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Some properties of soft topologies and group soft topologies

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Abstract. This paper aims to study some properties of soft neighbourhood system of identity soft element of group soft topology. Also some results on soft topologies and enriched soft topologies are established using soft elements. For this we define finite soft set, pseudo constant soft set, cofinite soft topology etc. and study their properties.

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1. Introduction

After the introduction of soft set theory in 1999 by Molodtsov [17] for dealing with uncertainties inherent in the problems of physical science, biological science, engineering, economics, social science, medical science etc., many researchers have worked in the different fields of mathematics such as algebra, analysis, topology etc. in this setting. In 2003, Maji et al. [13] worked on some mathematical aspects of soft sets, and, in 2007, Aktas et al. [1] introduced a basic version of soft group theory. Feng [5] in 2008, dealt with the concept of soft semirings; Shabir et al. (2009) [22] studied soft semigroups and soft ideals; Kharal et al. [12] as well as Majumdar et al. [15] defined soft mappings, Babitha et al. [3] worked on soft relations and functions. Recently Feng et al. [8] ascertained the relationships among five different types of soft subsets and considered the free soft algebras associated with soft product operations. In [8] it has been shown that soft sets have some non-classical algebraic properties which are distinct from those of crisp sets or fuzzy sets. It is to be noted that soft set theory has applications in different real life problems including decision making problems [14, 24, 6, 7, 16].

In 2011, Shabir et al. [23] came up with an idea of soft topological spaces. Later Aygun et al. [2], Zorlutuna et al. [25], Cagman et al. [4], Hussain et al. [11] and Hazra et al. [9] studied on soft topological spaces. As a continuation of this, it is natural to investigate the behaviour of a combination of algebraic and topological structures in soft-set theoretic form. In view of this, and also considering the importance of topological groups in developing Haar measure and Haar integral [10], our aim is to extend the theory of topological groups to soft setting. In this direction we have studied some continuity and separation properties of soft topologies, enriched soft topologies and the neighbourhood properties of group soft topologies which was introduced by us in [18] using the concept of soft elements. For this, we define finite soft set, pseudo constant soft set, cofinite soft topology etc. and investigate their properties. In this connection, it is also worth noting that, in soft setting, some significant works have been done on topological groups in [19, 20] approaching from different perspectives than that of the present paper. The organization of the paper is as follows:

Section 2 is the preliminary part where definitions and some properties of soft sets and soft topologies are given and features of soft topologies are studied. In section 3, we study some continuity properties of soft topological spaces and enriched soft topological spaces. In section 4, some basic facets of the system of neighbourhood of the identity soft elements of group soft topology are established. Section 5, concludes the paper. For the sake of the economy of space, few straightforward proofs are omitted throughout this paper.

2. Preliminaries

This section consists of some definitions and results which will be used in the rest of the paper.

Definition 2.1 ([13])**.** Let U be the initial universal set, $P(U)$ be the power set of U and E be the set of parameters. A pair (F, A) is called a $soft\ set$ over U, where F is a mapping given by $F : A \to P(U)$ and $A \subseteq E$.

Let (F, A) and (G, B) be two soft sets over U. Then their union is a soft set (H, C) over U where $C = A \cup B$ and $\forall\ \alpha \in C$,

$$H(\alpha) = \begin{cases} F(\alpha) & \text{if } \alpha \in A/B \\ G(\alpha) & \text{if } \alpha \in B/A \\ F(\alpha) \cup G(\alpha) & \text{if } \alpha \in A \cap B \end{cases}$$

This relationship is written as $(F, A)\ \widetilde{\cup}\ (G, B) = (H, C)$.

Let (F, A) and (G, B) be two soft sets over U. Then their intersection is a soft set (H, C) over U where $C = A \cap B$ and $\forall\ \alpha \in C,\ H(\alpha) = F(\alpha) \cap G(\alpha)$.

This relationship is written as $(F, A)\ \widetilde{\cap}\ (G, B) = (H, C)$.

In order to simplify the fundamental operations on soft sets in [13], redefined the soft sets as follows:

Let E be the set of parameters and $A \subseteq E$. Then for each soft set (F, A) over U a soft set (H, E) is constructed over U, where $\forall\ \alpha \in E$,

$$H(\alpha) = \begin{cases} F(\alpha) & \text{if } \alpha \in A \\ \phi & \text{if } \alpha \in E/A \end{cases}$$

Thus the soft sets (F, A) and (H, E) are equivalent to each other and the usual set operations of the soft sets $(F_i, A_i),\ i \in \Delta$ is the same as those of the soft sets $(H_i, E),\ i \in \Delta$. For this reason, in this paper, we have considered our soft sets over the same parameter set A.

Following Molodtsov [17] and Maji et al. [13] some definitions and preliminary results of soft sets are presented in this section in our form. Unless otherwise stated, U will be assumed to be an initial universal set and A will be taken to be a set of parameters. Let $P(U)$ denote the power set of U and $S(U, A)$ denote the set of all soft sets over U. In particular, if U is a group then $P(U)$ will denote the set of all subgroups of U.

A pair (F, A), where F is a mapping from A to $P(U)$, is called a *soft set* over U. Let (F_1, A) and (F_2, A) be two soft sets over a common universe U. Then (F_1, A) is said to be a *soft subset* of (F_2, A) if $F_1(\alpha) \subseteq F_2(\alpha),\ \forall\ \alpha \in A$. This relation is denoted by $(F_1, A)\widetilde{\subseteq}(F_2, A)$. (F_1, A) is said to be *soft equal* to (F_2, A) if $F_1(\alpha) = F_2(\alpha),\ \forall\ \alpha \in A$. It is denoted by $(F_1, A) = (F_2, A)$.

The *complement* of a soft set (F, A) is defined as $(F, A)^c = (F^c, A)$, where $F^c(\alpha) = (F(\alpha))^c = U - F(\alpha),\ \forall\ \alpha \in A$.

A soft set (F, A) over U is said to be *a null soft set* $\Big($*an absolute soft set*$\Big)$ if $F(\alpha) = \phi\ \Big(F(\alpha) = U\Big),\ \forall\ \alpha \in A$. This is denoted by $\widetilde{\Phi}\Big(\widetilde{A}\Big)$.

Definition 2.2. Let $\{(F_i, A); i \in \Delta\}$ be a nonempty family of soft sets over a common universe U. Then their

(i) *Intersection*, denoted by $\widetilde{\cap}_{i\in\Delta}$, is defined by $\widetilde{\cap}_{i\in\Delta}(F_i, A) = (\widetilde{\cap}_{i\in\Delta}F_i, A)$, where $(\widetilde{\cap}_{i\in\Delta}F_i)(\alpha) = \cap_{i\in\Delta}(F_i(\alpha)),\ \forall\ \alpha \in A$.

(ii) *Union*, denoted by $\widetilde{\cup}_{i\in\Delta}$, is defined by $\widetilde{\cup}_{i\in\Delta}(F_i, A) = (\widetilde{\cup}_{i\in\Delta}F_i, A)$, where $(\widetilde{\cup}_{i\in\Delta}F_i)(\alpha) = \cup_{i\in\Delta}(F_i(\alpha)),\ \forall\ \alpha \in A$.

Definition 2.3. Let X and Y be two nonempty sets and $f : X \to Y$ be a mapping. Then

(i) the *image* of a soft set $(F, A) \in S(X, A)$ under the mapping f is defined by $f[(F, A)] = (f(F), A)$, where $[f(F)](\alpha) = f[F(\alpha)],\ \forall\ \alpha \in A$.

(ii) the *inverse image* of a soft set $(G, A) \in S(Y, A)$ under the mapping f is defined by $f^{-1}[(G, A)] = (f^{-1}(G), A)$, where $[f^{-1}(G)](\alpha) = f^{-1}[G(\alpha)],\ \forall\ \alpha \in A$.

Definition 2.4 ([21])**.** A soft set (E, A) over X is said to be a soft element if $\exists\ \alpha \in A$ such that $E(\alpha)$ is a singleton, say,$\{x\}$ and $E(\beta) = \phi,\ \forall\ \beta(\neq \alpha)\ \in A$. Such a soft element is denoted by E^x_α. Let $\mathcal{E}$ be the set of all soft elements of the universal set X.

Also we denote the soft set (E, A) where $E(\alpha) = \{x\},\ \forall\ \alpha \in A$ by E_x.

Definition 2.5 ([21])**.** The soft element E^x_α is said to be in the soft set (G, A), denoted by $E^x_\alpha\ \widetilde{\in}\ (G, A)$, if $x \in G(\alpha)$.

Proposition 2.6 ([21])**.** $E^x_\alpha\ \widetilde{\in}\ (F, A)$ *iff* $E^x_\alpha\ \widetilde{\notin}(F^c, A)$.

Definition 2.7 ([18])**.** Let (F, A) and (G, A) be two soft sets over X. The parallel product of (F, A) and (G, A) is defined as $(F, A)\ \widetilde{\times}\ (G, A) = (F\ \widetilde{\times}\ G, A)$ where

$[F \,\widetilde{\times}\, G](\alpha) = F(\alpha) \times G(\alpha),\ \forall\ \alpha \in A$. It is clear that $(F \,\widetilde{\times}\, G, A)$ is a soft set over $X \times X$.

Definition 2.8. A soft set (F, A) over X is said to be a finite soft set if $F(\alpha)$ is a finite subset of $X,\ \forall\ \alpha \in A$.

Definition 2.9. A soft set $(F, A) \in S(X, A)$ is said to be pseudo constant soft set if $F(\alpha) = X$ *or* $\phi,\ \forall\ \alpha \in A$. Let $CS(X, A)$ denote the set of all pseudo constant soft sets over X under the parameter set A.

Definition 2.10 ([23])**.** Let τ be the collection of soft sets over X. Then τ is said to be a soft topology on X if

(i) $(\widetilde{\phi}, A),\ (\widetilde{X}, A) \in \tau$ where $\widetilde{\phi}(\alpha) = \phi$ and $\widetilde{X}(\alpha) = X,\ for\ all\ \alpha \in A$.
(ii) the intersection of any two soft sets in τ belongs to τ.
(iii) the union of any number of soft sets in τ belongs to τ.

The triplet (X, A, τ) is called a soft topological space over X.

Definition 2.11 ([2])**.** A sub collection τ of $S(X, A)$ is said to be an enriched soft topology on X if

(i) $(F, A) \in \tau,\ \forall\ (F, A) \in CS(X, A)$;
(ii) the intersection of any two soft sets in τ belongs to τ;
(iii) the union of any number of soft sets in τ belongs to τ.

The triplet (X, A, τ) is called an enriched soft topological space over X.

