Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 8, No. 4, (October 2014), pp. 505–510
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

A note on fuzzy Volterra spaces

Ganesan Thangaraj, Srinivasan Soundararajan

Received 14 October 2013; Accepted 27 March 2014

Abstract. In this paper we investigate several characterizations of fuzzy Volterra spaces and study the conditions under which a fuzzy topological space is a fuzzy Volterra space.

2010 AMS Classification: 54A40, 03E72

Keywords: Fuzzy dense set, Fuzzy G_δ-set, fuzzy F_σ-set, Fuzzy first category set, Fuzzy Baire, Fuzzy σ-Baire, Fuzzy submaximal, Fuzzy hyperconnected, Fuzzy strongly irresolvable, Fuzzy P-space, Fuzzy Volterra.

Corresponding Author: G. Thangaraj (g.thangaraj@rediffmail.com)

1. Introduction

The concept of fuzzy sets and fuzzy set operations were first introduced by L.A.Zadeh in his classical paper [18] in the year 1965. Thereafter the paper of C.L.Chang [3] in 1968 paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Since then much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed. The concepts of Volterra spaces have been studied extensively in classical topology in [4], [5], [6], [8] and [10]. The concept of Volterra spaces in fuzzy setting was introduced and studied by the authors in [17]. In this paper we investigate several characterizations of fuzzy Volterra spaces and study under what conditions a fuzzy topological space becomes a fuzzy Volterra space? and fuzzy Baire space, fuzzy σ-Baire space, fuzzy submaximal space, fuzzy hyperconnected space, fuzzy strongly irresolvable Baire space and fuzzy P-space are considered for this work.

2. Preliminaries

Now we introduce some basic notions and results used in the sequel. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang.

Definition 2.1. Let λ and μ be any two fuzzy sets in a fuzzy topological space (X, T). Then we define $\lambda \vee \mu : X \to [0, 1]$ as follows : $(\lambda \vee \mu)(x) = Max\{\lambda(x), \mu(x)\}$. Also we define $\lambda \wedge \mu : X \to [0, 1]$ as follows : $(\lambda \wedge \mu)(x) = Min\{\lambda(x), \mu(x)\}$.

Definition 2.2. Let (X, T) be a fuzzy topological space and λ be any fuzzy set in (X, T). We define

(i). $int(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \in T\}$

(ii). $cl(\lambda) = \wedge\{\mu/\lambda \leq \mu, 1 - \mu \in T\}$.

For any fuzzy set λ in a fuzzy topological space (X, T), it is easy to see that $1 - cl(\lambda) = int(1 - \lambda)$ and $1 - int(\lambda) = cl(1 - \lambda)$. [1]

Definition 2.3 ([15])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called a *fuzzy dense set* if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$.

Definition 2.4 ([15])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called a *fuzzy nowhere dense set* if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is, $intcl(\lambda) = 0$.

Definition 2.5 ([2])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called a *fuzzy G_δ-set* in (X, T) if $\lambda = \wedge_{i=1}^{\infty}\lambda_i \;\; where \;\; \lambda_i \in T, \;\; for \;\; i \in I$.

Definition 2.6 ([2])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called a *fuzzy F_σ-set* in (X, T) if $\lambda = \vee_{i=1}^{\infty}\lambda_i \;\; where \;\; 1 - \lambda_i \in T, \;\; for \;\; i \in I$.

Definition 2.7 ([15])**.** A fuzzy set λ in a fuzzy topological space (X, T) is called a *fuzzy first category* if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where λ_i's are fuzzy nowhere dense sets in (X, T). Any other fuzzy set in (X, T) is said to be of *fuzzy second category.*

Definition 2.8 ([11])**.** Let λ be a fuzzy first category set in a fuzzy topological space (X, T). Then $1 - \lambda$ is called a *fuzzy residual set* in (X, T).

Definition 2.9 ([16])**.** Let (X, T) be a fuzzy topological space. A fuzzy set λ in (X, T) is called a *fuzzy σ-nowhere dense set* if λ is a fuzzy F_σ-set in (X, T) such that $int(\lambda) = 0$.

