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A comparison of two formulations of soft compactness

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Abstract. When we try to generalize the concept of compactness to soft topological spaces, we encounter the following problem: Given a subset of the initial universe, when should we say that the subset is soft compact? In this paper, we give two definitions of soft compactness. The two concepts are then compared in relation with several soft topological concepts such as soft closedness, soft continuity and soft product.

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1. Introduction

We are surrounded by vagueness and uncertainties, which make our understanding of physical objects incomplete. Unfortunately, in most cases, it is almost impossible to eliminate vagueness and uncertainties. It is therefore very important to have a mathematical framework which allows us to handle problems precisely even in the presence of vagueness and uncertainties.

Such a mathematical framework should be sufficiently universal, i.e., useful in solving wide range of problems. In order to provide such a system, a great deal of efforts have been made. Examples include the theory of Fuzzy sets by Zadeh [11] and the theory of Rough sets by Pawlak [8]. The theory of soft sets is another mathematical tool to deal with uncertainties introduced by Molodtsov in 1999 [7]. It aims at modeling wide range of problems from, e.g, physical science, economics and engineering. Indeed, Molodtsov [7] has already presented applications of his theory to operations research, game theory, probability theory and so on.

The definition of soft set is simple, so the theory of soft sets finds lots of areas for its application. From theoretical point of view, its mathematical aspects are of interest as well. For these reasons, the research of the theory of soft sets is becoming more and more active.

Maji et al. [6] defined and studied basic notions of soft set theory. The study of soft topology was started by Shabir and Naz [9], independently by Çağman et al. [2]. The purpose of this paper is to obtain a deeper understanding of soft topology.

Since compactness is a very important concept in topology, it is natural to think about its soft analogue. But then we face the following question: Given a subset of the initial universe, when should we say the subset is soft compact? In this paper, we give two different definitions of soft compactness, and compare them in relation with several soft topological properties.

2. Preliminaries

Definition 2.1 ([7]). Let U be an initial universe and E be a set of parameters. Then a *soft set* over U is a function $F : E \to \mathcal{P}(U)$.

Definition 2.2. Let $\phi : U \to U'$ be a function and F (resp. F') be a soft set over U (resp. U') with its parameter set E. Then $\phi(F)$ (resp. $\phi^{-1}(F')$) is the soft set on U' (resp. U) defined by $(\phi(F))(e) = \phi(F(e))$ (resp. $(\phi^{-1}(F))(e) = \phi^{-1}(F(e))$).

We will identify a soft set $F : E \to \mathcal{P}(U)$ with a subset of $E \times U$. We shall use symbols $F, F', \dots$ for soft sets.

Soft versions of basic relations (resp. operations) on sets are obtained by requiring the relations (resp. applying the operations) at each parameter.

Definition 2.3 ([6]). Let F and F' be soft sets over U. Then

- (Soft subset) F is a *soft subset* of F', denoted by $F \tilde{\subset} F'$, if $F(e)$ is a subset of $F'(e)$ for every $e \in E$.
  In this case, we also say that F' is a *soft superset* of F.
- (Soft equality) F is *soft equal* to F', denoted by $F \tilde{=} F'$, if both $F \tilde{\subset} F'$ and $F' \tilde{\subset} F$ hold.
- (Soft intersection) The *soft intersection* of F and F', denoted by $F \tilde{\cap} F'$, is defined by $(F \tilde{\cap} F')(e) = F(e) \cap F'(e)$ for every $e \in E$.
- (Soft union) The *soft union* of F and F', denoted by $F \tilde{\cup} F'$, is defined by $(F \tilde{\cup} F')(e) = F(e) \cup F'(e)$ for every $e \in E$.
- (Soft complement) The *soft complement* of F, denoted by $F^{\tilde{c}}$, is defined by $F^{\tilde{c}}(e) = U \setminus F(e)$ for every $e \in E$.

Here and subsequently, the tilde $\widetilde{(\cdot)}$ is used to distinguish "soft" objects from classical (usual) ones.

For properties of these relations and operations, we refer the reader to [2].

Definition 2.4. Let X be a subset of U. Then $\tilde{X}$ is the soft set given by $\tilde{X}(e) = X$ for all parameters $e \in E$.

Therefore we have, for example, $E \times U \tilde{=} \tilde{U}$, $\tilde{X}^{\tilde{c}} \tilde{=} \widetilde{(X^c)}$ and $\widetilde{X \cup Y} \tilde{=} \tilde{X} \tilde{\cup} \tilde{Y}$.

Definition 2.5. Let x be an element of U and F be a soft set over U. We say that x is a *soft element* of F, denoted by $x \tilde{\in} F$, if $x \in F(e)$ holds for every $e \in E$.

The reader should keep in mind that $x \tilde{\notin} F$ and $\forall e \in E\,(x \notin F(e))$ are different in general. This simple fact plays an important role in the theory of soft sets.

3. Soft topology

This section introduces several soft topological concepts and studies their basic properties.

Definition 3.1 ([2, 9]). A family τ of soft sets over U is called a *soft topology* on U if the following three conditions are satisfied:

- $\tilde{\varnothing}$ and $\tilde{U}$ are in τ,
- τ is closed under finite soft intersection,
- τ is closed under (arbitrary) soft union.

We refer to a triplet $\langle U, \tau, E\rangle$ as a *soft topological space*. Each member of τ is called a *soft open* set. Throughout this paper, $\langle U, \tau, E\rangle$ stands for a soft topological space.

Definition 3.2 ([9]). Let x be an element of the universe U. A soft set F is called a *soft neighborhood* of x if there exists a soft open set F' such that $x \tilde{\in} F' \tilde{\subset} F$.

Definition 3.3. Let F be a soft set over U. Then

- (Soft closure [9]) The *soft closure* of F, denoted by $\widetilde{\mathrm{Cl}}(F)$, is the soft intersection of all soft closed supersets of F.
- (Soft interior [12]) The *soft interior* of F, denoted by $\widetilde{\mathrm{Int}}(F)$, is the soft union of all soft open subsets of F.

(Classical) topological concepts, such as boundary and limit points, are generalized to the setting of soft sets in a natural way. Interested reader will find more information from [2, 4, 9].

There are generalizations also of separation axioms.

Definition 3.4. Let $\langle U, \tau, E\rangle$ be a soft topological space. Then

- (Soft Hausdorff space [9]) $\langle U, \tau, E\rangle$ is called a *soft Hausdorff space* provided that there exist, for each pair $x, y \in U$ of distinct points, soft open sets F_x, F_y satisfying $x \tilde{\in} F_x, y \tilde{\in} F_y$ and $F_x \tilde{\cap} F_y \tilde{=} \tilde{\varnothing}$.
- (Soft normal space) $\langle U, \tau, E\rangle$ is called a *soft normal* if for every disjoint $X, Y \subset U$ with $\tilde{X}, \tilde{Y}$ soft closed, there exist soft open sets F_X, F_Y such that $\tilde{X} \tilde{\subset} F_X, \tilde{Y} \tilde{\subset} F_Y$ and $F_X \tilde{\cap} F_Y \tilde{=} \tilde{\varnothing}$.

