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New characterizations of h-semisimple hemirings

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Abstract. In this paper, the concept of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideals over a hemiring is introduced and investigated. Some characterization theorems of h-semisimple hemirings are derived in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left (right) h-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideals.

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1. Introduction

Dealing with uncertainties is a major problem in many areas such as economics, engineering, environmental science, medical science and social sciences. These kinds of problems cannot be dealt with using classical methods, because classical methods have inherent difficulties. To overcome these difficulties, Molodtsov [10] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Up till now, colorblack research on soft sets has been very active and many important results have been achieved in the theoretical aspect. Maji et al. [9] introduced several algebraic operations in soft set theory and published a detailed theoretical study on soft sets. Ali et al. [1] further presented and investigated some new algebraic operations for soft sets. Aygünoğlu and Aygün [2] discussed the applications of fuzzy soft sets to group theory and introduced the concept of (normal) fuzzy soft groups. Feng et al. [3] introduced the concept of soft semirings by using the soft set theory. Jun [4] introduced and investigated the notion of soft BCK/BCI-algebras. Jun and Park [6] and Jun et al. [5] discussed the applications of soft sets in ideal theory of BCK/BCI-algebras and in d-algebras, respectively. Koyuncu and Tanay [7] introduced and studied soft rings. Zhan and Jun [17] characterized the (implicative, positive implicative and fantastic) filteristic soft BL-algebras based on $\in$-soft sets and q-soft sets. Yin et al. [15] investigated the concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left h-ideals (right h-ideals, h-bi-ideals, and h-quasi-ideals) and present some characterization theorems of (left) h-hemiregular and (left) duo hemirings in terms of such $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-ideals.

As a continuation of the work of Yin et al. [15], this paper introduces and investigates the concept of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideals over a hemiring, and characterizes h-semisimple hemirings in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left (right) h-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideals. The rest of this paper is organized as follows. In Section 2, we summarize some basic concepts which will be used throughout the paper. In Section 3, we define and investigate $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-interior ideals. The characterization of h-semisimple hemirings in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left (right) h-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideals is discussed in Section 4. Some conclusions are given in the last Section.

2. Preliminaries

In this section, we recall some basic notions and results about hemirings, fuzzy sets and fuzzy soft sets (see [12, 16, 8]) which will be used in the sequel.

2.1. **Hemirings.** A *semiring* is an algebraic system $(S, +, \cdot)$ consisting of a non-empty set S together with two binary operations on S called addition and multiplication (denoted in the usual manner) such that $(S, +)$ and $(S, \cdot)$ are semigroups and the following distributive laws

$$a \cdot (b + c) = a \cdot b + b \cdot c \text{ and } (a + b) \cdot c = a \cdot c + b \cdot c$$

are satisfied for all $a, b, c \in S$.

By *zero* of a semiring $(S, +, \cdot)$ we mean an element $0 \in S$ such that $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. A semiring $(S, +, \cdot)$ with zero is called a *hemiring* if $(S, +)$ is commutative. For the sake of simplicity, we shall omit the symbol " $\cdot$ ", writing ab for $a \cdot b$ $(a, b \in S)$.

A subset A in a hemiring S is called a *left* (resp., *right*) ideal of S if A is closed under addition and $SA \subseteq A$ (resp., $AS \subseteq A$). Furthermore, A is called an *ideal* of S if it is both a left ideal and a right ideal of S. A subset A in a hemiring S is called an *interior ideal* if A is closed under addition and multiplication such that $SAS \subseteq A$.

A left ideal A of S is called a *left h-ideal* if $x, z \in S, a, b \in A$, and $x + a + z = b + z$ implies $x \in A$. *Right h-ideals, h-ideals* and *h-interior ideals* are defined similarly.

The h-closure $\overline{A}$ of a subset A in a hemiring S is defined as

$$\overline{A} = \{x \in S | x + a + z = b + z \text{ for some } a, b \in A,\ z \in S\}.$$

Clearly, if A is a left ideal (resp., right ideal, interior ideal) of S, then $\overline{A}$ is the smallest left h-ideal (resp., right h-ideal, h-interior ideal) of S containing A.

Note that every left h-ideal (resp., right h-ideal, h-interior ideal) of a hemiring S is a left ideal (resp., right ideal, interior ideal) of S. Also, every h-ideal of S is an h-interior ideal of S. However, the converse of these properties do not hold in general as shown in [12].

2.2. **Fuzzy sets.** Let X be a non-empty set. A fuzzy subset μ of X is defined as a mapping from X into $[0,1]$, where $[0,1]$ is the usual interval of real numbers. We denote by $\mathscr{F}(X)$ the set of all fuzzy subsets of X. For $\mu,\nu \in \mathscr{F}(X)$, by $\mu \subseteq \nu$ we mean $\mu(x) \le \nu(x)$ for all $x \in X$. And the *union* and *intersection* of μ and ν, denoted by $\mu \cup \nu$ and $\mu \cap \nu$, are defined as the fuzzy subsets of X by $(\mu \cup \nu)(x) = \max\{\mu(x), \nu(x)\}$ and $(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\}$ for all $x \in X$.

A fuzzy subset μ of X of the form

$$\mu(y) = \begin{cases} r(\neq 0) & \text{if } y = x, \\ 0 & \text{otherwise} \end{cases}$$

is said to be a *fuzzy point with support x and value r* and is denoted by x_r, where $r \in (0,1]$.

In what follows let $\gamma, \delta \in [0,1]$ be such that $\gamma < \delta$. For any $Y \subseteq X$, define $\chi^\delta_{\gamma Y}$ be the fuzzy subset of X by $\chi^\delta_{\gamma Y}(x) \ge \delta$ for all $x \in Y$ and $\chi^\delta_{\gamma Y}(x) \le \gamma$ otherwise. Clearly, $\chi^\delta_{\gamma Y}$ is the characteristic function of Y if $\gamma = 0$ and $\delta = 1$.