Proposition 2.12 ([23])**.** *Let (X, A, τ) be a soft topological space (an enriched soft topological space) over X. Then the collection $\tau^\alpha = \{F(\alpha) :\ (F, A) \in \tau\}$ for each $\alpha \in A$, defines a topology on X.*

Proposition 2.13 ([21])**.** *If (X, A, τ) be a soft topological space (an enriched soft topological space) and if $\tau^* = \{(G, A) \in S(X) :\ G(\alpha) \in \tau^\alpha,\ \forall\ \alpha \in A\}$ then τ^* is an enriched soft topology on X such that $\tau \subseteq \tau^*$ and $[\tau^*]^\alpha = \tau^\alpha,\ \forall \alpha \in A$.*

Definition 2.14. Let (X, A, τ) and (Y, A, ν) be soft topological spaces. The mapping $f : (X, A, \tau) \to (Y, A, \nu)$ is said to be soft continuous if $f^{-1}[(F, A)] \in \tau,\ \forall\ (F, A) \in \nu$.

Proposition 2.15 ([18])**.** *Let $(X, A, \tau),\ (Y, A, \nu)$ and (Z, A, ω) be soft topological spaces. If $f : (X, A, \tau) \to (Y, A, \nu)$ and $g : (Y, A, \nu) \to (Z, A, \omega)$ are soft continuous and $f(X) \subseteq Y$, then the mapping $gf : (X, A, \tau) \to (Z, A, \omega)$ is soft continuous.*

Proposition 2.16 ([18])**.** *Let (X, A, τ) and (Y, A, ν) be two soft topological spaces. Then $\mathcal{F} = \{(F, A) \,\widetilde{\times}\, (G, A) :\ (F, A) \in \tau,\ (G, A) \in \nu\}$ forms an open base of the soft topology on $X \times Y$.*

Definition 2.17 ([18])**.** The soft topology in $X \times Y$ induced by the open base $\mathcal{F}$ is said to be the product soft topology of the soft topologies τ and ν. It is denoted by $\tau \,\widetilde{\times}\, \nu$. The soft topological space $[X \times Y, A, \tau \widetilde{\times} \nu]$ is said to be the soft topological product of the soft topological spaces (X, A, τ) and (Y, A, ν).

Proposition 2.18 ([18])**.** *Let (X, A, τ) and (Y, A, ν) be two soft topological spaces. Then the projection mappings $\pi_X : (X \times Y, A, \tau \widetilde{\times} \nu) \to (X, A, \tau)$ and $\pi_Y : (X \times$*

$Y, A, \tau\widetilde{\times}\nu) \to (Y, A, \nu)$ are soft continuous and soft open. Also $\tau\widetilde{\times}\nu$ is the smallest soft topology in $X \times Y$ for which the projection mappings are soft continuous.

Proposition 2.19 ([18]). *Let (X, A, τ) be the product space of two soft topological spaces (X_1, A, τ_1) and (X_2, A, τ_2) respectively and $\pi_i : (X, A, \tau) \to (X_i, A, \tau_i),\ i = 1, 2$ be the projection mappings. If (Y, A, ν) be any soft topological space then the mapping $f : (Y, A, \nu) \to (X, A, \tau)$ is soft continuous iff the mappings $\pi_i f : (Y, A, \nu) \to (X_i, A, \tau_i),\ i = 1, 2$ are soft continuous.*

Proposition 2.20 ([18]). *Let (X, A, τ) be a soft topological space. Then for each $\alpha \in A,\ (\tau\widetilde{\times}\tau)^\alpha = \tau^\alpha \times \tau^\alpha$.*

Proposition 2.21 ([18]). *Let (X, A, τ) be a soft topological space and define $T^* = \{(F, A) \in S(X \times X)$ such that $F(\alpha) \in (\tau^\alpha \times \tau^\alpha),\ \forall\ \alpha \in A\}$. Then T^* is a soft topology over $X \times X$ and $T^* = \tau^*\ \widetilde{\times}\ \tau^*$, where τ^* is as in Proposition 2.13.*

Definition 2.22 ([21]). Let τ be a soft topology on X. Then a soft set (F, A) is said to be a $\tau - soft$ *neighbourhood* (shortly soft nbd) of the soft element E^x_α if there exists a soft set $(G, A) \in \tau$ such that $E^x_\alpha \widetilde{\in} (G, A) \widetilde{\subseteq} (F, A)$.

The soft nbd system of a soft element E^x_α in (X, A, τ), denoted by $N_\tau(E^x_\alpha)$, is the family of all its soft nbds.

Proposition 2.23 ([21]). *If $\{N_\tau(E^x_\alpha) :\ E^x_\alpha \in \mathcal{E}\}$ be the system of soft nbds then*

(i) $N_\tau(E^x_\alpha) \neq \phi,\ \forall E^x_\alpha \in \mathcal{E}$
(ii) $E^x_\alpha \widetilde{\in} (F, A),\ \forall\ (F, A) \in N_\tau(E^x_\alpha)$
(iii) $(F, A) \in N_\tau(E^x_\alpha)$ *and* $(F, A) \widetilde{\subseteq} (G, A) \ \Rightarrow\ (G, A) \in N_\tau(E^x_\alpha)$
(iv) $(F, A), (G, A) \in N_\tau(E^x_\alpha) \Rightarrow\ (F, A) \widetilde{\cap} (G, A) \in N_\tau(E^x_\alpha)$
(v) $(F, A) \in N_\tau(E^x_\alpha) \Rightarrow\ \exists\ (G, A) \in N_\tau(E^x_\alpha)$ *such that* $(G, A) \widetilde{\subseteq} (F, A)$ *and* $(G, A) \in N_\tau(E^y_\beta),\ \forall\ E^y_\beta \widetilde{\in} (G, A)$.

Proposition 2.24 ([18]). *Let (X, A, τ) be a soft topological space and $G \subseteq X$. Then $\tau_G = \{(F, A) \widetilde{\cap} (\widetilde{G}, A) :\ (F, A) \in \tau\}$, where $\widetilde{G}(\alpha) = G,\ \forall\ \alpha \in A$ is a soft topology on G.*

Definition 2.25 ([18]). A mapping $f : (X, A, \tau) \to (Y, A, \nu)$ is said to be soft homeomorphism if f is bijective and $f,\ f^{-1}$ are soft continuous.

Definition 2.26 ([21]). Let (X, A, τ) and (Y, A, ν) be two soft topological spaces. A mapping $f : (X, A, \tau) \to (Y, A, \nu)$ is said to be

(i) soft open if $(F, A) \in \tau \Rightarrow f[(F, A)] \in \nu$;
(ii) soft closed if (F, A) is soft closed in $(X, A, \tau) \Rightarrow f[(F, A)]$ is soft closed in (Y, A, ν).

Proposition 2.27 ([21]). *Let (X, A, τ) and (Y, A, ν) be two soft topological spaces. For a bijective mapping $f : (X, A, \tau) \to (Y, A, \nu)$, the following statements are equivalent.*

(i) $f : (X, A, \tau) \to (Y, A, \nu)$ *is soft homeomorphism;*
(ii) $f : (X, A, \tau) \to (Y, A, \nu)$ *and* $f^{-1} : (Y, A, \nu) \to (X, A, \tau)$ *are soft continuous;*
(iii) $f : (X, A, \tau) \to (Y, A, \nu)$ *is both soft continuous and soft open;*
(iv) $f : (X, A, \tau) \to (Y, A, \nu)$ *is both soft continuous and soft closed;*

(v) $f[\overline{(F,A)}] = \overline{f[(F,A)]},\ \forall\ (F,A) \in S(X,A)$.

Definition 2.28 ([23])**.** Let (X, A, τ) be a soft topological space over X and $x, y \in X$ such that $x \neq y$. If there exist soft open sets (F, A) and (G, A) such that $[E_x \widetilde{\in} (F, A)$ and $E_y \widetilde{\notin} (F, A)]$ or $[E_y \widetilde{\in} (G, A)$ and $E_x \widetilde{\notin} (G, A)]$, then (X, A, τ) is called a soft T_0-space.

Definition 2.29 ([23])**.** Let (X, A, τ) be a soft topological space over X and $x, y \in X$ such that $x \neq y$. If there exist soft open sets (F, A) and (G, A) such that $[E_x \widetilde{\in} (F, A)$ and $E_y \widetilde{\notin} (F, A)]$ and $[E_y \widetilde{\in} (G, A)$ and $E_x \widetilde{\notin} (G, A)]$, then (X, A, τ) is called a soft T_1-space.

Definition 2.30 ([23])**.** Let (X, A, τ) be a soft topological space over X and $x, y \in X$ such that $x \neq y$. If there exist soft open sets (F, A) and (G, A) such that $E_x \widetilde{\in} (F, A)$ and $E_y \widetilde{\in} (G, A)$ and $(F, A) \widetilde{\cap} (G, A) = (\widetilde{\phi}, A)$, then (X, A, τ) is called a soft T_2-space.

3. Some properties of soft topological spaces

In this section we study some continuity properties of soft topological spaces; the soft topology that we have considered is in the sense of Shabir and Naz [23]. Unless otherwise stated, X is an initial universal set, A is the nonempty set of parameters and $S(X, A)$ denotes the collection of all soft sets over X under the parameter set A.

Proposition 3.1. *Let X be an infinite set and τ consist of $(\widetilde{\phi}, A)$ and all those soft sets (F, A) over X such that $[(\widetilde{X}, A) - (F, A)]$ i.e. (F^c, A) are finite soft sets. Then (X, A, τ) is a soft topological space.*

Proof. Since $(\widetilde{X}^c, A) = (\widetilde{\phi}, A)$ is a finite soft set, it follows that $(\widetilde{X}, A) \in \tau$. Again let $(F_1, A),\ (F_2, A) \in \tau$. If any one of $(F_1, A),\ (F_2, A)$ be $(\widetilde{\phi}, A)$, then $[(F_1, A) \widetilde{\cap} (F_2, A)] \in \tau$. Now let $(F_1, A),\ (F_2, A) \neq (\widetilde{\phi}, A)$. So, $(F_1^c, A),\ (F_2^c, A)$ are finite soft sets and hence $[(F_1, A) \widetilde{\cap} (F_2, A)]^c = (F_1^c, A) \widetilde{\cup} (F_2^c, A)$ is a finite soft set and therefore $(F_1, A) \widetilde{\cap} (F_2, A) \in \tau$. Next let $\{(F_i, A),\ i \in \Delta\}$ be any collection of members of τ and without any loss of generality we assume that $(F_i, A) \neq (\widetilde{\phi}, A),\ \forall\ i \in \Delta$. So, (F_i^c, A) is finite soft set, $\forall\ i \in \Delta$. Thus $[\widetilde{\cup}_{i\in\Delta}(F_i, A)]^c = \widetilde{\cap}_{i\in\Delta}[(F_i^c, A)]$ is finite soft set and hence $\widetilde{\cup}_{i\in\Delta}(F_i, A) \in \tau$. Therefore (X, A, τ) is a soft topological space. □

Definition 3.2. The soft topology τ of Proposition 3.1, is called the cofinite soft topology over X.