Remark 3.5. The reader should note that some literature, including [9], uses the terminology “soft normal space” for a different concept.

It is interesting to see whether or not familiar results about Hausdorff spaces hold also for soft Hausdorff spaces. Our first example of such a result is related to product, so we give a definition of soft product here:

Definition 3.6. Let $\langle U, \tau, E\rangle, \langle U', \tau', E\rangle$ be soft spaces. Then their *soft product* is the soft space $\langle U \times U', \tau_\times, E\rangle$, where $\tau_\times$ is the soft topology on $U \times U'$ generated by $\{F \times F' \mid F \in \tau, F' \in \tau'\}$.

In the above definition, $F \times F'$ is the Cartesian product of F and F' defined by $(F \times F')(e) := F(e) \times F'(e)$ $(\forall e \in E)$.

Remark 3.7. Some literature (e.g., [1, 10]) employ a different definition of the Cartesian product of soft topological spaces: If $F : E \to X$ and $F' : E' \to X'$ are soft sets, then $F \times F'$ is a soft set over $X \times X'$ with its parameter set $E \times E'$ given by $(F \times F')(e, e') = F(e) \times F'(e')$.

Proposition 3.8. *Let $\langle U, \tau, E\rangle$ and $\langle U', \tau', E\rangle$ be soft spaces.*
(i) Both $\langle U, \tau, E\rangle$ and $\langle U', \tau', E\rangle$ are a soft Hausdorff space $\Longrightarrow$ Their soft product is a soft Hausdorff space.
(ii) The converse of the above implication is not true.

Proof. (i) Take distinct $(x, y), (x', y')$ from $U \times U'$. Without loss of generality, we assume $x \neq x'$. Then, by assumption, there exist disjoint soft open sets F_x and $F_{x'}$ separating x and x'. It is then clear that $F_x \times \tilde{U}'$ and $F_{x'} \times \tilde{U}'$ are disjoint soft open sets separating (x, y) from (x', y').
(ii) Put $\mathcal{S}_1 := \langle \mathbb{Z}_2, \tau, E\rangle$ and $\mathcal{S}_2 := \langle \{u\}, \mathcal{P}(E \times \{u\}), E\rangle$, where $E := \{e_1, e_2\}$ and $\tau := \{\tilde{\varnothing}, \{(e_1, \bar{0}), (e_2, \bar{1})\}, \{(e_1, \bar{1}), (e_2, \bar{0})\}, \widetilde{\mathbb{Z}_2}\}$. An easy computation shows that the soft product of $\mathcal{S}_1$ and $\mathcal{S}_2$ is soft Hausdorff. However, no soft open sets from τ separate $\bar{0}$ from $\bar{1}$. So, $\mathcal{S}_1$ is not a soft Hausdorff space. □

Remark 3.9. Varol and Aygün proved that U and U' are soft Hausdorff if and only if $U \times U'$ is a soft Hausdorff space [10, Theorem 3.29]. Our result here is not in conflict with their result, as our definition of soft product is different from their's. (cf. Remark 3.7)

Our next example of a well-known result on Hausdorff space is the following:

Proposition 3.10. *For any topological space X, the following are equivalent:*
(i) X is a Hausdorff space;
(ii) Δ is closed, where $\Delta := \{(x, x) \mid x \in X\}$ is the diagonal set;
(iii) For every $x \in X$, we have $\{x\} = \bigcap\{C \mid C$ is closed and $x \in \mathrm{Int}(C)\}$. □

The relationship between soft versions of these three conditions is more complicated:

Theorem 3.11. *The soft versions of the above conditions*
(i) $\langle U, \tau, E\rangle$ is a soft Hausdorff space;
(ii) $\tilde{\Delta}$ is soft closed;
(iii) For every $x \in U$, we have $\widetilde{\{x\}} = \tilde{\bigcap}\{F \mid F$ is soft closed and $x \,\tilde{\in}\, \widetilde{\mathrm{Int}}(F)\}$;
are related as follows:

- *(i) implies (ii) and (iii);*
- *(ii) implies neither (i) nor (iii);*
- *(iii) implies (ii);*
- *If the parameter set E is finite, (iii) implies (i). If E is infinite, (iii) does not imply (i).*

Proof.
• For any distinct $x, y \in U$, take disjoint soft open sets F_x, F_y such that $x \,\tilde{\in}\, F_x$ and $y \,\tilde{\in}\, F_y$. Clearly $\tilde{\Delta} \,\tilde{\cap}\, (F_x \times F_y) \,\tilde{=}\, \tilde{\varnothing}$ holds, as F_x and F_y are disjoint. Since $F_x \times F_y$ is a soft open neighborhood of (x, y), we conclude that $\tilde{\Delta}$ is soft closed.

We then deduce (iii) from (i). For each y from $U \setminus \{x\}$, pick up a pair F_x, F_y of disjoint soft open sets with $y \tilde{\in} F_y, x \tilde{\in} F_x$. Then $F_y^{\tilde{c}}$ is a soft closed neighborhood of x, so $\widetilde{\bigcap}\{F \mid F \text{ is soft closed and } x \tilde{\in} \widetilde{\mathrm{Int}}(F)\} \tilde{\subset} F_y^{\tilde{c}}$ holds for every y. Hence

$$\widetilde{\bigcap}\{F \mid F \text{ is soft closed and } x \tilde{\in} \widetilde{\mathrm{Int}}(F)\} \tilde{\subset} \widetilde{\bigcap}_{y \in U \setminus \{x\}} F_y^{\tilde{c}} \tilde{=} \widetilde{\{x\}}.$$

Since the other inclusion is obvious, the proof is now complete.

• Since we have proven that (i) implies (iii), it suffices to show that (ii) does not imply (iii). For this purpose, consider a soft topological space $\mathcal{S} := \langle \mathbb{Z}_2, \tau, E \rangle$, where $E := \{e_1, e_2\}$ and $\tau := \{\tilde{\varnothing}, \{(e_1, \bar{0}), (e_2, \bar{1})\}, \{(e_1, \bar{1}), (e_2, \bar{0})\}, \widetilde{\mathbb{Z}_2}\}$. In the soft product of $\mathcal{S}$ and $\mathcal{S}$, one can readily check that $\tilde{\Delta}$ is soft closed. An easy computation shows that $\widetilde{\bigcap}\{F \mid F \text{ is soft closed and } x \tilde{\in} \widetilde{\mathrm{Int}}(F)\} \tilde{=} E \times \mathbb{Z}_2$ holds for both $x = \bar{0}$ and $\bar{1}$. Therefore, this soft space does not satisfy the condition (iii).

• Take distinct x, y from the universe U. Since $\langle U, \tau, E \rangle$ satisfies (iii) by assumption, there exist, for each $e \in E$, a soft closed set F_e such that $x \tilde{\in} \widetilde{\mathrm{Int}}(F_e)$ and $y \notin F_e(e)$. If we put $F' := \widetilde{\bigcup}_{e \in E} \left(\widetilde{\mathrm{Int}}(F_e) \times F_e^{\tilde{c}} \right)$, then F' is a soft open neighborhood of (x, y) and $\tilde{\Delta} \tilde{\cap} F' \tilde{=} \tilde{\varnothing}$. As x and y were arbitrary, we now conclude that $\tilde{\Delta}$ is soft closed.

