Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 8, No. 4, (October 2014), pp. 549–559
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Anti-homomorphism on rough prime fuzzy ideals and rough primary fuzzy ideals

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Received 2 February 2014; Revised 16 February 2014; Accepted 29 March 2014

Abstract. In this paper, we shall study the concepts of rough fuzzy ideal, rough prime fuzzy ideal and rough primary fuzzy ideal in a ring and prove some properties of anti-homomorphism on these rough fuzzy ideals.

2010 AMS Classification: 13Axx, 08A72

Keywords: Rough ring, Rough ideal, Rough fuzzy ideal, Rough prime fuzzy ideal, Rough primary fuzzy ideal, Ring anti-homomorphism.

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1. Introduction

The fuzzy set introduced by L.A. Zadeh [13] in 1965 and the rough set introduced by Z Pawlak [12] in 1982 are generalisations of the classical set theory. Both these set theories are new mathematical tools to deal the uncertain, vague and imprecise data. In Zadeh's fuzzy set theory, the degree of membership of elements of a set plays the key role, whereas in Pawlak's rough set theory, equivalence classes of a set are the building blocks for the upper and lower approximations of the set, in which a subset of universe is approximated by the pair of ordinary sets, called upper and lower approximations. Combining the theory of rough set with abstract algebra is one of the trends in the theory of rough set. Some authors studied the concept of rough algebraic structures. On the other hand, some authors substituted an algebraic structure for the universal set and studied the roughness in algebraic structure. Biswas and Nanda introduced the notion of rough subgroups. The concept of rough ideal in a semigroup was introduced by Kuroki. B. Davvaz studied relationship between rough sets and ring theory and considered ring as a universal set and introduced the notion of rough ideals of a rings in [2]. Further study of this work is done by Osman Kazanci and B. Davaaz in [8]. The authors studied some properties of rough rings, rough ideals and rough groups in [5, 6, 10].Some properties of rough prime ideals and rough primary ideals are studied in [7]. The concept of rough semi-prime ideals and rough bi-ideals are studied in [11]. Extensive researches has also been carried out to compare the theory of rough sets with other theories of uncertainty such as fuzzy sets and conditional events. There have been many papers studying the connections and differences of fuzzy set theory and rough set theory. Dubois and Prade [3] were among the first few who combined fuzzy sets and rough sets in a fruitful way by defining rough fuzzy sets and fuzzy rough sets.

This paper concerns a relationship between rough sets, fuzzy sets and ring theory. In section 2, we review some basic definitions. In section 3 we give anti-homomorphic properties of rough fuzzy ideals. Section 4 deals with anti-homomorphic properties of rough prime fuzzy ideals. In section 5 we give anti-homomorphic properties of rough primary fuzzy ideals.

2. Rough ideals

Throughout this paper R is a ring and θ is an equivalence relation on R. One may refer the books by Gallian [4] or Artin [1] for basic concepts and theories in abstract algebra.

Definition 2.1. Let θ be an equivalence relation on R, then θ is called a full congruence relation if
$(a, b) \in \theta$ implies$(a + x, b + x)$, (ax, bx), $(xa, xb) \in \theta$ for all $x \in R$.

Definition 2.2. Let θ be a full congruence relation on R and A a subset of R. Then the sets

- $\theta_-(A) = \{ x \in R \mid [x]_\theta \subseteq A\}$ and
- $\theta^-(A) = \{x \in R \mid [x]_\theta \cap A \neq \phi\}$

are called, respectively , the θ - lower and θ - upper approximations of the set A.
$\theta(A) = (\theta_-(A)\ ,\ \theta^-(A))$ is called a rough set with respect to θ if $\theta_-(A) \neq \theta^-(A)$

Definition 2.3. A non-empty subset A of R is called an upper rough right(left) ideal of R if $\theta^-(A)$ is a right(left) ideal of R and A is called a lower rough right(left) ideal of R if $\theta_-(A)$ is a right(left) ideal of R.

Let A be a subset of a ring R and $(\theta_-(A), \theta^-(A))$ be a rough set. If both $\theta_-(A)$ and $\theta^-(A)$ are right(left) ideals of R, then we call $(\theta_-(A), \theta^-(A))$ a rough right(left) ideal. If $(\theta_-(A), \theta^-(A))$ is both a rough right and left ideal then it is simply called a rough ideal.

