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Solutions of fuzzy wave-like equations by variational iteration method

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Abstract. In this paper we give sufficient condition for the Buckley-Feuring solution to exist by the variation iteration method are used for find the exact fuzzy solution of the fuzzy wave-like equation in one and two dimensions with variable coefficients and fuzzy parameters. Some examples are given to show the reliability and the efficiency of the sufficient condition..

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1. Introduction

The fuzzy partial differential equation method is used for solving many problems in several applied fields like economics, finance, engineering and physics. These problems often boil down to the solution of a fuzzy equation. Therefore, various approaches for solving these problems have been reported in the last years.
In present paper, we assume wave-like models which can exactly describe some nonlinear phenomena, for example, wave-like equation can describe earthquake stresses [11], coupling currents in a flat multi-strand two-layer super conducting cable [1] and non-homogeneous elastic waves in soils [13]. We suppose the existence of imprecise parameters in wave-like equations with variable coefficients. Since fuzzy sets theory [17] is a powerful tool for modeling imprecise and processing vague in mathematical models, hence, the our idea is solving wave-like equations with fuzzy parameters via the same strategy as Buckley and Feuring [3] using Variational Iteration Method (VIM) [3, 9, 10].
In comparison with the paper [2], we investigate problems with fuzzy parameters, fuzzy initial value and fuzzy forcing functions, we propose a new theorem for finding the exact fuzzy solutions, witch extended to the Buckley-Feuring for the proposed models .

We begin section 2 by defining the notation where we will use in the paper and then in Sections 3 and 4, fuzzy wave-like equations and the VIM are illustrated, respectively. In Section 5, the same strategy as in Buckley-Feuring is presented for two-dimensional fuzzy wave-like equation. Some examples in Section 6 are illustrated.

2. Preliminaries

We place a bar over a capital letter to denote a fuzzy number of $\mathbb{R}^n$. So, $\overline{A}$, $\overline{K}$, $\overline{\gamma}$, $\overline{\beta}$ etc. all represent fuzzy numbers of $\mathbb{R}^n$ for some n. We write $\mu_{\overline{A}}(t)$, a number in $[0,1]$, for the membership function of $\overline{A}$ evaluated at $t \in \mathbb{R}^n$. An $\alpha-$cut of $\overline{A}$ is always a closed and bounded interval that written $\overline{A}[\alpha]$, is defined as $\{t \mid \mu_{\overline{A}}(t) \geq \alpha\}$ for $0 < \alpha < 1$. We separately specify $\overline{A}[0]$ as the closure of the union of all the $\overline{A}[\alpha]$ for $0 < \alpha \leq 1$

Definition 2.1 ([6])**.** Let $\mathbb{R}_{\mathcal{F}} = \left\{\overline{A} \mid \overline{A} : \mathbb{R} \to [0,1], \text{ satisfies } (1) - (4)\right\}$:

(1) $\forall \overline{A} \in \mathbb{R}_{\mathcal{F}}$, $\overline{A}$ is normal.
(2) $\forall \overline{A} \in \mathbb{R}_{\mathcal{F}}$, $\overline{A}$ is a fuzzy convex set.
(3) $\forall \overline{A} \in \mathbb{R}_{\mathcal{F}}$, $\overline{A}$ is upper semi-continuous on $\mathbb{R}$.
(4) $\overline{A}[0]$ is a compact set.

Then $\mathbb{R}_{\mathcal{F}}$ is called fuzzy number space and $\forall \overline{A} \in \mathbb{R}_{\mathcal{F}}$, $\overline{A}$ is called a fuzzy number.

Definition 2.2 ([6, 12])**.** We represent an arbitrary fuzzy number by an ordered pair of functions $\overline{A}[\alpha] = [A_1(\alpha), A_2(\alpha)]$, $\alpha \in [0,1]$, which satisfy the following requirements :

(1) $A_1(\alpha)$ is a nondecreasing function over $[0,1]$,
(2) $A_2(\alpha)$ is a nonincreasing function on $[0,1]$
(3) $A_1(\alpha)$ and $A_2(\alpha)$ are bounded left continuous on $(0,1]$, and right continuous at $\alpha = 0$, and
(4) $A_1(\alpha) \leq A_2(\alpha)$, for $0 \leq \alpha \leq 1$

Definition 2.3. Let $\overline{A} = (a_1, a_2, a_3)$, $(a_1 < a_2 < a_3)$. $\overline{A}$ is called triangular fuzzy number with peak (center) a_2, left width $a_2 - a_1 > 0$ and right width $a_3 - a_2 > 0$, if its membership function has the following form :

$$\mu_{\overline{A}}(t) = \begin{cases} 1 - \dfrac{(a_2 - t)}{a_2 - a_1}, & a_1 \leq t \leq a_2 \\ 1 - \dfrac{(t - a_2)}{a_3 - a_2}, & a_2 \leq t \leq a_3 \\ 0, & \text{otherwise.} \end{cases}$$

The support of $\overline{A}$ is $[a_1, a_3]$. We can write :

(1) $\overline{A} > 0$ if $a_1 > 0$,
(2) $\overline{A} \geq 0$ if $a_1 \geq 0$,
(3) $\overline{A} < 0$ if $a_3 < 0$,
(4) $\overline{A} \leq 0$ if $a_3 \leq 0$.

Definition 2.4. For arbitrary fuzzy numbers $\overline{A}[\alpha] = \Big[a_1(\alpha), a_2(\alpha)\Big]$ and $\overline{B}[\alpha] = \Big[b_1(\alpha), b_2(\alpha)\Big]$ we have algebraic operations as follows :

(1) $(\overline{A}+\overline{B})[\alpha] = [a_1(\alpha)+b_1(\alpha), a_2(\alpha)+b_2(\alpha)]$
(2) $(\overline{A}-\overline{B})[\alpha] = [a_1(\alpha)-b_2(\alpha), a_2(\alpha)-b_1(\alpha)]$
(3)
$$k\overline{A}[\alpha] = \begin{cases} [ka_1(\alpha), ka_2(\alpha)] & k \geq 0 \\ [ka_2(\alpha), ka_1(\alpha)] & k < 0 \end{cases}$$
(4) $(\overline{A}.\overline{B})[\alpha] = \{\min z, \max z\}$ with
$$z = \Big\{a_1(\alpha).b_1(\alpha),\ a_1(\alpha).b_2(\alpha),\ a_2(\alpha).b_1(\alpha),\ a_2(\alpha).b_2(\alpha)\Big\}$$
(5) If $0 \notin [b_1(\alpha), b_2(\alpha)]$
$$\frac{\overline{A}}{\overline{B}}[\alpha] = [(\frac{a_1}{b_1})(\alpha), (\frac{a_2}{b_2})(\alpha)]$$
where
$$(\frac{a_1}{b_1})(\alpha) = \min\left\{\frac{a_1(\alpha)}{b_1(\alpha)}, \frac{a_1(\alpha)}{b_2(\alpha)}, \frac{a_2(\alpha)}{b_1(\alpha)}, \frac{a_2(\alpha)}{b_2(\alpha)}\right\}$$
$$(\frac{a_2}{b_2})(\alpha) = \max\left\{\frac{a_1(\alpha)}{b_1(\alpha)}, \frac{a_1(\alpha)}{b_2(\alpha)}, \frac{a_2(\alpha)}{b_1(\alpha)}, \frac{a_2(\alpha)}{b_2(\alpha)}\right\}$$

We adopt the general definition of a fuzzy number given in [7].

3. Fuzzy wave-like equations

We consider the wave-like equations in one and tow dimensional cases which can be written in the forms

- One-dimensional [2] :

$$U_{tt}(t,x) + P(x,\gamma)U_{xx}(t,x) = F(t,x,k) \tag{3.1}$$

- Two-dimensional [2] :

$$U_{tt}(t,x,y) + P(x,\gamma)U_{xx}(t,x,y) + Q(y,\beta)U_{yy}(t,x,y) = F(t,x,y,k) \tag{3.2}$$

or

$$U_{tt}(t,x,y) + Q(y,\beta)U_{xx}(t,x,y) + P(x,\gamma)U_{yy}(t,x,y) = F(t,x,k) \tag{3.3}$$

subject to certain initial and boundary conditions.
These initial and boundary conditions, in state two-dimensional, can come in a variety of forms such as

$$U(0,x,y) = c_1 \quad \text{or} \quad U(0,x,y) = g_1(x,y,c_2) \quad \text{or} \quad U(M_1,x,y) = g_2(x,y,c_3,c_4), \ \ldots$$

In this paper the method is applied for the wave-like equation (3.2). For (3.1) and (3.3), the same discussion can be made. In following lines, the components of (3.2) are enumerated :

- $I_1 = [0, M_1]$, $I_2 = [M_2, M_3]$ and $I_3 = [M_4, M_5]$ are three intervals, which M_{n_1} $(n_1 = 2,3,4,5)$ is negative or positive and $M_1 > 0$.

- $F(t,x,y,k)$, $U(t,x,y)$, $P(x,\gamma)$ and $Q(y,\beta)$ will be continuous functions for $(t,x,y)\in\prod_{j=1}^{3} I_j$.
- $P(x,\gamma)$ and $Q(y,\beta)$ have a finite number of roots for each $(x,y)\in I_2\times I_3$
- $k=\Big(k_1,\dots,k_n\Big)$, $c=\Big(c_1,\dots,c_m\Big)$, $\gamma=\Big(\gamma_1,\dots,\gamma_s\Big)$ and $\beta=\Big(\beta_1,\dots,\beta_e\Big)$ are vectors of constants with $k_j\in J_j$, $c_i\in L_i$ and $\gamma_r\in H_r$ and $\beta_l\in D_l$.