3. Fuzzy Volterra spaces

Definition 3.1 ([17])**.** A fuzzy topological space (X, T) is called a *fuzzy Volterra space* if $cl\big(\wedge_{i=1}^{N} (\lambda_i)\big) = 1$, where λ_i's are fuzzy dense and fuzzy G_δ-sets in (X, T).

Definition 3.2 ([16])**.** Let (X, T) be a fuzzy topological space. Then (X, T) is called a *fuzzy σ-Baire space* if $int\big(\vee_{i=1}^{\infty} (\lambda_i)\big) = 0$, where λ_i's are fuzzy σ-nowhere dense sets in (X, T).

Definition 3.3 ([11])**.** Let (X, T) be a fuzzy topological space. Then (X, T) is called a *fuzzy Baire space* if $int\big(\vee_{i=1}^{\infty} (\lambda_i)\big) = 0$, where λ_i's are fuzzy nowhere dense sets in (X, T).

Theorem 3.4 ([12])**.** *If λ is a fuzzy dense and fuzzy G_δ-set in a fuzzy topological space (X, T), then $1 - \lambda$ is a fuzzy first category set in (X, T).*

Proposition 3.5. *If the fuzzy first category sets μ_i are formed from the fuzzy dense and fuzzy G_δ-sets λ_i in a fuzzy Volterra space (X, T), then $int\big(\vee_{i=1}^{N} (\mu_i)\big) = 0$.*

Proof. Let λ_i's ($i = 1$ *to* N) be fuzzy dense and fuzzy G_δ-sets in a fuzzy Volterra space (X,T). Then $cl\big(\wedge_{i=1}^{N}(\lambda_i)\big) = 1$. Now $1 - cl\big(\wedge_{i=1}^{N}(\lambda_i)\big) = 0$, implies that $int\big(\vee_{i=1}^{N}(1-\lambda_i)\big) = 0$. Since the fuzzy sets λ_i's ($i = 1$ *to* N) are fuzzy dense and fuzzy G_δ-sets in (X,T), by Theorem 3.4, $(1-\lambda_i)$'s are fuzzy first category sets in (X,T). Let $\mu_i = 1-\lambda_i$. Hence $int\big(\vee_{i=1}^{N}(\mu_i)\big) = 0$, where μ_i's are fuzzy first category sets in (X,T). □

Theorem 3.6 ([16])**.** *In a fuzzy topological space (X,T), a fuzzy set λ is fuzzy σ-nowhere dense if and only if $1-\lambda$ is a fuzzy dense and fuzzy G_δ-set.*

Proposition 3.7. *If the fuzzy topological space (X,T) is a fuzzy σ-Baire space, then (X,T) is a fuzzy Volterra space.*

Proof. Let λ_i's ($i = 1$ *to* ∞) be fuzzy dense and fuzzy G_δ-sets in (X,T). Consider the fuzzy set $cl\big(\wedge_{i=1}^{N}(\lambda_i)\big)$. Now, $1 - cl\big(\wedge_{i=1}^{N}(\lambda_i)\big) = int\big(1-\wedge_{i=1}^{N}(\lambda_i)\big) = int\big(\vee_{i=1}^{N}(1-\lambda_i)\big)$. But $int\big(\vee_{i=1}^{N}(1-\lambda_i)\big) \leq int\big(\vee_{i=1}^{\infty}(1-\lambda_i)\big) \longrightarrow (1)$. Since the fuzzy sets λ_i's ($i = 1$ *to* ∞) are fuzzy dense and fuzzy G_δ-sets in (X,T), by Theorem 3.6, $(1-\lambda_i)$'s are fuzzy σ-nowhere dense sets in (X,T). Also since (X,T) is a fuzzy σ-Baire space, $int\big(\vee_{i=1}^{\infty}(1-\lambda_i)\big) = 0 \longrightarrow (2)$. Hence, from (1) and (2), we have $int\big(\vee_{i=1}^{N}(1-\lambda_i)\big) = 0$. Then $int\big(1-\wedge_{i=1}^{N}(\lambda_i)\big) = 0$, implies that $1-cl\big(\wedge_{i=1}^{N}(\lambda_i)\big) = 0$. Then we have $cl\big(\wedge_{i=1}^{N}(\lambda_i)\big) = 1$. Hence (X,T) is a fuzzy Volterra space. □

