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Generalized $(\mathcal{I}, \mathcal{T})$-$L$-fuzzy rough sets based on $\mathcal{T}L$-fuzzy relational morphisms on semigroups

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Abstract. In this paper, we study some properties of generalized $(\mathcal{I}, \mathcal{T})$-$L$-fuzzy rough sets on semigroups with respect to $\mathcal{T}L$-fuzzy relational morphisms which were introduced by Ignjatović et al. [Jelena Ignjatović, Miroslav Ćirić, Stojan Bogdanović, Fuzzy homomorphisms of algebras, Fuzzy Sets and Systems 160 (2009) 2345-2365].

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1. Introduction

The rough set theory originally proposed [29] in 1982 by Pawlak is a powerful mathematical method to handle imprecision, vagueness, and uncertainty in data analysis. Owing to the explorations on itself or the usefulness on computer sciences, the rough set theory is an expanding research area. The theory has recently received wide attention in real-life applications and theoretical research. In this theory any subset of a universe which has uncertainty concepts is stated by a pair of ordinary sets called the lower and upper approximations. Thus knowledge hidden in information systems may be unraveled and expressed in the form of decision rules. So the starting point of this theory is an observation that objects having the same description are indiscernible with respect to available information. In Pawlak rough sets, the properties of elements are examined via equivalence classes, and the equivalence classes are the building blocks for the construction of the lower and upper approximations. The lower approximation of a given set is the union of all the equivalence classes which are subsets of the set, and the upper approximation of a given set is the union of all the equivalence classes which have a non-empty intersection with the set.

Rough sets have been studied the aspects of constructive and algebraic. In constructive approach, the lower and upper approximation operators are defined by using the binary relations and in algebraic (or axiomatic) approach, dual approximation operators providing some predetermined axioms are defined. The extensions of rough set theory have gained importance in the sense of the both of these two approaches [1, 2, 9, 17, 30, 34, 38, 40].

Some possible connections between rough sets and algebraic systems have been investigated considering the congruence relations on algebraic systems in place of the equivalence relations on universal sets. Biswas and Nanda [4] applied the notion of rough sets to algebra and introduced the notion of rough subgroups. Kuroki [20], introduced the notion of a rough ideal in a semigroup. In [7], Davvaz concerned the relationship between rough sets and ring theory considering a ring as a universal set. In [6], he also introduced the concept of a set-valued homomorphism for groups, which is a generalization of ordinary homomorphism. Yamak et al. [37] investigated the lower and upper approximations defined by the set-valued homomorphism in a ring with respect to an ideal of the ring. In [36], they also investigated the approximations in a module by using set-valued homomorphisms.

Fuzzy set theory which was introduced by Zadeh [39] in 1965 is an other mathematical tool to cope with the trouble of grading the turbidity in some universes. Dubois and Prade [10] introduced the problem that communicate with the fuzzy sets and the rough sets. Li et al. [24] studied $(\mathcal{I}, \mathcal{T})$-fuzzy rough approximation operators on a ring as a universal set with respect to a $\mathcal{T}L$-fuzzy ideal of a ring. Recently, Li and Yin [23] investigated the properties of ν-lower and $\mathcal{T}$-upper fuzzy rough approximation operators with respect to a $\mathcal{T}$-congruence L-fuzzy relation on a semigroup as a universal set. Wu, Leung and Mi [31] expanded $(\mathcal{I}, \mathcal{T})$-fuzzy rough set into two different universal sets. Since $\mathcal{T}$-congruence L-fuzzy relations are not suitable for generalized $(\mathcal{I}, \mathcal{T})$-fuzzy rough set, Ekiz et al. [11] applied generalized $(\mathcal{I}, \mathcal{T})$-fuzzy rough set to the theory of ring via $\mathcal{T}L$-fuzzy relational morphism introduced by Ignjatović et al. [15]. Fuzzy rough approximations, as a fuzzy generalization of the rough sets, received great attention in terms of the theoretical studies [5, 8, 12, 13, 18, 21, 25, 26, 27, 28, 30, 31, 32, 33, 35]. In this theoretical studies, much attention has been paid to set approximation by fuzzy relation, while little work has been done on extensions to the two universes. Thus, it is interesting to extend the universes to much wider classes of mathematical objects, for example to semigroups. This paper explores the theoretical study of generalized fuzzy rough approximation operators within the framework of semigroup theory. The widest possible context, we investigate relationship among L-fuzzy sets, rough sets and semigroup theory.

This paper is an attendance of ideas presented by some authors such as Davvaz [6], Kuroki [20], Yamak et al. [36, 37] and, Li and Yin [23]. Ignjatović et al.[15] introduced the notions of relational morphism and $\mathcal{T}L$-fuzzy relational morphism. We consider two semigroups as the universal sets and we shall introduce the notion of $(\mathcal{I}, \mathcal{T})$-$L$-fuzzy generalized rough set with respect to the $\mathcal{T}L$-fuzzy relational morphisms.

2. Preliminaries

In this paper we will use complete lattices as the truth values. $(L, \wedge, \vee, 0, 1)$ is denote a lattice with the least element 0 and the greatest element 1. Let X be a non-empty set called universe of discourse. L-fuzzy subsets (or L-fuzzy sets) was introduced by Goguen [14] as a generalization of the notion of Zadeh's fuzzy subsets. An L-fuzzy subset of X is any function from X into L. The class of L-fuzzy subsets (all subsets) of X will be denoted by $F(X, L)$ $(\mathcal{P}(X))$. In particular, if $L = [0, 1]$, then it is appropriate to replace fuzzy subset with L-fuzzy subset [39]. In this case the set of all fuzzy subsets of X is denoted by $F(X)$. For any $\mu \in F(X, L)$, the α-cut (or level) set of μ will be denoted by μ_α, that is, $\mu_\alpha = \{x \in X \mid \mu(x) \geq a\}$, where $\alpha \in L$. Any L-fuzzy subset μ of X has $\vee$-property if there exists an element $x_0 \in A$ such that $\mu(x_0) = \bigvee_{x \in A} \mu(x)$ for all non-empty subset A of X. In what follows, α_y will denote the fuzzy singleton with value α at y and 0 elsewhere. The characteristic function of a set $A \subseteq X$ is a function with value 1 if $y \in A$ and 0 if otherwise, and it is denoted by 1_A. Let μ and ν be any two L-fuzzy subsets of X. The symbols $\mu \vee \nu$ and $\mu \wedge \nu$ will means the following fuzzy subsets of X, for all x in X

$$(\mu \vee \nu)(x) = \mu(x) \vee \nu(x),$$
$$(\mu \wedge \nu)(x) = \mu(x) \wedge \nu(x).$$

Some lattice structures which are used in certain part of this paper are depicted in the following Hasse diagrams.

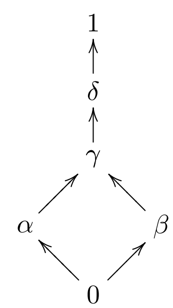

Figure 1. A lattice structure

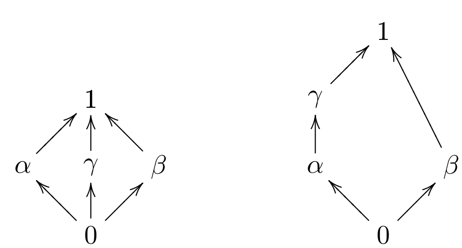

Figure 2. Lattices M_5 and N_5, respectively

2.1. **Fuzzy logical operators.** Let L be a complete lattice. A triangular norm, or t-norm in short, is an increasing, associative and commutative mapping $\mathcal{T} : L \times L \to L$ that satisfies the boundary condition: for all $\alpha \in L$, $\mathcal{T}(\alpha, 1) = \alpha$.

If, for two t-norms $\mathcal{T}_1$ and $\mathcal{T}_2$, the ordering $\mathcal{T}_1(\alpha, \beta) \leq \mathcal{T}_2(\alpha, \beta)$ holds for all $\alpha, \beta \in L$, then one can be said that $\mathcal{T}_1$ is weaker than $\mathcal{T}_2$ or, equivalently, that $\mathcal{T}_2$ stronger than $\mathcal{T}_1$, and it is written in this case $\mathcal{T}_1 \leq \mathcal{T}_2$. A t-norm $\mathcal{T}$ on L is called $\vee$-distributive if $\mathcal{T}(\alpha, \beta_1 \vee \beta_2) = \mathcal{T}(\alpha, \beta_1) \vee \mathcal{T}(\alpha, \beta_2)$ for all $\alpha, \beta_1, \beta_2 \in L$. $\mathcal{T}$ is also called infinitely $\vee$-distributive if $\mathcal{T}(\alpha, \bigvee_{i\in\Lambda} \beta_i) = \bigvee_{i\in\Lambda} \mathcal{T}(\alpha, \beta_i)$ for all $\alpha, \beta_i \in L$, where Λ is an index set. Any $a \in L$ is called an idempotent element of L with respect to the t-norm T if $\mathcal{T}(a, a) = a$. All of the idempotent elements of L is denoted by the set $D_{\mathcal{T}} = \{a \in L \mid \mathcal{T}(a, a) = a\}$ and for any t-norm $\mathcal{T}$, the operation $\mathcal{T}$ is a binary operation on $D_{\mathcal{T}}$. Two of the most popular t-norms on the lattice $L = [0, 1]$ are:

- the standard minimum operator $\mathcal{T}_M(\alpha, \beta) = min\{\alpha, \beta\}$ (the largest t-norm),
- the algebraic product $\mathcal{T}_p(\alpha, \beta) = \alpha \cdot \beta$

The standard minimum operator $\mathcal{T}_M$ defined by $\mathcal{T}(\alpha, \beta) = \alpha \wedge \beta$ for all $\alpha, \beta \in L$, is also a t-norm on an arbitrary complete lattice L.

A triangular conorm, or t-conorm in short, is an increasing, associative and commutative mapping $\mathcal{S} : L \times L \to L$ that satisfies the boundary condition: for all $\alpha \in L$, $\mathcal{S}(\alpha, 0) = \alpha$.