Assume that (3.2) has a solution

$$U(t,x,y)=G(t,x,y,k,c,\gamma,\beta) \tag{3.4}$$

for G and $G_{tt}(t,x,y,k,c,\gamma,\beta)+P(x,\gamma)G_{xx}(t,x,y,k,c,\gamma,\beta)+Q(y,\beta)G_{yy}(t,x,y,k,c,\gamma,\beta)$ are continuous with $(t,x,y)\in\prod\limits_{j=1}^{3} I_j$, $k\in J=\prod\limits_{j=1}^{n} J_j$, $c\in L=\prod\limits_{i=1}^{m} L_i$, $\gamma\in H=\prod\limits_{r=1}^{s} H_r$ and $\beta\in D=\prod\limits_{l=1}^{e} D_l$.

Suppose the constant k_j, c_i ,γ_r and β_l are imprecise in their values. We will model this uncertainty by substituting triangular fuzzy numbers for the k_j, c_i , γ_r and β_l. If we fuzzify (3.2), then we obtain the fuzzy wave-like equation. Using the extension principle, we compute $\overline{F}$, $\overline{P}$ and $\overline{Q}$ from F, P and Q where $\overline{F}(t,x,y,\overline{K})$ has $\overline{K}=(\overline{k}_1,\dots,\overline{k}_n)$, $\overline{P}(x,\overline{\gamma})$ has $\overline{\gamma}=(\overline{\gamma}_1,\dots,\overline{\gamma}_s)$ and $\overline{Q}(y,\overline{\beta})$ a $\overline{\beta}=(\overline{\beta}_1,\dots,\overline{\beta}_e)$ for k_j, γ_r and β_l a triangular fuzzy numbers in J_j $(0\le j\le n)$, H_r $(0\le r\le s)$ and D_l $(0\le l\le e)$.

The function U is changed to $\overline{U}$ where $\overline{U}:\prod\limits_{j=1}^{3} I_j\to\mathcal{F}(\mathbb{R})$. That is, $\overline{U}(t,x,y)$ is a fuzzy function. The fuzzy wave-like equation is

$$\overline{U}_t(t,x,y)+\overline{P}(x,\overline{\gamma})\overline{U}_{xx}(t,x,y)+\overline{Q}(y,\overline{\beta})\overline{U}_{yy}(t,x,y)=\overline{F}(t,x,y,\overline{K}) \tag{3.5}$$

subject to certain initial and boundary conditions. The initial and boundary conditions can be of the form

$$\overline{U}(0,x,y)=\overline{C}_1 \text{ or } \overline{U}(0,x,y)=\overline{g}_1(x,y,\overline{C}_2) \text{ or } \overline{U}(M_1,x,y)=\overline{g}_2(x,y,\overline{C}_3,\overline{C}_4)$$

The $\overline{g}_j$ is the fuzzification g_i via extension principle. We wish to solve the problem given in (3.5). Finally, we fuzzify G in (3.4).
Let $\overline{Z}(t,x,y)=\overline{G}(t,x,y,\overline{K},\overline{C},\overline{\gamma},\overline{\beta})$ where $\overline{Z}$ is computed using the extension principle and is a fuzzy solution. In section 5, we will discuss the concept solution with the same strategy as Buckley-Feuring for fuzzy wave-like equation.
Let $\overline{K}[\alpha]=\prod_{j=1}^{n}\overline{K}_j[\alpha]$, $\overline{\gamma}[\alpha]=\prod_{r=1}^{s}\overline{\gamma}_r[\alpha]$, $\overline{C}[\alpha]=\prod_{i=1}^{m}\overline{C}_i[\alpha]$
and $\overline{\beta}[\alpha]=\prod_{l=1}^{e}\overline{\beta}_l[\alpha]$

4. The variational iteration method

To illustrate the basic idea of the VIM we consider the following PDE model

$$L_tU+L_xU+L_yU+NU=F(t,x,y,k) \tag{4.1}$$

where L_t, L_x and L_y are linear operators of t, x and y, respectively, and N is a nonlinear operator, also $F(t,x,y,k)$ is the source non-homogeneous term. According to the VIM [15, 16], we can express the following correction function for (4.1) in t, x and y directions can be written as

$$\begin{aligned} U_{n+1}(t,x,y) &= U_n(t,x,y) + \int_0^t \lambda_1\{L_sU_n + (L_x+L_y+N)\widetilde{U}_n - F(s,x,y,k)\}ds \\ U_{n+1}(t,x,y) &= U_n(t,x,y) + \int_0^x \lambda_2\{L_sU_n + (L_t+L_y+N)\widetilde{U}_n - F(s,x,y,k)\}ds \\ U_{n+1}(t,x,y) &= U_n(t,x,y) + \int_0^y \lambda_3\{L_sU_n + (L_t+L_x+N)\widetilde{U}_n - F(s,x,y,k)\}ds \end{aligned}$$

where λ_i, $1 \le i \le 3$ are general Lagrange multipliers, which can be identified optimally via the variational theory [8, 16], and $\widetilde{U}_n$ is a restricted variation which means $\delta\widetilde{U}_n = 0$. It is required first to determine the Lagrange multipliers λ_i that will be identified optimally via integration by parts.The approximations U_{n+1}, $n \ge 0$, of the solution $U(t,x,y)$ will immediately follow upon using any selective function U_0. The initial values $U(0,x,y)$ and $U_t(0,x,y)$ are usually used for the selected zeroth approximations U_0. With the Lagrange multipliers λ_i determined, then several approximation $u_i(t,x,y)$, $i \ge 0$, can be determined. Consequently, the solution is given as

$$U(t,x,y) = \lim_{n\to\infty} U_n(t,x,y)$$

According to the VIM, we construct a correction functional for (3.2) in t-direction as follows

$$\begin{aligned} U_{n+1}(t,x,y) &= U_n(t,x,y) \\ &+ \int_0^t \lambda(s)\Big\{(U_n)_{ss} + P(x,\gamma)(\widetilde{U}_n)_{xx} + Q(y,\beta)(\widetilde{U}_n)_{yy} - F(s,x,y,k)\Big\}ds \end{aligned} \tag{4.2}$$

where $n \ge 0$ and λ is a lagrange multiplier. We now determine the lagrange multiplier

$$\begin{aligned} \delta U_{n+1}(t,x,y) &= \delta U_n(t,x,y) \\ &+ \delta\int_0^t \lambda(s)\Big\{(U_n)_{ss} + P(x,\gamma)(\widetilde{U}_n)_{xx} + Q(y,\beta)(\widetilde{U}_n)_{yy} - F(s,x,y,k)\Big\}ds \end{aligned}$$

$$\begin{aligned} \delta U_{n+1}(t,x,y) &= \delta U_n(t,x,y) \\ &+ \lambda(s)\delta\Big((U_n)_s\Big)|_{s=t} - \lambda'(s)\delta U_n|_{s=t} + \int_0^t \lambda''(s)\delta U_n ds \end{aligned}$$

Therefore, the stationary conditions are :

$$\begin{aligned} \delta U_n &: \lambda''(s) = 0, \\ \delta U_n &: 1 - \lambda'(s)|_{s=t} = 0, \\ \delta\Big((U_n)_s\Big) &: \lambda(s)|_{s=t} = 0. \end{aligned}$$

So, the lagrange multiplier is $\lambda = s - t$. Submitting the results into (4.2) leads to the following iteration formula

$$U_{n+1}(t,x,y) = U_n(t,x,y) + \int_0^t (s-t)\{(U_n)_{ss} + P(x,\gamma)(\widetilde{U}_n)_{xx} + Q(y,\beta)(\widetilde{U}_n)_{yy} - F(s,x,y,k)\}ds \tag{4.3}$$

Iteration formula start with initial approximation, for example $U_0(t,x,y) = U(0,x,y)$. Also the VIM used for system of linear and nonlinear partial differential equation [16] which handled in obtain Seikkala solution.

5. Buckley-Feuring Solution (BFS) and Seikkala solution (SS)

5.1. Buckley-Feuring solution. Buckley-Feuring first present the BFS [3, 4]. They define for all t, x, y and $\alpha \in [0,1]$,

$$\overline{Z}(t,x,y)[\alpha] = \Big[z_1(t,x,y,\alpha), z_2(t,x,y,\alpha)\Big], \quad \overline{F}\Big(t,x,y,\overline{k}\Big)[\alpha] = \Big[F_1(t,x,y,\alpha), F_2(t,x,y,\alpha)\Big]$$

and to check (3.5) we must compute $\overline{P}\Big(x,\overline{\gamma}\Big)$ and $\overline{Q}\Big(y,\overline{\beta}\Big)$. The α-cuts of $\overline{P}\Big(x,\overline{\gamma}\Big)$ and $\overline{Q}\Big(y,\overline{\beta}\Big)$ can be found as follows :
$\forall\alpha \in [0,1]$

$$\overline{P}(x,\overline{\gamma})[\alpha] = \Big[P_1(x,\alpha), P_2(x,\alpha)\Big], \qquad \overline{Q}(y,\overline{\beta})[\alpha] = \Big[Q_1(y,\alpha), Q_2(y,\alpha)\Big]$$

