Ideal convergence of nets in fuzzy topological spaces

B M Uzzal Afsan

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Abstract. In this paper, we have initiated the concept of convergence of nets of fuzzy points in a fuzzy topological space \((X, \sigma)\) via an ideal \(I\) (namely, \(I\)-convergence) and investigated its several properties. Two new limits, namely fuzzy upper and lower \(I\)-limits of nets of fuzzy sets are being introduced; several properties and their mutual relationships have been investigated. Finally, applications of the concept of the upper \(I\)-limit of nets are given to characterize various fuzzy covering properties.

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Corresponding Author: B M Uzzal Afsan (uzlafsan@gmail.com)

1. Introduction

In real line, the statistical convergence of sequences, introduced by Fast [13], has been developed by a good number of researchers, namely Červeňanský [4], Connor [6], Connor and Kline [7], Frídy [14, 15], Frídy et. al. [16], Kostyrko et. al. [25], Miller [30] and Šalát and Tijdeman [38]. The concept of \(I\)-convergence of sequences of real numbers was introduced by Kostyrko et. al. [26] and \(I\)-limit superior and inferior were initiated by Demirci [10]. Those concepts were further studied by Das and Lahiri [9] in general topological spaces. Some recent research works related to \(I\)-convergence are found in the papers of Das et. al. [8], Komisarski [24], Kumar [27], Mursaleen and Alotaibi [31], Mursaleen et. al. [34], Mursaleen and Mohiuddine [32, 33], Šahiner et. al. [37] and Šalát et. al. [39].

After the invention of fuzzy sets by Zadeh [44], a major task for the mathematicians was to fuzzify different existing concepts of pure mathematics. In this fashion, Nuray and Savaş [36] defined the concepts of statistical convergence and statistical
In this paper, the cardinalities of an ordinary set $K$ is called non-trivial if $N \not\in I$ and $S$ is defined as a non-empty family of numbers.

Throughout this paper, spaces $(X, \sigma)$ and $(Y, \delta)$ (or simply $X$ and $Y$) represent non-empty fuzzy topological spaces due to Chang [5] and the symbols $I$ and $I^X$ have been used for the unit closed interval $[0,1]$ and the set of all functions with domain $X$ and codomain $I$ respectively. The support of a fuzzy set $A$ is the set $\{x \in X : A(x) > 0\}$ and is denoted by $supp(A)$. A fuzzy set with only non-zero value $\lambda \in [0,1]$ at only one element $x \in X$ is called a fuzzy point and is denoted by $x_\lambda$ and the set of all fuzzy points of a fuzzy topological space is denoted by $FP(X)$. For any two fuzzy sets $A, B$ of $X$, $A \leq B$ if and only if $A(x) \leq B(x)$ for all $x \in X$. A fuzzy point $x_\lambda$ is said to be in a fuzzy set $A$ (denoted by $x_\lambda \in A$) if $x_\lambda \leq A$, that is, if $\lambda \leq A(x)$. The constant fuzzy sets of $X$ with values 0 and 1 are denoted by $0$ and $1$ respectively. A fuzzy set $A$ is said to be quasi-coincident with $B$ (written as $AqB$) [41] if $A(x) + B(x) > 1$ for some $x \in X$. A fuzzy set $A$ is said to be not quasi-coincident with $B$ (written as $A q B$) [26] if $A(x) + B(x) \leq 1$, for all $x \in X$. A fuzzy open set $A$ of $X$ is called fuzzy quasi-neighborhood of a fuzzy point $x_\lambda$ if $x_\lambda q A$. It is well-known that a function $\psi : X \rightarrow Y$ is fuzzy continuous [5] if for every fuzzy point $x_\lambda$ and every fuzzy quasi-neighborhood $V$ of a fuzzy point $\psi(x_\lambda)$, there exists a fuzzy quasi-neighborhood $U$ of a fuzzy point $x_\lambda$ such that $\psi(U) \subseteq V$.