As a generalization of the notions "belongingness ($\in$)" and "quasi-coincidence (q)" of a fuzzy point with a fuzzy set introduced by Pu and Liu [11], Yin and Zhan [14] introduced the following notions: For a fuzzy point x_r and a fuzzy subset μ of X, we say that

(1) $x_r \in_\gamma \mu$ if $\mu(x) \ge r > \gamma$.
(2) $x_r \, q_\delta \, \mu$ if $\mu(x) + r > 2\delta$.
(3) $x_r \in_\gamma \vee q_\delta \mu$ if $x_r \in_\gamma \mu$ or $x_r \, q_\delta \, \mu$.

Let us now introduce a new ordering relation on $\mathscr{F}(X)$, denoted as "$\subseteq \vee q_{(\gamma,\delta)}$", as follows.

For any $\mu, \nu \in \mathscr{F}(X)$, by $\mu \subseteq \vee q_{(\gamma,\delta)} \nu$ we mean that $x_r \in_\gamma \mu$ implies $x_r \in_\gamma \vee q_\delta \, \nu$ for all $x \in X$ and $r \in (\gamma, 1]$. Moreover, μ and ν are said to be (γ,δ)-equal, denoted by $\mu =_{(\gamma,\delta)} \nu$, if $\mu \subseteq \vee q_{(\gamma,\delta)} \nu$ and $\nu \subseteq \vee q_{(\gamma,\delta)} \mu$.

In the sequel, unless otherwise stated, $\overline{\alpha}$ means α does not hold, where $\alpha \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \subseteq \vee q_{(\gamma,\delta)}\}$.

Lemma 2.1 ([15])**.** *Let $\mu, \nu \in \mathscr{F}(X)$. Then $\mu \subseteq \vee q_{(\gamma,\delta)} \nu$ if and only if $\max\{\nu(x), \gamma\} \ge \min\{\mu(x), \delta\}$ for all $x \in X$.*

Lemma 2.2 ([15])**.** *Let $\mu, \nu, \omega \in \mathscr{F}(X)$. If $\mu \subseteq \vee q_{(\gamma,\delta)} \nu$ and $\nu \subseteq \vee q_{(\gamma,\delta)} \omega$. Then $\mu \subseteq \vee q_{(\gamma,\delta)} \omega$.*

Lemmas 2.1 and 2.2 give that "$=_{(\gamma,\delta)}$" is an equivalence relation on $\mathscr{F}(X)$. It is also worth noticing that $\mu =_{(\gamma,\delta)} \nu$ if and only if $\max\{\min\{\mu(x), \delta\}, \gamma\} = \max\{\min\{\nu(x), \delta\}, \gamma\}$ for all $x \in X$ by Lemma 2.1.

Yin et al. [13] and Yin and Li [12] have introduced the definitions of h-sum and h-intrinsic product of two fuzzy subsets of a hemiring S, respectively, as follows.

Definition 2.3 ([13])**.** Let μ and ν be fuzzy subsets of a hemiring S. Then the *h-sum* of μ and ν is defined by

$$(\mu +_h \nu)(x) = \sup_{x+a_1+b_1+z=a_2+b_2+z} \min\{\mu(a_1), \mu(a_2), \nu(b_1), \nu(b_2)\}.$$

Definition 2.4 ([12]). Let μ and ν be fuzzy subsets of a hemiring S. Then the *h-intrinsic product* of μ and ν is defined by

$$(\mu \odot_h \nu)(x) = \sup_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \min\{\mu(a_i), \mu(a'_j), \nu(b_i), \nu(b'_j)\}$$

for all $i = 1, \cdots, m; j = 1, \cdots, n$ and $(\mu \odot_h \nu)(x) = 0$ *if* x cannot be expressed as $x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z$.

Lemma 2.5 ([15]). *Let S be a hemiring and $X, Y \subseteq S$. Then we have*

(1) $X \subseteq Y$ if and only if $\chi^{\delta}_{\gamma X} \subseteq \vee q_{(\gamma,\delta)} \chi^{\delta}_{\gamma Y}$.

(2) $\chi^{\delta}_{\gamma X} \cap \chi^{\delta}_{\gamma Y} =_{(\gamma,\delta)} \chi^{\delta}_{\gamma (X \cap Y)}$.

(3) $\chi^{\delta}_{\gamma X} +_h \chi^{\delta}_{\gamma Y} =_{(\gamma,\delta)} \chi^{\delta}_{\gamma \overline{X+Y}}$.

(4) $\chi^{\delta}_{\gamma X} \odot_h \chi^{\delta}_{\gamma Y} =_{(\gamma,\delta)} \chi^{\delta}_{\gamma \overline{XY}}$.

2.3. **Fuzzy soft sets.** Let U be an initial universe set and E the set of all possible parameters under consideration with respect to U. As a generalization of soft set introduced in Molodtsov [10], Maji et al. [8] defined fuzzy soft set in the following way.

Definition 2.6. A pair $\langle F, A \rangle$ is called a *fuzzy soft set* over U if and only if $A \subseteq E$ and F is a mapping given by $F : A \to \mathscr{F}(U)$.

In general, for every $\varepsilon \in A$, $F(\varepsilon)$ is a fuzzy set of U and it is called *fuzzy value set* of parameter ε. The set of all fuzzy soft sets over U with parameters from E is called a *fuzzy soft classes*, and it is denoted by $\mathscr{F}\mathscr{S}(U, E)$.

Definition 2.7 ([1]). The *extended intersection* of two fuzzy soft sets $\langle F, A \rangle$ and $\langle G, B \rangle$ over U is a fuzzy soft set denoted by $\langle H, C \rangle$, where $C = A \cup B$ and

$$H(\varepsilon) = \begin{cases} F(\varepsilon) & \text{if } \varepsilon \in A - B, \\ G(\varepsilon) & \text{if } \varepsilon \in B - A, \\ F(\varepsilon) \cap G(\varepsilon) & \text{if } \varepsilon \in A \cap B, \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $\langle H, C \rangle = \langle F, A \rangle \tilde{\cap} \langle G, B \rangle$.