Let $W = \overline{K}[\alpha] \times \overline{C}[\alpha] \times \overline{\gamma}[\alpha] \times \overline{\beta}[\alpha]$. By definition

$$z_1(t,x,y,\alpha) = \min\Big\{G(t,x,y,k,c,\gamma,\beta) \,:\, (k,c,\gamma,\beta) \in W\Big\} \tag{5.1}$$

$$z_2(t,x,y,\alpha) = \max\Big\{G(t,x,y,k,c,\gamma,\beta) \,:\, (k,c,\gamma,\beta) \in W\Big\} \tag{5.2}$$

and

$$F_1(t,x,y,\alpha) = \min\Big\{F(t,x,y,k) \,:\, k \in \overline{K}[\alpha]\Big\}, \tag{5.3}$$

$$F_2(t,x,y,\alpha) = \max\Big\{F(t,x,y,k) \,:\, k \in \overline{K}[\alpha]\Big\} \tag{5.4}$$

$\forall(t,x,y) \in \prod_{j=1}^3 I_j$ *and* $\alpha \in [0,1]$
and

$$P_1(x,\alpha) = \min\{P(x,\gamma)|\gamma \in \overline{\gamma}[\alpha]\}, \quad P_2(x,\alpha) = \max\{P(x,\gamma)|\gamma \in \overline{\gamma}[\alpha]\} \tag{5.5}$$

$\forall x \in I_2$ and $\alpha \in [0,1]$
and

$$Q_1(y,\alpha) = \min\{Q(y,\beta)|\beta \in \overline{\beta}[\alpha]\}, \quad Q_2(y,\alpha) = \max\{Q(y,\beta)|\beta \in \overline{\beta}[\alpha]\} \tag{5.6}$$

$\forall y \in I_3$ and $\alpha \in [0,1]$
Assume that $P(x,\gamma) > 0$, $(P_1(x,\alpha) > 0)$, $Q(y,\beta) > 0$, $(Q_1(y,\alpha) > 0)$ and the $z_i(t,x,y,\alpha)$ $i = 1,2$, has continuous partial derivatives so $(z_i)_{tt} + P_i(z_i)_{xx} + Q_i(z_i)_{yy}$

is continuous for all $t, x, y \in \prod_{j=1}^{3} I_j$ and all $\alpha \in [0,1]$.
Define

$$\Gamma(t,x,y,\alpha) = \Big[(z_1)_{tt} + P_1(x,\alpha)(z_1)_{xx} + Q_1(y,\beta)(z_1)_{yy}, (z_2)_{tt} + P_2(x,\alpha)(z_2)_{xx} + Q_2(y,\beta)(z_2)_{yy}\Big]$$

for all $(t,x,y) \in \prod_{j=1}^{3} I_j$ and all α.
If, for each fixed $t, x, y \in \prod_{j=1}^{3} I_j$, $\Gamma(t,x,y,\alpha)$ defines the $\alpha-$cut of a fuzzy number, then will be said that $\overline{Z}(t,x,y)$ is differentiable and is written

$$\overline{Z}_{tt}[\alpha] + \overline{P}[\alpha]\overline{Z}_{xx}[\alpha] + \overline{Q}[\alpha]\overline{Z}_{yy}[\alpha] = \Gamma(t,x,y,\alpha)$$

for all $(t,x,y) \in \prod_{j=1}^{3} I_j$ and all α

Sufficient conditions for $\Gamma(t,x,y,\alpha)$ to define $\alpha-$cut of a fuzzy number are [7] :

(i) $(z_1)_{tt}(t,x,y,\alpha) + P_1(x,\alpha)(z_1)_{xx}(t,x,y,\alpha) + Q_1(y,\alpha)(z_1)_{yy}(t,x,y,\alpha)$ is an increasing function of α for each $(t,x,y) \in \prod_{j=1}^{3} I_j$

(ii) $(z_2)_{tt}(t,x,y,\alpha) + P_2(x,\alpha)(z_2)_{xx}(t,x,y,\alpha) + Q_2(y,\alpha)(z_2)_{yy}(t,x,y,\alpha)$ is an decreasing function of α for each $(t,x,y) \in \prod_{j=1}^{3} I_j$ and

(iii) for $(t,x,y) \in \prod_{j=1}^{3} I_j$

$$\big(z_1\big)_{tt}(t,x,y,1) + P_1(x,1)\big(z_1\big)_{xx}(t,x,y,1) + Q_1(y,1)\big(z_1\big)_{yy}(t,x,y,1) \leq \big(z_2\big)_{tt}(t,x,y,1) + P_2(x,1)\big(z_2\big)_{xx}(t,x,y,1) + Q_2(y,1)\big(z_2\big)_{yy}(t,x,y,1)$$

Now we assume that the $z_i(t,x,y,\alpha)$ has continuous partial derivatives so $(z_i)_{tt} + P_i(x,\alpha)(z_i)_{xx} + Q_i(y,\alpha)(z_i)_{yy}$ is continuous on $\prod_{j=1}^{3} I_j \times [0,1]$ $i = 1,2$.
Hence, if conditions (i)-(iii) above are hold, $\overline{Z}(t,x,y)$ is differentiable.

For $\overline{Z}(t,x,y)$ to be a BFS of the fuzzy wave-like equation we need

(a) $\overline{Z}(t,x,y)$ differentiable
(b) (3.5) hold for $\overline{U}(t,x,y) = \overline{Z}(t,x,y)$,
(c) $\overline{Z}(t,x,y)$ satisfies the initial and boundary conditions. Since no exist specified any particular initial and boundary conditions, then only is checked if (3.5) hold.

$\overline{Z}(t,x,y)$ is a BFS (without the initial and boundary conditions) if $\overline{Z}(t,x,y)$ is differentiable and $(\overline{Z})_{tt} + \overline{P}(x,\overline{\gamma})(\overline{Z})_{xx} + \overline{Q}(y,\overline{\beta})(\overline{Z})_{yy} = \overline{F}(t,x,y,\overline{k})$ or the following equations must hold

$$(z_1)_{tt} + P_1(x,\alpha)(z_1)_{xx} + Q_1(y,\alpha)(z_1)_{yy} = F_1(t,x,y,\alpha) \tag{5.7}$$

$$(z_2)_{tt} + P_2(x,\alpha)(z_2)_{xx} + Q_2(y,\alpha)(z_2)_{yy} = F_2(t,x,y,\alpha) \tag{5.8}$$

for all $(t,x,y) \in \prod_{j=1}^{3} I_j$ and $\alpha \in [0,1]$.
Now we will present a sufficient condition for the BFS to exist such as Buckley and Feuring. Since there are such a variety of possible initial and boundary conditions, so we will omit them from the following theorem. One must separately check out the initial and boundary conditions. So, we will omit the constants c_i, $1 \leq i \leq m$,

from the problem. Therefore, (3.4) becomes $U(t,x,y) = G(t,x,y,k,\gamma,\beta)$, so $\overline{Z}(t,x,y) = \overline{G}\Big(t,x,y,\overline{K},\overline{\gamma},\overline{\beta}\Big)$.

Theorem 5.1. *Assume $\overline{Z}(t,x,y)$ is differentiable.*

(a)

$$\text{if } P(x,\gamma_i) > 0 \quad \text{and} \quad \frac{\partial P}{\partial \gamma_i}\frac{\partial G}{\partial \gamma_i} > 0 \quad x \in I_2 \quad \text{for} \quad i = 1,2,\dots,m \tag{5.9}$$

and

$$\text{if} \quad Q(y,\beta_l) > 0 \quad \text{and} \quad \frac{\partial Q}{\partial \beta_l}\frac{\partial G}{\partial \beta_l} > 0 \quad y \in I_3 \quad \text{for} \quad l = 1,2,\dots,e \tag{5.10}$$

and

$$\text{if} \quad \frac{\partial G}{\partial k_j}\frac{\partial F}{\partial k_j} > 0 \quad \text{for} \quad j = 1,2,\dots,n \tag{5.11}$$

Then $BFS = \overline{Z}(t,x,y)$

(b) *If relations (5.9) does not hold for some i or relation (5.10) does not hold for some l, or relation (5.11) does not hold for some j, then $\overline{Z}(t,x,y)$ is not a BFS.*

Proof.
(a) For simplicity assume $k_j = k$, $\gamma_i = \gamma$, $\beta_l = \beta$ and $\frac{\partial G}{\partial k} < 0$, $\frac{\partial F}{\partial k} < 0$, $\frac{\partial P}{\partial \gamma} > 0$, $\frac{\partial G}{\partial \gamma} > 0$, $\frac{\partial Q}{\partial \beta} < 0$ and $\frac{\partial G}{\partial \beta} < 0$. The proof for $\frac{\partial G}{\partial k} > 0$, $\frac{\partial F}{\partial k} > 0$, $\frac{\partial P}{\partial \gamma} < 0$, $\frac{\partial G}{\partial \gamma} < 0$, $\frac{\partial Q}{\partial \beta} > 0$ and $\frac{\partial G}{\partial \beta} > 0$ is similar and omitted.
Since $\frac{\partial G}{\partial k} < 0$, $\frac{\partial G}{\partial \gamma} > 0$ and $\frac{\partial G}{\partial \beta} < 0$, then from (5.1) and (5.2) we have

$$z_1(t,x,y,\alpha) = G\Big(t,x,y,k_2(\alpha),\gamma_1(\alpha),\beta_2(\alpha)\Big),$$

$$z_2(t,x,y,\alpha) = G\Big(t,x,y,k_1(\alpha),\gamma_2(\alpha),\beta_1(\alpha)\Big)$$

from (5.3), (5.4) and $\frac{\partial F}{\partial k} < 0$ we have

$$F_1(t,x,y,\alpha) = F\Big(t,x,y,k_2(\alpha)\Big) \qquad F_2(t,x,y,\alpha) = F\Big(t,x,y,k_1(\alpha)\Big)$$

since (5.5) and $\frac{\partial P}{\partial \gamma} > 0$ we have

$$P_1(x,\alpha) = P\Big(x,\gamma_1(\alpha)\Big) \qquad P_2(x,\alpha) = P\Big(x,\gamma_2(\alpha)\Big)$$

from (5.6) and $\frac{\partial Q}{\partial \beta} < 0$ we have

$$Q_1(y,\alpha) = Q\Big(y,\beta_2(\alpha)\Big) \qquad Q_2(y,\alpha) = Q\Big(y,\beta_1(\alpha)\Big)$$

for all $\alpha \in [0,1]$ where $\overline{K}[\alpha] = \Big[k_1(\alpha),k_2(\alpha)\Big]$, $\overline{\gamma}[\alpha] = \Big[\gamma_1(\alpha),\gamma_2(\alpha)\Big]$ and $\overline{\beta}[\alpha] = \Big[\beta_1(\alpha),\beta_2(\alpha)\Big]$.
Now $G(t,x,y,k,\gamma,\beta)$ solves (3.2), which means

$$G_{tt} + P(x,\gamma)G_{xx} + Q(y,\beta)G_{yy} = F(t,x,y,k)$$

for all $(t,x,y) \in \prod_{j=1}^{3} I_j$, $k \in J$, $\gamma \in H$ and $\beta \in D$