Throughout this paper, $\mathcal{N}$ stands for a directed set. An ideal on a non-empty set $S$ is defined as a non-empty family $I$ of subsets of $S$ satisfying (i) $\emptyset \in I$, (ii) $A \in I$ and $B \subseteq A \Rightarrow B \in I$ and (iii) $A \in I$ and $B \in I \Rightarrow A \cup B \in I$. An ideal $I$ on $\mathcal{N}$ is called non-trivial if $\mathcal{N} \notin I$. A non-trivial ideal $I$ on $\mathcal{N}$ is called admissible [9] if $\mathcal{N}$ and $\mathcal{N}$.
Suppose if possible, let \( A_n = \{ n \in \mathcal{N} : n \geq \lambda \} \) for all \( \lambda \in \mathcal{N} \). Throughout this paper, \( \mathcal{I} \) stands for an admissible ideal on \( \mathcal{N} \).

3. Fuzzy \( \mathcal{I} \)-convergence

**Definition 3.1.** A net \( \{ S_n : n \in \mathcal{N} \} \) of fuzzy points of a fuzzy topological space \( X \) is said to fuzzy \( \mathcal{I} \)-converge to a fuzzy point \( x_\lambda \) if for every fuzzy quasi-neighborhood \( U \) of a fuzzy point \( x_\lambda \), \( \{ n \in \mathcal{N} : S_n \mathcal{q} U \} \in \mathcal{I} \).

**Remark 3.2.** If a net \( \{ S_n : n \in \mathcal{N} \} \) of fuzzy points of a fuzzy topological space fuzzy converges to a fuzzy point \( x_\lambda \), then the net \( \{ S_n : n \in \mathcal{N} \} \) is fuzzy \( \mathcal{I} \)-converges to \( x_\lambda \). For, if \( U \) be a fuzzy quasi-neighborhood of \( x_\lambda \) with \( S_n \mathcal{q} U \) for all \( n \in M_{m_0} \), then \( \mathcal{N} - M_{m_0} \in \mathcal{I} \) and so \( \{ n \in \mathcal{N} : S_n \mathcal{q} U \} \in \mathcal{I} \).

**Theorem 3.3.** In Hausdorff spaces, if a net fuzzy \( \mathcal{I} \)-converges to two distinct fuzzy points, then their supports are same.

**Proof.** If possible, let \( \{ S_n : n \in \mathcal{N} \} \) be a net in a fuzzy Hausdorff topological space \( X \) fuzzy \( \mathcal{I} \)-converging to two distinct fuzzy points \( x_\lambda \) and \( x_\mu \) such that \( \text{supp}(x_\lambda) \neq \text{supp}(x_\mu) \). Then for any two fuzzy quasi-neighborhood \( U \) and \( V \) of the fuzzy points \( x_\lambda \) and \( x_\mu \), \( \{ n \in \mathcal{N} : S_n \mathcal{q} U \} \in \mathcal{I} \) and \( \{ n \in \mathcal{N} : S_n \mathcal{q} V \} \in \mathcal{I} \) respectively and so \( \{ n \in \mathcal{N} : S_n \mathcal{q} (U \cap V) \} \in \mathcal{I} \). Since \( \mathcal{I} \) is non-trivial, there exists a positive integer \( k \in \mathcal{N} \) such that \( S_k \mathcal{q} U \) and \( S_k \mathcal{q} V \). Now suppose \( \text{supp}(S_k) = x \). Then \( U(x) > 0 \) and \( V(x) > 0 \), which is a contradiction. \( \square \)

**Theorem 3.4.** Let \( \psi : X \to X \) be a fuzzy continuous function and \( \{ S_n : n \in \mathcal{N} \} \) be a fuzzy \( \mathcal{I} \)-convergent net in \( X \). Then \( \{ \psi(S_n) \} \) is fuzzy \( \mathcal{I} \)-convergent.