Definition 2.8 ([15]). Let $\langle F, A \rangle$ and $\langle G, B \rangle$ be two fuzzy soft sets over U. We say that $\langle F, A \rangle$ is an (γ, δ)-*fuzzy soft subset* of $\langle G, B \rangle$ and write $\langle F, A \rangle \Subset_{(\gamma,\delta)} \langle G, B \rangle$ if

(i) $A \subseteq B$;

(ii) For any $\varepsilon \in A$, $F(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} G(\varepsilon)$.

$\langle F, A \rangle$ and $\langle G, B \rangle$ are said to be (γ, δ)-*fuzzy soft equal* and write $\langle F, A \rangle \asymp_{(\gamma,\delta)} \langle G, B \rangle$ if $\langle F, A \rangle \Subset_{(\gamma,\delta)} \langle G, B \rangle$ and $\langle G, B \rangle \Subset_{(\gamma,\delta)} \langle F, A \rangle$.

Lemma 2.9 ([15]). *Let $\langle F, A \rangle$, $\langle G, B \rangle$ and $\langle H, C \rangle$ be fuzzy soft sets over U. If $\langle F, A \rangle \Subset_{(\gamma,\delta)} \langle G, B \rangle$ and $\langle G, B \rangle \Subset_{(\gamma,\delta)} \langle H, C \rangle$. Then $\langle F, A \rangle \Subset_{(\gamma,\delta)} \langle F, C \rangle$.*

Definition 2.10 ([15]). The *h-sum* of two fuzzy soft sets $\langle F, A \rangle$ and $\langle G, B \rangle$ over a hemiring S is a fuzzy soft set over S, denoted by $\langle F +_h G, C \rangle$, where $C = A \cup B$ and

$$(F +_h G)(\varepsilon) = \begin{cases} F(\varepsilon) & \text{if } \varepsilon \in A - B, \\ G(\varepsilon) & \text{if } \varepsilon \in B - A, \\ F(\varepsilon) +_h G(\varepsilon) & \text{if } \varepsilon \in A \cap B, \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $\langle F +_h G, C \rangle = \langle F, A \rangle +_h \langle G, B \rangle$.

Definition 2.11 ([15]). The *h-intrinsic product* of two fuzzy soft sets $\langle F, A\rangle$ and $\langle G, B\rangle$ over a hemiring S is a fuzzy soft set over S, denoted by $\langle F \odot_h G, C\rangle$, where $C = A \cup B$ and

$$(F \odot_h G)(\varepsilon) = \begin{cases} F(\varepsilon) & \text{if } \varepsilon \in A - B, \\ G(\varepsilon) & \text{if } \varepsilon \in B - A, \\ F(\varepsilon) \odot_h G(\varepsilon) & \text{if } \varepsilon \in A \cap B, \end{cases}$$

for all $\varepsilon \in C$. This is denoted by $\langle F \odot_h G, C\rangle = \langle F, A\rangle \odot_h \langle G, B\rangle$.

The following result can be easily deduced.

Lemma 2.12 ([15]). *Let $\langle F_1, A\rangle$, $\langle F_2, A\rangle$, $\langle G_1, B\rangle$ and $\langle G_2, B\rangle$ be fuzzy soft sets over a hemiring S such that $\langle F_1, A\rangle \Subset_{(\gamma,\delta)} \langle F_2, A\rangle$ and $\langle G_1, B\rangle \Subset_{(\gamma,\delta)} \langle G_2, B\rangle$. Then*

(1) $\langle F_1, A\rangle +_h \langle G_1, B\rangle \Subset_{(\gamma,\delta)} \langle F_2, A\rangle +_h \langle G_2, B\rangle$ and $\langle F_1, A\rangle \odot_h \langle G_1, B\rangle \Subset_{(\gamma,\delta)} \langle F_2, A\rangle \odot_h \langle G_2, B\rangle$.

(2) $\langle F_1, A\rangle \tilde{\cap} \langle G_1, B\rangle \Subset_{(\gamma,\delta)} \langle F_2, A\rangle \tilde{\cap} \langle G_2, B\rangle$.

3. $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-ideals over a hemiring

In this section, we will introduce the concept of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideals over a hemiring and investigate its fundamental properties.

Definition 3.1 ([15]). A fuzzy soft set $\langle F, A\rangle$ over a hemiring S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft left* (resp., *right*) *h-ideal* over S if it satisfies:

(F1a) $\langle F, A\rangle +_h \langle F, A\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle$,
(F2a) $\Sigma(S, A) \odot_h \langle F, A\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle$,
(F3a) $x + a + z = b + z, a_r, b_s \in_\gamma F(\varepsilon)$ implies $x_{\min\{r,s\}} \in_\gamma \vee q_\delta\ F(\varepsilon)$ for all $a, b, x, z \in S$, $\varepsilon \in A$ and $r, s \in (\gamma, 1]$.

Definition 3.2 ([15]). A fuzzy soft set $\langle F, A\rangle$ over a hemiring S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft right h-ideal* over S if it satisfies conditions (F1a), (F3a) and
(F4a) $\langle F, A\rangle \odot_h \Sigma(S, A) \Subset_{(\gamma,\delta)} \langle F, A\rangle$).

Definition 3.3 ([15]). A fuzzy soft set over S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft h-ideal* over S if it is both an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft right h-ideal and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left h-ideal over S.

Definition 3.4. A fuzzy soft set $\langle F, A\rangle$ over a hemiring S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-*fuzzy soft h-interior ideal* if it satisfies conditions (F1a), (F3a) and

(F5a) $\langle F, A\rangle \odot_h \langle F, A\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle$,
(F6a) $\Sigma(S, A) \odot_h \langle F, A\rangle \odot_h \Sigma(S, A) \Subset_{(\gamma,\delta)} \langle F, A\rangle$.

Lemma 3.5 ([15]). *Let $\langle F, A\rangle$ be a fuzzy soft set over a hemiring S. Then* (F3a) *holds if and only if the following condition holds:*

(F3b) $x + a + z = b + z$ *implies* $\max\{F(\varepsilon)(x), \gamma\} \geq \min\{F(\varepsilon)(a), F(\varepsilon)(b), \delta\}$ *for all $a, b, x, z \in S$ and $\varepsilon \in A$.*

As a consequence of Lemmas 4.4 and 4.5 in Yin et al. [15], the following result holds.