Suppose $\overline{Z}(t,x,y)$ is differentiable and $P(x,\gamma) > 0$ and $Q(y,\beta) > 0$ so

$$\partial_{tt} z_1(t,x,y,\alpha) + P_1(x,\alpha)\partial_{xx} z_1(t,x,y,\alpha) + Q_1(y,\alpha)\partial_{yy} z_1(t,x,y,\alpha) = F_1(t,x,y,\alpha)$$

$$\partial_{tt} z_2(t,x,y,\alpha) + P_2(x,\alpha)\partial_{xx} z_2(t,x,y,\alpha) + Q_2(y,\alpha)\partial_{yy} z_2(t,x,y,\alpha) = F_2(t,x,y,\alpha)$$

for all $(t,x,y) \in \prod_{j=1}^{3} I_j$ and $\alpha \in [0,1]$

Hence, (5.7) and (5.8) holds and $\overline{Z}(t,x,y)$ is a BFS.

(b) Now consider the situation where (5.9) or (5.10) or (5.11) does not hold. Let us only look at one case where $\frac{\partial Q}{\partial \beta} < 0$ $\Big($ assume $\frac{\partial G}{\partial k} > 0$, $\frac{\partial F}{\partial k} > 0$, $\frac{\partial G}{\partial \gamma} > 0$, $\frac{\partial P}{\partial \gamma} > 0$ and $\frac{\partial G}{\partial \beta} > 0$, $P(x,\gamma) > 0$ and $Q(y,\beta) > 0\Big)$. Then we have

$$z_1(t,x,y,\alpha) = G\Big(t,x,y,k_1(\alpha),\gamma_1(\alpha),\beta_1(\alpha)\Big)$$
$$z_2(t,x,y,\alpha) = G\Big(t,x,y,k_2(\alpha),\gamma_2(\alpha),\beta_2(\alpha)\Big)$$
$$F_1(t,x,y,\alpha) = F\Big(t,x,y,k_1(\alpha)\Big), \quad F_2(t,x,y,\alpha) = F\Big(t,x,y,k_2(\alpha)\Big)$$

and

$$P_1(x,\alpha) = P\Big(x,\gamma_1(\alpha)\Big) \qquad P_2(x,\alpha) = P\Big(x,\gamma_2(\alpha)\Big)$$
$$Q_1(y,\alpha) = Q\Big(y,\beta_2(\alpha)\Big) \qquad Q_2(y,\alpha) = Q\Big(y,\beta_1(\alpha)\Big)$$

then we have

$$\partial_{tt} z_1(t,x,y,\alpha) + P_1(x,\alpha)\partial_{xx} z_1(t,x,y,\alpha) + Q_1(y,\alpha)\partial_{yy} z_1(t,x,y,\alpha) = F_1(t,x,y,\alpha)$$

$$\partial_{tt} z_2(t,x,y,\alpha) + P_2(x,\alpha)\partial_{xx} z_2(t,x,y,\alpha) + Q_2(y,\alpha)\partial_{yy} z_2(t,x,y,\alpha) = F_2(t,x,y,\alpha)$$

which is not true, because

$$\begin{aligned} G_{tt}\Big(t,x,y,k_1(\alpha),\gamma_1(\alpha),\beta_1(\alpha)\Big) + P\Big(x,\gamma_1(\alpha)\Big)G_{xx}\Big(t,x,y,k_1(\alpha),\gamma_1(\alpha),\beta_1(\alpha)\Big) \\ + Q\Big(x,\beta_2(\alpha)\Big)G_{yy}\Big(t,x,y,k_1(\alpha),\gamma_1(\alpha),\beta_1(\alpha)\Big) = F\Big(t,x,y,k_1(\alpha)\Big) \end{aligned}$$

$$\begin{aligned} G_{tt}\Big(t,x,y,k_2(\alpha),\gamma_2(\alpha),\beta_2(\alpha)\Big) + P\Big(x,\gamma_1(\alpha)\Big)G_{xx}\Big(t,x,y,k_2(\alpha),\gamma_2(\alpha),\beta_2(\alpha)\Big) \\ + Q\Big(y,\beta_1(\alpha)\Big)G_{yy}\Big(t,x,k_1(\alpha),\gamma_1(\alpha),\beta_2(\alpha)\Big) = F\Big(t,x,y,k_2(\alpha)\Big) \end{aligned}$$

□

Therefore, if $\overline{Z}(t,x,y)$ is a BFS and it satisfies the initial and boundary conditions we will say that $\overline{Z}(t,x,y)$ is a BFS satisfying the initial and boundary conditions. If $\overline{Z}(t,x,y)$ is not a BFS, then we will consider the SS.

5.2. **Seikkala solution (SS).** Now let us define the SS [14]. Let

$$\overline{U}(t,x,y)[\alpha] = \Big[u_1(t,x,y,\alpha), u_2(t,x,y,\alpha)\Big]$$

For example suppose $P(x,\gamma) < 0$ and $Q(y,\beta) > 0$, so consider the system of wave-like equations

$$(u_1)_{tt} + P_1(x,\alpha)(u_2)_{xx} + Q_1(y,\alpha)(u_1)_{yy} = F_1(t,x,y,\alpha) \tag{5.12}$$

$$(u_2)_{tt} + P_2(x,\alpha)(u_1)_{xx} + Q_2(y,\alpha)(u_2)_{yy} = F_2(t,x,y,\alpha) \tag{5.13}$$

Or if $P(x,\gamma) > 0$, $Q(y,\beta) > 0$, $\frac{\partial P}{\partial \gamma} > 0$, $\frac{\partial G}{\partial \gamma} < 0$, $\frac{\partial Q}{\partial \beta} > 0$, $\frac{\partial G}{\partial \beta} > 0$

$$(u_1)_{tt} + P_1(x,\alpha)(u_1)_{xx} + Q_1(y,\alpha)(u_1)_{yy} = F_1(t,x,y,\alpha)$$

$$(u_2)_{tt} + P_2(x,\alpha)(u_2)_{xx} + Q_2(y,\alpha)(u_2)_{yy} = F_2(t,x,y,\alpha)$$

for all $(t,x,y) \in \prod_{j=1}^{3} I_j$ and $\alpha \in [0,1]$. We append to Eqs. (5.12) and (5.13) any initial and boundary conditions. For example, if it was $\overline{U}(0,x,y) = \overline{C}$ then we add

$$u_1(0,x,y,\alpha) = c_1(\alpha)$$

$$u_2(0,x,y,\alpha) = c_2(\alpha)$$

where $\overline{C}[\alpha] = \Big[c_1(\alpha), c_2(\alpha)\Big]$.
Let $u_i(t,x,y,\alpha)$ i=1,2 solve Eqs. (5.12) and (5.13) plus initial and boundary conditions.
If

$$\Big[u_1(t,x,y,\alpha), u_2(t,x,y,\alpha)\Big],$$

defines the $\alpha-$cut of a fuzzy number, for all $(t,x,y) \in \prod_{j=1}^{3} I_j$, then $\overline{U}(t,x,y)$ is the SS.
We will say that derivative condition holds for fuzzy wave-like equation when Eqs.(5.9),(5.10) and (5.11) are true.

Theorem 5.2.

(1) *If BFS* $= \overline{Z}(t,x,y)$, *then SS* $= \overline{Z}(t,x,y)$.
(2) *If SS* $= \overline{Z}(t,x,y)$ *and the derivative condition holds, then BFS* $= \overline{U}(t,x,y)$.

Proof.

