, $y = 0$ and $z = b$, but $a \circ (0 \circ (0 \circ a)) = \{a\} \nsubseteq A$.

Lemma 3.11. *Let A be a subset of H containing 0. For any subset B of H with $B \ll A$, the following is valid:*

$$b \in B \;\Rightarrow\; \exists\, a \in A \text{ such that } (b \circ a) \cap A \neq \varnothing.$$

Proof. The proof is straightward. □

Theorem 3.12. *Every strong-special hyper BCK-ideal is a special hyper BCK-ideal.*

Proof. Let A be a strong-special hyper BCK-ideal of H. Suppose that the condition (3.4) is valid. For any $x, y \in H$ and $z \in A$, let $b \in ((x \circ y) \circ y) \circ z$. Then there exists $a \in A$ such that $(b \circ a) \cap A \neq \varnothing$ by Lemma 3.11. Since A is a strong hyper BCK-ideal of H (see Theorem 3.9), it follows from (2.3) that $b \in A$. Hence $((x \circ y) \circ y) \circ z \subseteq A$, and so $((x \circ y) \circ y) \circ z \cap A \neq \varnothing$, that is, (3.2) is valid. Therefore $x \circ (y \circ (y \circ x)) \subseteq A$, and thus A is a special hyper BCK-ideal of H. □

In Example 3.2, the special hyper BCK-ideal $A = \{0, a\}$ is not a strong-special hyper BCK-ideal of H since the condition (3.2) is valid for $x = b$, $y = 0$ and $z = a$, but $b \circ (0 \circ (0 \circ b)) = \{b\} \nsubseteq A$.

4. Fuzzifications of special hyper BCK-ideals

Definition 4.1. A fuzzy set μ in H is called a *fuzzy special hyper BCK-ideal* of H if it satisfies the condition (2.5) and

$$a \in x \circ (y \circ (y \circ x)) \;\Rightarrow\; \mu(a) \geq \min\left\{ \inf_{b \in ((x \circ y) \circ y) \circ z} \mu(b), \mu(z) \right\} \tag{4.1}$$

for all $x, y, z \in H$.

Example 4.2. (1) Consider the hyper BCK-algebra $H = \{0, a, b\}$ which is given in Example 3.2. Let μ be a fuzzy set in H given by $\mu(0) = \mu(a) = 0.7$ and $\mu(b) = 0.2$. It is routine to verify that μ is a fuzzy special hyper BCK-ideal of H.

(2) Consider a hyper BCK-algebra $H = \{0, a, b\}$ with the hyper operation "$\circ$" which is given by Table 9.
A fuzzy set μ in H defined by $\mu(0) = 0.8$, $\mu(a) = 0.6$ and $\mu(b) = 0.4$ is a fuzzy special hyper BCK-ideal of H.

Theorem 4.3. *Every fuzzy special hyper BCK-ideal is a fuzzy hyper BCK-ideal.*

Table 9. Cayley table for the hyper operation "$\circ$"

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a, b\}$

Proof. Let μ be a fuzzy special hyper BCK-ideal of H. Let $x, y, z \in H$. Using (a13) and (a9), we know that $x \in x \circ 0 = x \circ (0 \circ (0 \circ x))$ for all $x \in H$. It follows from (4.1) that

$$\mu(x) \geq \min \left\{ \inf_{b \in ((x \circ 0) \circ 0) \circ z} \mu(b), \mu(z) \right\} = \min \left\{ \inf_{b \in x \circ z} \mu(b), \mu(z) \right\} \tag{4.2}$$

for all $x, z \in H$. Therefore μ is a fuzzy hyper BCK-ideal of H. □

The following example shows that the converse of Theorem 4.3 is not true in general.

Example 4.4. Let $H = \{0, a, b, c\}$ be a hyper BCK-algebra which is given in Example 3.3. Let μ be a fuzzy set in H defined by $\mu(0) = \mu(a) = 0.9$ and $\mu(b) = \mu(c) = 0.2$. Then μ is a fuzzy hyper BCK-ideal of H. But it is not a fuzzy special hyper BCK-ideal of H since $b \in b \circ (c \circ (c \circ b))$ and

$$\mu(b) = 0.2 < 0.9 = \min \left\{ \inf_{w \in ((b \circ c) \circ c) \circ a} \mu(w), \mu(a) \right\}.$$