3. Rough fuzzy ideals

As it is well known in the fuzzy set theory established by Zadeh, a fuzzy subset μ of R is defined as a map from R to the unit interval $[0, 1]$.

Definition 3.1 ([8])**.** Let θ be an equivalence relation on R and μ a fuzzy subset of R. Then we define the fuzzy sets $\theta_-(\mu)$, $\theta^-(\mu)$ as follows:

$$\theta_-(\mu)(x) = \bigwedge_{z\in[x]_\theta} \mu(z) \quad \text{and} \quad \theta^-(\mu)(x) = \bigvee_{z\in[x]_\theta} \mu(z).$$

The fuzzy sets $\theta_-(\mu)$ and $\theta^-(\mu)$ are called, respectively the θ- lower and θ-upper approximations of the fuzzy set μ. $\theta(\mu) = (\theta_-(\mu)\ ,\ \theta^-(\mu))$ is called a rough fuzzy set with respect to θ if $\theta_-(\mu) \neq \theta^-(\mu)$.

Definition 3.2 ([9]). A fuzzy subset μ of a ring R is called a fuzzy ideal of R if

(1) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$
(2) $\mu(xy) \geq \mu(x) \vee \mu(y)$

for all $x, y \in R$.
A fuzzy subset μ of a ring R is called a fuzzy left (right) ideal of R if

(1) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$
(2) $\mu(xy) \geq \mu(y) \quad (\mu(xy) \geq \mu(x))$

for all $x, y \in R$.

Definition 3.3 ([8]). A fuzzy subset μ of a ring R is called an upper rough fuzzy left (right) ideal of R if if $\theta^{-}(\mu)$ is a fuzzy left (right) ideal of R, and a lower rough fuzzy left (right) ideal of R if $\theta_{-}(\mu)$ is a fuzzy left (right) ideal of R.

A fuzzy subset μ of a ring R is called an upper rough fuzzy ideal of R if $\theta^{-}(\mu)$ is a fuzzy ideal of R and a lower rough fuzzy ideal of R if $\theta_{-}(\mu)$ is a fuzzy ideal of R.

Let μ be a fuzzy subset of R and $\theta(\mu) = (\theta_{-}(\mu)\ ,\ \theta^{-}(\mu))$ a rough fuzzy set. If $\theta_{-}(\mu)$ and $\theta^{-}(\mu)$ are fuzzy ideals of R, then $(\theta_{-}(\mu)\ ,\ \theta^{-}(\mu))$ is called a rough fuzzy ideal.

Definition 3.4 ([9]). Let X and Y be two non-empty sets, $f : X \to Y$, μ and σ be fuzzy subsets of X and Y respectively. Then
$f(\mu)$, the image of μ under f is a fuzzy subset of Y defined by

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x);\ f(x) = y\} & \text{if } f^{-1}(y) \neq \phi \\ 0 \quad \text{otherwise} \end{cases}$$

$f^{-1}(\sigma)$, the pre-image of σ under f is a fuzzy subset of X defined by

$$f^{-1}(\sigma)(x) = \sigma(f(x)) \ \ \forall x \in X.$$

Definition 3.5 ([9]). For a function $f : R_1 \to R_2$, a fuzzy subset μ of a ring R_1 is called f-invariant if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$, $x,\ y \in R_1$.

We say that a fuzzy subset μ of a ring R_1 has the sup property if for any subset T of R_1, there exists $t_0 \in T$ such that $\mu(t_0) = \sup_{t \in T} \mu(t)$.

Remark 3.6. From here onwards θ, θ_1 and θ_2 denote full congruence relations on the rings R, R_1 and R_2 respectively.

Theorem 3.7. *Let f be a homomorphism (anti-homomorphism) from a ring R_1 onto a ring R_2 and let μ be a fuzzy subset of R_1 . Then*

(1) $f(\theta_1^{-}(\mu)) = \theta_2^{-}(f(\mu))$
(2) $f(\theta_{1-}(\mu)) \subseteq \theta_{2-}(f(\mu))$. *If f is one to one* $f(\theta_{1-}(\mu)) = \theta_{2-}(f(\mu))$