(1) Follows from the definition of BFS and SS.
(2) If SS $= \overline{U}(t,x,y)$ then the Seikkala derivative [4] exists and since the derivative condition holds, therefore, Eqs. following holds

$$(u_1)_{tt} + P_1(x,\alpha)(u_1)_{xx} + Q_1(y,\alpha)(u_1)_{yy} = F_1(t,x,y,\alpha)$$

$$(u_2)_{tt} + P_2(x,\alpha)(u_2)_{xx} + Q_2(y,\alpha)(u_2)_{yy} = F_2(t,x,y,\alpha)$$

Also suppose one $k_j = k$, $\gamma_i = \gamma$, $\beta_l = \beta$, $\frac{\partial G}{\partial \gamma} < 0$, $\frac{\partial P}{\partial \gamma} < 0$, $\frac{\partial G}{\partial k} < 0$ and $\frac{\partial F}{\partial k} < 0$, $\frac{\partial G}{\partial \beta} > 0$, $\frac{\partial Q}{\partial \beta} > 0$ (the other cases are similar and are omitted). We see

$$z_1(t,x,y,\alpha) = G\Big(t,x,y,k_2(\alpha),\gamma_2(\alpha),\beta_1(\alpha)\Big)$$
$$z_2(t,x,y,\alpha) = G\Big(t,x,y,k_1(\alpha),\gamma_1(\alpha),\beta_2(\alpha)\Big)$$
$$F_1(t,x,y,\alpha) = F\Big(t,x,y,k_2(\alpha)\Big), \quad F_2(t,x,y,\alpha) = F\Big(t,x,y,k_1(\alpha)\Big)$$
$$P_1(x,\alpha) = P\Big(x,\gamma_2(\alpha)\Big), \quad P_2(x,\alpha) = P\Big(x,\gamma_1(\alpha)\Big)$$
$$Q_1(y,\alpha) = Q\Big(y,\beta_1(\alpha)\Big), \quad Q_2(y,\alpha) = Q\Big(y,\beta_2(\alpha)\Big)$$

Now look at Eqs. (5.7), (5.8) also Eqs. (5.1) and (5.2), implies that

$$u_1(t,x,y,\alpha) = G\Big(t,x,y,k_2(\alpha),\gamma_2(\alpha),\beta_1(\alpha)\Big) = z_1(t,x,y,\alpha)$$

$$u_2(t,x,y,\alpha) = G\Big(t,x,y,k_1(\alpha),\gamma_1(\alpha),\beta_2(\alpha)\Big) = z_2(t,x,y,\alpha)$$

Therefore $BFS = \overline{U}(t,x,y)$

□

Lemma 5.3. *Consider (3.1) suppose $\overline{Z}(t,x)$ is differentiable.*

(a)

$$\text{(5.14)} \qquad \textit{if } P(x,\gamma_i) > 0 \textit{ and } \frac{\partial P}{\partial \gamma_i}\frac{\partial G}{\partial \gamma_i} > 0 \quad x \in I_2 \textit{ for } i = 1,2,\ldots,m$$

and

$$\text{(5.15)} \qquad \textit{if } \frac{\partial G}{\partial k_j}\frac{\partial F}{\partial k_j} > 0 \textit{ for } j = 1,2,\ldots,n$$

Then BFS=$\overline{Z}(t,x)$

(b) *If relations (5.14) does not hold for some i or relation (5.15) does not hold for some j, then $\overline{Z}(t,x)$ is not a BFS.*

Proof. It is similar to theorem (5.1) □

6. Examples

We consider the following examples ([2],[15]) and we added fuzzy parameters to these references.

Example 6.1. We first consider the one-dimensional wave-like equation with variable coefficients as

$$\text{(6.1)} \qquad U_{tt} + \frac{\gamma}{2}x^2U_{xx} = kxt$$

with the initial conditions

$$U(0,x) = cx^2 \qquad (U(0,x))_t = 1$$

where $x \in [0,1]$, $t \in]0, \pi/2]$, $k \in [0, J]$, $\gamma \in]0,1]$ and $c \in [L, 0[$ are constants.
According to the VIM, a correct functional for (6.1) from (4.3) can be constructed as follows

$$U_{n+1}(t,x) = U_n(t,x) + \int_0^t (s-t)\{(U_n(s,x))_{ss} + \frac{\gamma}{2}x^2(\widetilde{U}_n(s,x))_{xx} - kxs\}ds$$

Beginning with an initial approximation $U_0(t,x) = U(0,x) = cx^2 + t$, we can obtain the following successive approximations
$U_1(t,x) = cx^2(1 - \gamma\frac{t^2}{2!}) + kx\frac{t^3}{6} + t$
$U_2(t,x) = cx^2(1 - \gamma\frac{t^2}{2!} + \gamma^2\frac{t^4}{4!}) + kx\frac{t^3}{6} + t$
and $U_n(t,x) = cx^2(1 - \gamma\frac{t^2}{2!} + \gamma^2\frac{t^4}{4!} + \ldots + (-1)^n\gamma^n\frac{t^{2n}}{(2n)!}) + kx\frac{t^3}{6} + t, \quad n \geq 1$
The VIM admits the use of $U(t,x) = \lim_{n\to\infty} U_n(t,x)$, which gives the exact solution

$$U(t,x) = cx^2\cos(\sqrt{\gamma}t) + kx\frac{t^3}{6} + t$$

Now we fuzzify $F(t,x,k)$, $P(x,\gamma)$ and

$$G(t,x,k,c,\gamma) = cx^2\cos(\sqrt{\gamma}t) + kx\frac{t^3}{6} + t$$

Clearly

$$\begin{aligned}\overline{F}(t,x,\overline{K}) &= \overline{K}xt\\ \overline{P}(x,\overline{\gamma}) &= \frac{\overline{\gamma}}{2}x^2\end{aligned}$$

so that

$$F_1(t,x,\alpha) = k_1(\alpha)xt, \quad F_2(t,x,\alpha) = k_2(\alpha)xt$$
$$P_1(x,\alpha) = \frac{\gamma_1(\alpha)}{2}x^2, \quad P_2(x,\alpha) = \frac{\gamma_2(\alpha)}{2}x^2$$

Also $\overline{G}(t,x,\overline{K},\overline{C},\overline{\gamma}) = \overline{C}x^2\cos(\sqrt{\overline{\gamma}}t) + \overline{K}x\frac{t^3}{6} + t$, therefore

$$z_i(t,x,\alpha) = c_i(\alpha)x^2\cos(\sqrt{\gamma_i(\alpha)}t) + k_i(\alpha)x\frac{t^3}{6} + t$$

for $i = 1, 2$ and $\overline{C} < 0$ ($\overline{C} = (c_1, c_2, c_3)$ also with $c_3 < 0$), $\overline{K}[\alpha] = [k_1(\alpha), k_2(\alpha)]$, $\overline{C}[\alpha] = [c_1(\alpha), c_2(\alpha)]$, and $\overline{\gamma}[\alpha] = [\gamma_1(\alpha), \gamma_2(\alpha)]$.
$\overline{Z}(t,x)$ is differentiable because $(z_i(t,x,\alpha))_{tt} + \frac{\gamma_i(\alpha)}{2}x^2(z_i(t,x,\alpha))_{xx}$ for $i = 1, 2$ are $\alpha-$cuts of $\overline{K}xt$ i.e. $\alpha-$cuts of a fuzzy number. Due to

$$P(x,\gamma) > 0$$
$$\frac{\partial G}{\partial k_1} > 0, \quad \frac{\partial F}{\partial k_1} > 0$$
$$\frac{\partial P}{\partial \gamma} > 0, \quad \frac{\partial G}{\partial \gamma} = -cx^2\frac{t}{2\sqrt{\gamma}}\sin(\sqrt{\gamma}t) > 0$$

That is, $(\overline{Z})_{tt} + \frac{\gamma}{2}x^2(\overline{Z})_{xx} = \overline{K}xt$, a fuzzy number.
So Lemma 5.3 implies the result that $\overline{Z}(t,x)$ is a BFS. We easily see that

$$z_i(0,x,\alpha) = c_i(\alpha)x^2 \quad \Big(z_i(0,x,\alpha)\Big)_t = 1$$

for $i = 1, 2$, so $\overline{Z}(t,x)$ also satisfies the initial condition. The BFS that satisfies the initial condition may be written as

$$\overline{Z}(t,x) = \overline{C}x^2\cos(\sqrt{\gamma}t) + \overline{K}x\frac{t^3}{6} + t$$

for all $t \in]0, \pi/2]$, $x \in [0,1]$

Example 6.2. Consider the two-dimensional wave-like equation with variable coefficients as

$$(6.2) \qquad \begin{cases} U_{tt} + \frac{\gamma}{2}x^2U_{xx} + \frac{\beta}{2}y^2U_{yy} = k_1x^2 - k_2y^2 \\ U(0,x,y) = c_1x^2 \\ \Big(U(0,x,y)\Big)_t = c_2y \end{cases}$$

which $t \in [\frac{3\pi}{2}, 2\pi]$, x, $y \in [0,1]$, $k_1 \in [J_1, 0[$, $k_2 \in]0, J_2]$, $\gamma \in [\frac{1}{2}, 1]$, $c_1 \in]0, L_1]$, $c_2 \in [0, L_2]$ and $\beta \in [\frac{1}{2}, 1]$
Similarly we can establish an iteration formula in the form

$$(6.3) \quad U_{n+1}(t,x,y) = U_n(t,x,y) + \int_0^t (s-t)\Big\{(U_n(s,x,y))_{ss} + \frac{\gamma}{2}x^2(\widetilde{U}_n(s,x,y))_{xx} + \frac{\beta}{2}y^2(\widetilde{U}_n(s,x,y))_{yy} - k_1x^2 + k_2y^2\Big\}ds$$

We begin with an initial arbitrary approximation :

$$U_0(t,x,y) = U(0,x,y) = c_1x^2 + c_2yt$$

and using the iteration formula (6.3), we obtain the following successive approximations

$U_1(t,x,y) = c_1x^2(1 - \gamma\frac{t^2}{2!}) + \frac{k_1}{\gamma}x^2(\frac{\gamma t^2}{2!}) - \frac{k_2}{\beta}y^2(\frac{\beta t^2}{2!}) + c_2yt$

$U_2(t,x,y) = c_1x^2(1 - \gamma\frac{t^2}{2!} + \gamma^2\frac{t^4}{4!}) + \frac{k_1}{\gamma}x^2(\frac{\gamma t^2}{2!} - \gamma^2\frac{t^4}{4!}) - \frac{k_2}{\beta}y^2(\frac{\beta t^2}{2!} - \beta^2\frac{t^4}{4!}) + c_2yt$