Lemma 4.5 ([2])**.** *Every fuzzy hyper BCK-ideal μ of H satisfies the following inequality:*

$$(\forall x \in H)\,(\mu(0) \geq \mu(x)). \tag{4.3}$$

Lemma 4.6 ([1])**.** *Let K be a subset of H. If A is a hyper BCK-ideal of H such that $K \ll A$, then K is contained A.*

Theorem 4.7. *Let μ be a fuzzy set in H. Then μ is a fuzzy special hyper BCK-ideal of H if and only if the set*

$$\mu_t := \{x \in H \mid \mu(x) \geq t\}$$

is a special hyper BCK-ideal of H for all $t \in (0, 1]$ with $\mu_t \neq \varnothing$.

Proof. Assume that μ is a fuzzy special hyper BCK-ideal of H and let $t \in (0, 1]$ be such that $\mu_t \neq \varnothing$. Then there exists $a \in \mu_t$ and hence $\mu(a) \geq t$. By Lemma 4.5, $\mu(0) \geq \mu(a) \geq t$ and so $0 \in \mu_t$. Let $x, y, z \in H$ be such that $((x \circ y) \circ y) \circ z \ll \mu_t$ and $z \in \mu_t$. Then $\mu(z) \geq t$, and

$$(\forall a \in ((x \circ y) \circ y) \circ z)\,(\exists\, a_0 \in \mu_t))\,(a \ll a_0).$$

It follows from (2.5) that $\mu(a) \geq \mu(a_0) \geq t$ for all $a \in ((x \circ y) \circ y) \circ z$ and so that $\inf_{b \in ((x \circ y) \circ y) \circ z} \mu(b) \geq t$. Thus if $w \in x \circ (y \circ (y \circ x))$ then

$$\mu(w) \geq \min \left\{ \inf_{b \in ((x \circ y) \circ y) \circ z} \mu(b), \mu(z) \right\} \geq t$$

by (4.1). Hence $w \in \mu_t$, and so $x \circ (y \circ (y \circ x)) \subseteq \mu_t$. Consequently, μ_t is a special hyper BCK-ideal of H for all $t \in (0, 1]$ with $\mu_t \neq \varnothing$.

Conversely, suppose that the set μ_t is a special hyper BCK-ideal of H for all $t \in (0, 1]$ with $\mu_t \neq \varnothing$. Then μ_t is a hyper BCK-ideal of H. Let $x, y \in H$ be such that $x \ll y$ and $\mu(y) = t$. Then $y \in \mu_t$, and so $x \ll \mu_t$. It follows from Lemma 4.6 that $x \in \mu_t$, that is, $\mu(x) \geq t = \mu(y)$. For every $x, y, z \in H$, put $t = \min\left\{\inf_{b \in ((x \circ y) \circ y) \circ z} \mu(b), \mu(z)\right\}$. Then $z \in \mu_t$ and for each $a \in ((x \circ y) \circ y) \circ z$ we have

$$\mu(a) \geq \inf_{b \in ((x \circ y) \circ y) \circ z} \mu(b) \geq \min\left\{\inf_{b \in ((x \circ y) \circ y) \circ z} \mu(b), \mu(z)\right\} = t.$$

Hence $a \in \mu_t$, and so $((x \circ y) \circ y) \circ z \subseteq \mu_t$. Using (a8) induces $((x \circ y) \circ y) \circ z \ll \mu_t$. Combining $z \in \mu_t$ and μ_t being a special hyper BCK-ideal of H, we get

$$x \circ (y \circ (y \circ x)) \subseteq \mu_t.$$

Thus if $a \in x \circ (y \circ (y \circ x))$, then $a \in \mu_t$ and so

$$\mu(a) \geq t = \min\left\{\inf_{b \in ((x \circ y) \circ y) \circ z} \mu(b), \mu(z)\right\}.$$

Hence μ is a fuzzy special hyper BCK-ideal of H. □

Definition 4.8. A fuzzy set μ in H is called a *fuzzy weak-special hyper BCK-ideal* of H if it satisfies the conditions (4.1) and

$$(4.4) \qquad (\forall x \in H)\,(\mu(0) \geq \mu(x)).$$

Example 4.9. Consider the hyper BCK-algebra $H = \{0, a, b\}$ in Example 3.5. Let μ be a fuzzy set in H defined by $\mu(0) = \mu(b) = 0.7$ and $\mu(a) = 0.5$. It is routine to verify that μ is a fuzzy weak-special hyper BCK-ideal of H.