Theorem 3.8 ([16])**.** *If the fuzzy topological space (X,T) is a fuzzy Baire space and the fuzzy nowhere dense sets in (X,T) are fuzzy F_σ-sets in (X,T), then (X,T) is a fuzzy σ-Baire space.*

Proposition 3.9. *If the fuzzy nowhere dense sets are fuzzy F_σ-sets in a fuzzy Baire space (X,T), then (X,T) is a fuzzy Volterra space.*

Proof. Suppose that each fuzzy nowhere dense set is a fuzzy F_σ-set in the fuzzy Baire space (X,T). Then, by Theorem 3.8, (X,T) is a fuzzy σ-Baire space. Then, by Proposition 3.7, (X,T) is a fuzzy Volterra space. □

Definition 3.10. A fuzzy topological space (X,T) is said to be a *fuzzy strongly irresolvable space* if $clint(\lambda) = 1$ for each fuzzy dense set λ in (X,T).

Proposition 3.11. *If the fuzzy topological space (X,T) is a fuzzy strongly irresolvable Baire space, then (X,T) is a fuzzy Volterra space.*

Proof. Let λ_i's ($i = 1$ *to* N) be fuzzy dense and fuzzy G_δ-sets in (X,T). Since (X,T) is a fuzzy strongly irresolvable space, $cl(\lambda_i) = 1$, implies that $clint(\lambda_i) = 1$. Then $1 - clint(\lambda_i) = 0$. This implies that $intcl(1-\lambda_i) = 0$. Hence $(1-\lambda_i)$'s are fuzzy nowhere dense sets in (X,T). Suppose that μ_i's are fuzzy nowhere dense sets in (X,T) in which $\mu_i = 1-\lambda_i$, ($i = 1$ *to* N). Now $\vee_{i=1}^{N}(1-\lambda_i) \leq \vee_{i=1}^{\infty}(\mu_i)$. Then we have $int\big(\vee_{i=1}^{N}(1-\lambda_i)\big) \leq int\big(\vee_{i=1}^{\infty}(\mu_i)\big)$. Since the fuzzy topological space (X,T) is a fuzzy Baire space, $int\big(\vee_{i=1}^{\infty}(\mu_i)\big) = 0$. This implies that $int\big(\vee_{i=1}^{N}(1-\lambda_i)\big) = 0$. Now $int\big(1-\wedge_{i=1}^{N}(\lambda_i)\big) = 0$, implies that $1 - cl\big(\wedge_{i=1}^{N}(\lambda_i)\big) = 0$. Then we have $cl\big(\wedge_{i=1}^{N}(\lambda_i)\big) = 1$. Therefore (X,T) is a fuzzy Volterra space. □

Definition 3.12 ([14]). A fuzzy topological space (X, T) is called a *fuzzy P-space* if countable intersection of fuzzy open sets in (X, T) is fuzzy open. That is, every non-zero fuzzy G_δ-set in (X, T) is fuzzy open in (X, T).

Definition 3.13 ([7]). A fuzzy topological space (X, T) is called a *fuzzy hyperconnected space* if every fuzzy open set λ is fuzzy dense in (X, T). That is, $cl(\lambda) = 1$ for all $0 \neq \lambda \in T$.

Proposition 3.14. *If the fuzzy topological P-space (X, T) is a fuzzy hyperconnected space, then (X, T) is a fuzzy Volterra space.*

Proof. Let λ_i's $(i = 1 \text{ to } N)$ be fuzzy dense and fuzzy G_δ-sets in (X, T). Since (X, T) is a fuzzy P-space, λ_i's are fuzzy G_δ-sets in (X, T), implies that λ_i's are fuzzy open sets in (X, T). Then $\wedge_{i=1}^{N}(\lambda_i) \in T$. Also since (X, T) is a fuzzy hyperconnected space, $\wedge_{i=1}^{N}(\lambda_i) \in T$, implies that $cl\big(\wedge_{i=1}^{N}(\lambda_i)\big) = 1$. Therefore (X, T) is a fuzzy Volterra space. □

Definition 3.15 ([2]). A fuzzy topological space (X, T) is called a *fuzzy submaximal space* if for each fuzzy set λ in (X, T) such that $cl(\lambda) = 1$, then $\lambda \in T$ in (X, T).