**Proof.** Suppose \( \psi : X \to X \) be a fuzzy continuous function at \( x_\lambda \) and \( \{ S_n : n \in \mathcal{N} \} \) be a net in \( X \) fuzzy \( \mathcal{I} \)-converging to \( x_\lambda \). Consider \( V \) be a fuzzy quasi-neighborhood of \( \psi(x_\lambda) \). Then there exists a fuzzy quasi-neighborhood \( U \) of \( x_\lambda \) such that \( \psi(U) \leq V \). Since \( \{ S_n : n \in \mathcal{N} \} \) fuzzy \( \mathcal{I} \)-converges to \( x_\lambda \), we have \( \{ n \in \mathcal{N} : S_n \mathcal{q} U \} \in \mathcal{I} \). So the inclusion \( \{ n \in \mathcal{N} : \psi(S_n) \mathcal{q} V \} \in \mathcal{I} \) ensures that \( \{ n \in \mathcal{N} : \psi(S_n) \mathcal{q} V \} \in \mathcal{I} \). \( \square \)

**Theorem 3.5.** Let \( A \) be a fuzzy subset of \( X \). If a net \( \{ S_n : n \in \mathcal{N} \} \) in \( A \)-converges to a fuzzy point \( x_\lambda \), then \( x_\lambda \in \text{cl}(A) \).

**Proof.** Let \( U \) be any fuzzy quasi-neighborhood of \( x_\lambda \). Then \( \{ n \in \mathcal{N} : S_n \mathcal{q} U \} \in \mathcal{I} \). Since \( \mathcal{I} \) is non-trivial, there exists an \( m \in \mathcal{N} \) such that \( S_m \mathcal{q} U \). Let \( S_m = x_\mu \). Then \( U(x) + \mu > 1 \) and \( A(x) \geq \mu \) and so \( U(x) + A(x) > 1 \). Thus \( A \mathcal{q} U \) and so \( x_\lambda \in \text{cl}(A) \). \( \square \)

4. Fuzzy upper \( \mathcal{I} \)-limits and lower \( \mathcal{I} \)-limits

**Definition 4.1.** Let \( \{ A_n : n \in \mathcal{N} \} \) be a net of fuzzy sets of a fuzzy topological space \( X \). Then the fuzzy upper \( \mathcal{I} \)-limit of \( \{ A_n : n \in \mathcal{N} \} \) is defined and denoted by \( F\text{IU}L(A_n) = \vee \{ x_\lambda \in FP(X) : \text{for every fuzzy quasi-neighborhood } U \text{ of } x_\lambda, \{ n \in \mathcal{N} : A_n \mathcal{q} U \} \notin \mathcal{I} \} \).
**Theorem 4.2.** Let \( \{A_n : n \in \mathcal{N}\} \) be a net of fuzzy sets of a fuzzy topological space \( X \). Then the following properties hold:

(i) \( \text{FIUL}(A_n) \) is a closed set.

(ii) \( \text{FIUL}(A_n) = \text{FIUL}(cl(A_n)) \).

(iii) \( \text{FIUL}(A_n) \leq cl(\bigvee_{i=1}^{\infty} A_i) \).

(iv) If for each \( n \in \mathcal{N} \), \( A_n = A \in \mathcal{I}^X \), then \( \text{FIUL}(A_n) = cl(A) \).

**Proof.** (i) Let \( x_\lambda \in cl(\text{FIUL}(A_n)) \) and \( U \) be any fuzzy quasi-neighborhood of \( x_\lambda \) with \( \text{FIUL}(A_n)qU \). Then there exists an \( y \in X \) such that \( \text{FIUL}(A_n)(y)+U(y) > 1 \). Consider \( \text{FIUL}(A_n)(y) = \mu \). Then \( y_\mu qU \) and \( y_\mu \in \text{FIUL}(A_n) \). Hence \( \{n \in \mathcal{N} : A_nqU \} \notin \mathcal{I} \) and so \( x_\lambda \in \text{FIUL}(A_n) \).