Let μ be a fuzzy special hyper BCK-ideal of H. Then μ is a fuzzy hyper BCK-ideal of H by Theorem 4.3, and so it is a fuzzy weak hyper BCK-ideal of H (see [2, Corollary 3.11(i)]). Hence the condition (4.4) is valid. Therefore we have the following theorem.

Theorem 4.10. *Every fuzzy special hyper BCK-ideal is a fuzzy weak-special hyper BCK-ideal.*

The fuzzy weak-special hyper BCK-ideal μ of H in Example 4.9 is not a fuzzy special hyper BCK-ideal of H since $a \ll b$ and $\mu(a) = 0.5 < 0.7 = \mu(b)$, that is, the condition (4.4) is not valid. Therefore the converse of Theorem 4.10 is not true in general.

Theorem 4.11. *Let μ be a fuzzy set in H. Then μ is a fuzzy weak-special hyper BCK-ideal of H if and only if the set*

$$\mu_t := \{x \in H \mid \mu(x) \geq t\}$$

is a weak-special hyper BCK-ideal of H for all $t \in (0, 1]$ with $\mu_t \neq \varnothing$.

Proof. The proof is similar to the proof of Theorem 4.7. □

Definition 4.12. A fuzzy set μ in H is called a *fuzzy strong-special hyper BCK-ideal* of H if it satisfies the following conditions:

(4.5) $(\forall x \in H)\left(\inf_{a\in x\circ x}\mu(a) \geq \mu(x)\right)$,

(4.6) $(\forall x, y, z \in H)\left(a \in x \circ (y \circ (y \circ x)) \Rightarrow \mu(a) \geq \min\left\{\sup_{b\in((x\circ y)\circ y)\circ z}\mu(b), \mu(z)\right\}\right)$.

Lemma 4.13. *Every fuzzy strong-special hyper BCK-ideal is a fuzzy strong hyper BCK-ideal.*

Proof. Let μ be a fuzzy strong-special hyper BCK-ideal of H. Let $x, y, z \in H$. Using (a13) and (a9), we know that $x \in x \circ 0 = x \circ (0 \circ (0 \circ x))$ for all $x \in H$. It follows from (4.6) that

$$\mu(x) \geq \min\left\{\sup_{b\in((x\circ 0)\circ 0)\circ z}\mu(b), \mu(z)\right\} = \min\left\{\sup_{b\in x\circ z}\mu(b), \mu(z)\right\} \tag{4.7}$$

for all $x, z \in H$. Therefore μ is a fuzzy strong hyper BCK-ideal of H. □

Theorem 4.14. *Every fuzzy strong-special hyper BCK-ideal is a fuzzy special hyper BCK-ideal.*

Proof. Let μ be a fuzzy strong-special hyper BCK-ideal of H. Then μ is a fuzzy strong hyper BCK-ideal of H, and so it is a fuzzy hyper BCK-ideal of H by [2, Corollary 3.9]. Thus the condition (2.5) is true. Now let $a \in x \circ (y \circ (y \circ x)$ for all $x, y \in H$. Using (4.6), we have

$$\begin{aligned}\mu(a) &\geq \min\left\{\sup_{b\in((x\circ y)\circ y)\circ z}\mu(b), \mu(z)\right\}\\ &\geq \min\left\{\inf_{b\in((x\circ y)\circ y)\circ z}\mu(b), \mu(z)\right\}\end{aligned}$$

for all $x, y, z \in H$. Therefore μ is a fuzzy special hyper BCK-ideal of H. □

In Example 4.2(2), the fuzzy special hyper BCK-ideal μ is not a fuzzy strong-special hyper BCK-ideal of H since $b \circ (a \circ (a \circ b)) = \{a, b\}$ and

$$\mu(b) = 0.4 < 0.8 = \min\left\{\sup_{w\in((b\circ a)\circ a)\circ 0}\mu(w), \mu(0)\right\}.$$

This shows that the converse of Theorem 4.14 is not true in general.

Theorem 4.15. *If μ is a fuzzy strong-special hyper BCK-ideal of H, then the set*

$$\mu_t := \{x \in H \mid \mu(x) \geq t\}$$

is a strong-special hyper BCK-ideal of H for all $t \in (0, 1]$ with $\mu_t \neq \varnothing$.