• Firstly, suppose that E is finite. Take distinct x, y. For each $e \in E$, take a soft closed neighborhood F_e of x such that $y \notin F_e(e)$. Then, since E is finite, $\widetilde{\bigcap}_{e \in E} F_e$ is a soft closed neighborhood of x such that $y \notin (\widetilde{\bigcap}_{e \in E} F_e)(e)$ for every $e \in E$. $\widetilde{\mathrm{Int}}(\widetilde{\bigcap}_{e \in E} F_e)$ and $(\widetilde{\bigcap}_{e \in E} F_e)^{\tilde{c}}$ are soft open sets which soft separate x and y. Hence we conclude that (iii) implies (i).

Next we deal with the case where E is infinite: Let us take $U := [0, 1) \cup \{2, 3\}$ and $E := \mathbb{N}$. We induce a soft topology τ on U using the following subbase:

$$\begin{aligned}
&\{\tilde{\varnothing}, \tilde{U}\} \cup \{\widetilde{\{r\}} \mid 0 \leq r < 1\} \\
&\cup \{\{(m, s) \mid r < s < 1, n \leq m\} \tilde{\cup} \widetilde{\{2\}} \mid 0 \leq r < 1, n \in \mathbb{N}\} \\
&\cup \{\{(m, s) \mid r < s < 1, n \leq m\} \tilde{\cup} \widetilde{\{3\}} \mid 0 \leq r < 1, n \in \mathbb{N}\} \\
&\cup \{\{(m, s) \mid r < s < 1, m \leq n\} \tilde{\cup} \{(m, 2) \mid m \leq n\} \mid 0 \leq r < 1, n \in \mathbb{N}\} \\
&\cup \{\{(m, s) \mid r < s < 1, m \leq n\} \tilde{\cup} \{(m, 3) \mid m \leq n\} \mid 0 \leq r < 1, n \in \mathbb{N}\}.
\end{aligned}$$

One can check at once that there is no pair of disjoint soft open sets that separate 2 from 3. Thus this soft space is not a soft Hausdorff space. Now we claim that the condition (iii) holds for this soft space. Since this claim is readily verified at any point $r \in [0, 1)$, we prove it only for the point 2. $\widetilde{\{2\}}$ is clearly contained in $\widetilde{\bigcap}\{F \mid F \text{ is soft closed and } 2 \tilde{\in} \widetilde{\mathrm{Int}}(F)\}$, so our claim follows from the next lemma.

Lemma 3.12. *$\widetilde{\bigcap}\{F \mid F$ is soft closed and $2 \tilde{\in} \widetilde{\mathrm{Int}}(F)\}$ is a soft subset of $\widetilde{\{2\}}$.*

Proof. Note first that, for any $r \in [0, 1)$ and $n \in \mathbb{N}$, the following soft set is soft closed:

$$\{(m, s) \mid r < s < 1, n \leq m\} \tilde{\cup} \widetilde{\{2\}} \tilde{\cup} \{(m, 3) \mid n \leq m\}.$$

This soft set clearly contains the following soft open neighborhood of 2:

$$\{(m, s) \mid r < s < 1, n \leq m\} \tilde{\cup} \widetilde{\{2\}}.$$

Therefore, we see that $\widetilde{\bigcap}\{F \mid F \text{ is soft closed and } 2 \mathrel{\tilde{\in}} \widetilde{\mathrm{Int}}(F)\}$ is a soft subset of $\{(m, s) \mid n \le m, r < s < 1\} \mathbin{\tilde{\cup}} \widetilde{\{2\}} \mathbin{\tilde{\cup}} \{(m, 3) \mid n \le m\}$ for any $r \in [0, 1)$ and $n \in \mathbb{N}$. Observe that, for any $x \ne 2$ and $i \in \mathbb{N}$, there exist sufficiently large $n \in \mathbb{N}$ and r close enough to 1 so that

$$(i, x) \mathrel{\tilde{\notin}} \{(m, s) \mid n \le m, r < s < 1\} \mathbin{\tilde{\cup}} \widetilde{\{2\}} \mathbin{\tilde{\cup}} \{(m, 3) \mid n \le m\}.$$

It is then easy to see that $\widetilde{\bigcap}\{F \mid F \text{ is soft closed and } 2 \mathrel{\tilde{\in}} \widetilde{\mathrm{Int}}(F)\} \mathrel{\tilde{\subset}} \widetilde{\{2\}}$. □

A similar argument proves that the condition (iii) holds at the point 3 as well. Hence we have proved the claim. □

4. Soft compactness

Given (not a soft set but) a subset X of the initial universe U, when should we say that X is soft compact? — In this section, we first give two definitions of soft compactness. We then make a comparative study of the two definitions in relation with several soft topological concepts introduced so far.

Definition 4.1. Let X be a subset of the universe U, and $\mathcal{C}$ be a family of soft open sets over U. Then

- **(First definition):** $\mathcal{C}$ is a *soft covering* (SCV1) of X if for every $x \in X$ there exists an $F \in \mathcal{C}$ satisfying $x \mathrel{\tilde{\in}} F$.
- **(Second definition):** $\mathcal{C}$ is a *soft covering* (SCV2) of X if it satisfies $\widetilde{\bigcup}\mathcal{C} \mathrel{\tilde{=}} \tilde{X}$.

Definition 4.2.

- **(First definition):** A subset $X \subset U$ is called *soft compact* (SCPT1) if every SCV1 of X has a finite subfamily which is again an SCV1 of X.
- **(Second definition):** A subset $X \subset U$ is called *soft compact* (SCPT2) if every SCV2 of X has a finite subfamily which is again an SCV2 of X.

A soft space $\langle U, \tau, E\rangle$ is called *SCPT1* (resp. *SCPT2*) if U is *SCPT1* (resp. *SCPT2*).

Remark 4.3. The second definitions in Definitions 4.1 and 4.2 are essentially the same to the concepts defined in [12], where soft covering and soft compactness are defined not for subsets of the initial universe but for soft sets. We again emphasis that our concern here is the following question: Given not a soft set but a subset X of U, when should we say that X is soft compact?