Lemma 3.6. *A fuzzy soft set $\langle F, A\rangle$ over a hemiring S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left h-ideal if and only if (F3b) and the following conditions hold: for all $x, y \in S$ and $\varepsilon \in A$*

(F1b) $\max\{F(\varepsilon)(x+y), \gamma\} \geq \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\}$;
(F2b) $\max\{F(\varepsilon)(xy), \gamma\} \geq \min\{F(\varepsilon)(y), \delta\}$.

For $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideal, we have the following characterization theorem.

Lemma 3.7. *A fuzzy soft set $\langle F, A\rangle$ over a hemiring S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideal if and only if (F1b), (F3b) and the following conditions hold: for all $x, y, z \in S$ and $\varepsilon \in A$*

(F5b) $\max\{F(\varepsilon)(xy), \gamma\} \geq \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\}$;
(F6b) $\max\{F(\varepsilon)(xyz), \gamma\} \geq \min\{F(\varepsilon)(y), \delta\}$.

Proof. Assume that $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideal of S. By Lemmas 3.5 and 3.6, (F1b) and (F3b) are satisfied. We only show (F6b). (F5b) can be similarly proved. If there exist $x, y, z \in S$, $\varepsilon \in A$ and $r \in (0, 1]$ such that $\max\{F(\varepsilon)(xyz), \gamma\} < r < \min\{F(\varepsilon)(y), \delta\}$, then $F(\varepsilon)(y) > r$ and $F(\varepsilon)(xyz) < r < \delta$, that is, $(xyz)_r \overline{\in_\gamma \vee q_\delta}\, F(\varepsilon)$. On the other hand, we have $\max\{F(\varepsilon)(0), \gamma\} \geq \min\{F(\varepsilon)(y), \delta\}$ and so $F(\varepsilon)(0) \geq \min\{F(\varepsilon)(y), \delta\}$ since $\gamma < \min\{F(\varepsilon)(y), \delta\}$. By (F6a) and Lemma 2.1, we have

$$\begin{aligned}
&\max\{F(\varepsilon)(xyz), \gamma\} \geq \min\{(\chi^\delta_{\gamma S} \odot_h F(\varepsilon) \odot_h \chi^\delta_{\gamma S})(xyz), \delta\}\\
&= \min\left\{ \sup_{xyz+\sum_{i=1}^m a_ib_i+z'=\sum_{j=1}^n a'_jb'_j+z'} \min\{(\chi^\delta_{\gamma S} \odot_h F(\varepsilon))(a_i), (\chi^\delta_{\gamma S} \odot_h F(\varepsilon))(a'_j),\right.\\
&\qquad \left.\chi^\delta_{\gamma S}(b_i), \chi^\delta_{\gamma S}(b'_j)\}, \delta \right\}\\
&\geq \min\{(\chi^\delta_{\gamma S} \odot_h F(\varepsilon))(0), (\chi^\delta_{\gamma S} \odot_h F(\varepsilon))(xy), \delta\}\\
&\quad (\text{since } xyz + 00 + z' = (xy)z + z')\\
&= \min\left\{ \sup_{0+\sum_{i=1}^m a_ib_i+z'=\sum_{j=1}^n a'_jb'_j+z'} \min\{F(\varepsilon)(b_i), F(\varepsilon)(b'_j), \delta\},\right.\\
&\qquad \left. \sup_{xy+\sum_{i=1}^m a_ib_i+z'=\sum_{j=1}^n a'_jb'_j+z'}, \min\{F(\varepsilon)(b_i), F(\varepsilon)(b'_j), \delta\}, \delta \right\}\\
&\geq \min\{F(\varepsilon)(0), F(\varepsilon)(y), \delta\} \geq \min\{F(\varepsilon)(y), \delta\}\\
&\quad (\text{since } 0 + 00 + z' = 00 + z' \text{ and } xy + 00 + z' = xy + z')\\
&\geq \min\{r, \delta\} > \gamma.
\end{aligned}$$

Hence, $(xyz)_r \in_\gamma \chi^\delta_{\gamma S} \odot_h F(\varepsilon) \odot_h \chi^\delta_{\gamma S}$, which contradicts $\chi^\delta_{\gamma S} \odot_h F(\varepsilon) \odot_h \chi^\delta_{\gamma S} \subseteq \vee q_{(\gamma,\delta)} F(\varepsilon)$.

Conversely, assume that the given conditions hold. By Lemmas 3.5 and 3.6, conditions F(1a) and (F3a) is satisfied. We only show (F6a). (F5a) can be similarly proved. Let $x, y, z \in S$ and $\varepsilon \in A$. If $\Sigma(S, A) \odot_h \langle F, A\rangle \odot_h \Sigma(S, A) \overline{\Subset_{(\gamma,\delta)}} \langle F, A\rangle$, then there exist $\varepsilon \in A$ and $x_r \in_\gamma \chi^\delta_{\gamma S} \odot_h F(\varepsilon) \odot_h \chi^\delta_{\gamma S}$ such that $x_r \overline{\in_\gamma \vee q_\delta}\ F(\varepsilon)$. Hence $F(\varepsilon)(x) < r$ and $F(\varepsilon)(x)+r \leq 2\delta$, which gives $F(\varepsilon)(x) < \delta$. For $x+\sum_{i=1}^{m'} a_i b_i c_i + z = \sum_{j=1}^{n'} a'_j b'_j c'_j + z$, by conditions (F6b) and (F3b), we have

$$\begin{aligned}
\delta > \max\{F(\varepsilon)(x),\gamma\} &\geq \max\left\{\min\left\{F(\varepsilon)\left(\sum_{i=1}^{m'} a_i b_i c_i\right), F(\varepsilon)\left(\sum_{j=1}^{n'} a'_j b'_j c'_j\right), \delta\right\}, \gamma\right\} \\
&\geq \max\{\min\{\min\{F(\varepsilon)(a_i b_i c_i), \delta\}, \min\{F(\varepsilon)(a'_j b'_j c'_j), \delta\}, \delta\}, \gamma\} \\
&= \max\{\min\{F(\varepsilon)(a_i b_i c_i), F(\varepsilon)(a'_j b'_j c'_j), \delta\}, \gamma\} \\
&\geq \min\{\min\{F(\varepsilon)(b_i), \delta\}, \min\{F(\varepsilon)(b'_j), \delta\}, \delta\} \\
&= \min\{F(\varepsilon)(b_i), F(\varepsilon)(b'_j), \delta\}.
\end{aligned}$$