Proof. Assume that μ is a fuzzy strong-special hyper BCK-ideal of H. Then μ is a fuzzy strong hyper BCK-ideal of H by Lemma 4.13. Let $t \in (0, 1]$ be such that $\mu_t \neq \varnothing$. Then $\mu(a) \geq t$ for some $a \in H$. It follows from [2, Proposition 3.7(i)] that $\mu(0) \geq \mu(a) \geq t$ and so that $0 \in \mu_t$. Let $x, y, z \in H$ be such that $(((x\circ y)\circ y)\circ z)\cap\mu_t \neq$

$\varnothing$ and $z \in \mu_t$. Then there exists $a_0 \in (((x \circ y) \circ y) \circ z) \cap \mu_t$, and so $\mu(a_0) \geq t$. Let $w \in x \circ (y \circ (y \circ x))$. Then

$$\begin{aligned}\mu(w) &\geq \min\left\{\sup_{b\in((x\circ y)\circ y)\circ z}\mu(b), \mu(z)\right\}\\ &\geq \min\{\mu(a_0), \mu(z)\} \geq t,\end{aligned}$$

and so $w \in \mu_t$. Thus $x \circ (y \circ (y \circ x)) \subseteq \mu_t$. Therefore μ_t is a strong-special hyper BCK-ideal of H for all $t \in (0, 1]$ with $\mu_t \neq \varnothing$. □

Theorem 4.16. *Let μ be a fuzzy set in H satisfying the sup-property, that is, for every subset T of H there exists $x_0 \in T$ such that $\mu(x_0) = \sup_{x\in T}\mu(x)$. If the set*

$$\mu_t := \{x \in X \mid \mu(x) \geq t\}$$

is a strong-special hyper BCK-ideal of H for all $t \in (0, 1]$ with $\mu_t \neq \varnothing$, then μ is a fuzzy strong-special hyper BCK-ideal of H.

Proof. Let $t \in (0, 1]$ be such that $\mu_t \neq \varnothing$ and μ_t is a strong-special hyper BCK-ideal of H. Then there exists $x \in \mu_t$. Since $x \circ x \ll x$, it follows that $x \circ x \ll \mu_t$ and so from Lemma 4.6 that $x \circ x \subseteq \mu_t$. Thus for each $a \in x \circ x$, we have $a \in \mu_t$ and hence $\mu(a) \geq t$. Thus $\inf_{a\in x\circ x}\mu(a) \geq t = \mu(x)$. For any $x, y, z \in H$, let

$$k = \min\left\{\sup_{b\in((x\circ y)\circ y)\circ z}\mu(b), \mu(z)\right\}.$$

Then μ_k is a strong-special hyper BCK-ideal of H by hypothesis, and $z \in \mu_k$. Since μ satisfies the sup-property, there exists $a_0 \in ((x \circ y) \circ y) \circ z$ such that $\mu(a_0) = \sup_{b\in((x\circ y)\circ y)\circ z}\mu(b)$. Hence

$$\begin{aligned}\mu(a_0) &= \sup_{b\in((x\circ y)\circ y)\circ z}\mu(b)\\ &\geq \min\left\{\sup_{b\in((x\circ y)\circ y)\circ z}\mu(b), \mu(z)\right\}\\ &= k,\end{aligned}$$

and so $a_0 \in \mu_k$. This shows that $(((x \circ y) \circ y) \circ z) \cap \mu_k \neq \varnothing$. Thus $x \circ (y \circ (y \circ x)) \subseteq \mu_k$. If $w \in x \circ (y \circ (y \circ x))$, then $w \in \mu_k$ and hence

$$\mu(w) \geq k = \min\left\{\sup_{b\in((x\circ y)\circ y)\circ z}\mu(b), \mu(z)\right\}.$$

Therefore μ is a fuzzy strong-special hyper BCK-ideal of H. □

Corollary 4.17. *Assume that H is finite. If the set*

$$\mu_t := \{x \in X \mid \mu(x) \geq t\}$$

is a strong-special hyper BCK-ideal of H for all $t \in (0, 1]$ with $\mu_t \neq \varnothing$, then μ is a fuzzy strong-special hyper BCK-ideal of H.

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G. Muhiuddin (chishtygm@gmail.com)
Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia

Abdullah M. Al-roqi (alroqi10@yahoo.com)
Department of Mathematics, King Abdulaziz University, Jeddah 21589, Saudi Arabia