A note on fuzzy Volterra spaces

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ABSTRACT. In this paper we investigate several characterizations of fuzzy Volterra spaces and study the conditions under which a fuzzy topological space is a fuzzy Volterra space.

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Keywords: Fuzzy dense set, Fuzzy $G_\delta$-set, fuzzy $F_\sigma$-set, Fuzzy first category set, Fuzzy Baire, Fuzzy $\sigma$-Baire, Fuzzy submaximal, Fuzzy hyperconnected, Fuzzy strongly irresolvable, Fuzzy $P$-space, Fuzzy Volterra.

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1. INTRODUCTION

The concept of fuzzy sets and fuzzy set operations were first introduced by L.A.Zadeh in his classical paper [18] in the year 1965. Thereafter the paper of C.L.Chang [3] in 1968 paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. Since then much attention has been paid to generalize the basic concepts of general topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed. The concepts of Volterra spaces have been studied extensively in classical topology in [4], [5], [6], [8] and [10]. The concept of Volterra spaces in fuzzy setting was introduced and studied by the authors in [17]. In this paper we investigate several characterizations of fuzzy Volterra spaces and study under what conditions a fuzzy topological space becomes a fuzzy Volterra space? and fuzzy Baire space, fuzzy $\sigma$-Baire space, fuzzy submaximal space, fuzzy hyperconnected space, fuzzy strongly irresolvable Baire space and fuzzy $P$-space are considered for this work.

2. PRELIMINARIES

Now we introduce some basic notions and results used in the sequel. In this work by $(X, T)$ or simply by $X$, we will denote a fuzzy topological space due to Chang.
Definition 2.1. Let $\lambda$ and $\mu$ be any two fuzzy sets in a fuzzy topological space $(X, T)$. Then we define $\lambda \lor \mu : X \rightarrow [0, 1]$ as follows: $(\lambda \lor \mu)(x) = Max\{\lambda(x), \mu(x)\}$. Also we define $\lambda \land \mu : X \rightarrow [0, 1]$ as follows: $(\lambda \land \mu)(x) = Min\{\lambda(x), \mu(x)\}$.

Definition 2.2. Let $(X, T)$ be a fuzzy topological space and $\lambda$ be any fuzzy set in $(X, T)$. We define

(i). $int(\lambda) = \bigvee\{\mu/\mu \leq \lambda, \mu \in T\}$

(ii). $cl(\lambda) = \bigwedge\{\mu/\lambda \leq \mu, 1 - \mu \in T\}$.

For any fuzzy set $\lambda$ in a fuzzy topological space $(X, T)$, it is easy to see that $1 - cl(\lambda) = int(1 - \lambda)$ and $1 - int(\lambda) = cl(1 - \lambda)$. [1]

Definition 2.3 ([15]). A fuzzy set $\lambda$ in a fuzzy topological space $(X, T)$ is called a fuzzy dense set if there exists no fuzzy closed set $\mu$ in $(X, T)$ such that $\lambda < \mu < 1$.

Definition 2.4 ([15]). A fuzzy set $\lambda$ in a fuzzy topological space $(X, T)$ is called a fuzzy nowhere dense set if there exists no non-zero fuzzy open set $\mu$ in $(X, T)$ such that $\mu < cl(\lambda)$. That is, $int cl(\lambda) = 0$.

Definition 2.5 ([2]). A fuzzy set $\lambda$ in a fuzzy topological space $(X, T)$ is called a fuzzy $G_\delta$-set in $(X, T)$ if $\lambda = \bigwedge_{i=1}^{\infty} \lambda_i$ where $\lambda_i \in T$, for $i \in I$.

Definition 2.6 ([2]). A fuzzy set $\lambda$ in a fuzzy topological space $(X, T)$ is called a fuzzy $F_\sigma$-set in $(X, T)$ if $\lambda = \bigvee_{i=1}^{\infty} \lambda_i$ where $\lambda_i \in T$, for $i \in I$.