It follows that $\max\{F(\varepsilon)(x),\gamma\} \geq \min\{F(\varepsilon)(b_i), F(\varepsilon)(b'_j)\}$. Thus we have

$$\begin{aligned}
r &\leq (\Sigma(S,A) \odot_h \langle F,A\rangle \odot_h \Sigma(S,A))(x) \\
&= \sup_{x+\sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z} \min\{(\chi^\delta_{\gamma S} \odot_h F(\varepsilon))(a_i), (\chi^\delta_{\gamma S} \odot_h F(\varepsilon))(a'_j), \\
&\qquad\qquad \chi^\delta_{\gamma S}(b_i), \chi^\delta_{\gamma S}(b'_j)\} \\
&= \sup_{x+\sum_{i=1}^m a_i b_i + z = \sum_{j=1}^n a'_j b'_j + z} \\
&\qquad \min\left\{\sup_{a_i+\sum_{k=1}^{m_i} a_{ik} b_{ik} + z_1 = \sum_{l=1}^{n_i} a'_{il} b'_{il} + z_1} \min\{F(\varepsilon)(b_{ik}), F(\varepsilon)(b'_{il})\},\right. \\
&\qquad \left.\sup_{a'_j+\sum_{p=1}^{m_j} a_{jp} b_{jp} + z_2 = \sum_{q=1}^{n_j} a'_{jq} b'_{jq} + z_2} \min\{F(\varepsilon)(b_{jp}), F(\varepsilon)(b'_{jq})\}\right\} \\
&= \sup_{x+\sum_{i=1}^{m'} a_i b_i c_i + z' = \sum_{j=1}^{n'} a'_j b'_j c'_j + z'} \min\{F(\varepsilon)(b_i), F(\varepsilon)(b'_j)\} \\
&\leq \sup_{x+\sum_{i=1}^{m'} a_i b_i c_i + z' = \sum_{j=1}^{n'} a'_j b'_j c'_j + z'} \max\{F(\varepsilon)(x),\gamma\} = \max\{F(\varepsilon)(x),\gamma\},
\end{aligned}$$

a contradiction. Therefore condition (F6a) is satisfied. □

For any fuzzy soft set $\langle F,A\rangle$ over a hemiring S, $\varepsilon \in A$ and $r \in (\gamma,1]$. Denote $F(\varepsilon)_r = \{x \in S | x_r \in_\gamma F(\varepsilon)\}$, $\langle F(\varepsilon)\rangle_r = \{x \in S | x_r\, q_\delta F(\varepsilon)\}$ and $[F(\varepsilon)]_r = \{x \in S | x_r \in_\gamma \vee q_\delta F(\varepsilon)\}$. The next theorem presents the relationships between $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideals and crisp h-interior ideals of a hemiring S.

Theorem 3.8. *Let S be a hemiring and $\langle F,A\rangle$ a fuzzy soft set over S. Then:*

(1) $\langle F,A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideal over S if and only if non-empty subset $F(\varepsilon)_r$ is an h-interior ideal of S for all $\varepsilon \in A$ and $r \in (\gamma,\delta]$.

(2) If $2\delta = 1+\gamma$, then $\langle F,A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideal over S if and only if non-empty subset $\langle F(\varepsilon)\rangle_r$ is an h-interior ideal of S for all $\varepsilon \in A$ and $r \in (\delta,1]$.

(3) $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideal over S if and only if non-empty subset $[F(\varepsilon)]_r$ is an h-interior ideal of S for all $\varepsilon \in A$ and $r \in (\gamma, \min\{2\delta - \gamma, 1\}]$.

Proof. The proof is similar to that of Theorem 4.8 in Yin et al. [15]. For integrality, we only present the proof of (3). Let $\langle F, A\rangle$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideal over S and assume that $[F(\varepsilon)]_r \neq \varnothing$ for some $\varepsilon \in A$ and $r \in (\gamma, \min\{2\delta-\gamma, 1\}]$. Let $x, y \in [F(\varepsilon)]_r$. Then $x_r \in_\gamma \vee q_\delta F(\varepsilon)$ and $y_r \in_\gamma \vee q_\delta F(\varepsilon)$, that is, $F(\varepsilon)(x) \geq r > \gamma$ or $F(\varepsilon)(x) > 2\delta - r \geq 2\delta - (2\delta - \gamma) = \gamma$, and $F(\varepsilon)(y) \geq r > \gamma$ or $F(\varepsilon)(y) > 2\delta - r \geq 2\delta - (2\delta - \gamma) = \gamma$. Since $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideal over S, we have $\max\{F(\varepsilon)(xy), \gamma\} \geq \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\}$ and so $F(\varepsilon)(xy) \geq \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\}$ since $\gamma < \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\}$ in any case. Now we consider the following cases.

Case 1: $r \in (\gamma, \delta]$. Then $2\delta - r \geq \delta \geq r$.

(1) If $F(\varepsilon)(x) \geq r$ or $F(\varepsilon)(y) \geq r$, then $F(\varepsilon)(xy) \geq \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\} \geq r$. Hence $(xy)_r \in_\gamma F(\varepsilon)$.

(2) If $F(\varepsilon)(x) + r > 2\delta$ and $F(\varepsilon)(y) + r > 2\delta$, then

$$F(\varepsilon)(xy) \geq \min\{F(\varepsilon)(x), F(\varepsilon)(y),\, \delta\} = \delta \geq r.$$

Hence $(xy)_r \in_\gamma F(\varepsilon)$.

Case 2: $r \in (\delta, \min\{2\delta - \gamma, 1\}]$. Then $r > \delta > 2\delta - r$.