Proof. For $x \in R_2$

(1)

$$\begin{aligned}
f(\theta_1^-(\mu))(x) &= \bigvee_{f(a)=x} \theta_1^-(\mu)(a) = \bigvee_{f(a)=x} \bigvee_{z\in[a]_{\theta_1}} \mu(z) \\
&= \bigvee_{f(a)=x} \bigvee_{a\in[z]_{\theta_1}} \mu(a) = \bigvee_{a\in[z]_{\theta_1}} \bigvee_{f(a)=x} \mu(a) \\
&= \bigvee_{a\in[z]_{\theta_1}} f(\mu)(x) = \bigvee_{f(a)\in[f(z)]_{\theta_2}} f(\mu)(x) \\
&= \bigvee_{x\in[f(z)]_{\theta_2}} f(\mu)(x) = \bigvee_{f(z)\in[x]_{\theta_2}} f(\mu)(f(z)) \\
&= \theta_2^- f(\mu)(x)
\end{aligned}$$

Therefore, $f(\theta_1^-(\mu)) = \theta_2^-(f(\mu))$

(2)

$$\begin{aligned}
f(\theta_{1-}(\mu))(x) &= \bigvee_{f(a)=x} \theta_{1-}(\mu)(a) = \bigvee_{f(a)=x} \bigwedge_{z\in[a]_{\theta_1}} \mu(z) \\
&= \bigvee_{f(a)=x} \bigwedge_{a\in[z]_{\theta_1}} \mu(a) \\
&\leq \bigwedge_{a\in[z]_{\theta_1}} \bigvee_{f(a)=x} \mu(a) = \bigwedge_{a\in[z]_{\theta_1}} f(\mu)(x) \\
&= \bigwedge_{f(a)\in[f(z)]_{\theta_2}} f(\mu)(x) = \bigwedge_{x\in[f(z)]_{\theta_2}} f(\mu)(x) \\
&= \bigwedge_{f(z)\in[x]_{\theta_2}} f(\mu)(f(z)) = \theta_{2-}f(\mu)(x)
\end{aligned}$$

Therefore, $f(\theta_{1-}(\mu)) \subseteq \theta_{2-}(f(\mu))$. If f is one to one, $f(\theta_{1-}(\mu)) = \theta_{2-}(f(\mu))$ is clear. □

Example 3.8. Consider the onto ring homomorphism $f : Z_2 \to \{0\}$. Clearly f is not one-one. Define a fuzzy set $\mu : Z_2 \to [0,1]$ such that $\mu(0) = 0$ and $\mu(1) = 0.1$. Define an equivalence relation θ_1 on Z_2 and θ_2 on $\{0\}$ as $\theta_1 = \{(0,0),(0,1),(1,0),(1,1)\}$ and $\theta_2 = \{(0,0)\}$ respectively
For $x \in \{0\}$

$$\begin{aligned}
f(\theta_{1-}(\mu))(x) &= f(\theta_{1-}(\mu))(0) = \bigvee_{f(a)=0} \bigwedge_{(a,z)\in\theta_1} \mu(z) = \bigvee_{a=0,1} \bigwedge_{(a,z)\in\theta_1} \mu(z) \\
&= \bigvee\{\bigwedge_{(0,z)\in\theta_1} \mu(z), \bigwedge_{(1,z)\in\theta_1} \mu(z)\} = \bigvee\{\mu(0),\mu(0)\} = \mu(0) = 0 \\
\theta_{2-}f(\mu)(x) &= \theta_{2-}f(\mu)(0) = \bigwedge_{(z,0)\in\theta_2} \bigvee_{f(a)=z} \mu(a) = \bigwedge_{(0,0)\in\theta_2} \bigvee_{f(a)=0} \mu(a) \\
&= \bigvee_{a=0,1} \mu(a) = \mu(1) = 0.1
\end{aligned}$$

This shows that f is not one-one and $f(\theta_{1-}(\mu))(x) \neq \theta_{2-}f(\mu)(x)$.