Definition 3.3. Let (X, A, τ) be a soft topological space. A soft element $E_\alpha^x \in \mathcal{E}$ is said to be a limiting soft element of a soft set (F, A) over X if every open soft set containing E_α^x contains at least one soft element E_α^y of (F, A) other than E_α^x, i.e. if $\forall\ (G, A) \in \tau$ with $E_\alpha^x \widetilde{\in} (G, A),\ F(\alpha) \cap [G(\alpha) - \{x\}] \neq \phi$.

The union of all limiting soft elements of (F, A) is a soft set over X, called the derived soft set of (F, A) and is denoted by $(F, A)'$ or (F', A).

Example 3.4. Let $X = \{x, y, z\},\ A = \{\alpha, \beta\}$ and $\tau = \Big\{(\ \widetilde{\phi}, A) = \{\phi,\ \phi\},\ (F_1, A) = \big\{\{x\}, \{y\}\big\},\ (F_2, A) = \big\{\{y\}, \{z\}\big\},\ (F_3, A) = \big\{\{x, y\}, \{y, z\}\big\},\ (\ \widetilde{X}, A) = \{X,\ X\}\Big\}$.

Then τ is a soft topology over X. Let $(F,A)=\{\{x,y\},\{x,z\}\}$. Now $E_\alpha^x \,\widetilde{\in}\, (F_1,A)\in\tau$, but $F(\alpha)\cap[F_1(\alpha)-\{x\}]=\phi$. Therefore, E_α^x is not a limiting soft element of (F,A). The open soft sets containing E_α^y are $(F_2,A),(F_3,A)$ and $(\widetilde{X},A)$. But $F(\alpha)\cap[F_2(\alpha)-\{y\}]=\phi$. Therefore, E_α^y is not a limiting soft element of (F,A). The only open soft set containing E_α^z is $(\widetilde{X},A)$ and $F(\alpha)\cap[\widetilde{X}(\alpha)-\{z\}]=\{x,\ y\}\neq\phi$. Therefore, E_α^z is a limiting soft element of (F,A).

Similarly, we can show that E_β^x is a limiting soft element of (F,A) and $E_\beta^y,\ E_\beta^z$ are not the limiting soft elements of (F,A). Therefore, $(F',A)=\{\{z\},\{x\}\}$.

Proposition 3.5. *Let τ be a soft topology over X. A soft set (F,A) over X is said to be an $\tau-$open soft set iff $\forall\ E_\alpha^x\,\widetilde{\in}(F,A),\ \exists\ (G,A)\in\tau$ such that $E_\alpha^x\,\widetilde{\in}(G,A)\,\widetilde{\subseteq}\,(F,A)$.*

Proposition 3.6. *If the soft set (F,A) of a soft topological space (X,A,τ) is closed soft set then (F,A) contains all its limiting soft elements.*

Proof. Let (F,A) be a closed soft set and $E_\alpha^x\,\widetilde{\notin}(F,A)$. Since (F,A) is closed soft set, (F^c,A) is open soft set and $E_\alpha^x\,\widetilde{\in}\,(F^c,A)$. Since $F(\alpha)\cap F^c(\alpha)=\phi,\ E_\alpha^x$ cannot be a limiting soft element of (F,A). Therefore, (F,A) contains all its limiting soft elements. □

Remark 3.7. The converse of the above Proposition is not always true which is shown in the following example:

Example 3.8. Let $X,\ A,\ \tau$ be the same as in Example 3.4. If $(F,A)=\{\{x,y,z\},\{x,y\}\}$ then $(F',A)=\{\{z\},\ \{x\}\}$. Therefore, (F,A) contains all its limiting soft element. But (F,A) is not closed soft set.

Proposition 3.9. *If (X,A,τ) is an enriched soft topological space and (F,A) contains all its limiting soft elements, then (F,A) is a closed soft set.*

Proof. Let $E_\alpha^x\,\widetilde{\in}\,(F^c,A)$. Then $E_\alpha^x\,\widetilde{\notin}\,(F,A)$ and hence E_α^x is not a limiting soft element of (F,A). So $\exists\ (G,A)\in\tau$ such that $E_\alpha^x\,\widetilde{\in}\,(G,A)$ and $F(\alpha)\ \cap\ [G(\alpha)-\{x\}]=\phi$. Thus $F(\alpha)\ \cap\ G(\alpha)=\phi$, since $E_\alpha^x\,\widetilde{\notin}\,(F,A)$. Consider the soft set (H_α,A) where $H_\alpha(\alpha)=X$ and $H_\alpha(\beta)=\phi$. Since τ is enriched, $(H_\alpha,A)\in\tau$. So, $E_\alpha^x\,\widetilde{\in}\,[(G,A)\,\widetilde{\cap}\,(H_\alpha,A)]\in\tau$ and $[(G,A)\,\widetilde{\cap}\,(H_\alpha,A)]\,\widetilde{\subseteq}\,(F^c,A)$. Therefore (F^c,A) is open soft set and hence (F,A) is closed soft set. □

Remark 3.10. A soft set (F,A) of an enriched soft topological space (X,A,τ) is closed iff (F,A) contains all of its limiting soft elements.

Proposition 3.11. *For any two soft sets $(F,A),(G,A)$ of a soft topological space (X,A,τ), the following conditions hold.*

(i) $(\widetilde{\phi}',A)=(\widetilde{\phi},A)$;
(ii) $(F,A)\,\widetilde{\subseteq}\,(G,A)\Rightarrow(F',A)\,\widetilde{\subseteq}\,(G',A)$;
(iii) $([F\,\widetilde{\cup}\,G]',A)=([F'\,\widetilde{\cup}\,G'],A)$;
(iv) $([F\,\widetilde{\cup}\,F']',A)\,\widetilde{\subseteq}\,([F\,\widetilde{\cup}\,F'],A)$.

Proof. (i) and (ii) follow from definition.

(iii) From (ii), we have $(F',A)\,\widetilde{\subseteq}\,([F\,\widetilde{\cup}\,G]',A),\ (G',A)\,\widetilde{\subseteq}\,([F\,\widetilde{\cup}\,G]',A)$ and hence $([F'\,\widetilde{\cup}\,G'],A)\,\widetilde{\subseteq}\,([F\,\widetilde{\cup}\,G]',A)$...........(1).

Next, let $E_\alpha^x\,\widetilde{\notin}\,(F',A)\,\widetilde{\cup}\,(G',A)$. Then $\exists\ (H,A),\ (K,A)\in\tau$ such that

$$E_\alpha^x \,\widetilde{\in}\, (H, A),\ E_\alpha^x \,\widetilde{\in}\, (K, A),\ F(\alpha) \cap [H(\alpha) - \{x\}] = \phi$$

and $G(\alpha) \cap [K(\alpha) - \{x\}] = \phi$. Let $(V, A) = (H, A) \,\widetilde{\cap}\, (K, A)$. Then $E_\alpha^x \,\widetilde{\in}\, (V, A) \in \tau$. Also $V(\alpha) \cap [(F(\alpha) \cup G(\alpha)) - \{x\}] = [V(\alpha) \cap (F(\alpha) - \{x\})] \cup [V(\alpha) \cap (G(\alpha) - \{x\})] \subseteq [H(\alpha) \cap (F(\alpha) - \{x\})] \cup [K(\alpha) \cap (G(\alpha) - \{x\})] = \phi$. Therefore, E_α^x is not a limiting soft element of $(F, A) \,\widetilde{\cup}\, (G, A)$. Therefore, $E_\alpha^x \,\widetilde{\notin}\, ([F \,\widetilde{\cup}\, G]', A)$ and hence $([F \,\widetilde{\cup}\, G]', A) \,\widetilde{\subseteq}\, (F', A) \,\widetilde{\cup}\, (G', A)$..........(2).

From (1) and (2) we have $([F \,\widetilde{\cup}\, G]', A) = ([F' \,\widetilde{\cup}\, G'], A)$.

(iv) Let $E_\alpha^x \,\widetilde{\in}\, [(F, A) \,\widetilde{\cup} (F, A)']'$, we have to show that $E_\alpha^x \,\widetilde{\in}\, [(F, A) \,\widetilde{\cup} (F, A)']$.
Case - I: If $E_\alpha^x \,\widetilde{\in}\, (F, A)$, then $E_\alpha^x \,\widetilde{\in}\, [(F, A) \,\widetilde{\cup} (F, A)']$.
Case - II: If $E_\alpha^x \,\widetilde{\notin} (F, A)$, then $x \notin F(\alpha)$.
Also since $[(F, A) \,\widetilde{\cup} (F, A)']' = (F, A)' \,\widetilde{\cup} [(F, A)']'$ and hence
$E_\alpha^x \,\widetilde{\in}\, [(F, A) \,\widetilde{\cup} (F, A)']' \Rightarrow E_\alpha^x \,\widetilde{\in}\, (F, A)'$ or $E_\alpha^x \,\widetilde{\in}\, [(F, A)']'$.
Subcase II (a): If $E_\alpha^x \,\widetilde{\in}\, (F, A)'$, then $E_\alpha^x \,\widetilde{\in}\, [(F, A) \,\widetilde{\cup} (F, A)']$.
Subcase II (b): If $E_\alpha^x \,\widetilde{\notin} (F, A)'$, then $x \notin F'(\alpha)$. Also, $E_\alpha^x \,\widetilde{\in}\, [(F, A)']'$.
Let (G, A) be any soft open set containing E_α^x. Since E_α^x is a limiting soft element of $(F, A)'$, we have $G(\alpha) \cap [F'(\alpha) - \{x\}] \neq \phi$.
Let $y \in G(\alpha) \cap [F'(\alpha) - \{x\}]$. Then $E_\alpha^y \,\widetilde{\in}\, (G, A) \,\widetilde{\cap} [(F, A)' - E_\alpha^x]$ and (G, A) is an open soft set containing E_α^y, which is a limiting soft element of (F, A).
Therefore, $G(\alpha) \cap [F(\alpha) - \{y\}] \neq \phi$.
Let $z \in G(\alpha) \cap [F(\alpha) - \{y\}]$. Then $E_\alpha^z \,\widetilde{\in}\, (G, A) \,\widetilde{\cap} [(F, A) - E_\alpha^y]$.
Now $z \neq x$ [as $E_\alpha^x \,\widetilde{\notin} (F, A)$ from choice of E_α^x and $z \in \big[G(\alpha) \cap [F(\alpha) - \{x\}]\big]$.
Therefore, E_α^x is a limiting soft element of (F, A) i.e. $E_\alpha^x \,\widetilde{\in}\, (F, A)'$.
Thus in all Cases $E_\alpha^x \,\widetilde{\in}\, [(F, A) \,\widetilde{\cup}\, (F, A)']' \Rightarrow E_\alpha^x \,\widetilde{\in}\, [(F, A) \,\widetilde{\cup}\, (F, A)']$.
Hence, $[(F, A) \,\widetilde{\cup} (F, A)']' \,\widetilde{\subseteq}\, (F, A) \,\widetilde{\cup} (F, A)'$. □

Let us now define the closure of a soft set in terms of limiting soft element.