Definition 2.7 ([15]). A fuzzy set $\lambda$ in a fuzzy topological space $(X, T)$ is called a fuzzy first category if $\lambda = \bigvee_{i=1}^{\infty} (\lambda_i)$, where $\lambda_i$'s are fuzzy nowhere dense sets in $(X, T)$. Any other fuzzy set in $(X, T)$ is said to be of fuzzy second category.

Definition 2.8 ([11]). Let $\lambda$ be a fuzzy first category set in a fuzzy topological space $(X, T)$. Then $1 - \lambda$ is called a fuzzy residual set in $(X, T)$.

Definition 2.9 ([16]). Let $(X, T)$ be a fuzzy topological space. A fuzzy set $\lambda$ in $(X, T)$ is called a fuzzy $\sigma$-nowhere dense set if $\lambda$ is a fuzzy $F_\sigma$-set in $(X, T)$ such that $int(\lambda) = 0$.

3. FUZZY VOLterra SPACES

Definition 3.1 ([17]). A fuzzy topological space $(X, T)$ is called a fuzzy Volterra space if $cl\left(\bigwedge_{i=1}^{N} (\lambda_i)\right) = 1$, where $\lambda_i$'s are fuzzy dense and fuzzy $G_\delta$-sets in $(X, T)$.

Definition 3.2 ([16]). Let $(X, T)$ be a fuzzy topological space. Then $(X, T)$ is called a fuzzy $\sigma$-Baire space if $int\left(\bigvee_{i=1}^{\infty} (\lambda_i)\right) = 0$, where $\lambda_i$'s are fuzzy $\sigma$-nowhere dense sets in $(X, T)$.

Definition 3.3 ([11]). Let $(X, T)$ be a fuzzy topological space. Then $(X, T)$ is called a fuzzy Baire space if $int\left(\bigvee_{i=1}^{\infty} (\lambda_i)\right) = 0$, where $\lambda_i$'s are fuzzy nowhere dense sets in $(X, T)$.

Theorem 3.4 ([12]). If $\lambda$ is a fuzzy dense and fuzzy $G_\delta$-set in a fuzzy topological space $(X, T)$, then $1 - \lambda$ is a fuzzy first category set in $(X, T)$.

Proposition 3.5. If the fuzzy first category sets $\mu_i$ are formed from the fuzzy dense and fuzzy $G_\delta$-sets $\lambda_i$ in a fuzzy Volterra space $(X, T)$, then $int\left(\bigvee_{i=1}^{N} (\mu_i)\right) = 0$. 

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Let $\lambda_i$’s $(i = 1$ to $N)$ be fuzzy dense and fuzzy $G_\delta$-sets in a fuzzy Volterra space $(X, T)$. Then $\text{cl}(\bigwedge_{i=1}^N (\lambda_i)) = 1$. Now $1 - \text{cl}(\bigwedge_{i=1}^N (\lambda_i)) = 0$, implies that $\text{int}(\bigvee_{i=1}^N (1 - \lambda_i)) = 0$. Since the fuzzy sets $\lambda_i$’s $(i = 1$ to $N)$ are fuzzy dense and fuzzy $G_\delta$-sets in $(X, T)$, by Theorem 3.4, $(1 - \lambda_i)$’s are fuzzy first category sets in $(X, T)$. Let $\mu_i = 1 - \lambda_i$. Hence $\text{int}(\bigvee_{i=1}^N (\mu_i)) = 0$, where $\mu_i$’s are fuzzy first category sets in $(X, T)$.

**Theorem 3.6 ([16]).** In a fuzzy topological space $(X, T)$, a fuzzy set $\lambda$ is fuzzy $\sigma$-nowhere dense if and only if $1 - \lambda$ is a fuzzy dense and fuzzy $G_\delta$-set.

**Proposition 3.7.** If the fuzzy topological space $(X, T)$ is a fuzzy $\sigma$-Baire space, then $(X, T)$ is a fuzzy Volterra space.