$U_3(t,x,y) = c_1x^2(1 - \gamma\frac{t^2}{2!} + \gamma^2\frac{t^4}{4!} - \gamma^3\frac{t^6}{6!}) + \frac{k_1}{\gamma}x^2(\gamma\frac{t^2}{2!} - \gamma^2\frac{t^4}{4!} + \gamma^3\frac{t^6}{6!}) - \frac{k_2}{\beta}y^2(\beta\frac{t^2}{2!} - \beta^2\frac{t^4}{4!} + \beta^3\frac{t^6}{6!}) + c_2yt$

and

$$U_n(t,x,y) = c_1x^2(1 - \gamma\frac{t^2}{2!} + \gamma^2\frac{t^4}{4!} + \ldots + (-1)^n\gamma^n\frac{t^{2n}}{2n!}) + \frac{k_1}{\gamma}x^2(\gamma\frac{t^2}{2!} - \gamma^2\frac{t^4}{4!} + (-1)^{n+1}\gamma^n\frac{t^{2n}}{2n!}) - \frac{k_2}{\beta}y^2(\beta\frac{t^2}{2!} - \beta^2\frac{t^4}{4!} + (-1)^{n+1}\beta^n\frac{t^{2n}}{2n!}) + c_2yt$$

Then, the exact solution is given by

$$U(t,x,y) = c_1x^2\cos(\sqrt{\gamma}t) + \frac{k_1}{\gamma}x^2\Big(1 - \cos(\sqrt{\gamma}t)\Big) - \frac{k_2}{\beta}y^2\Big(1 - \cos(\sqrt{\beta}t)\Big) + c_2yt$$

Fuzzify $F(t,x,k_1,k_2)$, $P(x,\gamma)$, $Q(y,\beta)$ and

$$G\Big(t,x,k_1,k_2,c,\gamma,\beta\Big) = c_1x^2\cos(\sqrt{\gamma}t) + \frac{k_1}{\gamma}x^2\Big(1-\cos(\sqrt{\gamma}t)\Big) - \frac{k_2}{\beta}y^2\Big(1-\cos(\sqrt{\beta}t)\Big) + c_2yt$$

producing their α-cuts

$$z_1(t,x,y,\alpha) = c_{11}(\alpha)x^2\cos(\sqrt{\gamma_1(\alpha)}t) + \frac{k_{11}(\alpha)}{\gamma_1(\alpha)}x^2(1-\cos(\sqrt{\gamma_1(\alpha)}t)) - \frac{k_{22}(\alpha)}{\beta_1(\alpha)}y^2(1-\cos(\sqrt{\beta_1(\alpha)}t)) + c_{21}(\alpha)yt$$

$$z_2(t,x,y,\alpha) = c_{12}(\alpha)x^2\cos(\sqrt{\gamma_2(\alpha)}t) + \frac{k_{12}(\alpha)}{\gamma_2(\alpha)}x^2(1-\cos(\sqrt{\gamma_2(\alpha)}t)) - \frac{k_{21}(\alpha)}{\beta_2(\alpha)}y^2(1-\cos(\sqrt{\beta_2(\alpha)}t)) + c_{22}(\alpha)yt$$

$$F_1(t,x,y,\alpha) = k_{11}(\alpha)x^2 - k_{22}(\alpha)y^2, \quad F_2(t,x,y,\alpha) = k_{12}(\alpha)x^2 - k_{21}(\alpha)y^2$$
$$P_1(x,\alpha) = \frac{\gamma_1(\alpha)}{2}x^2, \quad P_2(x,\alpha) = \frac{\gamma_2(\alpha)}{2}x^2$$
$$Q_1(x,\alpha) = \frac{\beta_1(\alpha)}{2}y^2, \quad Q_2(x,\alpha) = \frac{\beta_2(\alpha)}{2}y^2$$

where $\overline{K_1}[\alpha] = \Big[k_{11}(\alpha),k_{12}(\alpha)\Big]$, $\overline{K_2}[\alpha] = \Big[k_{21}(\alpha),k_{22}(\alpha)\Big]$, $\overline{C}_1[\alpha] = \Big[c_{11}(\alpha),c_{12}(\alpha)\Big]$, $\overline{C}_2[\alpha] = \Big[c_{21}(\alpha),c_{22}(\alpha)\Big]$, $\overline{\gamma}[\alpha] = \Big[\gamma_1(\alpha),\gamma_2(\alpha)\Big]$ and $\overline{\beta}[\alpha] = \Big[\beta_1(\alpha),\beta_2(\alpha)\Big]$.
We first check to see if $\overline{Z}(t,x,y)$ is differentiable. We compute

$$\Big[(z_1)_{tt} + \frac{\gamma_1(\alpha)}{2}x^2(z_1)_{xx} + \frac{\beta_1(\alpha)}{2}y^2(z_1)_{yy}, (z_2)_{tt} + \frac{\gamma_2(\alpha)}{2}x^2(z_2)_{xx} + \frac{\beta_2(\alpha)}{2}y^2(z_2)_{yy}\Big]$$

which are $\alpha-$cuts of $\overline{K_1}x^2 - \overline{K_2}y^2$ i.e. $\alpha-$cuts of a fuzzy number. Hence, $\overline{Z}(t,x,y)$ is differentiable.
Since

$$P(x,\gamma) > 0, \quad Q(y,\beta) > 0$$
$$\frac{\partial G}{\partial k} > 0, \quad \frac{\partial F}{\partial k} > 0$$
$$\frac{\partial P}{\partial \gamma} > 0, \quad \frac{\partial G}{\partial \gamma} = -\frac{ctx^2}{2\sqrt{\gamma}}\sin(\sqrt{\gamma}t) + (-\frac{k_1}{\beta^2}(1-\cos(\sqrt{\gamma}t)) + \frac{k_1t}{2\gamma\sqrt{\gamma}}\sin(\sqrt{\gamma}t))x^2 > 0$$
$$\frac{\partial Q}{\partial \beta} > 0, \quad \frac{\partial G}{\partial \beta} = (\frac{k_2}{\beta^2}(1-\cos(\sqrt{\beta}t)) - \frac{k_2t}{2\beta\sqrt{\beta}}\sin(\sqrt{\beta}t))y^2 > 0$$

Then Theorem (5.1) tells us that $\overline{Z}(t,x,y)$ is a BFS. The initial condition

$$z_i(0,x,y,\alpha) = c_{1i}(\alpha)x^2$$
$$(z_i(0,x,y,\alpha))_t = c_{2i}(\alpha)y$$

Therefore $\overline{Z}(t,x,y)$ is a BFS which also satisfies the initial condition. This BFS may be written

$$\overline{Z}(t,x,y) = \overline{C_1}x^2\cos(\sqrt{\overline{\gamma}}t) + \frac{\overline{k_1}}{\overline{\gamma}}x^2\Big(1-\cos(\sqrt{\overline{\gamma}}t)\Big) - \frac{\overline{k_2}}{\overline{\beta}}y^2\Big(1-\cos(\sqrt{\overline{\beta}}t)\Big) + \overline{C_2}yt$$

for all $(x,y)\in[0,1]$, $t\in[\frac{3\pi}{2},2\pi]$

Example 6.3. We consider the one-dimensional wave-like model

$$(6.4)\qquad \begin{cases} \partial_{tt}U(t,x) - \gamma x\partial_{xx}U(t,x) = kxt^2 \\ U(0,x) = 0 \\ \partial_t U(0,x) = cx^2 \end{cases}$$

which $t\in[0,1]$, $x\in]0,1]$, and the value of parameters k, c and γ are in intervals $[0,J]$, $[0,L_1]$ and $[L_2,0[$, respectively.
We can obtain the following iteration formula for the Eq.(6.4)

$$(6.5)\quad U_{n+1}(t,x) = U_n(t,x) + \int_0^t (s-t)\Big\{(U_n(s,x))_{ss} - \gamma x(\widetilde{U}_n(s,x))_{xx} - kxt^2\Big\}ds$$

We begin with an initial approximation : $U(0,x) = cx^2t$. By Eq (6.5), after than two iterations the exact solution is given in the closed form as

$$U(t,x) = G(t,x,k,c,\gamma) = cx^2t + c\gamma x\frac{t^3}{3} + kx\frac{t^4}{4!}$$

Since

$$P(x,\gamma) > 0$$

$$\frac{\partial G}{\partial k} > 0,\quad \frac{\partial F}{\partial k} > 0$$

$$\frac{\partial P}{\partial \gamma} < 0,\quad \frac{\partial G}{\partial \gamma} = cx\frac{t^3}{3} > 0$$

then there is no BFS (lemma (5.3)). We proceed to look for a SS. We must solve

$$(u_1(t,x,\alpha))_{tt} - \gamma_2(\alpha)x(u_1(t,x,\alpha))_{xx} = k_1(\alpha)xt^2$$

$$(u_2(t,x,\alpha))_{tt} - \gamma_1(\alpha)x(u_2(t,x,\alpha))_{xx} = k_2(\alpha)xt^2$$

subject to

$$u_i(0,x,\alpha) = c_i(\alpha)x^2t$$

for i= 1,2 and

$$\widetilde{K}[\alpha] = \Big[k_1(\alpha),k_2(\alpha)\Big],\quad \widetilde{C}[\alpha] = \Big[c_1(\alpha),c_2(\alpha)\Big],\quad \text{and } \overline{\gamma}[\alpha] = \Big[\gamma_1(\alpha),\gamma_2(\alpha)\Big].$$