A negator $\mathcal{N}$ is a decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0) = 1$ and $\mathcal{N}(1) = 0$. A negator $\mathcal{N}$ is called involutive iff $\mathcal{N}(\mathcal{N}(\alpha)) = \alpha$ for all $\alpha \in L$. The negator $\mathcal{N}_s(\alpha) = 1 - \alpha$ for all $\alpha \in [0, 1]$ is usually referred to as the standard negator. Take the lattice M_5 specified in figure 2. There are two examples of negators in the following table

x	0	α	β	γ	1
$\mathcal{N}_1(x)$	1	β	γ	α	0
$\mathcal{N}_2(x)$	1	β	α	γ	0

where $\mathcal{N}_2$ is an involutive negator and $\mathcal{N}_1$ is not an involutive negator.
For a given negator $\mathcal{N}$, a t-norm $\mathcal{T}$ and a t-conorm S are called dual with respect to $\mathcal{N}$ iff the De Morgan's laws are satisfied, i.e.,

$$S(\mathcal{N}(\alpha), \mathcal{N}(\beta)) = \mathcal{N}(\mathcal{T}(\alpha, \beta))$$
$$\mathcal{T}(\mathcal{N}(\alpha), \mathcal{N}(\beta)) = \mathcal{N}(\mathcal{S}(\alpha, \beta)).$$

By an implicator we mean a function $\mathcal{I} : L \times L \to L$ satisfying $\mathcal{I}(1, 0) = 0$ and $\mathcal{I}(1, 1) = \mathcal{I}(0, 1) = \mathcal{I}(0, 0) = 1$. If, for two implicators $\mathcal{I}_1$ and $\mathcal{I}_2$, the ordering $\mathcal{I}_1(\alpha, \beta) \leq \mathcal{I}_2(\alpha, \beta)$ holds for all $\alpha, \beta \in L$, then one can be said that $\mathcal{I}_1$ is weaker than $\mathcal{I}_2$ or, equivalently, that $\mathcal{I}_2$ stronger than $\mathcal{I}_1$, and it is written in this case $\mathcal{I}_1 \leq \mathcal{I}_2$.

Given a t-norm $\mathcal{T}$, a t-conorm $\mathcal{S}$, and an implicator $\mathcal{I}$, and two L-fuzzy subsets

μ and ν of X, we can define the corresponding L-fuzzy subsets as

$$(\mu\mathcal{T}\nu)(x) = \mathcal{T}(\mu(x), \nu(x)),$$
$$(\mu\mathcal{S}\nu)(x) = \mathcal{S}(\mu(x), \nu(x)),$$
$$(\mu\mathcal{I}\nu)(x) = \mathcal{I}(\mu(x), \nu(x)).$$

for all $x \in X$. We recall here the definitions of two main cases of operators [3]. Let $\mathcal{T}$ be a t-norm, $\mathcal{S}$ be a t-conorm and $\mathcal{N}$ a negator. An implicator $\mathcal{I}$ is called

- an $\mathcal{S}$-implicator based on $\mathcal{S}$ and $\mathcal{N}$ iff
  $\mathcal{I}(x, y) = \mathcal{S}(\mathcal{N}(x), y)$ for all $x, y \in L$
- a R-implicator (residual implicator) based on $\mathcal{T}$ iff
  $\mathcal{I}(x, y) = \bigvee_{\mathcal{T}(x,\alpha)\leq y} \alpha$ for all $x, y \in L$

2.2. **L-fuzzy relations.** Let X, Y and Z be non-empty sets. An element $\varphi \in \mathcal{P}(X \times Y)$ is referred to as a (crisp) binary relation from X to Y. The inverse of the relation φ is the relation defined by the set $\{(x, y)|(y, x) \in \varphi\}$ and it is denoted by φ^{-1}. The relation φ is referred to as serial if there exists $y \in Y$ such that $(x, y) \in \varphi$ for all $x \in X$; if $X = Y$, then φ is referred to as a binary relation on X. φ is referred to as reflexive if $(x, x) \in \varphi$ for all $x \in X$; φ is referred to as symmetric if $(x, y) \in \varphi$ implies $(y, x) \in \varphi$ for all $x, y \in X$; φ is referred to as transitive if $(x, y) \in \varphi$ and $(y, z) \in \varphi$ imply $(x, z) \in \varphi$ for all $x, y, z \in X$. Recall that an equivalence relation φ is a reflexive, symmetric, and transitive binary relation on X. If φ is an equivalence relation on X then for every $x \in X$, $[x]_\varphi = \{y \in X|(x, y) \in \varphi\}$ denotes the equivalence class of the element x determined by φ. The compositions of the binary relations $\varphi \in \mathcal{P}(X \times Y)$ and $\theta \in \mathcal{P}(Y \times Z)$ is the set $\{(x, z) \in X \times Z|(x, y) \in \varphi \quad and \quad (y, z) \in \theta \quad for \quad some \quad y \in Y\}$ and is denoted by $\theta \circ \varphi$.

A fuzzy L-subset $R \in F(X \times Y, L)$ is referred to as an L-fuzzy binary relation from X to Y, and $R(x, y)$ is the degree of relation between x and y, where $(x, y) \in X \times Y$. If $L = [0, 1]$, then R is called a fuzzy relation from X to Y. If for each $x \in X$, there exists $y \in X$ such that $R(x, y) = 1$, then R is referred to as a serial L-fuzzy relation from X to Y. If $X = Y$, then R is referred to as an L-fuzzy relation on X; R is referred to as a reflexive L-fuzzy relation if $R(x, x) = 1$ for all $x \in X$; R is referred to as a symmetric L-fuzzy relation if $R(x, y) = R(y, x)$ for all $x, y \in X$; R is referred to as a T-transitive L-fuzzy relation if $R(x, z) \geq \bigvee_{y\in X} R(x, y)TR(y, z)$ for all $x, z \in X$. Let $R \in F(X \times Y, L)$ and $R^{-1} \in \mathcal{F}(Y \times X, L)$ be L-fuzzy relations which are satisfy the condition $R^{-1}(y, x) = R(x, y)$ for all $x \in X$ and $y \in Y$. Then R^{-1} is called the inverse L-fuzzy relation of R. The $\mathcal{T}$-compositions of any L-fuzzy relations $R \in F(X \times Y, L)$ and $P \in F(Y \times Z, L)$ is an L-fuzzy relation $P \circ_T R : X \times Y \to L$ defined by $P \circ_T R(x, z) = \bigvee_{y\in Y} R(x, y)TP(y, z)$ for all $(x, z) \in (X, Z)$.

2.3. **$\mathcal{T}L$-fuzzy subsemigroups.** Throughout this paper $(X, .)$ is referred to as a semigroup. In this section, we first recall some basic definitions which are used in the sequel. For any non-empty subsets A and B of the semigroup X the set $A \cdot B$ is defined by $\{a \cdot b \mid a \in A, b \in B\}$. If $A = \varnothing$ or $B = \varnothing$, then $A \cdot B = \varnothing$. Writing AB means $A \cdot B$. By a subsemigroup of X we mean a non-empty subset A of X such that $A^2 \subseteq A$, and by a left (right) ideal of X such that $XA \subseteq A$ $(AX \subseteq A)$. By two-sided ideal (ideal), we mean a non-empty subset of X which is both a left and right ideal of X. A subsemigoup A of a semigroup X is called a bi-ideal of X if $AXA \subseteq A$. Let X and Y be two semigroups. Then a function $f : X \to Y$ provided $f(x).f(y) = f(x.y)$, for all $x, y \in X$, is a homomorphism. An equivalence relation θ on X such that $(a, b) \in \theta$ implies $(a \cdot x, b \cdot x)$ and $(x \cdot a, x \cdot b) \in X$ is a congruence relation on X. An L-fuzzy binary relation R on X is called a $\mathcal{T}$-equivalence L-fuzzy relation if it satisfies the following conditions: for all $x, y, z \in X$,

- $R(x, x) = 1$,
- $R(x, y) = R(y, x)$,
- $R(x, y)\mathcal{T}R(y, z) \leq R(x, z)$.

An L-fuzzy binary relation R on X is called $\mathcal{T}L$-fuzzy compatible if it satisfies $R(x, y)\mathcal{T}R(a, b) \leq R(x.a, y.b)$ for all $x, y, a, b \in X$. R is called $\mathcal{T}$-congruence L-fuzzy relation if it is both a $\mathcal{T}$-equivalence L-fuzzy relation and a $\mathcal{T}L$-fuzzy compatible. The reader will find more information about the definitions which is given below in references [16, 19, 20, 22].

Definition 2.1. Let $(X, \cdot)$ be a semigroup and $\mu \in \mathcal{F}(X, L)$. Then μ is called

- $\mathcal{T}L$-fuzzy subsemigroup of X if $\mu(x)\mathcal{T}\mu(y) \leq \mu(x \cdot y)$ for all $x, y \in X$.
- $\mathcal{T}L$-fuzzy right (or left, two-sided, respectively) ideal of X if $\mu(x) \leq \mu(x \cdot y)$ $\big($or $\mu(x) \leq \mu(y \cdot x)$, $\mu(x) \vee \mu(y) \leq \mu(x \cdot y)$, respectively$\big)$ for all $x, y \in X$.
- $\mathcal{T}L$-fuzzy generalized bi-ideal of X if $\mu(x)\mathcal{T}\mu(y) \leq \mu(x \cdot a \cdot y)$ for all $x, a, y \in X$.

Definition 2.2. Let $(X, \cdot)$ be a semigroup and μ be a $\mathcal{T}L$-fuzzy subsemigroup of X. Then μ is called

- $\mathcal{T}L$-fuzzy bi-ideal of X if $\mu(x)\mathcal{T}\mu(y) \leq \mu(x \cdot a \cdot y)$ for all $x, a, y \in X$.
- $\mathcal{T}L$-fuzzy interior ideal of X if $\mu(a) \leq \mu(x \cdot a \cdot y)$ for all $x, a, y \in X$.

2.4. **Generalized rough sets.** Now we will give some information about generalized rough sets [6, 7, 29, 30, 31, 32, 33, 36, 37, 38].