Definition 3.12. The closure of a soft set (F, A) over X denoted by $\overline{(F, A)} = (\overline{F}, A)$ is defined by $\overline{(F, A)} = (F, A) \,\widetilde{\cup}\, (F', A)$.

Example 3.13. Let $X,\ A,\ \tau$ be the same as in Example 3.4.
If $(F, A) = \big\{\{x, y\}, \{x, z\}\big\}$ then $(F', A) = \big\{\{z\}, \{x\}\big\}$ and $\overline{(F, A)} = \big\{\{x, y, z\}, \{x, z\}\big\}$.

Proposition 3.14. *Let (X, A, τ) be a soft topological space. If (F, A) be a closed soft set then $(F, A) = \overline{(F, A)}$. But if (X, A, τ) is an enriched soft topological space and $(F, A) = \overline{(F, A)}$, then (F, A) is a closed soft set.*

Proof. Proof follows from Proposition 3.6 and Proposition 3.9. □

Proposition 3.15. *For any soft set (F, A), $\overline{F}(\alpha) = \overline{F(\alpha)}^{\alpha}$, $\forall\ \alpha \in A$, where $\overline{F(\alpha)}^{\alpha}$ is the closure of $F(\alpha)$ with respect to the topology τ^α.*

Proof. Let $x \in \overline{F}(\alpha)$. Then $x \in F(\alpha)$ or E_α^x is a limiting soft element of (F, A).If $x \in F(\alpha)$, then $x \in \overline{F(\alpha)}^{\alpha}$ and hence $\overline{F}(\alpha) \subseteq \overline{F(\alpha)}^{\alpha}$. Now let $x \notin F(\alpha)$ and E_α^x is a limiting soft element of (F, A). Let $G \in \tau^\alpha$ with $x \in G$. Then $\exists\ (H, A) \in \tau$ such that $H(\alpha) = G$, and, since E_α^x is a limiting soft element of (F, A), we have $F(\alpha) \cap [H(\alpha) - \{x\}] \neq \phi$. Hence x is a limit point of $F(\alpha)$ with respect to τ^α. So, $x \in \overline{F(\alpha)}^{\alpha}$. Therefore $\overline{F}(\alpha) \subseteq \overline{F(\alpha)}^{\alpha}$.

Conversely let $x \in \overline{F(\alpha)}^{\alpha}$. Then $x \in F(\alpha)$ or x is a limit point of $F(\alpha)$ with respect to τ^{α}. If $x \in F(\alpha)$, then $x \in \overline{F}(\alpha)$. If $x \notin F(\alpha)$ then x is a limit point of $F(\alpha)$ with respect to τ^{α}. Let $(K, A) \in \tau$ such that $E^x_{\alpha} \widetilde{\in} (K, A)$. Then $F(\alpha) \cap [K(\alpha) - \{x\}] \neq \phi$. Thus E^x_{α} is a limiting soft element of (F, A) and hence $x \in \overline{F}(\alpha)$. Therefore $\overline{F(\alpha)}^{\alpha} \subseteq \overline{F}(\alpha)$ and hence, $\overline{F}(\alpha) = \overline{F(\alpha)}^{\alpha}$, $\forall\ \alpha \in A$. □

Proposition 3.16. *For any two soft sets $(F, A), (G, A)$ of a soft topological space (X, A, τ), the following conditions hold.*

(i) $\overline{(\ \widetilde{\phi}, A)} = (\ \widetilde{\phi}, A)$;
(ii) $(F, A)\ \widetilde{\subseteq}\ \overline{(F, A)}$;
(iii) $\overline{[(F, A)\ \widetilde{\cup}\ (G, A)]} = \overline{(F, A)}\ \widetilde{\cup}\ \overline{(G, A)}$;
(iv) $\overline{[\overline{(F, A)}]} = \overline{(F, A)}$.

Proof. Proofs follows from Proposition 3.11. □

Proposition 3.17. *The closure of a soft set (F, A) w.r.t an enriched soft topological space (X, A, τ) is the smallest closed soft set containing (F, A), i.e. it is the meet of all closed soft sets containing (F, A).*

Proof. Since $\overline{[\overline{(F, A)}]} = \overline{(F, A)}$ and τ is enriched, therefore $\overline{(F, A)}$ is closed. Also $(F, A)\ \widetilde{\subseteq}\ \overline{(F, A)}$, therefore $\overline{(F, A)}$ is a closed soft set containing (F, A). Let (G, A) be any closed soft set containing (F, A). Then $(F, A)\ \widetilde{\subseteq}\ (G, A) \Rightarrow \overline{(F, A)}\ \widetilde{\subseteq}\ \overline{(G, A)} = (G, A)$. Thus $\overline{(F, A)}$ is a smallest closed soft set containing (F, A). □

Definition 3.18. Let (X, A, τ) be a soft topological space. If for $E^x_{\alpha},\ E^y_{\beta} \in \mathcal{E}$ with $E^x_{\alpha} \neq E^y_{\beta}$, there exists $(F, A) \in \tau$ such that $[E^x_{\alpha}\ \widetilde{\in}\ (F, A)$ and $E^y_{\beta}\ \widetilde{\notin}\ (F, A)]$ or $[E^y_{\beta}\ \widetilde{\in}\ (F, A)$ and $E^x_{\alpha}\ \widetilde{\notin}\ (F, A)]$, then (X, A, τ) is called a soft T_0-space.

Proposition 3.19. *An enriched soft topological space (X, A, τ) is soft T_0-space iff $\forall\ E^x_{\alpha},\ E^y_{\beta} \in \mathcal{E},\ E^x_{\alpha} \neq E^y_{\beta} \Rightarrow \overline{E^x_{\alpha}} \neq \overline{E^y_{\beta}}$.*

Proof. Let (X, A, τ) be a soft T_0-space and $E^x_{\alpha} \neq E^y_{\beta}$. Then $\exists\ (F, A) \in \tau$ such that one of $E^x_{\alpha},\ E^y_{\beta}$ belongs to (F, A) but not the other. Let $E^x_{\alpha}\ \widetilde{\in}\ (F, A)$ and $E^y_{\beta}\ \widetilde{\notin}\ (F, A)$. Then (F^c, A) is soft closed set containing E^y_{β} but $E^x_{\alpha}\ \widetilde{\notin}\ (F^c, A)$. Since τ is enriched, $\overline{E^y_{\beta}}$ is the meet of all soft closed set containing E^y_{β}, it follows that $E^x_{\alpha}\ \widetilde{\notin}\ \overline{E^y_{\beta}}$. But $E^y_{\beta}\ \widetilde{\in}\ \overline{E^y_{\beta}}$. Therefore, $\overline{E^x_{\alpha}} \neq \overline{E^y_{\beta}}$.

Conversely let the given condition be satisfied. Let $E^x_{\alpha},\ E^y_{\beta} \in \mathcal{E}$ such that $E^x_{\alpha} \neq E^y_{\beta}$. So, by the given condition $\overline{E^x_{\alpha}} \neq \overline{E^y_{\beta}}$. Then $\exists\ E^z_{\gamma} \in \mathcal{E}$ such that E^z_{γ} belongs to one of the soft sets $\overline{E^x_{\alpha}}$ and $\overline{E^y_{\beta}}$ but not the other. Let $E^z_{\gamma}\ \widetilde{\in}\ \overline{E^x_{\alpha}}$ but $E^z_{\gamma}\ \widetilde{\notin}\ \overline{E^y_{\beta}}$. If $E^x_{\alpha}\ \widetilde{\in}\ \overline{E^y_{\beta}}$, then $E^x_{\alpha}\ \widetilde{\subseteq}\ \overline{E^y_{\beta}}$ and hence $E^z_{\gamma}\ \widetilde{\in}\ \overline{E^x_{\alpha}}\ \widetilde{\subseteq}\ \overline{\overline{E^y_{\beta}}} = \overline{E^y_{\beta}}$. But $E^z_{\gamma}\ \widetilde{\notin}\ \overline{E^y_{\beta}}$. So, $E^x_{\alpha}\ \widetilde{\notin}\ \overline{E^y_{\beta}}$. Let $(U, A) = (\widetilde{X}, A) - \overline{E^y_{\beta}}$. Since τ is enriched, $(U, A) \in \tau$ such that $E^x_{\alpha}\ \widetilde{\in}\ (U, A)$ but $E^y_{\beta}\ \widetilde{\notin}\ (U, A)$. Therefore, (X, A, τ) is a soft T_0-space. □

Example 3.20. Let $X = \{x, y, z\},\ A = \{\alpha,\ \beta\}$ and

$$\tau = \{(\widetilde{\phi}, A),\ \{X,\ \phi\},\ \{\phi,\ X\},\ (F_1, A),\ (F_2, A),\ (F_3, A),\ (F_4, A),\ (\widetilde{X}, A)\},$$

where $(F_1, A) = \{\{x\},\ \phi\},\ (F_2, A) = \{\{x\},\ \{y\}\},\ (F_3, A) = \{\{y\},\ \{z\}\},$
$(F_4, A) = \{\{x, y\},\ \{y, z\}\}$.
Now $\overline{E^x_\alpha} = \{\{x, z\}, \phi\},\ \overline{E^y_\alpha} = \{\{y, z\}, \phi\},\ \overline{E^z_\alpha} = \{\{z\}, \phi\},\ \overline{E^x_\beta} = \{\phi, \{x\}\},$
$\overline{E^y_\beta} = \{\phi, \{x, y\}\},\ \overline{E^z_\beta} = \{\phi, \{x, z\}\}$. So, $\overline{E^x_\alpha} \neq \overline{E^y_\alpha} \neq \overline{E^z_\alpha} \neq \overline{E^x_\beta} \neq \overline{E^y_\beta} \neq \overline{E^z_\beta}$. Thus (X, A, τ) is a soft T_0-space.

Definition 3.21. Let (X, A, τ) be a soft topological space over X. If for any $E^x_\alpha,\ E^y_\beta \in \mathcal{E}$ with $E^x_\alpha \neq E^y_\beta$, $\exists\,(F, A),\ (G, A) \in \tau$ such that $[E^x_\alpha \,\widetilde{\in}\, (F, A),\ E^y_\beta \,\widetilde{\notin}\, (F, A)]$ and $[E^y_\beta \,\widetilde{\in}\, (G, A),\ E^x_\alpha \,\widetilde{\notin}\, (G, A)]$, then (X, A, τ) is called a soft T_1-space.