(ii) Let \( U \) be any fuzzy quasi-neighborhood of \( x_\lambda \). It is sufficient to show that \( \{n \in \mathcal{N} : cl(A_n)qU \} = \{n \in \mathcal{N} : A_nqU \} \). Let \( n \in \{n \in \mathcal{N} : A_nqU \} \). Then there exists an \( y \in X \) such that \( A_n(y)+U(y) > 1 \), that is, \( cl(A_n)(y)+U(y) > 1 \) and so \( n \in \{n \in \mathcal{N} : cl(A_n)qU \} \). Conversely, let \( n \in \{n \in \mathcal{N} : cl(A_n)qU \} \). Then there exists an \( y \in X \) such that \( cl(A_n)(y)+U(y) > 1 \). Suppose \( cl(A_n)(y) = \mu \). Then \( y_\mu \in cl(A_n) \) and \( U \) is fuzzy quasi-neighborhood of \( y_\mu \). Thus \( UqA_n \) and so \( n \in \{n \in \mathcal{N} : A_nqU \} \).

(iii) Let \( x_\lambda \in \text{FIUL}(A_n) \) and \( U \) be any fuzzy quasi-neighborhood of \( x_\lambda \). Then \( \{n \in \mathcal{N} : A_nqU \} \notin \mathcal{I} \). Since \( \mathcal{I} \) is non-trivial, there exists an \( m \in \mathcal{N} \) such that \( A_mqU \) and so \( \bigvee_{i=1}^{\infty} A_iqU \). Thus \( x_\lambda \in cl(\bigvee_{i=1}^{\infty} A_i) \).

(iv) (iii) implies that \( \text{FIUL}(A_n) \leq cl(A) \). So let \( x_\lambda \notin \text{FIUL}(A_n) \). Then there exists a fuzzy quasi-neighborhood \( U \) of \( x_\lambda \) such that \( \{n \in \mathcal{N} : A_nqU \} \notin \mathcal{I} \). Since \( \mathcal{N} \notin \mathcal{I} \), there exists an \( m \in \mathcal{N} \) satisfying \( A_mqU \), that is, \( AqU \) and so \( x_\lambda \notin cl(A) \).

**Theorem 4.3.** Let \( \{A_n : n \in \mathcal{N}\} \) and \( \{B_n : n \in \mathcal{N}\} \) be any two nets of fuzzy sets of a fuzzy topological space \( X \). Then:

(i) \( A_n \leq B_n \) for all \( n \in \mathcal{N} \) implies that \( \text{FIUL}(A_n) \leq \text{FIUL}(B_n) \).

(ii) \( \text{FIUL}(A_n \vee B_n) = \text{FIUL}(A_n) \vee \text{FIUL}(B_n) \).

(iii) \( \text{FIUL}(A_n \wedge B_n) \leq \text{FIUL}(A_n) \wedge \text{FIUL}(B_n) \).

**Proof.** (i) Let \( x_\lambda \in \text{FIUL}(A_n) \). Then for each fuzzy quasi-neighborhood \( U \) of a fuzzy point \( x_\lambda \), \( \{n \in \mathcal{N} : A_nqU \} \notin \mathcal{I} \). Since \( \{n \in \mathcal{N} : A_nqU \} \subset \{n \in \mathcal{N} : B_nqU \}, \{n \in \mathcal{N} : B_nqU \} \notin \mathcal{I} \).