Let X and Y be two non-empty universes and φ be binary relation from X to Y. Let a set-valued function $F_\varphi : X \to \mathcal{P}(Y)$ be defined by $F_\varphi(x) = \{y \in Y \mid (x, y) \in \varphi\}$ for all $x \in X$. Then the set $F_\varphi(x)$ is called successor neighborhood of x with respect to φ. Also the triple (X, Y, φ) is referred to as a generalized approximation space. Moreover, here, with the help of any set-valued function F from X into $\mathcal{P}(Y)$, a binary relation from X to Y can be defined by setting $\varphi_F = \{(x, y) \mid y \in F_\varphi(x)\}$. For any set $A \subseteq Y$, the lower and upper approximations, $\underline{\varphi}(A)$ and $\overline{\varphi}(A)$, are defined by

$$\underline{\varphi}(A) = \{x \in X \mid F_\varphi(x) \subseteq A\} \text{ and } \overline{\varphi}(A) = \{x \in X \mid F_\varphi(x) \cap A \neq \varnothing\}.$$

The pair $(\underline{\varphi}(A), \overline{\varphi}(A))$ is referred to as a generalized rough set of A in X, and $\underline{\varphi}$ and $\overline{\varphi}$ are referred to as lower and upper generalized approximation operators, respectively. A Pawlak approximation space is an ordered pair (X, φ), where X is a universe and φ is an equivalence relation on X. For each subset A of X, its lower and upper approximations are defined by

$$\underline{Apr}_\varphi(A) = \{x \mid [x]_\varphi \subseteq A\}, \quad \overline{Apr}_\varphi(A) = \{x \mid [x]_\varphi \cap A \neq \varnothing\},$$

respectively. The pair $Apr_\varphi(A) = (\underline{Apr}_\varphi(A), \overline{Apr}_\varphi(A))$ is called the rough set of A in X. It can be easily verified $\underline{\varphi}(A) = \underline{Apr}_\varphi(A)$ and $\overline{\varphi}(A) = \overline{Apr}_\varphi(A)$ when $X = Y$. Therefore every Pawlak's rough set may be considered as a generalized rough set.

3. Construction of generalized $(\mathcal{I}, \mathcal{T})$-$L$-fuzzy rough sets

Let X and Y be a non-empty universes of discourse and R be an L-fuzzy relation from X to Y. Then triple (X, Y, R) is called a generalized L-fuzzy approximation space. If R is an L-fuzzy relation on X, then (X, R) is called an L-fuzzy approximation space. Especially, if R is a fuzzy relation, i.e. $L = [0, 1]$, then some properties of (X, R) are investigated in references [5, 8, 10, 18, 21, 25, 26, 28] and some properties of (X, Y, R) are investigated in references [31, 32, 33].

Let $\mathcal{T}$ and $\mathcal{I}$ be a t-norm and implicator on L, respectively. For any L-fuzzy subset $\mu \in \mathcal{F}(Y, L)$, the $\mathcal{T}$-upper and $\mathcal{I}$-lower L-fuzzy rough approximations of μ, denoted as $\overline{R}^{\mathcal{T}}(\mu)$ and $\underline{R}_{\mathcal{I}}(\mu)$ respectively, with respect to the L-approximation space (X, Y, R) are L-fuzzy sets of X whose membership functions are defined respectively by

$$\overline{R}^{\mathcal{T}}(\mu)(x) = \bigvee_{y \in Y} \mathcal{T}(R(x, y), \mu(y)), \quad x \in X.$$
$$\underline{R}_{\mathcal{I}}(\mu)(x) = \bigwedge_{y \in Y} \mathcal{I}(R(x, y), \mu(y)), \quad x \in X.$$

The operators $\overline{R}^{\mathcal{T}}$ and $\underline{R}_{\mathcal{I}}$ from $\mathcal{F}(Y, L)$ to $\mathcal{F}(X, L)$ are referred to as $\mathcal{T}$-upper and $\mathcal{I}$-lower fuzzy rough approximation operators of (X, Y, R) respectively, and the pair $(\underline{R}_{\mathcal{I}}(\mu), \overline{R}^{\mathcal{T}}(\mu))$ is called the $(\mathcal{I}, \mathcal{T})$-$L$-fuzzy rough set of μ with respect to (X, Y, R). Especially, if $\mathcal{I}$ is an $\mathcal{S}$-implicator based on a t-conorm $\mathcal{S}$ and an involutive negator $\mathcal{N}$, and $\mathcal{T}$ and $\mathcal{S}$ are dual with respect to $\mathcal{N}$, then $(\underline{R}_{\mathcal{I}}(\mu), \overline{R}^{\mathcal{T}}(\mu))$ is called the $\mathcal{S}$-L-fuzzy rough set of μ with respect to (X, Y, R). If $\mathcal{I}$ is an $\mathcal{R}$-implicator based on a t-norm $\mathcal{T}$, then $(\underline{R}_{\mathcal{I}}(\mu), \overline{R}^{\mathcal{T}}(\mu))$ is called the $\mathcal{R}$-L-fuzzy rough set of μ with respect to (X, Y, R). In general, $\mathcal{S}$-L-fuzzy rough set of μ with respect to (X, Y, R) doesn't have to exist since there is not an involutive negator on any lattice L.

Example 3.1. Let $L = M_5$ in Figure 2. Let the sets $X = \{a, b, c\}$ and $Y = \{m, n, p\}$ be given and an L-fuzzy subset μ of Y be defined by

x	m	n	p
$\mu(x)$	0	1	α

Then it can be considered an involutive negator $\mathcal{N}$, a t-norm $\mathcal{T}$ and a t-conorm $\mathcal{S}$ which are dual with respect to $\mathcal{N}$ and an L-fuzzy relation R from X to Y to determine $\mathcal{R}$-L- and $\mathcal{S}$-L-fuzzy rough sets of μ. Consider the t-norm $\mathcal{T}$ in below if for all $x, y \in L$:

$$\mathcal{T}(x, y) = \begin{cases} y, & \text{if } x = 1; \\ x, & \text{if } y = 1; \\ 0, & \text{Otherwise} \end{cases}$$

Take the negator $\mathcal{N}_2$. Then the t-conorm S which is dual of the t-norm $\mathcal{T}$ with respect to $\mathcal{N}_2$ is, for all $x, y \in L$:

$$\mathcal{S}(x,y) = \begin{cases} y, & \textit{if } x = 0; \\ x, & \textit{if } y = 0; \\ 1, & \textit{Otherwise} \end{cases}$$

Consider the L-fuzzy relation R from X to Y defined by the following table

R	m	n	p
a	1	1	1
b	β	β	β
c	0	0	0

Thus the $\mathcal{T}$-upper L-fuzzy rough approximation of μ is

x	a	b	c
$\overline{R}^{\mathcal{T}}(\mu)(x)$	1	β	0

Let $\mathcal{I}_{\mathcal{S}}$ be taken as an $\mathcal{S}$-implicator based on the t-conorm $\mathcal{S}$ and the negator $\mathcal{N}_2$. Then the $\mathcal{I}$-lower L-fuzzy rough approximation of μ is

x	a	b	c
$\underline{R}_{\mathcal{I}_{\mathcal{S}}}(\mu)(x)$	0	α	1

Thus $\mathcal{S}$-L-fuzzy rough set of μ with respect to (X, Y, R) is $(\underline{R}_{\mathcal{I}_{\mathcal{S}}}(\mu), \overline{R}^{\mathcal{T}}(\mu))$. Let $\mathcal{I}_{\mathcal{R}}$ be taken as an $\mathcal{R}$-implicator based on the t-norm $\mathcal{T}$. Then the $\mathcal{I}$-lower L-fuzzy rough approximation of μ is

x	a	b	c
$\underline{R}_{\mathcal{I}_{\mathcal{R}}}(\mu)(x)$	0	1	1

Thus $\mathcal{R}$-L-fuzzy rough set of μ with respect to (X, Y, R) is $(\underline{R}_{\mathcal{I}_{\mathcal{R}}}(\mu), \overline{R}^{\mathcal{T}}(\mu))$.

Example 3.2. Let L be the lattice structure in Figure 1, and the sets $X = \{a, b, c\}$ and $Y = \{m, n, p\}$, and an L-fuzzy subset μ of Y be given as follows

x	m	n	p
$\mu(x)$	δ	1	α

Consider the t-norm $\mathcal{T}$ defined by

$$\mathcal{T}(x,y) = \begin{cases} \gamma, & x = \delta \textit{ and } y = \delta; \\ x \wedge y, & \textit{Otherwise} \end{cases}$$

Let R be an L-fuzzy relation from X to Y defined by the following table

R	m	n	p
a	0	α	β
b	β	δ	0
c	β	0	γ

Thus the $\mathcal{T}$-upper L-fuzzy rough approximation of μ is

x	a	b	c
$\overline{R}^{\mathcal{T}}(\mu)(x)$	α	δ	γ

Take the R-implicator based on the t-norm $\mathcal{T}$. The $\mathcal{I}$-lower L-fuzzy rough approximation of μ is

x	a	b	c
$\underline{R}_{\mathcal{I}}(\mu)(x)$	α	1	γ

$\mathcal{R}$-L-fuzzy rough set of μ is $\big(\underline{R}_{\mathcal{I}}(\mu), \overline{R}^{\mathcal{T}}(\mu)\big)$. $\mathcal{S}$-L-fuzzy rough set of μ does not calculable since there is no involutive negator on the lattice L.

Theorem 3.3 ([31])**.** *Let $B \subseteq Y$ and $\mu, \nu, 1_B, 1_y \in \mathcal{F}(Y, L)$. Then*

(i) $\overline{R}^{\mathcal{T}}(\bigvee_{i\in I} \mu_i) = \bigvee_{i\in I} \overline{R}^{\mathcal{T}}(\mu_i)$,
(ii) *If $\mu \leq \nu$, then $\overline{R}^{\mathcal{T}}(\mu) \leq \overline{R}^{\mathcal{T}}(\nu)$* ,
(iii) $\overline{R}^{\mathcal{T}}(\bigwedge_{i\in I} \mu_i) \leq \bigwedge_{i\in I} \overline{R}^{\mathcal{T}}(\mu_i)$,
(iv) *For all $(x, y) \in X \times Y$, $\overline{R}^{\mathcal{T}}(1_y)(x) = R(x, y)$* ,
(v) *For all $(x, y) \in X \times Y$, $\overline{R}^{\mathcal{T}}(1_B)(x) = \bigvee_{y\in B} R(x, y)$* .