Proposition 3.22. *A soft topological space (X, A, τ) is soft T_1-space iff $\forall\ E^x_\alpha \in \mathcal{E},\ \{E^x_\alpha\}$ is soft closed.*

Proof. Let (X, A, τ) be a soft topological space and $E^x_\alpha \,\widetilde{\in}\, \mathcal{E}$. Let $E^y_\beta \,\widetilde{\in}\, [(\widetilde{X}, A) - E^x_\alpha]$. Then $E^x_\alpha \neq E^y_\beta$. Since (X, A, τ) is a T_1-space, $\exists\,(U_y, A) \in \tau$ such that $E^y_\beta \,\widetilde{\in}\, (U_y, A)$ but $E^x_\alpha \,\widetilde{\notin}\, (U_y, A)$. Thus $(U_y, A) \,\widetilde{\subseteq}\, [(\widetilde{X}, A) - E^x_\alpha]$ and hence $E^y_\beta \,\widetilde{\subseteq}\, (U_y, A) \,\widetilde{\subseteq}\, [(\widetilde{X}, A) - E^x_\alpha]$, $\forall\ E^y_\beta \,\widetilde{\in}\, [(\widetilde{X}, A) - E^x_\alpha]$. So,

$$[(\widetilde{X}, A) - E^x_\alpha] = \widetilde{\bigcup}_{E^y_\beta\ \widetilde{\in}\ [(\widetilde{X},A)-E^x_\alpha]} E^y_\beta \,\widetilde{\subseteq}\, \widetilde{\bigcup}_{E^y_\beta\ \widetilde{\in}\ [(\widetilde{X},A)-E^x_\alpha]} (U_y, A) \,\widetilde{\subseteq}\, [(\widetilde{X}, A) - E^x_\alpha].$$

Therefore, $[(\widetilde{X}, A) - E^x_\alpha] = \widetilde{\bigcup}_{E^y_\beta\ \widetilde{\in}\ [(\widetilde{X},A)-E^x_\alpha]} (U_y, A)$. Since each (U_y, A) is open soft set, $[(\widetilde{X}, A) - E^x_\alpha]$ is open soft set and hence E^x_α is closed soft set.

Conversely let the given condition be satisfied. Let $E^x_\alpha,\ E^y_\beta \in \mathcal{E}$ such that $E^x_\alpha \neq E^y_\beta$. Let $(U, A) = [(\widetilde{X}, A) - E^x_\alpha]$ and $(V, A) = [(\widetilde{X}, A) - E^y_\beta]$. Since E^x_α and E^y_β are closed soft sets, it follows that $(U, A),\ (V, A)$ are open soft sets. Also $E^y_\beta \,\widetilde{\in}\, (U, A)$, $E^x_\alpha \,\widetilde{\notin}\, (U, A)$ and $E^x_\alpha \,\widetilde{\in}\, (V, A),\ E^y_\beta \,\widetilde{\notin}\, (V, A)$. Therefore, (X, A, τ) is a soft T_1-space. □

Remark 3.23. Every soft T_1-space is obiously a soft T_0-space. The following example shows that the converse is not always ture.

Example 3.24. Let $X,\ A$ and τ be the same as in Example 3.20. Then (X, A, τ) is a soft T_0-space. But $\{E^x_\alpha\}$ is not closed soft set since $\{\{y, z\}, \{x, y, z\}\} \notin \tau$ and hence (X, A, τ) is not a soft T_1-space.

Example 3.25. Let X be an infinite set and τ be the soft cofinite topology over X. Then for each $E^x_\alpha \in \mathcal{E},\ (\widetilde{X}, A) - [(\widetilde{X}, A) - \{E^x_\alpha\}] = \{E^x_\alpha\}$ is a finite soft set and hence $[(\widetilde{X}, A) - \{E^x_\alpha\}] \in \tau$. Thus, $\{E^x_\alpha\}$ is soft closed $\forall\ E^x_\alpha \in \mathcal{E}$. Therefore, (X, A, τ) is a soft T_1-space.

Definition 3.26. Let (X, A, τ) be a soft topological space over X. If for any $E^x_\alpha,\ E^y_\beta \in \mathcal{E}$ with $E^x_\alpha \neq E^y_\beta$, $\exists\,(F, A),\ (G, A) \in \tau$ such that $E^x_\alpha \,\widetilde{\in}\, (F, A),\ E^y_\beta \,\widetilde{\in}\, (G, A)$ and $(F, A) \,\widetilde{\cap}\, (G, A) = (\widetilde{\phi}, A)$, then (X, A, τ) is called a soft T_2-space.

Remark 3.27. Every soft T_2-space is obiously a soft T_1-space. The following example shows that the converse is not always ture.

Example 3.28. Let $X,\ A$ and τ be the same as in Example 3.25. Then (X, A, τ) is a soft T_1-space.
Let $(U, A),\ (V, A) \in \tau$ such that $(U, A),\ (V, A) \neq (\widetilde{\phi}, A)$ and $(U, A) \widetilde{\cap} (V, A) = (\widetilde{\phi}, A)$. Then $(U, A) \widetilde{\subseteq} [(\widetilde{X}, A) - (V, A)]$. This is not possible since (U, A) is infinite soft set and $[(\widetilde{X}, A) - (V, A)]$ is a finite soft set. Therefore, (X, A, τ) is not a soft T_2-space.

Example 3.29. Let R be the real number space and A be a non empty parameter set. Let for each $\alpha \in A,\ \tau^{\alpha}$ be the usual topology on R. Then (R, A, τ^*) where τ^* be the soft topology generated by $\{\tau^{\alpha},\ \alpha \in A\}$ as in Proposition 2.13, is a soft T_2-space.

Definition 3.30. A soft topological space (X, A, τ) is said to be *soft regular space* if for any soft closed set (F, A) and any soft element E^x_{α} such that $E^x_{\alpha} \widetilde{\notin} (F, A)$, $\exists$ open soft sets $(U, A),\ (V, A)$ such that $E^x_{\alpha} \widetilde{\in} (U, A),\ (F, A) \widetilde{\subseteq} (V, A)$ and $(U, A) \widetilde{\cap} (V, A) = (\widetilde{\phi}, A)$.

Proposition 3.31. *If a soft topological space (X, A, τ) is soft regular then $\forall\ E^x_{\alpha} \in \mathcal{E}$ and $\forall\ (U, A) \in \tau$ such that $E^x_{\alpha} \widetilde{\in} (U, A)$, $\exists\ (V, A) \in \tau$ such that $E^x_{\alpha} \widetilde{\in} (V, A)$ and $\overline{(V, A)} \widetilde{\subseteq} (U, A)$. The converse is true if τ is enriched.*

Proof. Let (X, A, τ) be a soft regular space. Let $E^x_{\alpha} \in \mathcal{E}$ and $(U, A) \in \tau$ such that $E^x_{\alpha} \widetilde{\in} (U, A)$. Consider $(F, A) = (\widetilde{X}, A) - (U, A)$. Then (F, A) is soft closed and $E^x_{\alpha} \widetilde{\notin} (F, A)$. Since (X, A, τ) is soft regular space, $\exists\ (V, A),\ (W, A) \in \tau$ such that $E^x_{\alpha} \widetilde{\in} (V, A),\ (F, A) \widetilde{\subseteq} (W, A)$ and $(V, A) \widetilde{\cap} (W, A) = (\widetilde{\phi}, A)$. Thus $(V, A) \widetilde{\subseteq} [(\widetilde{X}, A) - (W, A)]$ and hence $\overline{(V, A)} \widetilde{\subseteq} \overline{[(\widetilde{X}, A) - (W, A)]} = [(\widetilde{X}, A) - (W, A)] \widetilde{\subseteq} [(\widetilde{X}, A) - (F, A)] = (U, A)$. Therefore, the given condition is satisfied.

Conversely let the given condition is satisfied. Let $E^x_{\alpha} \in \mathcal{E}$ and (F, A) be soft closed set such that $E^x_{\alpha} \widetilde{\notin} (F, A)$. Consider $(U, A) = [(\widetilde{X}, A) - (F, A)]$. Then $(U, A) \in \tau$ such that $E^x_{\alpha} \widetilde{\in} (U, A)$. So, by the given condition $\exists\ (V, A) \in \tau$ such that $E^x_{\alpha} \widetilde{\in} (V, A)$ and $\overline{(V, A)} \widetilde{\subseteq} (U, A)$. Let $(W, A) = [(\widetilde{X}, A) - \overline{(V, A)}]$. Since τ is enriched, $(W, A) \in \tau$ and $(W, A) = [(\widetilde{X}, A) - \overline{(V, A)}] \widetilde{\supseteq} [(\widetilde{X}, A) - (U, A)] = (F, A)$, also $(V, A) \widetilde{\cap} (W, A) = (V, A) \widetilde{\cap} [(\widetilde{X}, A) - \overline{(V, A)}] = (\widetilde{\phi}, A)$ Therefore, (X, A, τ) is a soft regular space. □

Definition 3.32. A soft topological space (X, A, τ) is said to be *soft normal space* if for any two disjoint soft closed sets (F_1, A) and (F_2, A) $\exists$ soft open sets $(U, A),\ (V, A)$ such that $(F_1, A) \widetilde{\subseteq} (U, A),\ (F_2, A) \widetilde{\subseteq} (V, A)$ and $(U, A) \widetilde{\cap} (V, A) = (\widetilde{\phi}, A)$.

Proposition 3.33. *If a soft topological space (X, A, τ) is soft normal, then $\forall$ soft closed set (F, A) and $\forall\ (U, A) \in \tau$ such that $(F, A) \widetilde{\subseteq} (U, A)$, $\exists\ (V, A) \in \tau$ such that $(F, A) \widetilde{\subseteq} (V, A)$ and $\overline{(V, A)} \widetilde{\subseteq} (U, A)$. The converse is true if τ is enriched.*

Proof. Let (X, A, τ) be a soft normal space.
Let (F, A) be a soft closed set and $(U, A) \in \tau$ such that $(F, A) \widetilde{\subseteq} (U, A)$.
Consider $(F_1, A) = (\widetilde{X}, A) - (U, A)$.
Then (F_1, A) is soft closed set and $(F, A) \widetilde{\cap} (F_1, A) = (\widetilde{\phi}, A)$.
Since (X, A, τ) is soft normal space, $\exists\ (V, A),\ (W, A) \in \tau$ such that
$(F, A) \widetilde{\subseteq} (V, A),\ (F_1, A) \widetilde{\subseteq} (W, A)$ and $(V, A) \widetilde{\cap} (W, A) = (\widetilde{\phi}, A)$.