Proposition 3.16. *If the fuzzy topological space (X, T) is a fuzzy submaximal and fuzzy hyperconnected space, then (X, T) is a fuzzy Volterra space.*

Proof. Let λ_i's $(i = 1 \text{ to } N)$ be fuzzy dense and fuzzy G_δ-sets in (X, T). Since (X, T) is a fuzzy submaximal space, $cl(\lambda_i) = 1$, implies that $\lambda_i \in T$ in (X, T). Then we have $int(\lambda_i) = \lambda_i$. This implies that $clint(\lambda_i) = cl(\lambda_i)$. Thus $clint(\lambda_i) = 1$ for the fuzzy dense sets λ_i in (X, T). Thus (X, T) is a fuzzy strongly irresolvable space. Now $\lambda_i \in T$ implies that $\wedge_{i=1}^{N}(\lambda_i) \in T$. Also since (X, T) is a fuzzy hyperconnected space, $\wedge_{i=1}^{N}(\lambda_i) \in T$, implies that $cl\big(\wedge_{i=1}^{N}(\lambda_i)\big) = 1$. Therefore (X, T) is a fuzzy Volterra space. □

Definition 3.17 ([9]). A fuzzy topological space (X, T) is called a *fuzzy regular space* if each fuzzy open set λ in (X, T) is such that $\lambda = \vee_{\alpha=1}^{\infty}(\lambda_\alpha)$, where $\lambda_\alpha \in T$ and $cl(\lambda_\alpha) \leq \lambda$ for each α.

Definition 3.18 ([13]). A fuzzy topological space (X, T) is called a *totally fuzzy second category space* if every non-zero fuzzy closed set in (X, T) is a fuzzy second category set in (X, T).

Theorem 3.19 ([13]). *If (X, T) is a totally fuzzy second category, fuzzy regular space, then (X, T) is a fuzzy Baire space.*

Proposition 3.20. *If (X, T) is a totally fuzzy second category, fuzzy regular and fuzzy strongly irresolvable space, then (X, T) is a fuzzy Volterra space.*

Proof. Let λ_i's $(i = 1 \text{ to } N)$ be fuzzy dense and fuzzy G_δ-sets in (X, T). Since (X, T) is a totally fuzzy second category, fuzzy regular space, by Theorem 3.19, (X, T) is a fuzzy Baire space. Then (X, T) is a fuzzy strongly irresolvable Baire space. By Proposition 3.11, (X, T) is a fuzzy Volterra space. □

Theorem 3.21 ([11]). *If λ is a fuzzy dense and fuzzy open set in (X, T), then $1 - \lambda$ is a fuzzy nowhere dense set in (X, T).*

Theorem 3.22 ([11])**.** *Let (X, T) be a fuzzy topological space. Then the following are equivalent :*

(1) *(X, T) is a fuzzy Baire space.*
(2) *$int(\lambda) = 0$ for every fuzzy first category set λ in (X, T).*
(3) *$cl(\mu) = 1$ for every fuzzy residual set μ in (X, T).*

Proposition 3.23. *If (X, T) is a totally fuzzy second category, fuzzy regular and fuzzy P-space, then (X, T) is a fuzzy Volterra space.*