Theorem 3.4 ([31])**.** *Let $B \subseteq Y$ and $\mu, \nu, 1_{B\setminus\{y\}}, 1_B \in \mathcal{F}(Y, L)$. Then*

(i) $\underline{R}_{\mathcal{I}}(\bigwedge_{i\in I} \mu_i) = \bigwedge_{i\in I} \underline{R}_{\mathcal{I}}(\mu_i)$,
(ii) *If $\mu \leq \nu$, then $\underline{R}_{\mathcal{I}}(\mu) \leq \underline{R}_{\mathcal{I}}(\nu)$* ,
(iii) $\bigvee_{i\in I} \underline{R}_{\mathcal{I}}(\mu_i) \leq \underline{R}_{\mathcal{I}}(\bigvee_{i\in I} \mu_i)$,
(iv) *For all $(x, y) \in X \times Y$, $\underline{R}_{\mathcal{I}}(1_{B\setminus\{y\}})(x) = \mathcal{I}(R(x, y), 0)$* ,
(v) *For all $(x, y) \in X \times Y$, $\underline{R}_{\mathcal{I}}(1_B)(x) = \bigwedge_{y\notin B} \mathcal{I}(R(x, y), 0)$* .

Theorem 3.5. *Let (X, Y, R) be a L-fuzzy approximation space and $A \subseteq Y$. Then*

(i) *$\overline{R_\alpha}(A) \subseteq [\overline{R}^{\mathcal{T}}(1_A)]_\alpha$ for all $\alpha \in L$,*
(ii) *If R has $\vee$-property, then $\overline{R_\alpha}(A) = [\overline{R}^{\mathcal{T}}(1_A)]_\alpha$ for all $\alpha \in L$.*

Proof.

(i) Let $x \in \overline{R_\alpha}(A)$. Then there exists an element $t \in A$ such that $(x, t) \in R_\alpha$ since $F_{R_\alpha}(x) \cap A \neq \varnothing$. So $\alpha \leq R(x, t) = R(x, t)T1$. We have $\alpha \leq R(x, t)T1_A(t)$ since $t \in A$. And also we have $\alpha \leq \bigvee_{t\in Y} R(x, t)T1_A(t)$. Thus $\alpha \leq \overline{R}^{\mathcal{T}}(1_A)(x)$. Therefore $x \in [\overline{R}^{\mathcal{T}}(1_A)]_\alpha$. Consequently $\overline{R_\alpha}(A) \subseteq [\overline{R}^{\mathcal{T}}(1_A)]_\alpha$

(ii) Let R be L-fuzzy relation which has $\vee$-property and $x \in [\overline{R}^{\mathcal{T}}(1_A)]_\alpha$. Hence $\alpha \leq \overline{R}^{\mathcal{T}}(1_A)(x)$. Thus $\alpha \leq \bigvee_{y\in Y} R(x,y)\mathcal{T}1_A(y) = \bigvee_{y\in A} R(x,y)$. Owing to the fact that R has $\vee$-property there exists an element t of A such that $\alpha \leq R(x,t)$. So we have $(x,t) \in R_\alpha$. Thus it can be seen that $x \in \overline{R_\alpha}(A)$. Finally, we have $\overline{R_\alpha}(A) = [\overline{R}^{\mathcal{T}}(1_A)]_\alpha$ conjunction with (i). □

The following example shows that the equation in Theorem 3.5 (ii) is not true in general unless R has $\vee$-property.

Example 3.6. Let $L = M_5$ and the sets $X = \{k, l, m, n\}$ and $Y = \{a, b, c\}$ be given. Then the relation R defined by the following table is an L-fuzzy relation from X to Y.

R	a	b	c
k	0	1	β
l	γ	γ	β
m	1	α	1
n	0	γ	α

Let $A = \{l\}$ and $B = \{a, c\}$. Then $\bigvee_{(x,y)\in A\times B} R(x,y) = 1$. R has not have $\vee$-property since there is not exist any element $(x,y) \in A \times B$ such that $R(x,y) = 1$. And so we obtain the sets $\overline{R}_\alpha(B) = \{m, n\}$ and $[\overline{R}^{\mathcal{T}}(1_B)]_\alpha = \{l, m, n\}$ for B, where $\mathcal{T}$ is any t-norm on M_5.

Theorem 3.7. *Let (X, Y, R) be an L-fuzzy approximation space and $\mu \in \mathcal{F}(Y, L)$. Then $\overline{R}_\beta(\mu_\alpha) \subseteq [\overline{R}^{\mathcal{T}}(\mu)]_{\alpha\mathcal{T}\beta}$.*

Proof. Let $x \in \overline{R_\beta}(\mu_\alpha)$. Thus $F_{R_\beta}(x) \cap \mu_\alpha \neq \varnothing$. Thus there exists an element t of μ_α such that $(x,t) \in R_\beta$. In this case we have $\alpha \leq \mu(t)$ and $\beta \leq R(x,t)$. So $\alpha\mathcal{T}\beta \leq \mu(t)\mathcal{T}R(x,t) \leq \overline{R}^{\mathcal{T}}(\mu)(x)$. Hence $x \in [\overline{R}^{\mathcal{T}}(\mu)]_{\alpha\mathcal{T}\beta}$. We obtain $\overline{R_\beta}(\mu_\alpha) \subseteq [\overline{R}^{\mathcal{T}}(\mu)]_{\alpha\mathcal{T}\beta}$. □

According to next example Theorem 3.7 is not true with $'' \subseteq''$ replaced by $'' =''$ in general.

Example 3.8. Let $L = N_5$ and given the sets $X = \{k, l, m\}$ and $Y = \{a, b, c\}$. Then the relation R defined by the following table is an L-fuzzy relation from X to Y.

R	a	b	c
k	γ	β	1
l	α	0	β
m	1	α	0

Let $\mu \in \mathcal{F}(Y, L)$ be defined by

y	a	b	c
$\mu(y)$	β	α	β

and $\mathcal{T}$ be any t-norm on N_5. Then $\overline{R}_\beta(\mu_\alpha) = \{k\}$ and $[\overline{R}^{\mathcal{T}}(\mu)]_{\alpha\mathcal{T}\beta} = \{k, l, m\}$.

Theorem 3.9. *Let $\mu \in \mathcal{F}(Y, L)$ and, $\mathcal{T}_1$ and $\mathcal{T}_2$ are two t-norms on L. If $\mathcal{T}_1 \leq \mathcal{T}_2$, then $\overline{R}^{\mathcal{T}_1}(\mu) \leq \overline{R}^{\mathcal{T}_2}(\mu)$.*

Proof. Take any element x from X. Then we have the ordering $\overline{R}^{\mathcal{T}_1}(\mu)(x) = \bigvee_{y\in Y} R(x, y)\mathcal{T}_1\mu(y) \leq \bigvee_{y\in Y} R(x, y)\mathcal{T}_2\mu(y) = \overline{R}^{\mathcal{T}_2}(\mu)(x)$. □

Theorem 3.10. *Let $\mu \in \mathcal{F}(Y, L)$ and, $\mathcal{I}_1$ and $\mathcal{I}_2$ are two implicator on L. If $\mathcal{I}_1 \leq \mathcal{I}_2$, then $\underline{R}_{\mathcal{I}_1}(\mu) \leq \underline{R}_{\mathcal{I}_2}(\mu)$.*

Proof. Take any element x from X. Then we have the ordering $\underline{R}_{\mathcal{I}_1}(\mu)(x) = \bigwedge_{y\in Y} R(x, y)\mathcal{I}_1\mu(y) \leq \bigwedge_{y\in Y} R(x, y)\mathcal{I}_2\mu(y) = \underline{R}_{\mathcal{I}_2}(\mu)(x)$. □

4. Generalized $(\mathcal{I}, \mathcal{T})$-$L$-fuzzy rough sets in semigroups

In [20], Kuroki introduced the lower and the upper approximations in a semigroup with respect to fuzzy congruences. In [23], Li and Yin generalized the lower and the upper approximations to ν-lower and T-upper fuzzy rough approximations with respect to $\mathcal{T}$-congruence L-fuzzy relation on a semigroup. In this study, ν-lower and T-upper fuzzy rough approximations are examined from a larger angle than the others. We use TL-fuzzy relational morphisms which are special L-fuzzy relations from a semigroup to any other semigroup instead of $\mathcal{T}$-congruence L-fuzzy relation on a semigroup. If the semigroups are chosen to be the same and the TL-fuzzy relational morphism is make strong, then TL-fuzzy relational morphism is a $\mathcal{T}$-congruence L-fuzzy relation. We will construct an L-fuzzy approximation space with TL-fuzzy relational morphisms and investigate some properties of the relations between this approximation spaces and semigroups.

Throughout this section $(X, \bullet)$, $(Y, *)$ and $(Z, \star)$ are referred as three semigroups. And unless otherwise stated, $\mathcal{T}$ is taken as an arbitrary t-norm on a complete lattice L.

Definition 4.1 ([15])**.** *Let φ be a (crisp) binary relation from X to Y. Then φ is called a relational morphism from X to Y if the following condition hold:*

$$(x, y), (a, b) \in \varphi \quad \textit{imply} \quad (x \bullet a, y * b) \in \varphi \quad \textit{for all} \quad (x, y), (a, b) \in X \times Y$$

In particular every homomorphism of semigroups are relational morphism.