(ii) By (i), \( \text{FIUL}(A_n) \vee \text{FIUL}(B_n) \leq \text{FIUL}(A_n \vee B_n) \). Now let \( x_\lambda \notin \text{FIUL}(A_n) \) and \( x_\lambda \notin \text{FIUL}(B_n) \). Then there exist fuzzy quasi-neighborhoods \( U_1 \) and \( U_2 \) of the fuzzy point \( x_\lambda \) such that \( \{n \in \mathcal{N} : A_nqU_1 \} \notin \mathcal{I} \) and \( \{n \in \mathcal{N} : B_nqU_2 \} \notin \mathcal{I} \). Consider \( U = U_1 \cup U_2 \). Then \( \{n \in \mathcal{N} : (A_n \vee B_n)qU \} \subset \{n \in \mathcal{N} : A_nqU_1 \} \cup \{n \in \mathcal{N} : B_nqU_2 \} \). Thus \( \{n \in \mathcal{N} : (A_n \vee B_n)qU \} \notin \mathcal{I} \) and so \( x_\lambda \notin \text{FIUL}(A_n \vee B_n) \).

(iii) It follows from (i).
Theorem 4.4. Let \( \{A_n : n \in \mathcal{N}\} \) and \( \{B_n : n \in \mathcal{N}\} \) be any two nets of fuzzy sets of fuzzy topological spaces \( X \) and \( Y \) respectively. Then \( \text{FIUL}(A_n \times B_n) \leq \text{FIUL}(A_n) \times \text{FIUL}(B_n) \).

Proof. Let \((x, y)_\lambda \in \text{FIUL}(A_n \times B_n)\) and \(U_1\) (in \(X\)) and \(U_2\) (in \(Y\)) be fuzzy quasi-neighborhoods of the fuzzy points \(x_\lambda\) and \(y_\lambda\) respectively. Then \(U_1 \times U_2\) is fuzzy quasi-neighborhoods of \((x, y)_\lambda\) in \(X \times Y\). So \( \{n \in \mathcal{N} : (A_n \times B_n) \sqcap (U_1 \times U_2)_n\} \not\in \mathcal{I} \).

Clearly \( \{n \in \mathcal{N} : (A_n \times B_n)_n \sqcap (U_1 \times U_2) \} \subset \{n \in \mathcal{N} : A_n \sqcap U_1_n\} \) and \( \{n \in \mathcal{N} : (A_n \times B_n)_n \sqcap (U_1 \times U_2) \} \subset \{n \in \mathcal{N} : B_n \sqcap U_2_n\} \). So \( x_\lambda \in \text{FIUL}(A_n)\) and \( y_\lambda \in \text{FIUL}(B_n) \). \( \square \)

Definition 4.5. Let \( \{A_n : n \in \mathcal{N}\} \) be a net of fuzzy sets of a fuzzy topological space \( X \). Then its lower fuzzy \( \mathcal{I}\)-limit is denoted and defined by \( \text{FILL}(A_n) = \bigvee \{x_\lambda \in FP(X) : \text{for every fuzzy quasi-neighborhood } U \text{ of } x_\lambda, \{n \in \mathcal{N} : A_n \sqcap U\}_n \in \mathcal{I} \} \). If \( \text{FILL}(A_n) = \text{FIUL}(A_n) \), then the net \( \{A_n : n \in \mathcal{N}\} \) is said to converge to the limit \( \text{FL}(A_n)(= \text{FIUL}(A_n)) \).

Theorem 4.6. Let \( \{A_n : n \in \mathcal{N}\} \) be a net of fuzzy sets of a fuzzy topological space \( X \). Then the following properties hold:

(i) \( \text{FILL}(A_n) \) is a closed set.
(ii) \( \text{FIUL}(A_n) = \text{FILL}(\text{cl}(A_n)) \).
(iii) \( \text{FILL}(A_n) \leq \text{cl}(\bigvee_{i=1}^n A_i) \).
(iv) If for each \( n \in \mathcal{N} \), \( A_n = A \in I^X \), then \( \text{FILL}(A_n) = \text{cl}(A) \).
(v) \( \bigwedge_{i=1}^n A_i \leq \text{FILL}(A_n) \).
(vi) \( \text{FILL}(A_n) \leq \text{FIUL}(A_n) \).