**Proof.** Let $\lambda_i$’s $(i = 1$ to $\infty)$ be fuzzy dense and fuzzy $G_\delta$-sets in $(X, T)$. Consider the fuzzy set $\text{cl}(\bigwedge_{i=1}^N (\lambda_i))$. Now, $1 - \text{cl}(\bigwedge_{i=1}^N (\lambda_i)) = \text{int}(1 - \bigwedge_{i=1}^N (\lambda_i)) = \text{int}(\bigvee_{i=1}^N (1 - \lambda_i))$. But $\text{int}(\bigvee_{i=1}^N (1 - \lambda_i)) \leq \text{int}(\bigvee_{i=1}^\infty (1 - \lambda_i)) \longrightarrow (1)$. Since the fuzzy sets $\lambda_i$’s $(i = 1$ to $\infty)$ are fuzzy dense and fuzzy $G_\delta$-sets in $(X, T)$, by Theorem 3.6, $(1 - \lambda_i)$’s are fuzzy $\sigma$-nowhere dense sets in $(X, T)$. Also since $(X, T)$ is a fuzzy $\sigma$-Baire space, $\text{int}(\bigvee_{i=1}^\infty (1 - \lambda_i)) = 0 \longrightarrow (2)$. Hence, from (1) and (2), we have $\text{int}(\bigvee_{i=1}^N (1 - \lambda_i)) = 0$. Then $\text{int}(1 - \bigwedge_{i=1}^N (\lambda_i)) = 0$, implies that $1 - \text{cl}(\bigwedge_{i=1}^N (\lambda_i)) = 0$. Then we have $\text{cl}(\bigwedge_{i=1}^N (\lambda_i)) = 1$. Hence $(X, T)$ is a fuzzy Volterra space. □

**Theorem 3.8 ([16]).** If the fuzzy topological space $(X, T)$ is a fuzzy Baire space and the fuzzy nowhere dense sets in $(X, T)$ are fuzzy $F_\sigma$-sets in $(X, T)$, then $(X, T)$ is a fuzzy $\sigma$-Baire space.

**Proposition 3.9.** If the fuzzy nowhere dense sets are fuzzy $F_\sigma$-sets in a fuzzy Baire space $(X, T)$, then $(X, T)$ is a fuzzy Volterra space.

**Proof.** Suppose that each fuzzy nowhere dense set is a fuzzy $F_\sigma$-set in the fuzzy Baire space $(X, T)$. Then, by Theorem 3.8, $(X, T)$ is a fuzzy $\sigma$-Baire space. Then, by Proposition 3.7, $(X, T)$ is a fuzzy Volterra space. □

**Definition 3.10.** A fuzzy topological space $(X, T)$ is said to be a fuzzy strongly irresolvable space if $\text{cl}(\lambda) = 1$ for each fuzzy dense set $\lambda$ in $(X, T)$.

**Proposition 3.11.** If the fuzzy topological space $(X, T)$ is a fuzzy strongly irresolvable Baire space, then $(X, T)$ is a fuzzy Volterra space.

**Proof.** Let $\lambda_i$’s $(i = 1$ to $N)$ be fuzzy dense and fuzzy $G_\delta$-sets in $(X, T)$. Since $(X, T)$ is a fuzzy strongly irresolvable space, $\text{cl}(\lambda_i) = 1$, implies that $\text{cl}(\lambda_i) = 1$. Then $1 - \text{cl}(\lambda_i) = 0$. This implies that $\text{int}(1 - \lambda_i) = 0$. Hence $(1 - \lambda_i)$’s are fuzzy nowhere dense sets in $(X, T)$. Suppose that $\mu_i$’s are fuzzy nowhere dense sets in $(X, T)$ in which $\mu_i = 1 - \lambda_i$ $(i = 1$ to $N)$. Now $\bigvee_{i=1}^N (1 - \lambda_i) \leq \bigvee_{i=1}^\infty (\mu_i)$. Then we have $\text{int}(\bigvee_{i=1}^N (1 - \lambda_i)) \leq \text{int}(\bigvee_{i=1}^\infty (\mu_i))$. Since the fuzzy topological space $(X, T)$ is a fuzzy Baire space, $\text{int}(\bigvee_{i=1}^\infty (\mu_i)) = 0$. This implies that $\text{int}(\bigvee_{i=1}^N (1 - \lambda_i)) = 0$.