Definition 4.2 ([15])**.** Let $R \in \mathcal{F}(X \times Y, L)$. R is called $\mathcal{T}L$-fuzzy relational morphism if the following condition hold:

$$R(x, y)\mathcal{T}R(a, b) \leq R(x \bullet a, y * b) \quad \text{for all} \quad x, a \in X \quad \text{and} \quad y, b \in Y.$$

Example 4.3. Two binary operations $''\bullet''$ and $''*''$ on the sets $X = \{a, b\}$ and $Y = \{m, n, p\}$, respectively, are defined by the following tables:

$\bullet$	a	b
a	a	b
b	b	b

$*$	m	n	p
m	m	m	m
n	m	n	n
p	m	n	p

Then $(X, \bullet)$ and $(Y, *)$ are semigroups. Let $L = N_5$ and $R \in \mathcal{F}(X \times Y, L)$ be defined by

R	m	n	p
a	α	β	α
b	γ	γ	0

Then R is a TL-fuzzy relational morphism from X to Y for any t-norm $\mathcal{T}$.

Example 4.4. Let $(X, \bullet)$ and $(Y, *)$ be two semigroups and $f : X \to Y$ be a homomorphism of semigroups. For any $\alpha, \beta \in L$ which are $\beta \leq \alpha$, let $R \in \mathcal{F}(X \times Y, L)$ be defined by

$$R(x, y) = \begin{cases} \alpha, & \textit{if } f(x) = y; \\ \beta, & \textit{if } f(x) \neq y. \end{cases}$$

Then R is a $\mathcal{T}L$-fuzzy relational morphism from X to Y for any t-norm $\mathcal{T}$.

Theorem 4.5. *If R is a $\mathcal{T}L$-fuzzy relational morphism, then R^{-1} is a $\mathcal{T}L$-fuzzy relational morphism .*

Proof. It is trivial from Definition 4.2. □

Theorem 4.6. *Let $R \in \mathcal{F}(X \times Y, L)$. Then*

(i) *If R is a $\mathcal{T}L$-fuzzy relational morphism, then R_α is a relational morphism for all $\alpha \in D_\mathcal{T}$.*
(ii) *If R_α is a relational morphism for all $\alpha \in L$, then R is a $\mathcal{T}L$-fuzzy relational morphism.*

Proof.

(i) Let α be any element in $D_\mathcal{T}$ and $(x, y), (a, b) \in R_\alpha$. Then $\alpha \leq R(x, y)$ and $\alpha \leq R(a, b)$. Since $\alpha \in D_\mathcal{T}$, then it is acquired $\alpha = \alpha\mathcal{T}\alpha \leq R(x, y)\mathcal{T}R(a, b) \leq R(x \bullet a, y * b)$ via the monotonousness of t-norm $\mathcal{T}$. To this respect $(x \bullet a, y * b) \in R_\alpha$ is attained.
(ii) Let $\alpha = R(x, y)\mathcal{T}R(a, b)$. Since $\alpha = R(x, y)\mathcal{T}R(a, b) \leq R(x, y)$ and $\alpha = R(x, y)\mathcal{T}R(a, b) \leq R(a, b)$, then $(x, y), (a, b) \in R_\alpha$. Thus $(x \bullet a, y * b) \in R_\alpha$ by hypothesis. Hence $\alpha \leq R(x \bullet a, y * b)$. In conclusion R is a $\mathcal{T}L$-fuzzy relational morphism. □

Corollary 4.7. *Let $R \in \mathcal{F}(X \times Y, L)$, $\mathcal{T}$ be an arbitrary t-norm and $ImR \subseteq D_\mathcal{T}$. Then R is a $\mathcal{T}L$- fuzzy relational morphism if and only if R_α is a relational morphism for all $\alpha \in D_\mathcal{T}$.*

Proof. It is straightforward from Theorem 4.6 in view of $ImR \subseteq D_\mathcal{T}$. □

Theorem 4.8. *Let $\mathcal{T}$ be an infinitely $\vee$-distributive t-norm and $R \in \mathcal{F}(X \times Y, L)$ and $P \in \mathcal{F}(Y \times Z, L)$. If P and R are $\mathcal{T}L$-fuzzy relational morphisms, then $P \circ_\mathcal{T} R$ is a $\mathcal{T}L$-fuzzy relational morphism.*

Proof. Take any elements (x_1, z_1) and (x_2, z_2) from $X \times Z$. Thus

$$
\begin{aligned}
&(P \circ_{\mathcal{T}} R)(x_1, z_1)\mathcal{T}(P \circ_{\mathcal{T}} R)(x_2, z_2)\\
&= \big(\bigvee_{t\in Y} R(x_1, t)\mathcal{T}P(t, z_1)\big)\mathcal{T}\big(\bigvee_{k\in Y} R(x_2, k)\mathcal{T}P(k, z_2)\big)\\
&= \bigvee_{t\in Y}\bigvee_{k\in Y} R(x_1, t)\mathcal{T}R(x_2, k)\mathcal{T}P(t, z_1)\mathcal{T}P(k, z_2)\\
&\leq \bigvee_{t\in Y}\bigvee_{k\in Y} R(x_1 \bullet x_2, t * k)\mathcal{T}P(t * k, z_1 * z_2)\\
&\leq \bigvee_{y\in Y} R(x_1 \bullet x_2, y)\mathcal{T}R(y, z_1 * z_2)\\
&= (P \circ_{\mathcal{T}} R)(x_1 \bullet x_2, z_1 * z_2).
\end{aligned}
$$

Since $(P \circ_{\mathcal{T}} R)(x_1, z_1)\mathcal{T}(P \circ_{\mathcal{T}} R)(x_2, z_2) \leq (P \circ_{\mathcal{T}} R)(x_1 \bullet x_2, z_1 * z_2)$, then $P \circ_{\mathcal{T}} R$ is a $\mathcal{T}L$-fuzzy relational morphism. $\square$

Theorem 4.9. *Let $\mathcal{T}$ be an infinitely $\vee$-distributive t-norm, $R \in \mathcal{F}(X \times Y, L)$, $P \in \mathcal{F}(Y \times Z, L)$ and $\mu \in \mathcal{F}(Z, L)$. Then $\overline{P \circ_{\mathcal{T}} R}^{\mathcal{T}}(\mu) = \overline{R}^{\mathcal{T}}(\overline{P}^{\mathcal{T}}(\mu))$.*

Proof. Take any element x from X. Thus

$$
\begin{aligned}
\overline{P \circ_{\mathcal{T}} R}^{\mathcal{T}}(\mu)(x) &= \bigvee_{z\in Z} (P \circ_{\mathcal{T}} R)(x, z)\mathcal{T}\mu(z)\\
&= \bigvee_{z\in Z}\Big(\bigvee_{y\in Y} R(x, y)\mathcal{T}P(y, z)\Big)\mathcal{T}\mu(z)\\
&= \bigvee_{z\in Z}\bigvee_{y\in Y} R(x, y)\mathcal{T}P(y, z)\mathcal{T}\mu(z)\\
&= \bigvee_{y\in Y} R(x, y)\mathcal{T}\Big(\bigvee_{z\in Z} P(y, z)\mathcal{T}\mu(z)\Big)\\
&= \bigvee_{y\in Y} R(x, y)\mathcal{T}\overline{P}^{\mathcal{T}}(\mu)(y)\\
&= \overline{R}^{\mathcal{T}}\big(\overline{P}^{\mathcal{T}}(\mu)\big)(x).
\end{aligned}
$$

So we have $\overline{P \circ_{\mathcal{T}} R}^{\mathcal{T}}(\mu) = \overline{R}^{\mathcal{T}}\big(\overline{P}^{\mathcal{T}}(\mu)\big)$. $\square$

Theorem 4.10. *Let $R \in \mathcal{F}(X \times Y, L)$ be a $\mathcal{T}L$-fuzzy relational morphism and $\mathcal{T}$ be an infinitely $\vee$-distributive t-norm. If μ is a $\mathcal{T}L$-fuzzy subsemigroup of Y, then $\overline{R}^{\mathcal{T}}(\mu)$ is a $\mathcal{T}L$-fuzzy subsemigroup of X.*

Proof. Take any elements a and b from X. Thus

$$\begin{aligned}
\overline{R}^{\mathcal{T}}(\mu)(a)\mathcal{T}\overline{R}^{\mathcal{T}}(\mu)(b) &= \Big(\bigvee_{y_1\in Y} R(a,y_1)\mathcal{T}\mu(y_1)\Big)\mathcal{T}\Big(\bigvee_{y_2\in Y} R(b,y_2)\mathcal{T}\mu(y_2)\Big)\\
&= \bigvee_{y_1\in Y}\bigvee_{y_2\in Y} R(a,y_1)\mathcal{T}\mu(y_1)\mathcal{T}R(b,y_2)\mathcal{T}\mu(y_2)\\
&\leq \bigvee_{y_1\in Y}\bigvee_{y_2\in Y} R(a\bullet b,y_1*y_2)\mathcal{T}\mu(y_1)\mathcal{T}\mu(y_2)\\
&\leq \bigvee_{y_1\in Y}\bigvee_{y_2\in Y} R(a\bullet b,y_1*y_2)\mathcal{T}\mu(y_1*y_2)\\
&\leq \bigvee_{y\in Y} R(a\bullet b,y)\mathcal{T}\mu(y)\\
&= \overline{R}^{\mathcal{T}}(\mu)(a\bullet b).
\end{aligned}$$

So we have $\overline{R}^{\mathcal{T}}(\mu)(a)\mathcal{T}\overline{R}^{\mathcal{T}}(\mu)(b) \leq \overline{R}^{\mathcal{T}}(\mu)(a\bullet b)$. □

Theorem 4.11. *Let $R \in \mathcal{F}(X\times Y, L)$ be a serial $\mathcal{T}L$-fuzzy relational morphism and $\mathcal{T}$ be an infinitely $\vee$-distributive t-norm. If μ is a $\mathcal{T}L$-fuzzy left (resp. right or two-sided) ideal of Y, then $\overline{R}^{\mathcal{T}}(\mu)$ is a $\mathcal{T}L$-fuzzy left [resp.right or two-sided] ideal of X.*