Thus $(V, A) \widetilde{\subseteq} [(\widetilde{X}, A) - (W, A)]$ and hence
$\overline{(V, A)} \widetilde{\subseteq} \overline{[(\widetilde{X}, A) - (W, A)]} = [(\widetilde{X}, A) - (W, A)] \widetilde{\subseteq} [(\widetilde{X}, A) - (F_1, A)] = (U, A)$.
Therefore, the given condition is satisfied. Conversely let the given condition be satisfied.
Let $(F_1, A),\ (F_2, A)$ be two disjoint soft closed sets.
Consider $(U, A) = [(\widetilde{X}, A) - (F_2, A)]$. Then $(U, A) \in \tau$ such that $(F_1, A) \widetilde{\subseteq} (U, A)$.
So, by the given condition $\exists\, (V, A) \in \tau$ such that $(F_1, A) \widetilde{\subseteq} (V, A)$ and $\overline{(V, A)} \widetilde{\subseteq} (U, A)$.
Let $(W, A) = [(\widetilde{X}, A) - \overline{(V, A)}]$. Since τ is enriched, $(W, A) \in \tau$ and
$(W, A) = [(\widetilde{X}, A) - \overline{(V, A)}] \widetilde{\supseteq} [(\widetilde{X}, A) - (U, A)] = (F_2, A)$, also
$(V, A) \widetilde{\cap} (W, A) = (V, A) \widetilde{\cap} [(\widetilde{X}, A) - \overline{(V, A)}] = (\widetilde{\phi}, A)$
Therefore, (X, A, τ) is a soft normal space. □

4. Some properties of neighbourhood system of identity soft elements of group soft topology

Let us now turn our attention to the identity element of a group (with a soft topology defined on it) and look into basic facets of the system of neighbourhood of the identity soft elements of group soft topology. In this section, τ is a group soft topology over a group G and $E^e_\alpha,\ \alpha \in A$ are the identity soft elements of (G, A, τ) where e is the identity element of the group G.

Definition 4.1 ([18])**.** Let G be a group and τ be a soft topology on G. Then τ is said to be a group soft topology on G if the mappings

(i) $f : (G \times G, A, \tau \,\widetilde{\times}\, \tau) \to (G, A, \tau)$, defined by $f(x, y) = xy$ and
(ii) $g : (G, A, \tau) \to (G, A, \tau)$ defined by $g(x) = x^{-1}$
are soft continuous.

Proposition 4.2 ([18])**.** *Let τ be a group soft topology on G. Then*
$$\tau^\alpha = \{F(\alpha) :\ (F, A) \in \tau\}$$
is a group topology on G, $\forall\ \alpha \in A$.

Proposition 4.3 ([18])**.** *Let τ^α be a group topology on G, $\forall\ \alpha \in A$. Then $\tau^* = \{(F, A) \in S(X, A) :\ F(\alpha) \in \tau^\alpha,\ \forall\ \alpha \in A\}$ is a group soft topology on G.*

Definition 4.4 ([18])**.** Let $(F, A),\ (H, A)$ be two soft sets over a group $(G, \circ)$. Then

(i) $(F, A) \,\widetilde{\circ}\, (H, A) = (F \,\widetilde{\circ}\, H, A)$ where $(F \,\widetilde{\circ}\, H)(\alpha) = F(\alpha) \circ H(\alpha),\ \forall\ \alpha \in A$.
(ii) $(F, A)^{-1} = (F^{-1}, A)$ where $F^{-1}(\alpha) = [F(\alpha)]^{-1},\ \forall\ \alpha \in A$.

Proposition 4.5 ([18])**.** *Let τ be a group soft topology on a group G and τ contains all those soft sets (F, A) such that $F(\alpha) = \phi$ or G, $\forall\ \alpha \in A$ i.e. τ is an enriched soft topology. Then the mappings*

(i) *$R_a : (G, A, \tau) \to (G, A, \tau)$ defined by $R_a(x) = xa$*
(ii) *$L_a : (G, A, \tau) \to (G, A, \tau)$ defined by $L_a(x) = ax$*
(iii) *$\alpha : (G, A, \tau) \to (G, A, \tau)$ defined by $\alpha(x) = axa^{-1}$*
are all are soft homeomorphisms.

Proposition 4.6 ([18])**.** *Let τ be a group soft topology on a group G and τ contains all those soft sets (F, A) such that $F(\alpha) = \phi$ or G, $\forall\ \alpha \in A$ i.e. τ is an* *enriched soft topology. If $(P, A) \in \tau$, (H, A) is closed soft set and $a \in G$ then $(P \,\tilde{\circ}\, E_a, A)$, $(E_a \,\tilde{\circ}\, P, A)$, $(P, A)^{-1}$ are open soft sets and $(E_a \,\tilde{\circ}\, H, A)$, $(H, \,\tilde{\circ}\, E_a, A)$, (H^{-1}, A) are closed soft sets.*

Definition 4.7 ([18])**.** A soft nbd system $\mathcal{B}$ of a soft element E^x_α is said to be a fundamental soft nbd system or soft nbd base of E^x_α if for any soft nbd (F, A) of E^x_α, $\exists\ (H, A) \in \mathcal{B}$ such that $(H, A) \,\tilde{\subseteq}\, (F, A)$.

Proposition 4.8 ([18])**.** *Let $\mathcal{B}$ be a fundamental soft nbd system of the soft element E^e_α in (G, A, τ). Then $\forall\ (F, A) \in \mathcal{B}$, $(F, A)^{-1}$ is also a soft nbd of E^e_α.*

Definition 4.9 ([18])**.** A soft set (F, A) over a group G is said to be symmetric if $(F, A)^{-1} = (F, A)$.

Definition 4.10 ([18])**.** A soft nbd system is said to be a symmetric soft nbd system if all the members of that system are symmetric. Also a fundamental soft nbd system $\mathcal{B}$, every member of which is symmetric, is said to be a fundamental symmetric soft nbd system.

Proposition 4.11 ([18])**.** *Let τ be a group soft topology over a group G and e be the identity element of G. Then there exists a fundamental symmetric soft nbd system of the soft element E^e_α in (G, A, τ).*

Proposition 4.12 ([18])**.** *For each soft nbd (W, A) of E^e_α in (G, A, τ) where τ is an enriched soft topology and for each finite set $\{k_1,\ k_2, \ldots,\ k_n\}$ where $k_i = 1$ or -1, $i = 1, 2, \ldots, n$, $\exists$ a symmetric soft nbd (U, A) of E^e_α such that $(U, A)^{k_1} \,\tilde{\circ}\, (U, A)^{k_2} \,\tilde{\circ} \ldots \tilde{\circ}\, (U, A)^{k_n} \,\tilde{\subseteq}\, (W, A)$.*

Proposition 4.13. *If $\mathcal{B}$ be a soft nbd base of E^e_α in (G, A, τ) where τ is an enriched soft topology then $\mathcal{B}' = \{E_x \,\tilde{\circ}\, (F, A) :\ (F, A) \in \mathcal{B}\}$ is a soft nbd base of E^x_α.*

Proof. Let (G, A) be any soft nbd of E^x_α. Then $L_{x^{-1}}[(G, A)]$ is a soft nbd of $L_{x^{-1}}(E^x_\alpha)$ i.e. $E_{x^{-1}} \,\tilde{\circ}\, (G, A)$ is a soft nbd of $E_{x^{-1}} \,\tilde{\circ}\, E^x_\alpha = E^e_\alpha$.
So $\exists\ (F, A) \in \mathcal{B}$ such that $E^e_\alpha \,\tilde{\in}\, (F, A) \,\tilde{\subseteq}\, E_{x^{-1}} \,\tilde{\circ}\, (G, A)$.
Hence $L_x[(F, A)]$ is a soft nbd of $L_x(E^e_\alpha)$ such that

$$L_x(E^e_\alpha) \,\tilde{\in}\, L_x[(F, A)] \,\tilde{\subseteq}\, L_x(E_{x^{-1}} \,\tilde{\circ}\, (G, A)).$$

Thus $E^x_\alpha \,\tilde{\in}\, E_x \,\tilde{\circ}\, (F, A) \,\tilde{\subseteq}\, E_x \,\tilde{\circ}\, (E_{x^{-1}} \,\tilde{\circ}\, (G, A)) = (G, A)$. Therefore $\mathcal{B}'$ is a soft nbd base of E^x_α. □

Proposition 4.14. *If $\mathcal{B}$ be a soft nbd base of E^x_α in (G, A, τ) where τ is an enriched soft topology then $\mathcal{B}' = \{E_{x^{-1}} \,\tilde{\circ}\, (F, A) :\ (F, A) \in \mathcal{B}\}$ is a soft nbd base of E^e_α.*

Proof. Proof is similar to that of Proposition 4.13. □

Proposition 4.15. *If $\mathcal{B}$ be the system of all soft nbds of E^e_α, in (G, A, τ), $\alpha \in A$ where τ is an enriched soft topology. Then $\mathcal{B}_\alpha = \{U(\alpha) :\ (U, A) \in \mathcal{B}\}$ is the system of all nbds of e in (G, τ^α).*

Proof. Let $\mathcal{B}$ be the system of all soft nbds of E^e_α, $\alpha \in A$ in (G, A, τ) and $\mathcal{B}_\alpha = \{U(\alpha) :\ (U, A) \in \mathcal{B}\}$.
Let U be any nbd of e in (G, τ^α). Then $\exists\ V \in \tau^\alpha$ such that $e \in V \subseteq U$.