We first check that these two definitions actually give rise to different concepts:

Proposition 4.4. *$\langle U, \tau, E\rangle$ is SCPT2 $\not\Longrightarrow$ $\langle U, \tau, E\rangle$ is SCPT1.*

Proof. Let us borrow an example from [3]: Consider an ordinal number $\omega + 1$ and $E := \{e_1, e_2\}$. Let τ be a soft topology on $\omega + 1$ generated by a subbase

$$\{\tilde{\varnothing}, \widetilde{\omega + 1}, F'\} \cup \{F_n \mid n \in \omega\},$$

where $F' := \{(e_1, \alpha), (e_2, \omega) \mid 5 \le \alpha \le \omega\}$ and $F_n := \{(e_1, n), (e_2, \alpha) \mid \alpha \in \omega + 1\}$. It is not hard to see that both $\langle \omega + 1, \tau_{e_1}\rangle$ and $\langle \omega + 1, \tau_{e_2}\rangle$ are compact. Since E is a doubleton, this soft space is SCPT2. To see that $\langle \omega + 1, \tau, E\rangle$ is not SCPT1, observe that no finite subfamily of an SCV1 $\{F', F_0, F_1 \dots\}$ is an SCV1. □

Proposition 4.5. *For E finite, $\langle U, \tau, E\rangle$ is SCPT1 $\Longrightarrow$ $\langle U, \tau, E\rangle$ is SCPT2.*
For E infinite, $\langle U, \tau, E\rangle$ is SCPT1 $\not\Longrightarrow$ $\langle U, \tau, E\rangle$ is SCPT2.

Proof. Let $\langle U, \tau, E\rangle$ be SCPT1 with its parameter set finite, say $E = \{e_1, \ldots, e_n\}$. Take an SCV2 $\mathcal{C}$ of U arbitrarily. For each $x \in U$ and $i \in \{1, \ldots, n\}$, choose an $F_{e_i,x} \in \mathcal{C}$ so that $x \in F_{e_i,x}(e_i)$ holds. Put $F_x := \widetilde{\bigcup}_{i=1}^{n} F_{e_i,x}$. Then F_x is a soft open neighborhood of x, so $\mathcal{C}' := \{F_x \mid x \in U\}$ is an SCV1 of U. Then, by assumption, there exist $x_1, \ldots, x_k$ such that $\{F_{x_j}\}_{j=1}^{k}$ is an SCV1 of U. It is clear that $\{F_{e_i,x_j} \mid 1 \le i \le n, 1 \le j \le k\}$ is an SCV2 of U.

For the second assertion, we employ a soft space $\langle \{u\}, \mathcal{P}(\{u\} \times \mathbb{N}), \mathbb{N}\rangle$. Since there is only one SCV1, namely $\widetilde{\{u\}}$, this soft space is SCPT1. However, no finite subfamily of an SCV2 $\{\{(n, u)\} \mid n \in \mathbb{N}\}$ is an SCV2. □

The rest of this section is devoted to a comparative study of the two notions in relation with several soft topological properties.

4.1. **Compactness of e-parameter spaces.** It is natural to ask how the soft compactness is related to the compactness of e-parameter spaces $\langle U, \tau_e\rangle_{e\in E}$. Here are answers:

Proposition 4.6 ([3])**.** *(i) $\langle U, \tau, E\rangle$ is SCPT1 $\not\Longrightarrow$ $\langle U, \tau_e\rangle$ is compact for every $e \in E$.*
(ii) $\langle U, \tau_e\rangle$ is compact for every $e \in E$ $\not\Longrightarrow$ $\langle U, \tau, E\rangle$ is SCPT1.

Proposition 4.7 ([1])**.** *(i) $\langle U, \tau, E\rangle$ is SCPT2 $\not\Longrightarrow$ $\langle U, \tau_e\rangle$ is compact for every $e \in E$..*
(ii) For E finite, $\langle U, \tau_e\rangle$ is compact for every $e \in E$ $\Longrightarrow$ $\langle U, \tau, E\rangle$ is SCPT2.
For E infinite, $\langle U, \tau_e\rangle$ is compact for every $e \in E$ $\not\Longrightarrow$ $\langle U, \tau, E\rangle$ is SCPT2.

Proof. For later reference, we give a proof for (i): Consider the following soft space $\langle \mathbb{N}, \tau, \{e_1, e_2\}\rangle$, where

$$\tau := \{\tilde{\varnothing}, \widetilde{\mathbb{N}}\} \cup \{\{e_1\} \times \mathcal{S} \mid \mathcal{S} \text{ is a non-empty subset of } \mathbb{N}\}.$$

The reader will easily see that $\widetilde{\mathbb{N}}$ is an element of every SCV2 of this soft space. So, this soft space is trivially SCPT2. However, τ_{e_1} is the discrete topology on $\mathbb{N}$, which is non-compact. □

4.2. **Soft closedness.** A familiar result from general topology carries over to soft sets as follows:

Proposition 4.8. *Let $X \subset U$ be an SCPT1 subset and $A \subset X$ be such that $\tilde{A}$ is soft closed. Then A is SCPT1.*

Proof. Let $\mathcal{C}$ be an SCV1 of A. Since $\tilde{A}$ is soft closed, $\mathcal{C} \cup \{\tilde{A}^{\tilde{c}}\}$ is an SCV1 of X. Since X is SCPT1, the family $\mathcal{C} \cup \{\tilde{A}^{\tilde{c}}\}$ has a finite subfamily, say $\{F_1, \ldots, F_n\}$, which is still an SCV1 of X. Then $\{F_1, \ldots, F_n\} \setminus \{\tilde{A}^{\tilde{c}}\}$ is an SCV1 of A. Thus, we conclude that A is SCPT1. □

Similarly, we can prove:

Proposition 4.9. *Let $X \subset U$ be an SCPT2 subset and $A \subset X$ be such that $\tilde{A}$ is soft closed. Then A is SCPT2.* □

4.3. **Soft Hausdorff space.** A number of fundamental results regarding compactness in Hausdorff spaces extend naturally to soft spaces. Let us present a few examples:

Proposition 4.10. *Let $\langle U, \tau, E\rangle$ be a soft Hausdorff space. If $X \subset U$ is SCPT1 and $a \in U$ is not in X, then there exist soft open sets F_1, F_2 such that $\tilde{X} \tilde{\subset} F_1, a \tilde{\in} F_2$ and $F_1 \tilde{\cap} F_2 \tilde{=} \tilde{\varnothing}$.*

Proof. For each $x \in X$, pick up a pair $F_{x,1}, F_{x,2}$ of soft open sets satisfying that $x \tilde{\in} F_{x,1}, a \tilde{\in} F_{x,2}$ and $F_{x,1} \tilde{\cap} F_{x,2} \tilde{=} \tilde{\varnothing}$. Since X is SCPT2 by assumption, there exist finitely many elements $x_1, \ldots, x_n \in X$ such that $\{F_{x_1,1}, \ldots, F_{x_n,1}\}$ is an SCV1 of X. Then $F_1 := F_{x_1,1} \tilde{\cup} \cdots \tilde{\cup} F_{x_n,1}$ and $F_2 := F_{x_1,2} \tilde{\cap} \cdots \tilde{\cap} F_{x_n,2}$ have the desired properties. □

Corollary 4.11. *Let X be an SCPT1 subset of a soft Hausdorff space $\langle U, \tau, E\rangle$. Then $\tilde{X}$ is soft closed.* □

Corollary 4.12. *Let X_1, X_2 be disjoint SCPT1 sets from a soft Hausdorff space $\langle U, \tau, E\rangle$. Then we can separate $\tilde{X}_1$ and $\tilde{X}_2$ by soft open sets.*