Proof. Take any elements $a, b \in X$. Since R is serial $\mathcal{T}L$-fuzzy relational morphism, then there exists an element $t \in Y$ such that $R(a,t) = 1$. So we have

$$\begin{aligned}
\overline{R}^{\mathcal{T}}(\mu)(b) &= 1\mathcal{T}\overline{R}^{\mathcal{T}}(\mu)(b)\\
&= R(a,t)\mathcal{T}\bigvee_{y\in Y} R(b,y)\mathcal{T}\mu(y)\\
&= \bigvee_{y\in Y} R(a,t)\mathcal{T}R(b,y)\mathcal{T}\mu(y)\\
&\leq \bigvee_{y\in Y} R(a\bullet b,t*y)\mathcal{T}\mu(t*y)\\
&\leq \bigvee_{k\in Y} R(a\bullet b,k)\mathcal{T}\mu(k)\\
&= \overline{R}^{\mathcal{T}}(\mu)(a\bullet b).
\end{aligned}$$

Consequently $\overline{R}^{\mathcal{T}}(\mu)$ is a $\mathcal{T}L$-fuzzy left ideal of X. Similarly, if μ is a $\mathcal{T}L$-fuzzy right ideal of Y, then $\overline{R}^{\mathcal{T}}(\mu)$ is a $\mathcal{T}L$-fuzzy right ideal of X. □

Theorem 4.12. *Let $R \in \mathcal{F}(X\times Y, L)$ be a serial $\mathcal{T}L$-fuzzy relational morphism and T be an infinitely $\vee$-distributive t-norm. If μ is a $\mathcal{T}L$-fuzzy generalized bi-ideal of Y, then $\overline{R}^{\mathcal{T}}(\mu)$ is a $\mathcal{T}L$-fuzzy generalized bi-ideal of X.*

Proof. Take any elements x, a, y from X. Since R is serial, then there exists $b \in Y$

such that $R(a,b)=1$ for the $a \in X$. Thus

$$\begin{aligned}
\overline{R}^{\mathcal{T}}(\mu)(x)\mathcal{T}\overline{R}^{\mathcal{T}}(\mu)(y) &= \Big(\bigvee_{t\in Y} R(x,t)\mathcal{T}\mu(t)\Big)\mathcal{T}\Big(\bigvee_{k\in Y} R(y,k)\mathcal{T}\mu(k)\Big)\\
&= \bigvee_{t\in Y}\bigvee_{k\in Y} R(x,t)\mathcal{T}\mu(t)\mathcal{T}R(y,k)\mathcal{T}\mu(k)\\
&\leq \bigvee_{t,k\in Y} R(x,t)\mathcal{T}R(y,k)\mathcal{T}\mu(t*b*k)\\
&= \bigvee_{t,k\in Y} R(x,t)\mathcal{T}R(a,b)\mathcal{T}R(y,k)\mathcal{T}\mu(t*b*k)\\
&\leq \bigvee_{t,k\in Y} R(x\bullet a\bullet y, t*b*k)\mathcal{T}\mu(t*b*k)\\
&\leq \bigvee_{r\in Y} R(x\bullet a\bullet y, r)\mathcal{T}\mu(r)\\
&= \overline{R}^{\mathcal{T}}(\mu)(x\bullet a\bullet y).
\end{aligned}$$

So we obtain $\overline{R}^{\mathcal{T}}(\mu)(x)\mathcal{T}\overline{R}^{\mathcal{T}}(\mu)(y) \leq \overline{R}^{\mathcal{T}}(\mu)(x\bullet a\bullet y)$. □

The following example shows that Theorem 4.11 and Theorem 4.12 are not true in general unless R is serial.

Example 4.13. Let $gcd(x,y)$ be denoted the greatest common divisor of the integers x and y. Then the set $\mathbb{N}$ of all natural numbers is a semigroup, with binary operation given by $x*y = gcd(x,y)$. Let take $L=[0,1]$ and take any t-norm T on $[0,1]$. A fuzzy set μ of $\mathbb{N}$ defined by $\mu(n)=\frac{1}{n+1}$ for all $n\in\mathbb{N}$ is a $\mathcal{T}$-fuzzy left ideal and also $\mathcal{T}$-fuzzy generalized bi-ideal of $\mathbb{N}$. The $\mathcal{T}$-fuzzy relation R of $\mathbb{N}\times\mathbb{N}$ defined by

$$R(x,y)=\begin{cases}\frac{1}{y+1}, & \textit{if } 2\mid x;\\ 0, & \textit{if } 2\nmid x.\end{cases}$$

for all $x,y\in\mathbb{N}$ is a $\mathcal{T}$-fuzzy relational morphism which is not serial since there is not exist any $y\in\mathbb{N}$ for $3\in\mathbb{N}$ such that $R(3,y)=1$. $\mathcal{T}$-upper approximation of μ with respect to the approximation space $(\mathbb{N},R)$ is $\overline{R}^{\mathcal{T}}(\mu)(x) = \bigvee_{y\in\mathbb{N}} R(x,y)\mathcal{T}\mu(y)$ for all $x\in\mathbb{N}$.
So we have

$$\overline{R}^{\mathcal{T}}(\mu)(x)=\begin{cases}1, & \textit{if } 2\mid x;\\ 0, & \textit{if } 2\nmid x.\end{cases}$$

Hence $\overline{R}^{\mathcal{T}}(\mu)$ is not a $\mathcal{T}$-fuzzy left ideal of $\mathbb{N}$ since $\overline{R}^{\mathcal{T}}(\mu)(2)=1\nleq 0=\overline{R}^{\mathcal{T}}(\mu)(1)=\overline{R}^{\mathcal{T}}(\mu)(gcd(1,2))=\overline{R}^{\mathcal{T}}(\mu)(1*2)$ and also it is not a $\mathcal{T}$-fuzzy generalized bi-ideal of $\mathbb{N}$ since $\overline{R}^{\mathcal{T}}(\mu)(2)\mathcal{T}\overline{R}^{\mathcal{T}}(\mu)(2)=1\mathcal{T}1=1\nleq 0=\overline{R}^{\mathcal{T}}(\mu)(1)=\overline{R}^{\mathcal{T}}(\mu)(gcd(2,1,2))=\overline{R}^{\mathcal{T}}(\mu)(2*1*2)$.

Corollary 4.14. *Let $R\in\mathcal{F}(X\times Y,L)$ be a serial $\mathcal{T}L$-fuzzy relational morphism and T be an infinitely $\vee$-distributive t-norm. If μ is a $\mathcal{T}L$-fuzzy bi-ideal of Y, then*

$\overline{R}^{\mathcal{T}}(\mu)$ *is a* $\mathcal{T}L$*-fuzzy bi-ideal of* X*.*

Proof. It is straightforward from Theorem 4.10 and Theorem 4.12. □

Theorem 4.15. *Let* $R \in \mathcal{F}(X \times Y, L)$ *be a serial* $\mathcal{T}L$*-fuzzy relational morphism and* T *an infinitely* $\vee$*-distributive t-norm. If* μ *is a* $\mathcal{T}L$*-fuzzy interior ideal of* Y*, then* $\overline{R}^{\mathcal{T}}(\mu)$ *is a* $\mathcal{T}L$*-fuzzy interior ideal of* X*.*

Proof. Take any element a, x, b from X. Since R is serial, then there exist $y_1, y_2 \in Y$ such that $R(a, y_1) = 1$ and $R(b, y_2) = 1$ for all $a, b \in X$, thus we have

$$\begin{aligned} \overline{R}^{\mathcal{T}}(\mu)(x) &= \bigvee_{t \in Y} R(x,t)\mathcal{T}\mu(t) \\ &= \bigvee_{t \in Y} R(a,y_1)\mathcal{T}R(x,t)\mathcal{T}R(b,y_2)\mathcal{T}\mu(t) \\ &\leq \bigvee_{t \in Y} R(a,y_1)\mathcal{T}R(x,t)\mathcal{T}R(b,y_2)\mathcal{T}\mu(y_1 * t * y_2) \\ &\leq \bigvee_{t \in Y} R(a \bullet x \bullet b, y_1 * t * y_2)\mathcal{T}\mu(y_1 * t * y_2) \\ &\leq \bigvee_{r \in Y} R(a \bullet x \bullet b, r)\mathcal{T}\mu(r) \\ &= \overline{R}^{\mathcal{T}}(\mu)(a \bullet x \bullet b). \end{aligned}$$

So we obtain $\overline{R}^{\mathcal{T}}(\mu)$ is a $\mathcal{T}L$-fuzzy interior ideal of X by using Theorem 4.10. □

The following example shows that Theorem 4.15 is not true in general unless R to be serial.

Example 4.16. Let R be the $\mathcal{T}L$-fuzzy relational morphism which is given in Example 4.3. R is not serial since for $a \in X$ there does not exist any element in Y such that $R(x, y) = 1$. Let $\mu \in \mathcal{F}(Y, L)$ be defined by

y	m	n	p
$\mu(y)$	β	β	0

Then μ is a $\mathcal{T}L$-fuzzy interior ideal of Y, where $\mathcal{T}$ is the minimum t-norm. So we have

$$\overline{R}^{\mathcal{T}}(\mu)(x) = \begin{cases} \beta, & \textit{if } x = a; \\ 0, & \textit{if } x = b. \end{cases}$$

Since $\overline{R}^{\mathcal{T}}(\mu)(a) = \beta \nleq 0 = \overline{R}^{\mathcal{T}}(\mu)(b) = \overline{R}^{\mathcal{T}}(\mu)(b * a * b)$, then $\overline{R}^{\mathcal{T}}(\mu)$ is not a $\mathcal{T}L$-fuzzy interior ideal of X.

Definition 4.17. Let $R \in \mathcal{F}(X \times Y, L)$. Then R is called $\mathcal{T}L$-fuzzy complete relational morphism if $R(x, y)\mathcal{T}R(a, b) = R(x \bullet a, y * b)$ for all $x, a \in X$ and $y, b \in Y$.

Example 4.18. $\mathbb{N} \setminus \{0\}$ is a semigroup under ordinary multiplication of integers. Let take the algebraic product $\mathcal{T}_p$ as the t-norm. Then a fuzzy relation $R : \mathbb{N} \setminus \{0\} \times$

$\mathbb{N}\setminus\{0\} \to [0,1]$ defined by $R(a,b) = \frac{1}{a.b}$ for all $a,b \in \mathbb{N}\setminus\{0\}$, is a $\mathcal{T}_p$-fuzzy complete relational morphism.