Theorem 3.9. *Anti-homomorphic image of an upper rough fuzzy left(right) ideal with sup property is an upper rough fuzzy right(left) ideal. Moreover anti-homomorphic image of an upper rough fuzzy ideal with sup property is an upper rough fuzzy ideal.*

Proof. Let f be an anti-homomorphism from a ring R_1 onto a ring R_2 and let μ be an upper rough fuzzy left ideal of R_1 with sup property . Then $\theta_1^-(\mu)$ is a fuzzy left ideal of R_1. Given $f(x), f(y) \in R_2$, let $x_0 \in f^{-1}[f(x)]$ and $y_0 \in f^{-1}[f(y)]$ be such that

$$\theta_1^-(\mu)(x_0) = \sup_{t \in f^{-1}[f(x)]} \theta_1^-(\mu)(t), \quad \theta_1^-(\mu)(y_0) = \sup_{t \in f^{-1}[f(y)]} \theta_1^-(\mu)(t)$$

Now
$$\begin{aligned} f(\theta_1^-(\mu))(f(x)-f(y)) &= f(\theta_1^-(\mu))(f(x-y)) \\ &= \sup_{t \in f^{-1}(f(x-y))} \theta_1^-(\mu)(t) \\ &\geq \theta_1^-(\mu)(x_0 - y_0) \\ &\geq \theta_1^-(\mu)(x_0) \wedge \theta_1^-(\mu)(y_0) \quad (\because \theta_1^-(\mu) \text{ is fuzzy ideal}) \\ &= \sup_{t \in f^{-1}[f(x)]} \theta_1^-(\mu)(t) \wedge \sup_{t \in f^{-1}[f(y)]} \theta_1^-(\mu)(t) \\ &= f(\theta_1^-(\mu))f(x) \wedge f(\theta_1^-(\mu))f(y) \end{aligned}$$

Also
$$\begin{aligned} f(\theta_1^-(\mu))(f(x)f(y)) &= f(\theta_1^-(\mu))(f(yx)) \quad (\because \text{ f is an anti-homomorphism}) \\ &= \sup_{t \in f^{-1}[f(yx)]} \theta_1^-(\mu)(t) \\ &\geq \theta_1^-(\mu)(y_0 x_0) \\ &\geq \theta_1^-(\mu)(x_0) \quad (\because \theta_1^-(\mu) \text{ is fuzzy left ideal}) \\ &= \sup_{t \in f^{-1}[f(x)]} \theta_1^-(\mu)(t) \\ &= f(\theta_1^-(\mu))f(x) \end{aligned}$$

Therefore, $f(\theta_1^-(\mu))$ is a fuzzy right ideal. Since μ is an upper rough fuzzy left ideal, μ is a fuzzy set. Hence by theorem 3.7, $f(\theta_1^-(\mu)) = \theta_2^-(f(\mu))$ is a fuzzy right ideal. Similarly we can prove the other case. This completes the proof. □

Theorem 3.10. *Anti-isomorphic image of a lower rough fuzzy left(right) ideal with sup property is a lower rough fuzzy right(left) ideal. Moreover anti-homomorphic image of a lower rough fuzzy ideal with sup property is a lower rough fuzzy ideal.*

Proof. Let f be an anti-isomorphism from a ring R_1 onto a ring R_2 and let μ be a lower rough fuzzy left ideal of R_1 with sup property. Then $\theta_{1-}(\mu)$ is a fuzzy left ideal of R_1. Given $f(x), f(y) \in R_2$, let $x_0 \in f^{-1}[f(x)]$ and $y_0 \in f^{-1}[f(y)]$ be such that

$$\theta_{1-}(\mu)(x_0) = \sup_{t \in f^{-1}[f(x)]} \theta_{1-}(\mu)(t), \quad \theta_{1-}(\mu)(y_0) = \sup_{t \in f^{-1}[f(y)]} \theta_{1-}(\mu)(t)$$

As in theorem 3.9, we can show that

$$f(\theta_{1-}(\mu))(f(x)-f(y)) \geq f(\theta_{1-}(\mu))f(x) \wedge f(\theta_1^-(\mu))f(y)$$
$$\text{and} \quad f(\theta_{1-}(\mu))(f(x)f(y)) \geq f(\theta_{1-}(\mu))f(x)$$

Therefore, $f(\theta_{1-}(\mu))$ is a fuzzy right ideal. Since μ is a lower rough fuzzy left ideal, μ is a fuzzy set. Hence by theorem 3.7, $f(\theta_{1-}(\mu)) = \theta_{2-}(f(\mu))$ is a fuzzy right ideal. Similarly we can prove the other case. This completes the proof. □

Corollary 3.11. *Anti-isomorphic image of a rough fuzzy left(right) ideal with sup property is a rough fuzzy right(left) ideal. Moreover anti-homomorphic image of a rough fuzzy ideal with sup property is a rough fuzzy ideal.*