Proof. For each point a from X_2, apply Proposition 4.10 to $X = X_1$ and a. Let F_1^a, F_2^a be as in the proposition. The family $\{F_2^a \mid a \in X_2\}$ is then an SCV1 of X_2, so it has a finite subfamily, say $\{F_2^{a_1}, \ldots, F_2^{a_n}\}$, which is still an SCV1 of X_2. It is then obvious that $F_1 := F_{a_1,1} \tilde{\cap} \cdots \tilde{\cap} F_{a_n,1}$ and $F_2 := F_{a_1,2} \tilde{\cup} \cdots \tilde{\cup} F_{a_n,2}$ have the desired property, i.e., $\tilde{X}_i \tilde{\subset} F_i \, (i = 1, 2)$ and $F_1 \tilde{\cap} F_2 \tilde{=} \tilde{\varnothing}$. □

Corollary 4.13. *SCPT1 soft Hausdorff space is soft normal.*

Proof. This assertion follows easily from Proposition 4.8 and Corollary 4.12. □

Similarly, one can show that:

Proposition 4.14. ([10]) *Let $\langle U, \tau, E\rangle$ be a soft Hausdorff space. If $X \subset U$ is SCPT2 and $a \in U$ is not in X, then there exist soft open sets F_1, F_2 such that $\tilde{X} \tilde{\subset} F_1, a \tilde{\in} F_2$ and $F_1 \tilde{\cap} F_2 \tilde{=} \tilde{\varnothing}$.* □

Corollary 4.15. *Let X be an SCPT2 subset of a soft Hausdorff space $\langle U, \tau, E\rangle$. Then $\tilde{X}$ is soft closed.* □

Corollary 4.16. *Let X_1, X_2 be disjoint SCPT2 sets from a soft Hausdorff space $\langle U, \tau, E\rangle$. Then we can separate $\tilde{X}_1$ and $\tilde{X}_2$ by soft open sets.* □

Corollary 4.17. *SCPT2 soft Hausdorff space is soft normal.* □

Our next topic is related to the cardinality of a space. Recall first the following result from general topology:

Proposition 4.18. *A non-empty compact Hausdorff space without isolated point is uncountable.* □

In order to think about the soft version of the above proposition, we give a precise definition of isolated point in the setting of soft sets here:

Definition 4.19. Let x be a point from U and A be a subset of U. Then we say x is a *soft isolated point* of A if there exists a soft open neighborhood F of x such that no element $y\,(\neq x)$ from A satisfies $y \,\tilde{\in}\, F$.

We now prove the soft version of Proposition 4.18:

Proposition 4.20. *A non-empty SCPT1 soft Hausdorff space without soft isolated point is uncountable.*

Proof. To obtain a contradiction, suppose that $U = \{x_n\}_{n\in\mathbb{N}}$ is a non-empty SCPT1 soft Hausdorff space. We inductively construct sequences $\{F_n^x\}_{n\in\mathbb{N}}, \{F_n^y\}_{n\in\mathbb{N}}$ of soft sets having the following properties:

- Each F_n^x, F_n^y is soft open;
- $x_n \,\tilde{\in}\, F_n^x$ holds for every $n \in \mathbb{N}$;
- F_n^y has a soft element for every $n \in \mathbb{N}$;
- $F_1^y \,\tilde{\supset}\, F_2^y \,\tilde{\supset}\, \cdots$;
- $F_n^x \,\tilde{\cap}\, F_n^y \,\tilde{=}\, \tilde{\varnothing}$ for every $n \in \mathbb{N}$.

As x_1 is not a soft isolated point, we can choose a y_1 different from x_1. Then take disjoint soft open sets F_1^x, F_1^y such that $x_1 \,\tilde{\in}\, F_1^x, y_1 \,\tilde{\in}\, F_1^y$. Suppose that F_n^x and F_n^y have been constructed. Since x_{n+1} is not a soft isolated point, we have a $y_{n+1}\,(\neq x_{n+1})$ such that $y_{n+1} \,\tilde{\in}\, F_n^y$. Then take disjoint soft open sets F_{n+1}^x, F_{n+1}^y so that $x_{n+1} \,\tilde{\in}\, F_{n+1}^x$ and $y_{n+1} \,\tilde{\in}\, F_{n+1}^y \,\tilde{\subset}\, F_n^y$ holds. It is obvious that the resulting sequences $\{F_n^x\}_{n\in\mathbb{N}}, \{F_n^y\}_{n\in\mathbb{N}}$ have the desired properties.

Since $x_n \,\tilde{\in}\, F_n^x$ holds for every $n \in \mathbb{N}$, it follows that $\{F_n^x\}_{n\in\mathbb{N}}$ is an SCV1 of U. As $\langle U, \tau, E\rangle$ is SCPT1, there exists a finite subfamily $\{F_{n_1}^x, \ldots, F_{n_k}^x\}$ $(n_1 < \cdots < n_k)$ which is again an SCV1 of U. However, y_{n_k} is not a soft element of $\widetilde{\bigcup}_{i=1}^k F_{n_i}^x$, because $y_{n_k} \,\tilde{\in}\, F_{n_k}^y$ and $F_{n_i}^x \,\tilde{\cap}\, F_{n_k}^y \,\tilde{=}\, \tilde{\varnothing}$ $(i = 1, \ldots, k)$. □

By observing that the family $\{F_n^x\}_{n\in\mathbb{N}}$ in the above proof is not only an SCV1 but also an SCV2, we have

Proposition 4.21. *A non-empty SCPT2 soft Hausdorff space without soft isolated point is uncountable.* □

Problem. Is the above proposition true if we change the definition of soft isolated point as follows? x is a soft isolated point of A if there exists a soft open neighborhood F of x such that no element $y\,(\neq x)$ from A satisfies $y \in F(e)$ for some $e \in E$.

4.4. **Soft continuous mapping.** Continuous functions play an important role in the study of topological spaces. It would be interesting to investigate the notion of soft continuity in connection with soft compactness.

Definition 4.22. We say that $\phi : U \to U'$ is a *soft continuous function* from $\langle U, \tau, E\rangle$ to $\langle U', \tau', E\rangle$ if the following condition is satisfied:

(SC1): For every $x \in U$ and for every soft neighborhood F' of $\phi(x)$, there exists a soft neighborhood F of x such that $\phi(F) \,\tilde{\subset}\, F'$.

In order to make explicit the underlying soft topological structure, we sometimes write $\phi : \langle U, \tau, E\rangle \to \langle U', \tau', E\rangle$ instead of just $\phi : U \to U'$.

It is a fundamental fact from general topology that there are several equivalent ways to define the notion of continuity. It is then natural to ask whether or not this is the case for soft topology. In order to answer this question, let us introduce the following three conditions on a function $\phi : \langle U, \tau, E\rangle \to \langle U', \tau', E\rangle$:

(SC2): For every soft open set $F' \in \tau'$, the inverse image $\phi^{-1}(F')$ is also soft open.

(SC3): For every soft closed set F', the inverse image $\phi^{-1}(F')$ is also soft closed.