Example 4.19. Let R be the $\mathcal{T}L$-fuzzy relational morphism which is given in Example 4.3. Since $R(b,p)\mathcal{T}R(a,n) = 0 \neq \gamma = R(b,n) = R(b \bullet a, p * n)$, then it is not $\mathcal{T}L$-fuzzy complete relational morphism.

Theorem 4.20. *Let $R \in \mathcal{F}(X \times Y, L)$ be a $\mathcal{T}L$-fuzzy complete relational morphism, $\mathcal{T}$ be an infinitely $\vee$-distributive t-norm and $\mathcal{I}$ be a $\mathcal{R}$-implicator based on the t-norm $\mathcal{T}$. If μ be a $\mathcal{T}L$-fuzzy subsemigroup of Y, then $\underline{R}_{\mathcal{I}}(\mu)$ is a $\mathcal{T}L$-fuzzy subsemigroup of X.*

Proof. Take any elements $a,b \in X$.

$$\begin{aligned}
\underline{R}_{\mathcal{I}}(\mu)(a)\mathcal{T}\underline{R}_{\mathcal{I}}(\mu)(b) &= \Big(\bigwedge_{t\in Y} R(a,t)\mathcal{I}\mu(t)\Big)\mathcal{T}\Big(\bigwedge_{k\in Y} R(b,k)\mathcal{I}\mu(k)\Big)\\
&= \bigwedge_{t\in Y}\Big(\bigvee_{R(a,t)\mathcal{T}\alpha\leq\mu(t)} \alpha\Big)\mathcal{T}\bigwedge_{k\in Y}\Big(\bigvee_{R(b,k)\mathcal{T}\beta\leq\mu(k)} \beta\Big)\\
&\leq \bigwedge_{t,k\in Y}\Big(\bigvee_{R(a,t)\mathcal{T}\alpha\leq\mu(t)} \alpha \;\;\mathcal{T} \bigvee_{R(b,k)\mathcal{T}\beta\leq\mu(k)} \beta\Big)\\
&= \bigwedge_{t,k\in Y}\Big(\bigvee_{\substack{R(a,t)\mathcal{T}\alpha\leq\mu(t)\\ R(b,k)\mathcal{T}\beta\leq\mu(k)}} \alpha\mathcal{T}\beta\Big)\\
&\leq \bigwedge_{t*k\in Y}\Big(\bigvee_{R(a\bullet b,t*k)\mathcal{T}\alpha\mathcal{T}\beta\leq\mu(t*k)} \alpha\mathcal{T}\beta\Big)\\
&\leq \bigwedge_{p\in Y}\Big(\bigvee_{R(a\bullet b,p)\mathcal{T}\gamma\leq\mu(p)} \gamma\Big)\\
&= \bigwedge_{p\in Y} R(a\bullet b,p)\mathcal{I}\mu(p)\\
&= \underline{R}_{\mathcal{I}}(\mu)(a\bullet b).
\end{aligned}$$

Therefore we obtain that $\underline{R}_{\mathcal{I}}(\mu)$ is a $\mathcal{T}L$-fuzzy subsemigroup of X. □

The following example shows that Theorem 4.20 is not true in general for a $\mathcal{T}L$-fuzzy relational morphism R. So we need R to be a $\mathcal{T}L$-fuzzy complete relational morphism to satisfying the theorem.

Example 4.21. Let $\mathbb{Z}$ be the set of all integers and $M_2(\mathbb{Z})$ be the set of all 2×2 matrices over $\mathbb{Z}$. Then $(\mathbb{Z}_2,+)$ and $(M_2(\mathbb{Z}),.)$ are semigroups. Let a fuzzy subset μ of $M_2(\mathbb{Z})$ be defined by

$$\mu(A) = \begin{cases} \frac{1}{2}, & \textit{if } detA \neq 0,\\ \frac{1}{3}, & \textit{if } detA = 0,\end{cases}$$

for all $A \in M_2(\mathbb{Z})$. Let the fuzzy relaton $R : \mathbb{Z}_2 \times M_2(\mathbb{Z}) \to [0,1]$ be defined by

$$R(x,A) = \begin{cases} \mu(A), & \textit{if } x = \overline{0},\\ 0, & \textit{if } x = \overline{1},\end{cases}$$

for all $A \in M_2(\mathbb{Z})$. Let the t-norm $\mathcal{T} = \mathcal{T}_M$. Then μ is a fuzzy subsemigroup of $M_2(\mathbb{Z})$ and R is a fuzzy relational morphism.

Let t-conorm $\mathcal{S} = \mathcal{S}_M$ which is dual to the t-norm $\mathcal{T}$, and the negator $\mathcal{N} = \mathcal{N}_\mathcal{S}$ be considered. Then the related $\mathcal{S}$-imlicator is $\mathcal{I}(x, y) = \mathcal{S}(\mathcal{N}(x), y) = (1 - x) \vee y$. So $\underline{R}_\mathcal{I}(\mu)(\overline{0}) = \frac{1}{2}$ and $\underline{R}_\mathcal{I}(\mu)(\overline{1}) = 1$, whence $\underline{R}_\mathcal{I}(\mu)(\overline{1})\mathcal{T}\underline{R}_\mathcal{I}(\mu)(\overline{1}) = 1$ and $\underline{R}_\mathcal{I}(\mu)(\overline{1} + \overline{1}) = \underline{R}_\mathcal{I}(\mu)(\overline{0}) = \frac{1}{2}$. Since $1 \not\leq \frac{1}{2}$, then $\underline{R}_\mathcal{I}(\mu)$ is not a $\mathcal{T} = \mathcal{T}_M$-fuzzy subsemigroup of $\mathbb{Z}_2$.

Theorem 4.22. *Let $R \in \mathcal{F}(X \times Y, L)$ be a serial $\mathcal{T}L$-fuzzy complete relational morphism, $\mathcal{T}$ be a infinitely $\vee$-distributive t-norm and $\mathcal{I}$ be a $\mathcal{R}$-implicator based on the t-norm $\mathcal{T}$. If μ is a $\mathcal{T}L$-fuzzy left (resp. right or two-sided) ideal of Y, then $\underline{R}_\mathcal{I}(\mu)$ is a $\mathcal{T}L$-fuzzy left (resp. right or two-sided) ideal of X.*

Proof. Take any elements $a, b \in X$. Since R is serial, there exists $k \in Y$ for all $b \in X$ such that $R(b, k) = 1$. So we have

$$\begin{aligned}
\underline{R}_\mathcal{I}(\mu)(a) &= \bigwedge_{t \in Y} R(a, t)\mathcal{I}\mu(t) \\
&= \bigwedge_{t \in Y} \Big(\bigvee_{R(a,t)\mathcal{T}\alpha \leq \mu(t)} \alpha \Big) \\
&= \bigwedge_{t \in Y} \Big(\bigvee_{R(a,t)\mathcal{T}R(b,k)\mathcal{T}\alpha \leq \mu(t)} \alpha \Big) \\
&\leq \bigwedge_{t \in Y} \Big(\bigvee_{R(a \bullet b, t*k)\mathcal{T}\alpha \leq \mu(t*k)} \alpha \Big) \\
&\leq \bigwedge_{p \in Y} \Big(\bigvee_{R(a \bullet b, p)\mathcal{T}\alpha \leq \mu(p)} \alpha \Big) \\
&= \underline{R}_\mathcal{I}(\mu)(a \bullet b).
\end{aligned}$$

Therefore $\underline{R}_\mathcal{I}(\mu)$ is a $\mathcal{T}L$-fuzzy left ideal of X. Similarly, if μ is a $\mathcal{T}L$-fuzzy right ideal of Y, then $\underline{R}_\mathcal{I}(\mu)$ is a $\mathcal{T}L$-fuzzy right ideal of X. □

Theorem 4.22 is not true in general for a $\mathcal{T}L$-fuzzy relational morphism R even if R serial. So we need R to be a $\mathcal{T}L$-fuzzy complete relational morphism to satisfying the theorem.

Example 4.23. Let $gcd(x, y)$ be denoted the greatest common divisor of the integers x and y. Then the set $\mathbb{N}$ of all natural numbers is a semigroup, with binary operation given by $x * y = gcd(x, y)$. Let take $L = [0, 1]$ and take any t-norm $\mathcal{T}$ on $[0, 1]$. A fuzzy set μ of $\mathbb{N}$ defined by $\mu(n) = \frac{1}{n+1}$ for all $n \in \mathbb{N}$ is a $\mathcal{T}$-fuzzy left ideal of $\mathbb{N}$. Let take an $\mathcal{T}$-fuzzy relation R of $\mathbb{N} \times \mathbb{N}$ defined by

$$R(x, y) = \begin{cases} \frac{1}{y+1}, & \textit{if } 2 \mid x; \\ 1, & \textit{if } 2 \nmid x. \end{cases}$$

for all $x, y \in \mathbb{N}$. Although R is $\mathcal{T}$-fuzzy relational morphism, it is not a $\mathcal{T}$-fuzzy complete relational morphism since $R(2, 2)\mathcal{T}R(2, 1) = \frac{1}{3}\mathcal{T}\frac{1}{2} \leq \frac{1}{3} \neq \frac{1}{2} = R(2, 1) = R(2 * 2, 2 * 1)$. Moreover R is serial. Now let take the residual implication $\mathcal{I}$ of the t-norm $\mathcal{T}$. Then $\mathcal{I}$-lower approximation of μ with respect to the approximation space

$(\mathbb{N}, R)$ is $\underline{R}_{\mathcal{I}}(\mu)(x) = \bigwedge_{y\in\mathbb{N}} R(x,y)\mathcal{I}\mu(y) = \bigwedge_{y\in\mathbb{N}}(\bigvee_{R(x,y)\mathcal{T}\alpha\leq\mu(y)} \alpha)$ for all $x \in \mathbb{N}$. So we have

$$\underline{R}_{\mathcal{I}}(\mu)(x) = \begin{cases} 1, & \textit{if } 2 \mid x; \\ 0, & \textit{if } 2 \nmid x. \end{cases}$$

Hence $\underline{R}_{\mathcal{I}}(\mu)$ is not a $\mathcal{T}$-fuzzy left ideal of $\mathbb{N}$ since $\underline{R}_{\mathcal{I}}(\mu)(2) = 1 \nleq 0 = \underline{R}_{\mathcal{I}}(\mu)(1) = \underline{R}_{\mathcal{I}}(\mu)(gcd(1,2)) = \underline{R}_{\mathcal{I}}(\mu)(1 * 2)$.