Proof. Let (X, T) be a totally fuzzy second category, fuzzy regular and fuzzy P-space. Let λ_i's ($i = 1$ *to* ∞) be fuzzy dense and fuzzy G_δ-sets in (X, T). Since (X, T) is a P-space, λ_i's are fuzzy G_δ-sets in (X, T) implies that λ_i's are fuzzy open sets in (X, T). By Theorem 3.21, $(1 - \lambda_i)$'s are fuzzy nowhere dense sets in (X, T). Now the fuzzy set $\lambda = \vee_{i=1}^{\infty}(1 - \lambda_i)$ is a fuzzy first category set in (X, T). Since (X, T) is a totally fuzzy second category, fuzzy regular space, by Theorem 3.19, (X, T) is a fuzzy Baire space. Then by Theorem 3.22, $int(\lambda) = 0$ in (X, T). This implies that $int\big(\vee_{i=1}^{\infty}(1 - \lambda_i)\big) = 0$. Then $int\big(1 - \wedge_{i=1}^{\infty}(\lambda_i)\big) = 1 - cl\big(\wedge_{i=1}^{\infty}(\lambda_i)\big)$ implies that $1 - cl\big(\wedge_{i=1}^{\infty}(\lambda_i)\big) = 0$. That is, $cl\big(\wedge_{i=1}^{\infty}(\lambda_i)\big) = 1$. Then we have $cl\big(\wedge_{i=1}^{N}(\lambda_i)\big) = 1$. $\{$since $cl\big(\wedge_{i=1}^{\infty}(\lambda_i)\big) \leq cl\big(\wedge_{i=1}^{N}(\lambda_i)\big)\}$. Therefore (X, T) is a fuzzy Volterra space. □

4. Conclusions

In this paper we studied the conditions underwhich a fuzzy topological space becomes a fuzzy Volterra space.

References

[1] K. K. Azad, On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl. 82 (1981) 14–32.
[2] G. Balasubramanian, Maximal fuzzy topologies, Kybernetika (Prague) 31(5) (1995) 459–464.
[3] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.
[4] D. B. Gauld and Z. Piotrowski, On Volterra spaces, Far East J. Math. Sci. 1(2) (1993) 209–214.
[5] David Gauld, Sina Greenwood and Zbigniew Piotrowski, On Volterra spaces – II, Papers on General Topology and its Applications, Ann. New York Acad. Sci. 806 (1996) 169–173.
[6] David Gauld, Sina Greenwood and Zbigniew Piotrowski, On Volterra spaces – III, Topological operations. Proceedings of the 1998 Topology and Dynamics Conference (Fairfax, VA). Topology Proc. 23 (1998) 167–182.
[7] Erdal Ekici, Generalization of some fuzzy functions, Bull. Inst. Math. Acad. Sinica 33(3) (2005) 277–289.
[8] G. Gruenhage and D. Lutzer, Baire and Volterra paces, Proc. Amer. Soc. 128 (2000) 3115–3124.
[9] B. Hutton and I. L. Reilly, Separation axioms in fuzzy topological spaces, Fuzzy Sets and Systems 3 (1980) 99–104.
[10] Jiling Cao and David Gauld, Volterra spaces revisited, J. Aust. Math. Soc. 79(1) (2005) 61–76.
[11] G. Thangaraj and S. Anjalmose, On fuzzy Baire spaces, J. Fuzzy Math. 21(3) (2013) 667–676.
[12] G. Thangaraj and S. Anjalmose, A note on fuzzy Baire spaces, Inter. J. Fuzzy Math. Sys. 3(4) (2013) 269–274.
[13] G. Thangaraj and S. Anjalmose, Some remarks on fuzzy Baire spaces, Scientia Magna 9(1) (2013) 1–7.
[14] G. Thangaraj and G. Balasubramanian, On fuzzy basically disconnected spaces, J. Fuzzy Math. 9(1) (2001) 103–110.

[15] G. Thangaraj and G. Balasubramanian, On somewhat fuzzy continuous functions, J. Fuzzy Math. 11(2) (2003) 725–736.
[16] G. Thangaraj and E. Poongothai, On fuzzy σ-Baire spaces, Inter. J. Fuzzy Math. Sys. 3(4) (2013) 275–283.
[17] G. Thangaraj and S. Soundararajan, On fuzzy Volterra spaces, J. Fuzzy Math. 21(4) (2013) 895–904.
[18] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

G. Thangaraj (g.thangaraj@rediffmail.com)
Professor and Head, Department of Mathematics, Thiruvalluvar University, Serkkadu, Vellore-632 115, Tamilnadu, India

S. Soundararajan (soundar.rajan2002@yahoo.com)
Associate Professor, Department of Mathematics, Islamiah College (Autonomous), Vaniyambadi-635 752, Tamilnadu, India