(SC4): For every soft set F, we have $\phi(\widetilde{\mathrm{Cl}}(F)) \,\tilde{\subset}\, \widetilde{\mathrm{Cl}}(\phi(F))$.

These four conditions on a function are related as follows:

Theorem 4.23 ([3]). *(i) The conditions (SC2), (SC3) and (SC4) are equivalent. (ii) The condition (SC1) follows from but not imply (SC2).*

Remark 4.24 ([3]). also explains the reason for employing not (SC2) (or any equivalent ones) but (SC1) as the definition of soft continuity.

We are now in a position to investigate two notions of soft compactness in relation with soft continuity.

Proposition 4.25.
(i) The image of an SCPT1 set under a soft continuous function is also SCPT1.
(ii) The image of an SCPT1 set under a function satisfying the condition (SC2) is also SCPT1.

Proof. (i) See [3, Proposition 3.17].
(ii) This assertion follows from (i) and Theorem 4.23 (ii). □

For the other notion of soft compactness, we have:

Proposition 4.26.
(i) If E is finite, then the image of an SCPT2 set under a soft continuous function is also SCPT2.

If E is infinite, the image of an SCPT2 set under a soft continuous function is not necessarily SCPT2.
(ii) The image of an SCPT2 set under a function satisfying the condition (SC2) is also SCPT2.

Proof. (i) For the first assertion, let $\phi : \langle U, \tau, E\rangle \to \langle U', \tau', E\rangle$ be a soft continuous function with E finite, say $\{e_1, \dots, e_n\}$. Suppose $X \subset U$ is SCPT2. Take any SCV2 $\mathcal{C}'$ of $\phi(X)$. For each $\in X$ and $i \in \{1, \dots, n\}$, pick up an $F'_{e_i,x} \in \mathcal{C}'$ so that $\phi(x) \in F'_{e_i,x}(e_i)$ holds. Then $F'_x := \widetilde{\bigcup}_{i=1}^{n} F'_{e_i,x}$ is a soft neighborhood of $\phi(x)$. By the soft continuity of ϕ, there exists a soft neighborhood F_x of x such that $\phi(F_x) \,\tilde{\subset}\, F'_x$. Since $\{F_x \mid x \in X\}$ is an SCV1, we can choose $x_1, \dots, x_m \in X$ such that $\tilde{X} \,\tilde{\subset}\, \widetilde{\bigcup}_{j=1}^{m} F_{x_j}$. Then, we have

$$\widetilde{\phi(X)} \,\tilde{=}\, \phi(\tilde{X}) \,\tilde{\subset}\, \phi\big(\widetilde{\bigcup}_{j=1}^{m} F_{x_j}\big) \,\tilde{=}\, \widetilde{\bigcup}_{j=1}^{m} \phi(F_{x_j}) \,\tilde{\subset}\, \widetilde{\bigcup}_{j=1}^{m} F'_{x_j}.$$

Therefore, $\{F'_{e_i,x_j} \mid 1 \le i \le n, 1 \le j \le m\} \subset \mathcal{C}'$ is a finite SCV2 of X.

For the second assertion, consider the identity function

$$\mathrm{id} : \langle \{u\}, \{\tilde{\varnothing}, \widetilde{\{u\}}\}, \mathbb{N}\rangle \to \langle \{u\}, \mathcal{P}(\mathbb{N} \times \{u\}), \mathbb{N}\rangle.$$

It is clear that the identity function is soft continuous, but the set $\{u\}$ is not SCPT2 in the second space, as witnessed by an SCV2 $\{\{(n, u)\} \mid n \in \mathbb{N}\}$.
(ii) The proof is very similar to the standard proof for compactness, so left to the reader. □

Recall the following important result regarding a continuous function from a compact space to a Hausdorff space:

Proposition 4.27. *Let $\phi : X \to Y$ be a continuous function from a compact space X to a Hausdorff space Y. If ϕ is bijective, then it is a homeomorphism.* □

For neither notion of soft compactness, the soft version of this proposition holds:

Proposition 4.28.
(i) There exist soft spaces $\langle U, \tau, E\rangle, \langle U', \tau', E\rangle$ and a bijective soft continuous function $\phi : \langle U, \tau, E\rangle \to \langle U', \tau', E\rangle$ with the following properties:

- *$\langle U, \tau, E\rangle$ is both SCPT1 and SCPT2;*
- *$\langle U', \tau', E\rangle$ is a soft Hausdorff space;*
- *$\phi^{-1} : \langle U', \tau', E\rangle \to \langle U, \tau, E\rangle$ is not soft continuous.*

(ii) There exists a bijective function $\phi : \langle U, \tau, E\rangle \to \langle U', \tau', E\rangle$ from a soft space $\langle U, \tau, E\rangle$ to $\langle U', \tau', E\rangle$ satisfying the following properties:

- *$\langle U, \tau, E\rangle$ is both SCPT1 and SCPT2;*
- *$\langle U', \tau', E\rangle$ is a soft Hausdorff space;*
- *$\phi : \langle U, \tau, E\rangle \to \langle U', \tau', E\rangle$ does, but $\phi^{-1} : \langle U', \tau', E\rangle \to \langle U, \tau, E\rangle$ does not satisfy the condition (SC2).*

Proof. (i) Let U and U' be $[0, 1] \cup \{2\}\,(\subset \mathbb{R})$, E be $\{e_1, e_2\}$ and ϕ be the identity function. For soft topology τ, put $\tau := \{E \times O \mid O \in \mathcal{O}\}$. τ' is given by a subbase $\{E \times O \mid O \in \mathcal{O} \text{ and } 2 \notin O\} \cup \{\widetilde{\{2\}}\,\tilde{\cup}\,\{(e_2, x) \mid r < x\} \mid r < 1\}$. (In both definitions of soft topology, $\mathcal{O}$ denotes the relative topology on $[0, 1] \cup \{2\}$ given by the usual topology on $\mathbb{R}$.) Being a closed and bounded subset of $\mathbb{R}$, the topological space $\langle [0, 1] \cup \{2\}, \mathcal{O}\rangle$ is compact. By the definition of the soft topology τ, it is then clear that $\langle U, \tau, E\rangle$ is both SCPT1 and SCPT2. The reader will be able to verify that $\langle U', \tau', E\rangle$ is a soft Hausdorff space and that $\phi\,(= \mathrm{id}) : U \to U'$ is soft continuous.