Proof. The proofs of (i),(ii) and (iii) are parallel to the proofs of Theorem 4.2. So we prove only (iv)-(vi).

(iv) (iii) implies that \( \text{FILL}(A_n) \leq \text{cl}(A) \). So let \( x_\lambda \notin \text{FILL}(A_n) \). Then there exists a fuzzy quasi-neighborhood \( U \) of \( x_\lambda \) satisfying \( \{n \in \mathcal{N} : A_n \sqcap U\}_n \not\in \mathcal{I} \). Since \( \emptyset \in \mathcal{I} \), there exists an \( m \in \mathcal{N} \) such that \( A_m \sqcap U \), that is, \( AqU \) and so \( x_\lambda \notin \text{cl}(A) \).

(v) Suppose \( x_\lambda \notin \text{FIUL}(A_n) \). Then there exists a fuzzy quasi-neighborhood \( U \) of \( x_\lambda \) satisfying \( \{n \in \mathcal{N} : A_n \sqcap U\}_n \not\in \mathcal{I} \). So there exists an \( m \in \mathcal{N} \) such that \( A_m \sqcap U \). So, for every \( y \in X \), \( A_m(y) + U(y) \leq 1 \). Again since \( U \) is a fuzzy quasi-neighborhood of \( x_\lambda \), \( U(x) + \lambda > 1 \). Thus \( A_m(x) < \lambda \), that is, \( x_\lambda \notin A_m \).

(vi) Let \( x_\lambda \notin \text{FIUL}(A_n) \). There exists a fuzzy quasi-neighborhood \( U \) of \( x_\lambda \) such that \( \{n \in \mathcal{N} : A_n \sqcap U\}_n \in \mathcal{I} \). Since \( \mathcal{N} \not\in \mathcal{I} \), \( \{n \in \mathcal{N} : A_n \sqcap U\}_n \not\in \mathcal{I} \). \( \square \)

Theorem 4.7. Let \( \{A_n : n \in \mathcal{N}\} \) and \( \{B_n : n \in \mathcal{N}\} \) be any two nets of fuzzy sets of a fuzzy topological space \( X \). Then:

(i) For all \( n \in \mathcal{N} \), \( A_n \leq B_n \) implies that \( \text{FILL}(A_n) \leq \text{FILL}(B_n) \).
(ii) \( \text{FIUL}(A_n \lor B_n) \geq \text{FIUL}(A_n) \lor \text{FIUL}(B_n) \).
(iii) \( \text{FILL}(A_n \land B_n) \leq \text{FILL}(A_n) \land \text{FILL}(B_n) \).
(iv) \( \text{FIL}(A_n \land B_n) \leq \text{FIL}(A_n) \land \text{FIL}(B_n) \).
(v) \( \text{FIL}(A_n \lor B_n) = \text{FIL}(A_n) \lor \text{FIL}(B_n) \).

Proof. (iv) and (v) follow from Theorem 4.3 (ii) and (iii) follow from (i). So we prove only (i).
Let \( x_\lambda \in FILL(A_n) \). Then for each fuzzy quasi-neighborhood \( U \) of a fuzzy point \( x_\lambda, \{ n \in \mathcal{N} : A_n \check{\cap} U \} \in \mathcal{I} \). Since \( \{ n \in \mathcal{N} : B_n \check{\cap} U \} \subset \{ n \in \mathcal{N} : A_n \check{\cap} U \}, \{ n \in \mathcal{N} : B_n \check{\cap} U \} \in \mathcal{I} \).

\[\text{Theorem 4.8.} \ \text{Let} \{ A_n : n \in \mathcal{N} \} \ \text{and} \{ B_n : n \in \mathcal{N} \} \ \text{be any two nets of fuzzy sets of a fuzzy topological spaces} \ X \ \text{and} \ Y \ \text{respectively. Then} \]

(i) \( FILL(A_n \times B_n) \leq FILL(A_n) \times FILL(B_n) \).