(1) If $F(\varepsilon)(x) \geq r$ and $F(\varepsilon)(y) \geq r$, then $F(\varepsilon)(xy) \geq \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\} = \delta > 2\delta - r$. Hence $(xy)_r q_\delta F(\varepsilon)$.

(2) If $F(\varepsilon)(x)+r > 2\delta$ or $F(\varepsilon)(y)+r > 2\delta$, then $F(\varepsilon)(xy) \geq \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\} > 2\delta - r$. Hence $(xy)_r q_\delta F(\varepsilon)$.

Thus, in any case, $(xy)_r \in_\gamma \vee q_\delta F(\varepsilon)$, that is, $xy \in [F(\varepsilon)]_r$. Similarly, we can show that $x + y \in [F(\varepsilon)]_r$ for all $x, y \in [F(\varepsilon)]_r$, $xyz \in [F(\varepsilon)]_r$ for all $x, z \in S$ and $y \in [F(\varepsilon)]_r$, and that $x + a + z = b + z$ for $x, z \in S$ and $a, b \in [F(\varepsilon)]_r$ implies $x \in [F(\varepsilon)]_r$. Therefore, $[F(\varepsilon)]_r$ is an h-interior ideal of S.

Conversely, assume that the given conditions hold. If there exist $\varepsilon \in A$ and $x, y \in S$ such that $\max\{F(\varepsilon)(xy), \gamma\} < r = \min\{F(\varepsilon)(x), F(\varepsilon)(y), \delta\}$. Then $x_r, y_r \in_\gamma F(\varepsilon)$ but $(xy)_r \overline{\in_\gamma \vee q_\delta} F(\varepsilon)$, that is, $x, y \in [F(\varepsilon)]_r$ but $xy \notin [F(\varepsilon)]_r$, a contradiction. Hence $\langle F, A\rangle$ satisfies condition (F5b). Similarly we may show that $\langle F, A\rangle$ satisfies conditions (F1b), (F3b) and (F6b). Therefore, $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideal over S. □

4. h-semisimple hemirings

In this section, we will concentrate our study on the characterization of h-semisimple hemirings in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left (right) h-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideals. We start by introducing the following definition.

Definition 4.1 ([13])**.** A hemiring S is called *h-semisimple* if every h-ideal of S is weak idempotent, that is, $A = \overline{AA}$ for any h-ideal A.

Lemma 4.2 ([13])**.** *A hemiring S is h-semisimple if and only if one of the following conditions holds:*

(1) There exist $c_i, d_i, e_i, f_i, c'_j, d'_j, e'_j, f'_j, z \in S$ *such that*

$$x + \sum_{i=1}^{m} c_i x d_i e_i x f_i + z = \sum_{j=1}^{n} c'_j x d'_j e'_j x f'_j + z,$$

for all $x \in S$.
(2) $x \in \overline{SxSxS}$ *for all* $x \in S$.
(3) $A \subseteq \overline{SASAS}$ *for all* $A \subseteq S$.

Theorem 4.3. *Let* S *be an* h*-semisimple hemiring. Then* $\langle F, A\rangle$ *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft* h*-interior ideal over* S *if and only if* $\langle F, A\rangle$ *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft* h*-ideal over* S.

Proof. Assume that S is an h-semisimple hemiring. If $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-ideal over S, it follows directly from Lemmas 3.6 and 3.7 that $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideal over S. Now let $\langle F, A\rangle$ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideal over S, and x and y any elements of S. Since S is h-semisimple, there exist $c_i, d_i, e_i, f_i, c'_j, d'_j, e'_j, f'_j, z \in S$ such that

$$x + \sum_{i=1}^{m} c_i x d_i e_i x f_i + z = \sum_{j=1}^{n} c'_j x d'_j e'_j x f'_j + z,$$

and so

$$xy + \sum_{i=1}^{m} c_i x d_i e_i x f_i y + zy = \sum_{j=1}^{n} c'_j x d'_j e'_j x f'_j y + zy.$$

Then for any $\varepsilon \in A$, we have

$$\begin{aligned}
\max\{F(\varepsilon)(xy), \gamma\} &\geq \min\left\{F(\varepsilon)\left(\sum_{i=1}^{m} c_i x d_i e_i x f_i y\right), F(\varepsilon)\left(\sum_{j=1}^{n} c'_j x d'_j e'_j x f'_j y\right), \delta\right\} \\
&\geq \min\{\min\{F(\varepsilon)(c_i x d_i e_i x f_i y), \delta\}, \min\{F(\varepsilon)(c'_j x d'_j e'_j x f'_j y), \delta\}, \delta\} \\
&= \min\{F(\varepsilon)((c_i x d_i e_i) x (f_i y)), F(\varepsilon)((c'_j x d'_j e'_j) x (f'_j y)), \delta\} \\
&\geq \min\{\min\{F(\varepsilon)(x), \delta\}, \min\{F(\varepsilon)(x), \delta\}, \delta\} \\
&= \min\{F(\varepsilon)(x), \delta\}.
\end{aligned}$$

This implies that $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft right h-ideal over S. In a similar way we may prove that $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left h-ideal over S. Thus $\langle F, A\rangle$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-ideal over S. □

Theorem 4.4. *A hemiring* S *is* h*-semisimple if and only if for any* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft* h*-interior ideals* $\langle F, A\rangle$ *and* $\langle G, B\rangle$ *over* S*, we have* $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle \asymp_{(\gamma,\delta)} \langle F, A\rangle \odot_h \langle G, B\rangle$.

Proof. Assume that S is h-semisimple. Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideals over S. Then, by Theorem 4.3, $\langle F, A\rangle$ and $\langle G, B\rangle$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-ideals over S. Hence we have

$$\langle F, A\rangle \odot_h \langle G, B\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle \odot_h \Sigma(S, B)\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle$$

and

$$\langle F,A\rangle \odot_h \langle G,B\rangle \Subset_{(\gamma,\delta)} \Sigma(S,A)\odot_h \langle G,B\rangle \Subset_{(\gamma,\delta)} \langle G,B\rangle.$$

Thus $\langle F,A\rangle \odot_h \langle G,B\rangle \Subset_{(\gamma,\delta)} \langle F,A\rangle\widetilde{\cap}\langle G,B\rangle$.