Consider the inverse $\mathrm{id}^{-1} : U' \to U$ at the point 2. From the definition of τ and τ', we see at once that $\widetilde{\{2\}}$ is soft open with respect to τ, while it is not soft open with respect to τ'. Hence, no soft neighborhood F of 2 satisfies $F \,\tilde{\doteq}\, \mathrm{id}^{-1}(F) \,\tilde{\subset}\, \widetilde{\{2\}}$, witnessing the failure of soft continuity for ϕ^{-1}.
(ii) Consider the following two soft spaces:

$$\begin{aligned} \langle U, \tau, E\rangle &= \langle \{u\}, \{\tilde{\varnothing}, \{(e_2, u)\}, \widetilde{\{u\}}\}, \{e_1, e_2\}\rangle, \\ \langle U', \tau', E\rangle &= \langle \{u\}, \{\tilde{\varnothing}, \widetilde{\{u\}}\}, \{e_1, e_2\}\rangle. \end{aligned}$$

Take ϕ to be the identity function. Clearly, $\langle U, \tau, E\rangle$ is both SCPT1 and SCPT2, as $E \times U$ is finite. It is also evident that $\langle U', \tau', E\rangle$ is a soft Hausdorff space, as U' is a singleton. From the observation that $\tau \supsetneq \tau'$, one can easily see that $\mathrm{id} : U \to U'$ does, but $\mathrm{id}^{-1} : U' \to U$ does not, satisfy the condition (SC2). □

4.5. **Soft product.**

The relationship between soft compactness and soft product is not as clear as the relationship between compactness and product. We present two such results:

Our first concern is the soft version of the next well-known result (A proof can be found in, e.g., [5]):

Theorem 4.29. *Let A, B be compact subsets of a topological space T. If $A \times B \subset C$ holds for an open subset C of $T \times T$, then there exist open sets $V_A, V_B \subset T$ such that $V_A \times V_B \subset C$ and $A \subset V_A, B \subset V_B$.* □

Neither definition of soft compactness has this property:

Example 4.30. Consider a soft topological space $\langle \mathbb{Z}_2, \tau, E \rangle$, where $E := \{e_1, e_2\}$ and $\tau := \{\tilde{\varnothing}, \{(e_1, \bar{0}), (e_2, \bar{1})\}, \{(e_1, \bar{1}), (e_2, \bar{0})\}, \widetilde{\mathbb{Z}_2}\}$. Put $A := \{\bar{1}\}$ and $B := \{\bar{0}\}$. Then, as $\mathbb{Z}_2$ being finite, it is evident that both A and B are both SCPT1 and SCPT2. An easy computation shows that $\{\widetilde{(\bar{0}, \bar{1}), (\bar{1}, \bar{0})}\}$ is a soft open subset in the soft product space containing $\widetilde{A \times B}$. However, there are no soft open sets F_A, F_B such that $\tilde{A} \,\tilde{\subset}\, F_A, \tilde{B} \,\tilde{\subset}\, F_B$ and $F_A \times F_B \,\tilde{\subset}\, \{\widetilde{(\bar{0}, \bar{1}), (\bar{1}, \bar{0})}\}$.

It is an important result that compactness is preserved under product. This fact is provable without any extra assumption for the case of finite product. For the case of infinite product, it is equivalent to the Axiom of Choice (known as Tychonoff's theorem). For soft compactness, however, we have the following result:

Proposition 4.31. *(i) The property SCPT1 is not preserved under soft product.*
(ii) The property SCPT2 is not preserved under soft product.

Proof. (i) See [3, Example 3.27].
(ii) Consider the following soft spaces $\langle \mathbb{N}, \tau_1, \{e_1, e_2\} \rangle, \langle \mathbb{N}, \tau_2, \{e_1, e_2\} \rangle$, where

$$\tau_1 := \{\tilde{\varnothing}, \widetilde{\mathbb{N}}\} \cup \{\{e_1\} \times \mathcal{S} \mid \mathcal{S} \text{ is a non-empty subset of } \mathbb{N}\},$$

$$\tau_2 := \{\tilde{\varnothing}, \widetilde{\mathbb{N}}\} \cup \{\{e_2\} \times \mathcal{S} \mid \mathcal{S} \text{ is a non-empty subset of } \mathbb{N}\}.$$

A similar reasoning to the proof of Proposition 4.7 proves that both soft spaces are SCPT2. In their soft product, however, the following SCV2 has no finite subfamily which is again an SCV2:

$$\{\{e_1\} \times (\{n\} \times \mathbb{N}) \mid n \in \mathbb{N}\} \cup \{\{e_2\} \times (\mathbb{N} \times \{n\}) \mid n \in \mathbb{N}\}.$$

□

Remark 4.32. Aygünoğlu and Aygün proved that the product soft topology of (arbitrary many) soft compact topology is again soft compact [1, Theorem 4.4]. (Note that their notion of soft compactness corresponds to SCPT2 in this paper.) Our result above is not in conflict with their's, as we employed different definition of soft product. (cf. Remark 3.7)

4.6. **Finite soft intersection property.**

We can characterize compactness using finite intersection property. It is interesting to investigate whether or not this is the case also for soft compactness.

Definition 4.33. Let $\langle U, \tau, E\rangle$ be a soft space and $\mathcal{C}$ be a family of soft sets. We say that $\mathcal{C}$ has the *finite soft intersection property* (FSIP, for short) if $\widetilde{\bigcap}\, \mathcal{C}' \not\tilde{=} \tilde{\varnothing}$ holds for every finite subfamily $\mathcal{C}'$ of $\mathcal{C}$.

Proposition 4.34. *The following conditions are equivalent:*
(i) $\langle U, \tau, E\rangle$ is SCPT2;
(ii) Every non-empty family of soft closed sets which has the FSIP has a non-empty soft intersection.

Proof. (i) $\Rightarrow$ (ii): Let $\mathcal{C}$ be a non-empty family of soft closed sets having the FSIP. Assume for contradiction that we have $\widetilde{\bigcap}\, \mathcal{C} \tilde{=} \tilde{\varnothing}$. Then $\{F^{\tilde{c}} \mid F \in \mathcal{C}\}$ is an SCV2, so, by assumption, there exists a finite subfamily $\{F_1^{\tilde{c}}, \dots, F_n^{\tilde{c}}\}$ which is an SCV2. These soft closed sets then satisfy $F_1 \tilde{\cap} \cdots \tilde{\cap} F_n \tilde{=} \tilde{\varnothing}$, contradicting the assumption that $\mathcal{C}$ has the FSIP.
(ii) $\Rightarrow$ (i): Let $\mathcal{C}$ be an arbitrary SCV2. Then we have $\widetilde{\bigcap}\{F^{\tilde{c}} \mid F \in \mathcal{C}\} \tilde{=} \tilde{\varnothing}$, so the family $\{F^{\tilde{c}} \mid F \in \mathcal{C}\}$ of soft closed sets cannot have the FSIP. Put another way, $F_1^{\tilde{c}} \tilde{\cap} \cdots \tilde{\cap} F_n^{\tilde{c}} \tilde{=} \tilde{\varnothing}$ holds for some $F_1, \dots, F_n \in \mathcal{C}$. These soft open sets satisfy $F_1 \tilde{\cup} \cdots \tilde{\cup} F_n \tilde{=} \tilde{U}$, witnessing that $\mathcal{C}$ has a finite subfamily which is an SCV2. □

This proposition, together with Proposition 4.4 and 4.5, yields the next result:

Corollary 4.35. *Two conditions*
(i) $\langle U, \tau, E\rangle$ is SCPT1;
(ii)Every non-empty family of soft closed sets which has the FSIP has a non-empty soft intersection.
are related as follows:

- *(ii) does not imply (i).*
- *If E is finite, (i) implies (ii). If E is infinite, (i) does not imply (ii).* □

A natural question would be: Is there a variant of finite soft intersection property which allows us to characterize the property SCPT1 as in the form of Proposition 4.34? Below, we present two approaches:

Our first approach is to replace "having non-empty soft intersection" by "having a soft element".