(ii) \( FILL(A_n \times B_n) \leq FILL(A_n) \times FILL(B_n) \).

\[\text{Proof.} \ \text{The proof is parallel to the proof of Theorem 4.4.} \]

5. Applications

In this section, we have given some applications of the concepts studied in the earlier section.

Lowen [29] defined weakly fuzzy compactness in a fuzzy topological spaces as follows:

A fuzzy topological space \( X \) is called weakly fuzzy compact if for every fuzzy open cover \( \{ U_\alpha : \alpha \in \Delta \} \) of \( X \) and for each \( \epsilon > 0 \), there exists finite number of indices \( \alpha_1, \alpha_2, ..., \alpha_p \in \Delta \) such that \( \bigvee_{i=1}^p U_i \geq 1 - \epsilon \).

\[\text{Theorem 5.1.} \ \text{A fuzzy topological space} \ X \ \text{is weakly fuzzy compact if and only if for every net} \ \{ F_n : n \in \mathcal{N} \} \ \text{of fuzzy closed sets, for every ideal} \ \mathcal{I} \ \text{on} \ \mathcal{N} \ \text{with} \ FIU\mathcal{L}(F_n) = \emptyset \ \text{and for every} \ \epsilon > 0, \ \{ n \in \mathcal{N} : F_n \not\in \mathcal{I} \} \subset \mathcal{I} \]

\[\text{Proof.} \ \text{Let} \ X \ \text{be weakly fuzzy compact and} \ \{ F_n : n \in \mathcal{N} \} \ \text{be a net of fuzzy closed sets,} \ \mathcal{I} \ \text{be an ideal on} \ \mathcal{N} \ \text{with} \ FIU\mathcal{L}(F_n) = \emptyset \ \text{and} \ \epsilon > 0. \ \text{Then for each fuzzy point} \ x_\lambda \ \text{of} \ X, \ \text{there exists a fuzzy quasi-neighborhood} \ U_{x_\lambda} \ \text{of} \ x_\lambda \ \text{such that} \ \{ n \in \mathcal{N} : F_n \check{\cap} U_{x_\lambda} \} \in \mathcal{I} \text{. Since} \ X \ \text{is weakly fuzzy compact and} \ \{ U_{x_\lambda} : x_\lambda \in FP(X) \} \ \text{is a fuzzy open cover of} \ X, \ \text{there exists finite number of fuzzy points} \ e_1, e_2, ..., e_p \in FP(X) \ \text{such that} \ \bigvee_{i=1}^p U_{e_i} \geq 1 - \epsilon \ . \ \text{Here} \ \{ n \in \mathcal{N} : F_n \check{\cap} \bigvee_{i=1}^p U_{e_i} \} = \bigcup_{i=1}^p \{ n \in \mathcal{N} : F_n \check{\cap} U_{e_i} \} \in \mathcal{I} \ . \ \text{Since} \ \{ n \in \mathcal{N} : F_n \not\in \mathcal{I} \} \subset \mathcal{I} \ \text{and} \ \mathcal{I} \ \text{is an ideal on} \ \mathcal{N} \ \text{and for every ideal} \ \mathcal{I} \ \text{on} \ \mathcal{N} \ \text{with} \ FIU\mathcal{L}(F_n) = \emptyset \ . \]

\[\text{Gantner et. al. [17] introduced the concept of strongly fuzzy compact in fuzzy topological space as follows:}

A fuzzy topological space \( X \) is called strongly fuzzy compact if for each \( \alpha \in [0, 1) \) and each family \( \mathcal{U}_\alpha \) with the property that for each \( x \in X \), there exists an \( U \in \mathcal{U}_\alpha \) satisfying \( U(x) > \alpha \), has finite subfamily satisfying the same property.