Proof. This follows from theorems 3.9 and 3.10. □

Theorem 3.12. *Anti-isomorphic pre-image of an upper rough fuzzy left(right) ideal is an upper fuzzy right(left) ideal. Moreover anti-isomorphic pre-image of an upper rough fuzzy ideal is an upper fuzzy ideal.*

Proof. Let f be an anti-isomorphism from ring R_1 to a ring R_2 and let σ be an upper rough fuzzy left ideal of R_2. Then $\theta_2^-(\sigma)$ is a fuzzy left ideal of R_2.

$$\begin{aligned} \textit{Now} \quad f^{-1}(\theta_2^-(\sigma))(x-y) &= \theta_2^-(\sigma)(f(x-y)) \\ &= \theta_2^-(\sigma)(f(x)-f(y)) \\ &\geq \theta_2^-(\sigma)f(x) \wedge \theta_2^-(\sigma)f(y) \quad (\because \theta_2^-(\sigma) \text{ is fuzzy ideal}) \\ &= f^{-1}(\theta_2^-(\sigma))(x) \wedge f^{-1}(\theta_2^-(\sigma))(y) \end{aligned}$$

$$\begin{aligned} \textit{Also} \quad f^{-1}(\theta_2^-(\sigma))(xy) &= \theta_2^-(\sigma)(f(xy)) \\ &= \theta_2^-(\sigma)(f(y)f(x)) \quad (\because \text{ f is an anti-homomorphism}) \\ &\geq \theta_2^-(\sigma)f(x) \quad (\because \theta_2^-(\sigma) \text{ is fuzzy left ideal}) \\ &= f^{-1}(\theta_2^-(\sigma))(x) \end{aligned}$$

Therefore, $f^{-1}(\theta_2^-(\sigma))$ is a fuzzy right ideal. Since σ is an upper rough fuzzy left ideal, $f^-(\sigma)$ is a fuzzy set. Hence by theorem 3.7, $f^{-1}(\theta_2^-(\sigma)) = \theta_1^-(f^{-1}(\sigma))$ is a fuzzy right ideal. Similarly the other case also follows, completing the proof. □

Theorem 3.13. *Anti-isomorphic pre-image of a lower rough fuzzy left(right) ideal is a lower rough fuzzy right(left) ideal. Moreover anti-isomorphic pre-image of a lower rough fuzzy ideal is a lower rough fuzzy ideal.*

Proof. The proof is similar to that of Theorem 3.12. □

Corollary 3.14. *Anti-isomorphic pre-image of a rough fuzzy left(right) ideal is a rough fuzzy right(left) ideal. Moreover anti-isomorphic pre-image of a rough fuzzy ideal is a rough fuzzy ideal.*

Proof. This follows from Theorems 3.12 and 3.13. □

The following theorems can be proved in similar way as the corresponding theorems in anti-homomorphism.

Theorem 3.15. *Homomorphic image of an upper rough fuzzy left(right) ideal with sup property is an upper rough fuzzy left (right)ideal. Moreover homomorphic image of an upper rough fuzzy ideal with sup property is an upper rough fuzzy ideal.*

Theorem 3.16. *Isomorphic image of a lower rough fuzzy left(right) ideal with sup property is a lower rough fuzzy left (right) ideal. Moreover homomorphic image of a lower rough fuzzy ideal with sup property is a lower rough fuzzy ideal.*

Corollary 3.17. *Isomorphic image of a rough fuzzy left(right) ideal with sup property is a rough fuzzy left (right) ideal. Moreover homomorphic image of a rough fuzzy ideal with sup property is a rough fuzzy ideal.*

Theorem 3.18. *Isomorphic pre-image of an upper rough fuzzy left(right) ideal is an upper fuzzy left (right) ideal. Moreover isomorphic pre-image of an upper rough fuzzy ideal is an upper fuzzy ideal.*

Theorem 3.19. *Isomorphic pre-image of a lower rough fuzzy left(right) ideal is a lower rough fuzzy left (right) ideal. Moreover isomorphic pre-image of a lower rough fuzzy ideal is a lower rough fuzzy ideal.*

Corollary 3.20. *Isomorphic pre-image of a rough fuzzy left(right) ideal is a rough fuzzy right(left) ideal. Moreover anti-isomorphic pre-image of a rough fuzzy ideal is a rough fuzzy ideal.*