Now constract a soft set (U, A), where $U(\alpha) = U$ and $U(\beta) = \phi,\ \forall\ \beta \neq \alpha$.
Also since $V \in \tau^\alpha$ and τ is enriched, we have a soft set $(V, A) \in \tau$ such that $V(\alpha) = V$ and $V(\beta) = \phi,\ \beta \neq \alpha$. Clearly $E^e_\alpha \,\widetilde{\in}\, (U, A) \,\widetilde{\subseteq}\, (V, A)$.
Thus, $(U, A) \in \mathcal{B}$ and hence $U(\alpha) = U \in \mathcal{B}_\alpha$.
Therefore, $\mathcal{B}_\alpha = \{U(\alpha) :\ (U, A) \in \mathcal{B}\}$ is the system of all nbds of e in (G, τ^α). □

Proposition 4.16. *Let $\mathcal{B}$ be the system of all soft nbds of E^e_α in (G, A, τ) and (F, A) be any soft set over G. Then $\overline{F}(\alpha) = \bigcap_{(U,A)\in\mathcal{B}}[F(\alpha)\circ U(\alpha)] = \bigcap_{(U,A)\in\mathcal{B}}[U(\alpha)\circ F(\alpha)]$.*

Proof. Since $\mathcal{B}$ is the system of all soft nbds of E^e_α in (G, A, τ), it follows that $\mathcal{B}_\alpha$ is the system of all nbds of e in (G, τ^α). Also (F, A) is a soft set over G.
So $F(\alpha)$ is a subset of G. Then from the topological group theory we get
$\overline{F(\alpha)} = \overline{F}(\alpha) = \bigcap_{U(\alpha)\in\mathcal{B}_\alpha}[F(\alpha) \circ U(\alpha)] = \bigcap_{(U,A)\in\mathcal{B}}[F(\alpha) \circ U(\alpha)]$.
Similarly we get $\overline{F}(\alpha) = \bigcap_{(U,A)\in\mathcal{B}}[U(\alpha) \circ F(\alpha)]$. □

Proposition 4.17. *For any soft nbd (U, A) of E^e_α in (G, A, τ) where τ is an enriched soft topology, $\exists$ a soft nbd (V, A) of E^e_α such that $\overline{(V, A)} \,\widetilde{\subseteq}(U, A)$.*

Proof. Let (U, A) be any nbd of E^e_α. Then by Proposition 4.12, $\exists$ a nbd (W, A) of E^e_α such that $(W, A) \,\widetilde{\circ}(W, A) \,\widetilde{\subseteq}\, (U, A)$.
Also, by the Proposition 4.16, $\overline{W}(\alpha) = \overline{W(\alpha)}^\alpha \subseteq [W(\alpha) \circ W(\alpha)]$.
Now construct a soft set (V, A) where $V(\alpha) = W(\alpha)$ and $V(\beta) = \phi,\ \forall\ \beta \neq \alpha$.
Then $(V, A) \in \tau$ and $E^e_\alpha \,\widetilde{\in}\, (V, A) \,\widetilde{\subseteq}\, (W, A)$.
So (V, A) is a soft nbd of E^e_α such that $\overline{V}(\alpha) \subseteq \overline{W}(\alpha) \subseteq [W(\alpha) \circ W(\alpha)]$ and $\phi = \overline{V}(\beta) \subseteq [W(\beta) \circ W(\beta)],\ \forall\ \beta \neq \alpha$ and hence $\overline{(V, A)} \,\widetilde{\subseteq}\, [(W, A) \,\widetilde{\circ}\, (W, A)] \,\widetilde{\subseteq}\, (U, A)$. □

Proposition 4.18. *If an enriched soft topology τ be a group topology over a group G, then the soft topological space (G, A, τ) is soft regular.*

Proof. Let $E^x_\alpha \in \mathcal{E}$ and $(U, A) \in \tau$ such that $E^x_\alpha \,\widetilde{\in}\, (U, A)$.
Then (U, A) is a soft nbd of E^x_α and hence by Proposition 4.13,
$E_{x^{-1}} \,\widetilde{\circ}\, (U, A) = (W, A)$(say) is a soft nbd of E^e_α.
Then $\exists$ soft nbd (V, A) of E^e_α such that $\overline{(V, A)} \,\widetilde{\subseteq}\, (W, A)$.
So, $E_x \,\widetilde{\circ}\, (V, A)$ is a soft nbd of E^x_α and hence $\exists\ (P, A) \in \tau$ such that

$$E^x_\alpha \,\widetilde{\in}\, (P, A) \,\widetilde{\subseteq}\, E_x \,\widetilde{\circ}\, (V, A).$$

Thus, $\overline{(P, A)} \,\widetilde{\subseteq}\, \overline{E_x \,\widetilde{\circ}\, (V, A)} = \overline{L_x[(V, A)]} = L_x[\overline{(V, A)}]$ (since L_x is soft homeomorphism) $= E_x \,\widetilde{\circ}\, \overline{(V, A)} \,\widetilde{\subseteq}\, E_x \,\widetilde{\circ}\, E_{x^{-1}} \,\widetilde{\circ}\, (U, A) = (U, A)$.
Therefore, the soft topological space (G, A, τ) is soft regular. □

Proposition 4.19. *Let τ be a group soft topology over a group G where τ is an enriched soft topology. Then there exists a fundamental soft nbd system $\mathcal{B}$ of closed soft nbds of E^e_α such that*

(i) *each $(U, A) \in \mathcal{B}$ is symmetric;*
(ii) $\forall\ (U, A) \in \mathcal{B},\ \exists\ (V, A) \in \mathcal{B}$ *such that* $(V, A)\widetilde{\circ}(V, A) \,\widetilde{\subseteq}\, (U, A)$;
(iii) $\forall\ (U, A) \in \mathcal{B}$ *and* $\forall\ a \in G\ \exists\ (V, A) \in \mathcal{B}$ *such that* $(V, A) \,\widetilde{\subseteq}\, E^{a^{-1}}_\alpha \,\widetilde{\circ}\, (U, A) \,\widetilde{\circ}\, E^a_\alpha$ *i.e.* $E^a_\alpha \,\widetilde{\circ}\, (V, A) \,\widetilde{\circ}\, E^{a^{-1}}_\alpha \,\widetilde{\subseteq}\, (U, A)$;
(iv) $\forall\ (V, A) \in \mathcal{B}$ *and* $\forall\ E^a_\alpha \,\widetilde{\in}\, (V, A),\ \exists\ (U, A) \in \mathcal{B}$ *such that* $E^a_\alpha \,\widetilde{\circ}\, (U, A) \,\widetilde{\subseteq}(V, A)$.

Proof. (i) We know that $\exists$ a fundamental system $\mathcal{B}'$ of symmetric nbds of E^e_α.
Let $\mathcal{B} = \{\overline{(U,A)} : (U,A) \in \mathcal{B}'\}$. Since $(U,A) \in \mathcal{B}' \Rightarrow (U,A)$ is symmetric and hence $(U,A)^{-1} = (U,A)$. Again since the inverse mapping is soft homeomorphism and hence $\forall\ \alpha \in A$, $\overline{[U^{-1}]}(\alpha) = \overline{U^{-1}(\alpha)}^{\alpha} = \overline{[U(\alpha)]^{-1}}^{\alpha} = [\overline{U(\alpha)}^{\alpha}]^{-1} = [\overline{U}(\alpha)]^{-1} = \overline{U}^{-1}(\alpha)$. So $\overline{(U,A)} = \overline{(U,A)^{-1}} = [\overline{(U,A)}]^{-1}$. Thus $\overline{(U,A)}$ is a symmetric closed nbd of E^e_α. We shall now show that $\mathcal{B}$ is a fundamental system of nbds of E^e_α. Let (W,A) be any nbd of E^e_α. Then by Proposition 4.17, there exists a nbd (V,A) of E^e_α such that $\overline{(V,A)}\ \widetilde{\subseteq}(W,A)$. Since $\mathcal{B}'$ is a fundamental system of nbds of E^e_α, $\exists\ (U,A) \in \mathcal{B}'$ such that $(U,A)\ \widetilde{\subseteq}\ (V,A)$. Then $\overline{(U,A)} \in \mathcal{B}$ and $\overline{(U,A)}\ \widetilde{\subseteq}\ \overline{(V,A)}\ \widetilde{\subseteq}\ (W,A)$. Therefore $\mathcal{B}$ is a fundamental system of closed nbds of E^e_α such that each member of $\mathcal{B}$ is symmetric.

(ii) Proof follows from Proposition 4.12.

(iii) Let $(U,A) \in \mathcal{B}$. Since $E_a\ \widetilde{\circ}\ E^e_\alpha\ \widetilde{\circ}\ E_{a^{-1}} = E^e_\alpha$, (U,A) is a soft nbd of E^e_α and translation mapping is soft homeomorphism, $\exists$ a soft nbd (V_1,A) of E^e_α such that $E_a\ \widetilde{\circ}\ (V_1,A)\ \widetilde{\circ}\ E_{a^{-1}}\ \widetilde{\subseteq}\ (U,A)$. Since $\mathcal{B}$ is a fundamental soft nbd system of E^e_α, $\exists\ (V,A) \in \mathcal{B}$ such that $(V,A)\ \widetilde{\subseteq}\ (V_1,A)$. Therefore

$$E^a_\alpha\ \widetilde{\circ}\ (V,A)\ \widetilde{\circ}\ E^{a^{-1}}_\alpha\ \widetilde{\subseteq}\ E_a\ \widetilde{\circ}\ (V,A)\ \widetilde{\circ}\ E_{a^{-1}}\ \widetilde{\subseteq}\ E_a\ \widetilde{\circ}\ (V_1,A)\ \widetilde{\circ}\ E_{a^{-1}}\ \widetilde{\subseteq}\ (U,A).$$

(iv) Let $(V,A) \in \mathcal{B}$ and $E^a_\alpha\ \widetilde{\in}\ (V,A)$.
Since $E_a\ \widetilde{\circ}\ E^e_\alpha = E^a_\alpha$ i.e. $L_a(E^e_\alpha) = E^a_\alpha$, (V,A) is a soft nbd of E^a_α and left translation mapping is soft homeomorphism, there exists a soft nbd (U_1,A) of E^e_α such that $L_a[(U_1,A)]\ \widetilde{\subseteq}\ (V,A)$ i.e. $E_a\ \widetilde{\circ}\ (U_1,A)\ \widetilde{\subseteq}(V,A)$. Since $\mathcal{B}$ is a fundamental soft nbd system of E^e_α, $\exists\ (U,A) \in \mathcal{B}$ such that $(U,A)\ \widetilde{\subseteq}\ (U_1,A)$. Therefore $E^a_\alpha\ \widetilde{\circ}\ (U,A)\ \widetilde{\subseteq}\ E_a\ \widetilde{\circ}\ (U,A)\ \widetilde{\subseteq}\ E_a\ \widetilde{\circ}\ (U_1,A)\ \widetilde{\subseteq}\ (V,A)$. □

Proposition 4.20. *Let τ be a group soft topology over a group G where τ is an enriched soft topology and e as the identity element of G. Then the following statements are equivalent.*

(i) *(G,A,τ) is a soft T_0-space.*
(ii) *(G,A,τ) is a soft T_1-space.*
(iii) *(G,A,τ) is a soft T_2-space.*
(iv) *$\widetilde{\bigcap}\{(U,A) : (U,A) \in \mathcal{B}\} = \{E^e_\alpha\}$, where $\mathcal{B}$ is a fundamental system of nbds of E^e_α, $\alpha \in A$.*