\[\text{Theorem 5.2.} \ \text{A fuzzy topological space} \ X \ \text{is strongly fuzzy compact if and only if for each} \ \alpha \in [0, 1), \ \text{for each net} \ \{ F_n : n \in \mathcal{N} \} \ \text{of fuzzy closed sets and for every ideal} \ \mathcal{I} \ \text{on} \ \mathcal{N} \ \text{with} \ FIU\mathcal{L}(F_n) \leq 1 - \alpha, \ \{ n \in \mathcal{N} : F_n \not\in \mathcal{I} \} \in \mathcal{I} \]

\[\text{Proof.} \ \text{Let} \ X \ \text{be strongly fuzzy compact and} \ \{ F_n : n \in \mathcal{N} \} \ \text{be a net of fuzzy closed sets,} \ \mathcal{I} \ \text{be an ideal on} \ \mathcal{N} \ \text{with} \ FIU\mathcal{L}(F_n) \leq 1 - \alpha \ \text{and an} \ \alpha \in [0, 1) . \ \text{Then for each fuzzy point} \ x_\lambda \ \text{of} \ X \ \text{satisfying} \ FIU\mathcal{L}(F_n) < x_\lambda \leq 1 - \alpha, \ \text{there exists a fuzzy quasi-neighborhood} \ U_{x_\lambda} \ \text{of} \ x_\lambda \ \text{such that} \ \{ n \in \mathcal{N} : F_n \check{\cap} U_{x_\lambda} \} \in \mathcal{I} \ . \ \text{Since} \ X \ \text{is strongly fuzzy compact, there exists a finite subfamily of}\]
fuzzy compact and \( \{ U_{x,\lambda} : x_\lambda \in FP(X), FIUL(F_n) < x_\lambda \leq 1 - \alpha \} \) satisfies the property that for each \( x \in X \), \( U_{x,\lambda}(x) > 1 - \lambda \geq \alpha \), there exist finite number of fuzzy points \( e_1, e_2, \ldots, e_p \in FP(X) \) such that for each \( x \in X \), \( U_{e_i}(x) > \alpha \) for some \( i \in \{1, 2, \ldots, p\} \). Here \( \{ n \in \mathcal{N} : F_n \cap \bigvee_{i=1}^p U_{e_i} \} = \bigcup_{i=1}^p \{ n \in \mathcal{N} : F_n \cap U_{e_i} \} \in I \). Since \( \{ n \in \mathcal{N} : F_n \cap \bigvee_{i=1}^p U_{e_i} \} \subseteq \{ n \in \mathcal{N} : F_n \cap \bigvee_{i=1}^p U_{e_i} \} \), \( \{ n \in \mathcal{N} : F_n \cap \bigvee_{i=1}^p U_{e_i} \} \in I \).

Lowen [29] defined fuzzy compactness in a fuzzy topological spaces as follows:

A fuzzy topological spaces \( X \) is called fuzzy compact if for each family \( \mathcal{U} \) of fuzzy open sets of \( X \) and for each \( \alpha \in [0, 1] \) such that \( \forall \{ U : U \in \mathcal{U} \} \geq \alpha \) and for each \( \epsilon \in (0, \alpha] \), there exists a finite subfamily \( \mathcal{U}_0 \subseteq \mathcal{U} \) satisfying \( \forall \{ U : U \in \mathcal{U}_0 \} \geq \alpha - \epsilon \).

**Theorem 5.3.** A fuzzy topological space \( X \) is fuzzy compact if and only if for each \( \alpha \in [0, 1] \), for each net \( \{ F_n : n \in \mathcal{N} \} \) of fuzzy closed sets and for every ideal \( I \) on \( \mathcal{N} \) with \( FIUL(F_n) \leq 1 - \alpha \) and for each \( \epsilon \in (0, \alpha] \), \( \{ n \in \mathcal{N} : F_n \not\subseteq 1 - \alpha + \epsilon \} \in I \).

**Proof.** The proof is analogous to that of Theorem 5.2.

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**References**