4. Rough prime fuzzy ideals

Definition 4.1. A fuzzy ideal μ of R is called a prime fuzzy ideal if for all x, $y \in R$

$$\mu(xy) = \mu(x) \text{ or } \mu(xy) = \mu(y)$$

Definition 4.2. A fuzzy subset μ of a ring R is called an upper rough prime fuzzy ideal of R if $\theta^-(\mu)$ is a prime fuzzy ideal of R and a lower rough prime fuzzy ideal of R if $\theta_-(\mu)$ is a prime fuzzy ideal of R.

Let μ be a fuzzy subset of R and $\theta(\mu) = (\theta_-(\mu)\ ,\ \theta^-(\mu))$ a rough fuzzy set. If $\theta_-(\mu)$ and $\theta^-(\mu)$ are prime fuzzy ideals of R, then $(\theta_-(\mu)\ ,\ \theta^-(\mu))$ is called a rough prime fuzzy ideal.

Theorem 4.3. *Let f be an anti-isomorphism from a ring R_1 to a ring R_2 and let σ be an upper rough prime fuzzy ideal of R_2. Then $f^{-1}(\sigma)$ is an upper rough prime fuzzy ideal of R_1.*

Proof. Let σ be an upper rough prime fuzzy ideal of R_2. Then $\theta_2^-(\sigma)$ is a prime fuzzy ideal of R_2. By theorem 3.12, $f^{-1}(\theta_2^-(\sigma))$ is a fuzzy ideal of R_1. For x and $y \in R_1$,

$$\begin{aligned} f^{-1}(\theta_2^-(\sigma))(xy) &= \theta_2^-(\sigma)f(xy) \\ &= \theta_2^-(\sigma)(f(y)f(x)) \quad (\because \text{ f is an anti-homomorphism}) \\ &= \theta_2^-(\sigma)f(y) \quad \text{or} \quad \theta_2^-(\sigma)f(x) \quad (\because \theta_2^-(\sigma) \text{ is a prime fuzzy ideal}) \\ &= f^{-1}(\theta_2^-(\sigma))(y) \quad \text{or} \quad f^{-1}(\theta_2^-(\sigma))(x) \end{aligned}$$

Therefore, $f^{-1}(\theta_2^-(\sigma))$ is a prime fuzzy ideal of R_1. By theorem 3.7, $\theta_1^-(f^{-1}(\sigma))$ is a prime fuzzy ideal of R_1. Hence the theorem is proved. □

Theorem 4.4. *Let f be an anti-isomorphism from a ring R_1 to a ring R_2 and let σ be a lower rough prime fuzzy ideal of R_2. Then $f^{-1}(\sigma)$ is a lower rough prime fuzzy ideal of R_1.*

Proof. The proof is similar to that of theorem 4.3. □

Corollary 4.5. *Anti-isomorphic pre-image of a rough prime fuzzy ideal is a rough prime fuzzy ideal.*

Proof. This follows from Theorems 4.3 and 4.4. □

Theorem 4.6. *Let f be an anti-isomorphism from a ring R_1 to a ring R_2 and let μ be an upper rough f-invariant prime fuzzy ideal of R_1 with sup property. Then $f(\mu)$ is a upper rough prime fuzzy ideal of R_2.*

Proof. Let μ be an upper rough prime fuzzy ideal of R_1. Then $\theta_1^-(\mu)$ is a prime fuzzy ideal of R_1. By theorem 3.9, $f(\theta_1^-(\mu))$ is a fuzzy ideal of R_2. For $f(x)$,$f(y) \in R_2$, let $x_0 \in f^{-1}[f(x)]$ and $y_0 \in f^{-1}[f(y)]$ be such that

$$\theta_1^-(\mu)(x_0) = \sup_{t\in f^{-1}[f(x)]} \theta_1^-(\mu)(t), \quad \theta_1^-(\mu)(y_0) = \sup_{t\in f^{-1}[f(y)]} \theta_1^-(\mu)(t)$$