Proposition 4.36. *Two conditions*
(i) $\langle U, \tau, E\rangle$ is SCPT1;
(ii) The soft intersection of any non-empty family of soft closed sets has a soft element provided that the soft intersection of every finite subfamily has a soft element;
are related as follows:

- *(i) does not imply (ii).*
- *(ii) does not imply (i).*

Proof.
• Put $U := (0, 1)$ $(\subset \mathbb{R})$, $E := \{e_1, e_2\}$ and induce a soft topology on U by a subbase $\{\tilde{\varnothing}, \tilde{U}\} \cup \{\{e_2\} \times (1/2^i, 1) \mid i \in \mathbb{N}\}$. It is evident that this soft space is SCPT1, as

every SCV1 of U has $E \times U$ as an element. Then consider a family $\mathcal{C} := \{F_i \mid i \in \mathbb{N}\}$ of soft closed sets, where $F_i := (\{e_2\} \times (1/2^i, 1))^{\tilde{c}}$. We have $1/2^{i_k+1} \tilde{\in} F_{i_1} \tilde{\cap} \cdots \tilde{\cap} F_{i_k}$ for any $i_1 < \cdots < i_k$. As $\widetilde{\bigcap} \mathcal{C} \tilde{=} \{e_1\} \times (0, 1)$, no element $r \in (0, 1)$ satisfies that $r \tilde{\in} \widetilde{\bigcap} \mathcal{C}$. In other words, $\widetilde{\bigcap} \mathcal{C}$ does not have a soft element.

• We use again the soft space $\langle \omega + 1, \tau, E\rangle$ from the proof of Proposition 4.4. It is proved in the proposition that this soft space is not SCPT1.

We claim that the condition (ii) does not hold for this soft space. To prove this claim, let us recall how we gave a soft topology τ: We induced a soft topology τ on $\omega + 1$ by a subbase $\{\tilde{\varnothing}, \widetilde{\omega + 1}, F'\} \cup \{F_n \mid n \in \omega\}$, where

$$\begin{aligned} F' &:= \{(e_1, \alpha), (e_2, \omega) \mid 5 \leq \alpha \leq \omega\} \text{ and} \\ F_n &:= \{(e_1, n), (e_2, \alpha) \mid \alpha \in \omega + 1\}. \end{aligned}$$

We prepare a lemma.

Lemma 4.37. *Let F be a soft closed set such that $\alpha \tilde{\in} F$ holds for some $\alpha \in \omega + 1$. Then $n \tilde{\in} F$ holds for $n = 0, 1, \ldots, 4$.*

Proof. By the definition of subbase, $F^{\tilde{c}}$ is of the form $\widetilde{\bigcup}_{\lambda\in\Lambda} (F^{1,\lambda} \tilde{\cap} \cdots \tilde{\cap} F^{m_\lambda,\lambda})$, where $F^{1,\lambda}, \ldots, F^{m_\lambda,\lambda}$ are from the subbase. Since $\alpha \notin \left(F^{1,\lambda} \tilde{\cap} \cdots \tilde{\cap} F^{m_\lambda,\lambda}\right)(e_2)$ for every $\lambda \in \Lambda$, and since $F_n(e_2)$ is $\omega + 1$ for every n, it follows that for every $\lambda \in \Lambda$ there exists a $k_\lambda \leq m_\lambda$ such that $F^{k_\lambda,\lambda}$ is either $\tilde{\varnothing}$ or F'. Then, for every $\lambda \in \Lambda$, we have $F^{1,\lambda} \tilde{\cap} \cdots \tilde{\cap} F^{m_\lambda,\lambda} \tilde{\subset} F'$, and so

$$F^{\tilde{c}} \tilde{=} \widetilde{\bigcup}_{\lambda\in\Lambda} (F^{1,\lambda} \tilde{\cap} \cdots \tilde{\cap} F^{m_\lambda,\lambda}) \tilde{\subset} F'.$$

Therefore $n \tilde{\in} F'^{\tilde{c}} \tilde{\subset} F$ holds for $n = 0, 1, \ldots, 4$. □

Let $\mathcal{C}$ be a non-empty family of soft closed sets such that the soft intersection of every finite subfamily of $\mathcal{C}$ has a soft element. Then, in particular, every element $F \in \mathcal{C}$ has a soft element. From the above lemma, we conclude that every $F \in \mathcal{C}$ satisfies $n \tilde{\in} F$ $(n = 0, 1, \ldots, 4)$. Therefore, the soft intersection of $\mathcal{C}$ has soft elements $0, 1, \ldots, 4$. Hence we have shown that $\langle \omega + 1, \tau, E\rangle$ does not satisfy the condition (ii). □

The second approach is to restrict the form of soft closed sets in the families:

Proposition 4.38. *Two conditions*

(i) $\langle U, \tau, E\rangle$ is SCPT1;

(ii) The soft intersection of any non-empty family of soft closed sets of the form $\tilde{A}$ for $A \subset X$ is non-empty provided that it has the FSIP;

are related as follows:

- *(i) implies (ii).*
- *(ii) does not imply (i).*

Proof.

• Assume for the sake of contradiction that a non-empty family $\mathcal{C} = \{\tilde{A} \mid A \in \Phi\}$ witnesses the failure of (ii). Since $\widetilde{\bigcap}\, \mathcal{C} \tilde{=} \tilde{\varnothing}$, we see that $\widetilde{\bigcup}\, \{\tilde{A}^{\tilde{c}} \mid A \in \Phi\} \tilde{=} \tilde{U}$. It is readily seen that $\{\tilde{A}^{\tilde{c}} \mid A \in \Phi\}$ is not only an SCV2 but also an SCV1 of U. One can then deduce contradiction as in the proof of Proposition 4.34.

• Consider the following soft topological space $\langle \mathbb{Z}, \tau, E \rangle$, where $E := \{e_1, e_2\}$ and τ is generated by a subbase

$$\{\tilde{\varnothing}, \widetilde{\mathbb{Z}}\} \cup \{\{(e_1, i), (e_2, i-1), (e_2, i), (e_2, i+1)\} \mid i \in \mathbb{Z}\}.$$

From the way we gave a soft topology, it follows that if a family $\mathcal{C}$ of soft closed sets is of the form $\{\tilde{A} \mid A \in \Phi\}$, then it is $\{E \times \mathbb{Z}\}$. Hence $\langle \mathbb{Z}, \tau, E \rangle$ trivially satisfies the condition (ii). However, it is obvious that the following SCV1 does not have a finite subfamily which is still an SCV1:

$$\{\{(e_1, i), (e_2, i-1), (e_2, i), (e_2, i+1)\} \mid i \in \mathbb{Z}\}$$

Therefore, we conclude that $\langle \mathbb{Z}, \tau, E \rangle$ is not SCPT1. □

Problem. Find a suitable notion of finite soft intersection property so that we can characterize the property SCPT1.

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