By VIM, the solution is

$$u_1(t,x,\alpha) = c_1(\alpha)x^2t + c_1(\alpha)\gamma_2(\alpha)x\frac{t^3}{3} + k_1(\alpha)x\frac{t^4}{4!}$$

$$u_2(t,x,\alpha) = c_2(\alpha)x^2t + c_2(\alpha)\gamma_1(\alpha)x\frac{t^3}{3} + k_2(\alpha)x\frac{t^4}{4!}.$$

Now we show $\Big[u_1(t,x,\alpha),u_2(t,x,\alpha)\Big]$ defines $\alpha-$cut of a fuzzy number.
Thus we only need to check if $\frac{\partial u_1}{\partial\alpha} > 0$and $\frac{\partial u_2}{\partial\alpha} < 0$. Since $u_i(t,x,\alpha)$ are continuous

and $u_1(t,x,1) = u_2(t,x,1)$. There is a region $\mathfrak{R}$ contained in $[0,1]\times]0,1]$ for which the SS exists and in $[0,1]\times]0,1] - \mathfrak{R}$ there may be no SS.
Since $\overline{K}$, $\overline{C}$ and $\overline{\gamma}$ are triangular fuzzy numbers, hence, we pick simple fuzzy parameter so that $k_1^{'}(\alpha) = c_1^{'}(\alpha) = \gamma_1^{'}(\alpha) = \lambda$ and $k_2^{'}(\alpha) = c_2^{'}(\alpha) = \gamma_2^{'}(\alpha) = -\lambda$. Then, for the SS exists we need

$$\frac{\partial u_1}{\partial \alpha} = \lambda(x^2 t + \gamma_2(\alpha)x\frac{t^3}{3} - c_1(\alpha)x\frac{t^3}{3} + x\frac{t^4}{4!}) > 0$$
$$\frac{\partial u_2}{\partial \alpha} = -\lambda(x^2 t + \gamma_1(\alpha)x\frac{t^3}{3} - c_2(\alpha)x\frac{t^3}{3} + x\frac{t^4}{4!}) < 0$$

Therefore inequalities hold if

$$x^2 t + \gamma_1(\alpha)x\frac{t^3}{3} - c_2(\alpha)x\frac{t^3}{3} + x\frac{t^4}{4!} > 0 \tag{6.6}$$

for $t \in [0,1]$, $x \in]0,1]$. The inequality (6.6) holds if

$$0 \le t \le 1 \quad (c_2(\alpha) - \gamma_1(\alpha))\frac{t^2}{3} - \frac{t^3}{4!} < x \le 1 \quad for\ all\ \alpha \in [0,1]$$

So under the above assumptions we may choose

$$\mathfrak{R} = \left\{(t,x)|0 \le t \le 1 \quad (c_2(\alpha) - \gamma_1(\alpha))\frac{t^2}{3} - \frac{t^3}{4!} < x \le 1 \quad for\ all\ \alpha \in [0,1]\right\}$$

and the SS exists on $\mathfrak{R}$ in form Eqs.

$$\overline{U}(t,x) = \overline{C}x^2 t + \overline{C}\overline{\gamma}x\frac{t^3}{3} + \overline{K}x\frac{t^4}{4!}$$

for all $t \in [0,1]$, $x \in]0,1]$.

Example 6.4. We consider the one-dimensional wave-like model

$$\begin{cases} \partial_{tt}U(t,x) + \gamma x \partial_{xx}U(t,x) = -kx^2 \\ U(0,x) = c\sin(x) \end{cases} \tag{6.7}$$

which $t \in [0,1]$, $x \in [0,\pi]$,and the value of parameters k, c and γ are in intervals $[0,J]$, $[0,L]$ and $]0,H]$, respectively.
We can obtain the following iteration formula for the Eq.(6.7)

$$U_{n+1}(t,x) = U_n(t,x) + \int_0^t (s-t)\left\{(U_n(s,x))_{ss} + \gamma x(\widetilde{U}_n(s,x))_{xx} + kx^2\right\}ds \tag{6.8}$$

We begin with an initial approximation : $U(0,x) = c\sin(x)$. By Eq (6.8), after than two iterations the exact solution is given in the closed form as

$$U(t,x) = G(t,x,k,c,\gamma) = c\sin(x) + cx\sin(x)(\cosh(\sqrt{\gamma}t) - 1) + \gamma kx\frac{t^4}{12} - x^2\frac{t^2}{2}$$

since $\frac{\partial F}{\partial k} = -x^2 < 0$ and $\frac{\partial G}{\partial k} = \gamma x\frac{t^4}{12} - x^2\frac{t^2}{2} > 0$ for

$$\sqrt{\frac{6x}{\gamma}} < t \le 1 \text{ and } 0 < x < \frac{\gamma}{6}$$

then there is no BFS (lemma (5.3)). We proceed to look for a SS. We must solve

$$(u_1(t,x,\alpha))_{tt} + \gamma_1(\alpha)x(u_1(t,x,\alpha))_{xx} = -k_2(\alpha)x^2$$
$$(u_2(t,x,\alpha))_{tt} + \gamma_2(\alpha)x(u_2(t,x,\alpha))_{xx} = -k_1(\alpha)x^2$$

subject to

$$u_i(0,x,\alpha) = c_i(\alpha)\sin(x)$$

for $i = 1, 2$ and

$$\widetilde{k}[\alpha] = \Big[k_1(\alpha), k_2(\alpha)\Big], \quad \widetilde{c}[\alpha] = \Big[c_1(\alpha), c_2(\alpha)\Big] \text{ and } \overline{\gamma}[\alpha] = \Big[\gamma_1(\alpha), \gamma_2(\alpha)\Big].$$

By VIM, the solution is

$$\begin{aligned}(6.9)\quad u_1(t,x,\alpha) &= c_1(\alpha)\sin(x) \\ &+ c_1(\alpha)x\sin(x)\Big(\cosh(\sqrt{\gamma_1(\alpha)})t - 1\Big) + \gamma_1(\alpha)k_2(\alpha)x\frac{t^4}{12} - k_2(\alpha)x^2\frac{t^2}{2}\end{aligned}$$

$$\begin{aligned}u_2(t,x,\alpha) &= c_2(\alpha)\sin(x) \\ &+ c_2(\alpha)x\sin(x)\Big(\cosh(\sqrt{\gamma_2(\alpha)})t - 1) + \gamma_2(\alpha)k_1(\alpha)x\frac{t^4}{12} - k_1(\alpha)x^2\frac{t^2}{2}.\end{aligned}$$

Since $u_i(t,x,\alpha)$ are continuous and $u_1(t,x,1) = u_2(t,x,1)$ then we only require to check if $\frac{\partial u_1}{\partial \alpha} > 0$, $\frac{\partial u_2}{\partial \alpha} < 0$ and $\overline{K}$, $\overline{C}$, $\overline{\gamma}$ are triangular fuzzy numbers, hence, we pick simple fuzzy parameter so that $k_1^{'}(\alpha) = c_1^{'}(\alpha) = \gamma_1^{'}(\alpha) = \lambda$ and $k_2^{'}(\alpha) = c_2^{'}(\alpha) = \gamma_2^{'}(\alpha) = -\lambda$. Then, for the SS exists we need

$$\begin{aligned}\frac{\partial u_1}{\partial \alpha} &= \lambda\Big(\sin(x) + x\sin(x)\Big(\cosh(\sqrt{\gamma_1(\alpha)})t - 1\Big) \\ &+ c_1(\alpha)\frac{t}{2\sqrt{\gamma_1(\alpha)}}x\sin(x)\sinh\left(\sqrt{\gamma_1(\alpha)}\,t\right) - \gamma_1(\alpha)x\frac{t^4}{12} + k_2(\alpha)x\frac{t^4}{12} + x^2\frac{t^2}{2}\Big) > 0.\end{aligned}$$

$$\begin{aligned}\frac{\partial u_2}{\partial \alpha} &= -\lambda\Big(\sin(x) + x\sin(x)\Big(\cosh(\sqrt{\gamma_2(\alpha)})t - 1\Big) \\ &+ c_2(\alpha)\frac{t}{2\sqrt{\gamma_2(\alpha)}}x\sin(x)\sinh\left(\sqrt{\gamma_2(\alpha)}t\right) - \gamma_2(\alpha)x\frac{t^4}{12} + k_2(\alpha)x\frac{t^4}{12} + x^2\frac{t^2}{2}\Big) < 0.\end{aligned}$$

Therefore inequalities hold if

$$\begin{aligned}(6.10)\quad &\sin(x) + x\sin(x)\Big(\cosh(\sqrt{\gamma_1(\alpha)})t - 1\Big) \\ &+ c_1(\alpha)\frac{t}{2\sqrt{\gamma_1(\alpha)}}x\sin(x)\sinh\left(\sqrt{\gamma_1(\alpha)}t\right) - \gamma_1(\alpha)x\frac{t^4}{12} + k_2(\alpha)x\frac{t^4}{12} + x^2\frac{t^2}{2} > 0\end{aligned}$$

$$\begin{aligned}(6.11)\quad &\sin(x) + x\sin(x)\Big(\cosh(\sqrt{\gamma_2(\alpha)})t - 1\Big) \\ &+ c_2(\alpha)\frac{t}{2\sqrt{\gamma_2(\alpha)}}x\sin(x)\sinh\left(\sqrt{\gamma_2(\alpha)}t\right) - \gamma_2(\alpha)x\frac{t^4}{12} + k_1(\alpha)x\frac{t^4}{12} + x^2\frac{t^2}{2} > 0\end{aligned}$$

it is sufficient that

$$-\gamma_2(\alpha)x\frac{t^4}{12}+k_1(\alpha)x\frac{t^4}{12}+x^2\frac{t^2}{2}>0 \tag{6.12}$$

for $t\in[0,1]$, $x\in(0,\pi]$. The inequality (6.12) holds if

$$0\le t\le 1 \quad (\gamma_2(\alpha)-k_1(\alpha))\frac{t^2}{6}<x\le\pi \quad \text{for all} \quad \alpha\in[0,1]$$

So under the above assumptions we may choose

$$\Re=\left\{(t,x)|0\le t\le 1 \quad (\gamma_2(\alpha)-k_1(\alpha))\frac{t^2}{6}<x\le\pi \quad for\ all\ \alpha\in[0,1]\right\}$$

and the SS exists on $\Re$ in form Eqs.(6.9).