$$\begin{aligned} f(\theta_1^-(\mu))(f(x)f(y)) =& f(\theta_1^-(\mu))(f(yx)) \quad (\because \text{ f is an anti-homomorphism}) \\ =& \sup_{t\in f^{-1}[f(yx)]} \theta_1^-(\mu)(t) \\ =& \ \theta_1^-(\mu)(y_0x_0) \quad (\because \theta_1^-(\mu) \text{ is f-invariant}) \\ =& \ \theta_1^-(\mu)(y_0) \quad \text{or} \quad \theta_1^-(\mu)(x_0) \quad (\because \theta_1^-(\mu) \text{ is prime fuzzy ideal}) \\ =& \sup_{t\in f^{-1}[f(y)]} \theta_1^-(\mu)(t) \quad \text{or} \quad \sup_{t\in f^{-1}[f(x)]} \theta_1^-(\mu)(t) \\ =& \ f(\theta_1^-(\mu))f(y) \quad \text{or} \quad f(\theta_1^-(\mu))f(x) \end{aligned}$$

Therefore, $f(\theta_1^-(\mu))$ is a prime fuzzy ideal of R_2. By theorem 3.7, $\theta_2^-(f(\mu))$ is a prime fuzzy ideal R_2. This completes the theorem. □

Theorem 4.7. *Let f be an anti-isomorphism from a ring R_1 to a ring R_2 and let μ be a lower rough f-invariant prime fuzzy ideal of R_1 with sup property. Then $f(\mu)$ is a lower rough prime fuzzy ideal of R_2.*

Proof. The proof is similar to that of theorem 4.6. □

Corollary 4.8. *Let f be an anti-isomorphism from a ring R_1 to a ring R_2 and let μ be a rough f-invariant prime fuzzy ideal of R_1 with sup property. Then $f(\mu)$ is a rough prime fuzzy ideal of R_2.*

Proof. This follows from theorems 4.6 and 4.7. □

5. Rough primary fuzzy ideals

Definition 5.1. A fuzzy ideal μ of R is called a primary fuzzy ideal if for all x, $y \in R$

$$\mu(xy) = \mu(x) \text{ or } \mu(xy) = \mu(y^n) \text{ for some positive integer } n.$$

Definition 5.2. A fuzzy subset μ of a ring R is called an upper rough primary fuzzy ideal of R if $\theta^-(\mu)$ is a primary fuzzy ideal of R and a lower rough primary fuzzy ideal of R if $\theta_-(\mu)$ is a primary fuzzy ideal of R.

Let μ be a fuzzy subset of R and $\theta(\mu) = (\theta_-(\mu)\ ,\ \theta^-(\mu))$ a rough fuzzy set. If $\theta_-(\mu)$ and $\theta^-(\mu)$ are primary fuzzy ideals of R, then $(\theta_-(\mu)\ ,\ \theta^-(\mu))$ is called a rough primary fuzzy ideal.

Obviously from definitions it follows that every prime fuzzy ideal is a primary fuzzy ideal.

Theorem 5.3. *Let f be an anti-isomorphism from a ring R_1 to a ring R_2 and let σ be an upper rough primary fuzzy ideal of R_2. Then $f^{-1}(\sigma)$ is an upper rough primary fuzzy ideal of R_1.*

Proof. Let σ be an upper rough primary fuzzy ideal of R_2. Then $\theta_2^-(\sigma)$ is a primary fuzzy ideal of R_2. By theorem (3.12), $f^{-1}(\theta_2^-(\sigma))$ is a fuzzy ideal of R_1. For x and $y \in R_1$,

$$\begin{aligned} f^{-1}(\theta_2^-(\sigma))(xy) &= \theta_2^-(\sigma)f(xy) \\ &= \theta_2^-(\sigma)(f(y)f(x)) \quad (\because \text{ f is an anti-homomorphism}) \\ &= \theta_2^-(\sigma)f(y) \quad \text{or} \quad \theta_2^-(\sigma)(f(x))^n, \quad \text{for some positive integer } n \\ &\qquad\qquad (\because \theta_2^-(\sigma) \text{ is a primary fuzzy ideal} \\ &= \theta_2^-(\sigma)f(y) \quad \text{or} \quad \theta_2^-(\sigma)(f(x^n)) \\ &= f^{-1}(\theta_2^-(\sigma))(y) \quad \text{or} \quad f^{-1}(\theta_2^-(\sigma))(x^n) \end{aligned}$$