Now let x be any element of S, $\varepsilon \in A\cup B$ and $\langle F,A\rangle\widetilde{\cap}\langle G,B\rangle = \langle H, A\cup B\rangle$. We consider the following cases.

Case 1: $\varepsilon \in A-B$. Then $H(\varepsilon)=F(\varepsilon)=(F\odot_h G)(\varepsilon)$.

Case 2: $\varepsilon \in B-A$. Then $H(\varepsilon)=G(\varepsilon)=(F\odot_h G)(\varepsilon)$.

Case 3: $\varepsilon \in A\cap B$. Then $H(\varepsilon)=F(\varepsilon)\cap G(\varepsilon)$ and $(F\odot_h G)(\varepsilon)=F(\varepsilon)\odot_h G(\varepsilon)$. Now we show that $F(\varepsilon)\cap G(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}F(\varepsilon)\odot_h G(\varepsilon)$. Since S is h-semisimple, there exist $c_i, d_i, e_i, f_i, c'_j, d'_j, e'_j, f'_j, z\in S$ such that

$$x+\sum_{i=1}^{m'} c_i x d_i e_i x f_i + z = \sum_{j=1}^{n'} c'_j x d'_j e'_j x f'_j + z.$$

Thus we have

$$\begin{aligned}
&\max\{(F(\varepsilon)\odot_h G(\varepsilon))(x),\gamma\}\\
&=\max\left\{\sup_{x+\sum_{i=1}^{m} a_i b_i + z=\sum_{j=1}^{n} a'_j b'_j + z}\min\{F(\varepsilon)(a_i),F(\varepsilon)(a'_j),G(\varepsilon)(b_i),G(\varepsilon)(b'_j)\},\gamma\right\}\\
&\geq \min\{\max\{F(\varepsilon)(c_i x d_i),\gamma\},\max\{F(\varepsilon)(c'_j x d'_j),\gamma\},\max\{G(\varepsilon)(e_i x f_i),\gamma\},\\
&\quad \max\{G(\varepsilon)(e'_j x f'_j),\gamma\}\}\\
&\geq \min\{\min\{F(\varepsilon)(x),\delta\},\min\{F(\varepsilon)(x),\delta\},\min\{G(\varepsilon)(x),\delta\},\min\{G(\varepsilon)(x),\delta\}\}\\
&=\min\{F(\varepsilon)(x),G(\varepsilon)(x),\delta\}\\
&=\min\{(F(\varepsilon)\cap G(\varepsilon))(x),\delta\}.
\end{aligned}$$

It follows that $F(\varepsilon)\cap G(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}F(\varepsilon)\odot_h G(\varepsilon)$, that is, $H(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}(F\odot_h G)(\varepsilon)$. Thus, in any case, we have $H(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}(F\odot_h G)(\varepsilon)$, and so $\langle F,A\rangle\widetilde{\cap}\langle G,B\rangle \Subset_{(\gamma,\delta)} \langle F,A\rangle\odot_h\langle G,B\rangle$. Therefore, $\langle F,A\rangle\widetilde{\cap}\langle G,B\rangle \asymp_{(\gamma,\delta)} \langle F,A\rangle\odot_h\langle G,B\rangle$.

Conversely, assume that the given condition holds. Let H be any ideal of S. Then it is easy to see that the relative whole (γ,δ)-fuzzy soft set $\Sigma(H,A)$ is an $(\in,\in_\gamma\vee q_\delta)$-fuzzy h-ideal over S. Now by assumption and Lemma 2.5, we have

$$\chi^{\delta}_{\gamma\overline{H\cap H}}=\chi^{\delta}_{\gamma H}\cap\chi^{\delta}_{\gamma H}=_{(\gamma,\delta)}\chi^{\delta}_{\gamma H}\odot_h\chi^{\delta}_{\gamma H}=\chi^{\delta}_{\gamma\overline{HH}}.$$

It follows from Lemma 2.5 that $H=\overline{H}=\overline{HH}$. Thus, S is h-semisimple. □

Combining Theorems 4.3 and 4.4, we have the following result.

Theorem 4.5. *Let S be a hemiring. Then the following conditions are equivalent.*

(1) S is h-semisimple;

(2) $\langle F,A\rangle\widetilde{\cap}\langle G,B\rangle \asymp_{(\gamma,\delta)} \langle F,A\rangle\odot_h\langle G,B\rangle$ for any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft h-ideals $\langle F,A\rangle$ and $\langle G,B\rangle$ over S;

(3) $\langle F,A\rangle\widetilde{\cap}\langle G,B\rangle \asymp_{(\gamma,\delta)} \langle F,A\rangle\odot_h\langle G,B\rangle$ for any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft h-ideal $\langle F,A\rangle$ and any $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft h-interior ideal $\langle G,B\rangle$ over S;

(4) $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle \asymp_{(\gamma,\delta)} \langle F, A\rangle \odot_h \langle G, B\rangle$ *for any* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft* h*-interior ideal* $\langle F, A\rangle$ *and any* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft* h*-ideal* $\langle G, B\rangle$ *over* S*;*

(5) The set of all $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft* h*-ideals over* S *is a semilattice under the* h*-intrinsic product of two fuzzy soft sets and the relation "* $\asymp_{(\gamma,\delta)}$ *" on* $\mathscr{F}\mathscr{S}(U, E)$*, that is,* $\langle F, A\rangle \odot_h \langle G, B\rangle \asymp_{(\gamma,\delta)} \langle G, B\rangle \odot_h \langle F, A\rangle$ *and* $\langle F, A\rangle \odot_h \langle F, A\rangle \asymp_{(\gamma,\delta)} \langle F, A\rangle$ *for all* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft* h*-ideals* $\langle F, A\rangle$ *and* $\langle G, B\rangle$ *over* S*;*