Example 6.5. We consider the one-dimensional wave-like model

$$\begin{cases} U_{tt}(t,x)-\gamma U_{xx}=-k \\ U(0,x)=0 \\ U_t(0,x)=c\exp(x) \end{cases}$$

which $x\in[0,1]$, $t\in]0,\frac{1}{2}]$ and the value of parameters k, c and γ are in intervals $[0,J]$, $[0,10]$ and $]0,10]$ respectively.
We can obtain the following iteration formula

$$U_{n+1}(t,x)=U_n(t,x)+\int_0^t(s-t)\{(U_n)_{ss}(s,x)-\gamma(\widetilde{U}_n)_{xx}(s,x)+k\}ds \tag{6.13}$$

We begin with an initial approximation : $U_0(t,x)=U(0,x)=c\exp(x)t$. By (6.13), the following successive approximation are obtained

$$\begin{aligned}
&U_0(t,x)=U(0,x)=c\exp(x)t\\
&U_1(t,x)=\tfrac{c}{\sqrt{\gamma}}\exp(x)(t+(\sqrt{\gamma})^3\tfrac{t^3}{3!})-k\tfrac{t^2}{2}\\
&U_2(t,x)=\tfrac{c}{\sqrt{\gamma}}\exp(x)(t+(\sqrt{\gamma})^3\tfrac{t^3}{3!}+(\sqrt{\gamma})^5\tfrac{t^5}{5!})-k\tfrac{t^2}{2}\\
&\vdots\\
&U_n(t,x)=\tfrac{c}{\sqrt{\gamma}}\exp(x)(t+(\sqrt{\gamma})^3\tfrac{t^3}{3!})+...+(\sqrt{\gamma})^{2n+1}\tfrac{t^{2n+1}}{2n+1!})-k\tfrac{t^2}{2},\ n\ge 1
\end{aligned}$$

The VIM admits the use of $U(t,x)=\lim_{n\to\infty}U_n(t,x)$

$$U(t,x)=G(t,x,k,c,\gamma)=\frac{c}{\sqrt{\gamma}}\exp(x)\sinh(\sqrt{\gamma}t)-k\frac{t^2}{2}$$

which gives the exact solution. There is no BFS because $P(x,\gamma)=-\gamma<0$ with $\gamma\in]0,10]$ (lemma (5.3)). We proceed to look for a SS. We must solve

$$(u_1(t,x,\alpha))_{tt}-\gamma_2(u_2(t,x,\alpha))_{xx}=-k_2(\alpha)$$
$$(u_1(t,x,\alpha))_{tt}-\gamma_1(u_1(t,x,\alpha))_{xx}=-k_1(\alpha)$$

subject to $u_i(0,x,\alpha)=c_i(\alpha)\exp(x)t$ for $i=1,2$
and $\overline{K}[\alpha]=[k_1(\alpha),k_2(\alpha)]$, $\overline{C}[\alpha]=[c_1(\alpha),c_2(\alpha)]$ and $\overline{\gamma}[\alpha]=[\gamma_1(\alpha),\gamma_2(\alpha)]$.

We note

$$\xi_1 = \frac{c_1(\alpha)}{\sqrt[4]{\gamma_1(\alpha)\gamma_2(\alpha)}}, \quad \zeta_1 = \frac{c_2(\alpha)\sqrt[4]{\gamma_1(\alpha)\gamma_2(\alpha)}}{\gamma_1(\alpha)}$$
$$\xi_2 = \frac{c_2(\alpha)}{\sqrt[4]{\gamma_1(\alpha)\gamma_2(\alpha)}}, \quad \zeta_2 = \frac{c_1(\alpha)\sqrt[4]{\gamma_1(\alpha)\gamma_2(\alpha)}}{\gamma_2(\alpha)}$$

The solution is

$$(6.14) \quad u_1(t,x,\alpha) = \xi_1 \frac{\exp(x)}{2}\Big(\sinh\Big(\sqrt[4]{\gamma_1(\alpha)\gamma_2(\alpha)}t\Big) + \sin\Big(\sqrt[4]{\gamma_1(\alpha)\gamma_2(\alpha)}t\Big)\Big) + \zeta_1 \frac{\exp(x)}{2}\Big(\sinh\Big(\sqrt[4]{\gamma_1(\alpha)\gamma_2(\alpha)}t\Big) - \sin\Big(\sqrt[4]{\gamma_1(\alpha)\gamma_2(\alpha)}t\Big)\Big) - k_2(\alpha)\frac{t^2}{2}$$

$$u_2(t,x,\alpha) = \xi_2 \frac{\exp(x)}{2}\Big(\sinh\Big(\sqrt[4]{\gamma_1(\alpha)\gamma_2(\alpha)}t\Big) + \sin\Big(\sqrt[4]{\gamma_1(\alpha)\gamma_2(\alpha)}t\Big)\Big) + \zeta_2 \frac{\exp(x)}{2}\Big(\sinh\Big(\sqrt[4]{\gamma_1(\alpha)\gamma_2(\alpha)}t\Big) - \sin\Big(\sqrt[4]{\gamma_1(\alpha)\gamma_2(\alpha)}t\Big)\Big) - k_1(\alpha)\frac{t^2}{2}$$

We only need to check if $\frac{\partial u_1}{\partial \alpha} > 0$ and $\frac{\partial u_2}{\partial \alpha} < 0$, since the $u_i(t,x,\alpha)$ are continuous and $u_1(t,x,1) = u_2(t,x,1)$.
We pick simple fuzzy parameter $k_1'(\alpha) = c_1'(\alpha) = \gamma_1'(\alpha) = \lambda > 0$ and $k_2'(\alpha) = c_2'(\alpha) = \gamma_2'(\alpha) = -\lambda$.
Let $w = \sqrt[4]{\gamma_1(\alpha)\gamma_2(\alpha)}$, $s = \gamma_1(\alpha)\gamma_2(\alpha)$ and $b = \gamma_2(\alpha) - \gamma_1(\alpha)$
Now we need to check if $\frac{\partial u_1}{\partial \alpha} > 0$ and $\frac{\partial u_2}{\partial \alpha} < 0$, for all $t \in]0, \frac{1}{2}]$.
We note

$$\eta_1 = \frac{\Big(-4s\gamma_1(\alpha) + c_2(\alpha)\gamma_1(\alpha)b - 4c_2(\alpha)s\Big)}{4w^3\Big(\gamma_1(\alpha)\Big)^2}$$
$$\eta_2 = \frac{\Big(-4s\gamma_2(\alpha) - c_1(\alpha)\gamma_2(\alpha)b - 4c_1(\alpha)s\Big)}{4w^3\Big(\gamma_2(\alpha)\Big)^2}$$

$$\frac{\partial u_1}{\partial \alpha} = \frac{\lambda}{2}\Big(\Big((\frac{4s - c_1(\alpha)b}{4w^5})\Big(\sinh(wt) + \sin(wt)\Big) + \eta_1\Big(\sinh(wt) - \sin(wt)\Big) + \frac{c_1(\alpha)b}{4s}t\Big(\cosh(wt) + \cos(wt)\Big) + \frac{c_2(\alpha)b}{4\gamma_1(\alpha)\sqrt{s}}t\big(\cosh(wt) - \cos(wt)\big)\Big)\exp(x) + t^2\Big) > 0$$

$$(6.15)\quad \frac{\partial u_2}{\partial \alpha} = \frac{-\lambda}{2}\Bigg(\Big(\big(\frac{4s + c_2(\alpha)b}{4w^5}\big)\big(\sinh(wt) + \sin(wt)\big) + \eta_2\Big(\sinh(wt) - \sin(wt)\Big) - \frac{c_2(\alpha)b}{4s}t\Big(\cosh(wt) + \cos(wt)\Big) - \frac{c_1(\alpha)b}{4\gamma_2(\alpha)\sqrt{s}}t\big(\cosh(wt) - \cos(wt)\big)\Big)\exp(x) + t^2\Bigg) < 0$$

Since (6.15) holds for each $t \in]0, \frac{1}{2}]$, $x \in [0, 1]$, $c \in [0, 10]$ and $\gamma \in]0, 10]$, therefore, $\overline{U}(t, x)$ is SS in form Eqs.(6.14), for all $t \in]0, \frac{1}{2}]$, $x \in [0, 1]$, $c \in [0, 10]$ and $\gamma \in]0, 10]$

7. Conclusion

In this paper, we give sufficient condition for the Buckley-Feuring solution to exist by the VIM for the proposed models , we obtain the exact solution of various kinds of fuzzy wave-like equations. Application of this method is easy and calculation of successive approximations is direct and straightforward. We using the VIM and strategy based on [5] introduced two type of solutions, the Buckley-Feuring solution and the Seikkala solution. If the Buckley-Feuring solution fails to exist and when the Seikkala solution fails to exist we offer no solution to the fuzzy wave-like equations.

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