Therefore, $f^{-1}(\theta_2^-(\sigma))$ is a primary fuzzy ideal of R_1. By theorem 3.7, $\theta_1^-(f^{-1}(\sigma))$ is a primary fuzzy ideal of R_1. This proves the theorem. □

Theorem 5.4. *Let f be an anti-isomorphism from a ring R_1 to a ring R_2 and let σ be an lower rough primary fuzzy ideal of R_2. Then $f^{-1}(\sigma)$ is a lower rough primary fuzzy ideal of R_1.*

Proof. The proof is similar to that of theorem 5.3. □

Corollary 5.5. *Anti-isomorphic pre-image of a rough primary fuzzy ideal is a rough primary fuzzy ideal.*

Proof. This follows from Theorems 5.3 and 5.4. □

Theorem 5.6. *Let f be an anti-isomorphism from a ring R_1 to a ring R_2 and let μ be an upper rough f-invariant primary fuzzy ideal of R_1 with sup property. Then $f(\mu)$ is a upper rough primary fuzzy ideal of R_2.*

Proof. Let μ be an upper rough primary fuzzy ideal of R_1. Then $\theta_1^-(\mu)$ is a primary fuzzy ideal of R_1. By theorem 3.9, $f(\theta_1^-(\mu))$ is a fuzzy ideal of R_2. Now we prove that $f(\theta_1^-(\mu))$ is a primary fuzzy ideal of R_2. Let $x', y' \in R_2$ be arbitrary. Then there exist $x, y \in R_1$ such that $x' = f(x)$ and $y' = f(y)$. Let $x_0 \in f^{-1}[f(x)]$ and $y_0 \in f^{-1}[f(y)]$ be such that

$$\theta_1^-(\mu)(x_0) = \sup_{t \in f^{-1}[f(x)]} \theta_1^-(\mu)(t), \quad \theta_1^-(\mu)(y_0) = \sup_{t \in f^{-1}[f(y)]} \theta_1^-(\mu)(t)$$

Further we have,

$$\begin{aligned}
f(\theta_1^-(\mu))(x'y') =& f(\theta_1^-(\mu))(f(x)f(y)) \\
=& f(\theta_1^-(\mu))(f(yx)) \quad (\because\ f \text{ is an anti-homomorphism}) \\
=& \sup_{t\in f^{-1}[f(yx)]} \theta_1^-(\mu)(t) \\
=& \ \theta_1^-(\mu)(y_0x_0) \quad (\because \theta_1^-(\mu) \text{ is } f\text{-invariant}) \\
=& \ \theta_1^-(\mu)(y_0) \quad \text{or} \quad \theta_1^-(\mu)([x_0]^n), \quad \text{for some positive integer } n \\
& (\because \theta_1^-(\mu) \text{ is primary fuzzy ideal}) \\
=& \sup_{t\in f^{-1}[f(y)]} \theta_1^-(\mu)(t) \quad \text{or} \quad \sup_{t\in f^{-1}[f(x^n)]} \theta_1^-(\mu)(t) \\
=& \sup_{t\in f^{-1}[f(y)]} \theta_1^-(\mu)(t) \quad \text{or} \quad \sup_{t\in f^{-1}[f(x)]^n} \theta_1^-(\mu)(t) \\
=& \ f(\theta_1^-(\mu))f(y) \quad \text{or} \quad f(\theta_1^-(\mu))[f(x)]^n
\end{aligned}$$

Therefore, $f(\theta_1^-(\mu))$ is a primary fuzzy ideal of R_2. By theorem 3.7, $\theta_2^-(f(\mu))$ is a primary fuzzy ideal of R_2. Hence the theorem is proved. □

Theorem 5.7. *Let f be an anti-isomorphism from a ring R_1 to a ring R_2 and let μ be a lower rough f-invariant primary fuzzy ideal of R_1 with sup property. Then $f(\mu)$ is a lower rough primary fuzzy ideal of R_2.*

Proof. The proof is similar to that of theorem 5.6. □

Corollary 5.8. *Let f be an anti-isomorphism from ring R_1 to a ring R_2 and let μ be a rough f-invariant primary fuzzy ideal of R_1 with sup property. Then $f(\mu)$ is a rough primary fuzzy ideal of R_2.*

Proof. This follows from theorems 5.6 and 5.7. □

Acknowledgements. We express our heart-felt thanks to the Editor in chief and referees for their valuable remarks and suggestions to our paper.

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