(6) The set of all $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft* h*-interior ideals over* S *is a semilattice under the* h*-intrinsic product of two fuzzy soft sets and the relation "* $\asymp_{(\gamma,\delta)}$ *" on* $\mathscr{F}\mathscr{S}(U, E)$*, that is,* $\langle F, A\rangle \odot_h \langle G, B\rangle \asymp_{(\gamma,\delta)} \langle G, B\rangle \odot_h \langle F, A\rangle$ *and* $\langle F, A\rangle \odot_h \langle F, A\rangle \asymp_{(\gamma,\delta)} \langle F, A\rangle$ *for all* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft* h*-interior ideals* $\langle F, A\rangle$ *and* $\langle G, B\rangle$ *over* S*.*

Theorem 4.6. *Let* S *be a hemiring. Then the following conditions are equivalent.*

(1) S *is* h*-semisimple.*

(2) $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle \odot_h \langle G, B\rangle$ *for any* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft left* h*-ideal* $\langle F, A\rangle$ *and any* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft* h*-interior ideal* $\langle G, B\rangle$ *over* S*;*

(3) $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle \odot_h \langle G, B\rangle$ *for any* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft left* h*-ideal* $\langle F, A\rangle$ *and any* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft* h*-ideal* $\langle G, B\rangle$ *over* S*;*

(4) $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle \odot_h \langle G, B\rangle$ *for any* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft* h*-interior ideal* $\langle F, A\rangle$ *and any* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft right* h*-ideal* $\langle G, B\rangle$ *over* S*;*

(5) $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle \Subset_{(\gamma,\delta)} \langle F, A\rangle \odot_h \langle G, B\rangle$ *for any* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft* h*-ideal* $\langle F, A\rangle$ *and any* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy soft right* h*-ideal* $\langle G, B\rangle$ *over* S*.*

Proof. Assume that (1) holds. Let S be an h-semisimple hemiring, $\langle F, A\rangle$ any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft left h-ideal and $\langle G, B\rangle$ any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft h-interior ideal over S, respectively. Now let x be any element of S, $\varepsilon \in A \cup B$ and $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle = \langle H, A \cup B\rangle$. We consider the following cases.

Case 1: $\varepsilon \in A - B$. Then $H(\varepsilon) = F(\varepsilon) = (F \odot_h G)(\varepsilon)$.

Case 2: $\varepsilon \in B - A$. Then $H(\varepsilon) = G(\varepsilon) = (F \odot_h G)(\varepsilon)$.

Case 3: $\varepsilon \in A \cap B$. Then $H(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$ and $(F \odot_h G)(\varepsilon) = F(\varepsilon) \odot_h G(\varepsilon)$. Now we show that $F(\varepsilon) \cap G(\varepsilon) \subseteq \vee q_{(\gamma,\delta)} F(\varepsilon) \odot_h G(\varepsilon)$. Since S is h-semisimple, there exist $c_i, d_i, e_i, f_i, c'_j, d'_j, e'_j, f'_j, z \in S$ such that

$$x + \sum_{i=1}^{m} c_i x d_i e_i x f_i + z = \sum_{j=1}^{n} c'_j x d'_j e'_j x f'_j + z.$$

Thus, we have

$$\begin{aligned}
&\max\{(F(\varepsilon)\odot_h G(\varepsilon))(x),\gamma\}\\
&=\max\left\{\sup_{x+\sum_{i=1}^m a_ib_i+z=\sum_{j=1}^n a'_jb'_j+z}\min\{F(\varepsilon)(a_i),F(\varepsilon)(a'_j),G(\varepsilon)(b_i),G(\varepsilon)(b'_j)\},\gamma\right\}\\
&\geq \max\{\min\{F(\varepsilon)(c_ix),F(\varepsilon)(c'_jx),G(\varepsilon)(d_ie_ixf_i),G(\varepsilon)(d'_je'_jxf'_j)\},\gamma\}\\
&=\min\{\max\{F(\varepsilon)(c_ix),\gamma\},\max\{F(\varepsilon)(c'_jx),\gamma\},\max\{G(\varepsilon)(d_ie_ixf_i),\gamma\},\\
&\qquad\qquad \max\{G(\varepsilon)(d'_je'_jxf'_j),\gamma\}\}\\
&\geq \min\{\min\{F(\varepsilon)(x),\delta\},\min\{G(\varepsilon)(x),\delta\}\}=\min\{(F(\varepsilon)\cap G(\varepsilon)(x),\delta\}.
\end{aligned}$$

It follows that $F(\varepsilon)\cap G(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}F(\varepsilon)\odot_h G(\varepsilon)$, that is, $H(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}(F\odot_h G)(\varepsilon)$.

Thus, in any case, we have $H(\varepsilon)\subseteq\vee q_{(\gamma,\delta)}(F\odot_h G)(\varepsilon)$ and so (2) holds. Similarly, we may show that (4) holds. It is clear that (2)⇒(3) and (4)⇒(5).

Now assume that (3) holds. Let H be any ideal of S. Then it is easy to see that the relative whole (γ,δ)-fuzzy soft set $\Sigma(H,A)$ is an $(\in,\in_\gamma\vee q_\delta)$-fuzzy h-ideal over S. Now by assumption and Lemma 2.5, we have

$$\chi^\delta_{\gamma\overline{H\cap H}}=\chi^\delta_{\gamma H}\cap\chi^\delta_{\gamma H}\subseteq\vee q_{(\gamma,\delta)}\chi^\delta_{\gamma H}\odot_h\chi^\delta_{\gamma H}=\chi^\delta_{\gamma\overline{HH}}.$$

It follows from Lemma 2.5 that $H=\overline{H}\subseteq\overline{HH}$. On the other hand, it is clear that $\overline{HH}\subseteq\overline{H}=H$. Thus, S is h-semisimple. In a similar way, we may show that (5)⇒(1). This completes the proof. □

5. Conclusions

In this paper, we introduced the concept of $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft h-interior ideals over a hemiring and investigated some of its related properties. We also derived some characterization theorems of h-semisimple hemirings in terms of $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft left (right) h-ideals and $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy soft h-interior ideals. Our future work on this topic will focus on studying the characterization of other types of hemirings.

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