Theorem 4.24. *Let $R \in \mathcal{F}(X \times Y, L)$ be a serial $\mathcal{T}L$-fuzzy complete relational morphism, $\mathcal{T}$ be a infinitely $\vee$-distributive t-norm and $\mathcal{I}$ be a $\mathcal{R}$-implicator based on the t-norm $\mathcal{T}$. If μ is a $\mathcal{T}L$-fuzzy generalized bi-ideal of Y, then $\underline{R}_{\mathcal{I}}(\mu)$ is a $\mathcal{T}L$-fuzzy generalized bi-ideal of X.*

Proof. Take any elements $a, x, b \in X$. Since R is serial, then there exists $y \in Y$ such that $R(x, y) = 1$ for all $x \in X$. Thus we have

$$\begin{aligned}
\underline{R}_{\mathcal{I}}(\mu)(a)\mathcal{T}\underline{R}_{\mathcal{I}}(\mu)(b) &= (\bigwedge_{t\in Y} R(a,t)\mathcal{I}\mu(t))\mathcal{T}(\bigwedge_{k\in Y} R(b,k)\mathcal{I}\mu(k)) \\
&= \bigwedge_{t\in Y}(\bigvee_{R(a,t)\mathcal{T}\alpha\leq\mu(t)} \alpha)\mathcal{T}\bigwedge_{k\in Y}(\bigvee_{R(b,k)\mathcal{T}\beta\leq\mu(k)} \beta) \\
&\leq \bigwedge_{t,k\in Y}((\bigvee_{R(a,t)\mathcal{T}\alpha\leq\mu(t)} \alpha)\mathcal{T}(\bigvee_{R(b,k)\mathcal{T}\beta\leq\mu(k)} \beta)) \\
&= \bigwedge_{t,k\in Y}(\bigvee_{\substack{R(a,t)\mathcal{T}\alpha\leq\mu(t)\\ R(b,k)\mathcal{T}\beta\leq\mu(k)}} \alpha\mathcal{T}\beta) \\
&= \bigwedge_{t,k\in Y}(\bigvee_{\substack{R(a,t)\mathcal{T}1\mathcal{T}\alpha\leq\mu(t)\\ R(b,k)\mathcal{T}\beta\leq\mu(k)}} \alpha\mathcal{T}\beta) \\
&= \bigwedge_{t,k\in Y}(\bigvee_{\substack{R(a,t)\mathcal{T}R(x,r)\mathcal{T}\alpha\leq\mu(t)\\ R(b,k)\mathcal{T}\beta\leq\mu(k)}} \alpha\mathcal{T}\beta) \\
&\leq \bigwedge_{t,k\in Y}(\bigvee_{R(a\bullet x\bullet b, t*r*k)\mathcal{T}\alpha\mathcal{T}\beta\leq\mu(t*r*k)} \alpha\mathcal{T}\beta) \\
&\leq \bigwedge_{p\in Y}(\bigvee_{R(a\bullet x\bullet b, p)\mathcal{T}\gamma\leq\mu(p)} \gamma) \\
&= \bigwedge_{p\in Y} R(a \bullet b, p)\mathcal{I}\mu(p) \\
&= \underline{R}_{\mathcal{I}}(\mu)(a \bullet x \bullet b).
\end{aligned}$$

Consequently we obtain that $\underline{R}_{\mathcal{I}}(\mu)$ is a $\mathcal{T}L$-fuzzy generalized bi-ideal of X. □

Example 4.25. Let consider the $\mathcal{T}L$-fuzzy relational morphism R and the fuzzy subset μ of $\mathbb{N}$ in Example 4.23. Then μ is a $\mathcal{T}$-fuzzy generalized bi-ideal of $\mathbb{N}$. However $\underline{R}_{\mathcal{I}}(\mu)$ is not a $\mathcal{T}L$-fuzzy generalized bi-ideal of $\mathbb{N}$ since $\underline{R}_{\mathcal{I}}(\mu)(2)\mathcal{T}\underline{R}_{\mathcal{I}}(\mu)(2) = 1\mathcal{T}1 = 1 \nleq 0 = \underline{R}_{\mathcal{I}}(\mu)(1) = \underline{R}_{\mathcal{I}}(\mu)(gcd(2,1,2)) = \underline{R}_{\mathcal{I}}(\mu)(2 * 1 * 2)$.

Corollary 4.26. *Let $R \in \mathcal{F}(X \times Y, L)$ be a serial $\mathcal{T}L$-fuzzy complete relational morphism, $\mathcal{T}$ be a infinitely $\vee$-distributive t-norm and $\mathcal{I}$ be a $\mathcal{R}$-implicator based on the t-norm $\mathcal{T}$. If μ is a $\mathcal{T}L$-fuzzy bi-ideal of Y, then $\underline{R}_{\mathcal{I}}(\mu)$ is a $\mathcal{T}L$-fuzzy bi-ideal of X.*

Proof. It is straightforward from Theorem 4.20 and Theorem 4.24. □

Theorem 4.27. *Let $R \in \mathcal{F}(X \times Y, L)$ be a serial $\mathcal{T}L$-fuzzy complete relational morphism, $\mathcal{T}$ be a infinitely $\vee$-distributive t-norm, $\mathcal{I}$ be a $\mathcal{R}$-implicator based on the t-norm $\mathcal{T}$. If μ be a $\mathcal{T}L$-fuzzy interior ideal of Y, then $\underline{R}_{\mathcal{I}}(\mu)$ is a $\mathcal{T}L$-fuzzy interior ideal of X.*

Proof. Let μ be a $\mathcal{T}L$-fuzzy interior ideal of Y and take any elements $a, x, b \in X$. Since R is serial, there exists $y_1, y_2 \in Y$ such that $R(a, y_1) = 1$ and $R(b, y_2) = 1$ for all $a, b \in X$, then we have

$$\begin{aligned}
\underline{R}_{\mathcal{I}}(\mu)(x) &= \bigwedge_{t \in Y} R(x,t)\mathcal{I}\mu(t) \\
&= \bigwedge_{t \in Y} \Big(\bigvee_{R(x,t)\mathcal{T}\alpha \le \mu(t)} \alpha \Big) \\
&= \bigwedge_{t \in Y} \Big(\bigvee_{R(a,y_1)\mathcal{T}R(x,t)\mathcal{T}R(b,y_2)\mathcal{T}\alpha \le \mu(t)} \alpha \Big) \\
&\le \bigwedge_{t \in Y} \Big(\bigvee_{R(a \bullet x \bullet b, y_1 * t * y_2)\mathcal{T}\alpha \le \mu(y_1 * t * y_2)} \alpha \Big) \\
&\le \bigwedge_{p \in Y} \Big(\bigvee_{R(a \bullet x \bullet b, p)\mathcal{T}\alpha \le \mu(p)} \alpha \Big) \\
&= \underline{R}_{\mathcal{I}}(\mu)(a \bullet x \bullet b).
\end{aligned}$$

So we obtain $\underline{R}_{\mathcal{I}}(\mu)$ is a $\mathcal{T}L$-fuzzy interior ideal of X by using Theorem 4.20. □

The following example shows that Theorem 4.27 is not true in general unless R to be serial.

Example 4.28. Let X and Y be the semigroups which is given in Example 4.3. Let $L = N_5$ and $R \in \mathcal{F}(X \times Y, L)$ be defined by

R	m	n	p
a	β	β	β
b	0	0	0

Then R is a TL-fuzzy complete relational morphism for the t-norm $\mathcal{T} = \wedge$. However R is not serial since for $a \in X$ there does not exist any element in Y such that $R(a, y) = 1$. Let $\mu \in \mathcal{F}(Y, L)$ be defined by

y	m	n	p
$\mu(y)$	β	β	0

Then μ is a $\mathcal{T}L$-fuzzy interior ideal of Y. So we have

$$\overline{R}^{\mathcal{T}}(\mu)(x) = \begin{cases} \gamma, & \text{if } x = a; \\ 0, & \text{if } x = b. \end{cases}$$

Since $\overline{R}^{\mathcal{T}}(\mu)(a) = \gamma \nleq 0 = \overline{R}^{\mathcal{T}}(\mu)(b) = \overline{R}^{\mathcal{T}}(\mu)(b * a * b)$, then $\overline{R}^{\mathcal{T}}(\mu)$ is not a $\mathcal{T}L$-fuzzy interior ideal of X.

5. Conclusions

Since Zadeh and Pawlak proposed the notions of fuzzy sets and rough sets, respectively, their ideas have been applied to various fields. Some properties of rough sets and fuzzy rough sets are investigated on algebraic structures [4, 6, 7, 8, 18, 20, 21, 23, 24, 36, 37]. In [23], $\mathcal{I}$-lower and $\mathcal{T}$-upper fuzzy rough approximation operators with respect to $\mathcal{T}$-congruence L-fuzzy relations on a semigroup are investigated. In this paper, we consider two semigroups as the universal sets and investigate the $\mathcal{I}$-lower and $\mathcal{T}$-upper fuzzy rough approximation operators with respect to a $\mathcal{T}L$-fuzzy relational morphism. We have also studied relationships between $(\mathcal{I}, \mathcal{T})$-fuzzy generalized rough sets and the L-fuzzy relations. Our future work on this topic will focus on considering other L-fuzzy approximation spaces based on a $\mathcal{T}L$-fuzzy relational morphism and the algebraic structures such as groups and modules.

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