Proof. Let (G,A,τ) be a soft T_0-space and E^x_α, $E^y_\beta \in \mathcal{E}$ such that $E^x_\alpha \neq E^y_\beta$. If $\alpha \neq \beta$ and since τ is enriched there exists open soft sets (F,A), (G,A) where $F(\alpha) = X$, $F(\beta) = \phi$, $\forall\ \beta \neq \alpha$ and $G(\beta) = X$, $G(\alpha) = \phi$, $\forall\ \alpha \neq \beta$. So $E^x_\alpha\ \widetilde{\in}\ (F,A)$, $E^y_\beta\ \widetilde{\notin}\ (F,A)$, $E^x_\alpha\ \widetilde{\notin}\ (G,A)$, $E^y_\beta\ \widetilde{\in}\ (G,A)$. Thus we are to prove the case when $\alpha = \beta$ and $x \neq y$. Since (G,A,τ) is a soft T_0-space, it follows that there exists an open soft set (F,A) such that one of E^x_α and E^y_α belong to (F,A) but the other does not belong to (F,A). Let $E^x_\alpha\ \widetilde{\in}\ (F,A)$ but $E^y_\alpha\ \widetilde{\notin}\ (F,A)$. Consider $(V,A) = E_{x^{-1}}\ \widetilde{\circ}\ (F,A)$. Then (V,A) is a open soft set containing E^e_α and $(U,A) = (V,A)\ \widetilde{\cap}\ (V,A)^{-1}$ is a symmetric open soft set containing E^e_α.
So $E_y\ \widetilde{\circ}\ (U,A)$ is an open soft set containing E^y_α.
We shall now show that $E^x_\alpha\ \widetilde{\notin}\ E_y\ \widetilde{\circ}\ (U,A)$. Now if possible, $E^x_\alpha\ \widetilde{\in}\ E_y\ \widetilde{\circ}\ (U,A)$, then

$E_\alpha^{x^{-1}} \widetilde{\in} (U,A)^{-1} \widetilde{\circ} E_{y^{-1}} = (U,A) \widetilde{\circ} E_{y^{-1}} \widetilde{\subseteq} (V,A) \widetilde{\circ} E_{y^{-1}} = E_{x^{-1}} \widetilde{\circ} (F,A) \widetilde{\circ} E_{y^{-1}}$. Thus $E_\alpha^e = E_\alpha^x \widetilde{\circ} E_\alpha^{x^{-1}} \widetilde{\in} E_\alpha^x \widetilde{\circ} E_{x^{-1}} \widetilde{\circ} (F,A) \widetilde{\circ} E_{y^{-1}} = E_\alpha^e \widetilde{\circ} (F,A) \widetilde{\circ} E_{y^{-1}}$. Therefore $E_\alpha^y \widetilde{\in} (F,A)$, which is a contradiction. Therefore (G,A,τ) is a T_1-space. This shows that $(i) \Rightarrow (ii)$.

Next assume that (G,A,τ) be T_1-space. Let $E_\alpha^x, E_\alpha^y \in \mathcal{E}$ such that $E_\alpha^x \neq E_\alpha^y$. Since τ is T_1-space, it follows that $\{E_\alpha^x\}$ is closed soft set and hence $(P,A) = (\widetilde{G},A) - E_\alpha^x$ is open soft set containing E_α^y. So $E_{y^{-1}} \widetilde{\circ} (P,A)$ is a nbd of E_α^e. Then by Proposition 4.12, there exists an open soft nbd (V,A) of E_α^e such that $(V,A) \widetilde{\circ} (V,A)^{-1} \widetilde{\subseteq} E_{y^{-1}} \widetilde{\circ} (P,A)$. So $E_y \widetilde{\circ} (V,A)$ is an open soft nbd of E_α^y and $(Q,A) = (\widetilde{G},A) - \overline{[E_y \widetilde{\circ} (V,A)]}$ is an open soft set. If $E_\alpha^x \widetilde{\in} \overline{[E_y \widetilde{\circ} (V,A)]}$, then $[E_x \widetilde{\circ} (V,A)] \widetilde{\cap} [E_y \widetilde{\circ} (V,A)] \neq (\widetilde{\phi},A)$ and

$$E_\alpha^x \widetilde{\in} E_y \widetilde{\circ} (V,A) \widetilde{\circ} (V,A)^{-1} \widetilde{\subseteq} E_y \widetilde{\circ} E_{y^{-1}} \widetilde{\circ} (P,A) = (P,A),$$

which is a contradiction because $E_\alpha^x \widetilde{\notin} (P,A)$. Thus (Q,A) is a soft open containing E_α^x. Clearly $E_\alpha^x \widetilde{\in} (Q,A)$, $E_\alpha^y \widetilde{\in} [E_y \widetilde{\circ} (V,A)]$ and $(Q,A) \widetilde{\cap} [E_y \widetilde{\circ} (V,A)] = (\widetilde{\phi},A)$. Therefore $(ii) \Rightarrow (iii)$.

Next let (G,A,τ) be a soft T_2-space and $\mathcal{B}$ be a fundamental system of nbds of E_α^e. Let $E_\alpha^x \widetilde{\in} \widetilde{\bigcap}\{(U,A) : (U,A) \widetilde{\in} \mathcal{B}\}$. If possible let $E_\alpha^x \neq E_\alpha^e$. Since (G,A,τ) is a T_2-space, there exists a soft open set (F,A) such that $E_\alpha^e \widetilde{\in} (F,A)$ but $E_\alpha^x \widetilde{\notin} (F,A)$. Since $\mathcal{B}$ is a fundamental system of nbds of E_α^e, $\exists\ (U,A) \in \mathcal{B}$ such that $(U,A) \widetilde{\subseteq} (F,A)$. So $E_\alpha^x \widetilde{\notin} (U,A)$, which contradicts our assumption $E_\alpha^x \widetilde{\in} (U,A)$, $\forall\ (U,A) \in \mathcal{B}$. Thus $E_\alpha^x = E_\alpha^e$ and $\widetilde{\bigcap}\{(U,A) : (U,A) \in \mathcal{B}\} = \{E_\alpha^e\}$. This shows that $(iii) \Rightarrow (iv)$. Again let $\widetilde{\bigcap}\{(U,A) : (U,A) \in \mathcal{B}\} = \{E_\alpha^e\}$ and $E_\alpha^x, E_\alpha^y \in \mathcal{E}$ such that $E_\alpha^x \neq E_\alpha^y$. Then $E_\alpha^{y^{-1}} \widetilde{\circ} E_\alpha^x \neq E_\alpha^e$ and $\exists\ (U,A) \in \mathcal{B}$ such that $E_\alpha^{y^{-1}} \widetilde{\circ} E_\alpha^x \widetilde{\notin} (U,A)$.
So $E_\alpha^x \widetilde{\notin} E_y \widetilde{\circ} (U,A)$. Thus $E_y \widetilde{\circ} (U,A)$ is a soft open set containing E_α^y but $E_\alpha^x \widetilde{\notin} E_y \widetilde{\circ} (U,A)$. Therefore (G,A,τ) is a soft T_0-space and $(iv) \Rightarrow (i)$. Therefore the statements (i), (ii), (iii) and (iv) are equivalent. □

5. Conclusion

In this paper, we have studied separation properties of group soft topology [18]. In this context the neibourhood systems of soft elements play important role because of the homeomorphism property of the translation operations. It is just a beginning of this study. There is an ample scope of its generalization where the underlying mappings are soft mappings instead of ordinary mappings and parallel soft product is replaced by usual soft product.

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References

[1] H. Aktas and N. Cagman, Soft sets and soft groups, Inform. Sci. 177 (2007) 2726–2735.

[2] A. Aygunoglu and H. Aygun, Some notes on soft topological spaces, Neural Comput & Applic (2012) S113–S119.

[3] K. V. Babitha and J. J. Sunil, Soft set relations and functions, Comput. Math. Appl. 60 (2010) 1840–1849.

[4] N. Cagman, S. Karatas and S. Enginoglu, Soft topology, Comput. Math. Appl. 62 (2011) 351–358.

[5] F. Feng, Y. B. Jun and X. Zhao, Soft semirings, Comput. Math. Appl. 56 (2008) 2621–2628.

[6] F. Feng, Y. B. Jun, X. Y. Liu and L. F. Li, An adjustable approach to fuzzy soft set based decision making, J. Comput. Appl. Math. 234 (2010) 10–20.

[7] F. Feng, Y. M. Li and N. Cagman, Generalized uni-int decision making schemes based on choice value soft sets, European J. Oper. Res. 220 (2012) 162–170.

[8] F. Feng and Y. M. Li, Soft subsets and soft product operations, Inform. Sci. 232 (2013) 44–57.

[9] H. Hazra, P. Majumdar and S. K.Samanta, Soft topology, Fuzzy Inform. Eng. 4(1) (2012) 105–115.

[10] T. Husain, Introduction to topological groups, W. B. Saunders Company, Philadelphia 1966.

[11] S. Hussain and B. Ahmad, Some properties of soft topological spaces, Comput. Math. Appl. 62 (2011) 4058–4067.

[12] A. Kharal and B. Ahmad, Mappings on soft classes, New Math. Nat. Comput. 7(3) 471–481.

[13] P. K. Maji, R. Biswas and A. Roy, Soft set theory, Comput. Math. Appl. 45 (2003) 555–562.

[14] P. K. Maji, A. Roy and R. Biswas, An application of soft sets in a decision making problem, Comput. Math. Appl. 44 (2002) 1077–1083.

[15] P. Majumdar and S. K. Samanta, On soft mappings, Comput. Math. Appl. 60 (2010) 2666–2672.

[16] P. Majumdar and S. K. Samanta, Decision making based on similarity measure of vague soft sets, J. Intell. Fuzzy Systems 24 637–646.

[17] D. Molodtsov, Soft set theory-First results, Comput. Math. Appl. 37 (1999) 19–31.

[18] Sk. Nazmul and S. K. Samanta, Group soft topology, J. Fuzzy Math. 22(2) (2014) 435–450.

[19] Sk. Nazmul and S. K. Samanta, Soft topological groups, Kochi J. Math. 5 (2010) 151–161.

[20] Sk. Nazmul and S. K. Samanta, Soft topological soft groups, Mathematical Sciences 6:66 (2012) 1–10.

[21] Sk. Nazmul and S. K. Samanta, Neighbourhood properties of soft topological spaces, Ann. Fuzzy Math. Inform. 6(1) (2013) 1–15.

[22] M. Shabir and M. Ali, Soft ideals and generalized fuzzy ideals in semigroups, New Math. Nat. Comput. 5 599–615.

[23] M. Shabir and M. Naz, On soft toplogical spaces, Comput. Math. Appl. 61(7) (2011) 1786–1799.

[24] A. R. Roy and P. K. Maji, A fuzzy soft set theoretic approach to decision making problems, J. Comput. Appl. Math. 203 (2007) 412–418.

[25] I. Zorlutuna, M. Akdag, W. K. Min and S. Atmaca, Remarks on soft topological spaces, Ann. Fuzzy Math. Inform. 3(2) (2012) 171–185.

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