Now $\text{int}(1 - \bigwedge_{i=1}^N (\lambda_i)) = 0$, implies that $1 - \text{cl}(\bigwedge_{i=1}^N (\lambda_i)) = 0$. Then we have $\text{cl}(\bigwedge_{i=1}^N (\lambda_i)) = 1$. Therefore $(X, T)$ is a fuzzy Volterra space. □
Definition 3.12 ([14]). A fuzzy topological space \((X, T)\) is called a fuzzy P-space if countable intersection of fuzzy open sets in \((X, T)\) is fuzzy open. That is, every non-zero fuzzy \(G_\delta\)-set in \((X, T)\) is fuzzy open in \((X, T)\).

Definition 3.13 ([7]). A fuzzy topological space \((X, T)\) is called a fuzzy hyperconnected space if every fuzzy open set \(\lambda\) is fuzzy dense in \((X, T)\). That is, \(cl(\lambda) = 1\) for all \(0 \neq \lambda \in T\).

Proposition 3.14. If the fuzzy topological P-space \((X, T)\) is a fuzzy hyperconnected space, then \((X, T)\) is a fuzzy Volterra space.

Proof. Let \(\lambda_i\)'s \((i = 1 \text{ to } N)\) be fuzzy dense and fuzzy \(G_\delta\)-sets in \((X, T)\). Since \((X, T)\) is a fuzzy P-space, \(\lambda_i\)'s are fuzzy \(G_\delta\)-sets in \((X, T)\), implies that \(\lambda_i\)'s are fuzzy open sets in \((X, T)\). Then \(\bigwedge_{i=1}^{N}(\lambda_i) \in T\). Also since \((X, T)\) is a fuzzy hyperconnected space, \(\bigwedge_{i=1}^{N}(\lambda_i) \in T\), implies that \(cl(\bigwedge_{i=1}^{N}(\lambda_i)) = 1\). Therefore \((X, T)\) is a fuzzy Volterra space. \(\Box\)

Definition 3.15 ([2]). A fuzzy topological space \((X, T)\) is called a fuzzy submaximal space if for each fuzzy set \(\lambda\) in \((X, T)\) such that \(cl(\lambda) = 1\), then \(\lambda \in T\) in \((X, T)\).

Proposition 3.16. If the fuzzy topological space \((X, T)\) is a fuzzy submaximal and fuzzy hyperconnected space, then \((X, T)\) is a fuzzy Volterra space.

Proof. Let \(\lambda_i\)'s \((i = 1 \text{ to } N)\) be fuzzy dense and fuzzy \(G_\delta\)-sets in \((X, T)\). Since \((X, T)\) is a fuzzy submaximal space, \(cl(\lambda_i) = 1\), implies that \(\lambda_i \in T\) in \((X, T)\). Then we have \(int(\lambda_i) = \lambda_i\). This implies that \(cl(int(\lambda_i) = cl(\lambda_i))\). Thus \(cl(int(\lambda_i) = 1\) for the fuzzy dense sets \(\lambda_i\) in \((X, T)\). Thus \((X, T)\) is a fuzzy strongly irresolvable space. Now \(\lambda_i \in T\) implies that \(\bigwedge_{i=1}^{N}(\lambda_i) \in T\). Also since \((X, T)\) is a fuzzy hyperconnected space, \(\bigwedge_{i=1}^{N}(\lambda_i) \in T\), implies that \(cl(\bigwedge_{i=1}^{N}(\lambda_i)) = 1\). Therefore \((X, T)\) is a fuzzy Volterra space. \(\Box\)

Definition 3.17 ([9]). A fuzzy topological space \((X, T)\) is called a fuzzy regular space if each fuzzy open set \(\lambda\) in \((X, T)\) is such that \(\lambda = \bigvee_{\alpha=1}^{\infty}(\lambda_{\alpha})\), where \(\lambda_{\alpha} \in T\) and \(cl(\lambda_{\alpha}) \leq \lambda\) for each \(\alpha\).

Definition 3.18 ([13]). A fuzzy topological space \((X, T)\) is called a totally fuzzy second category space if every non-zero fuzzy closed set in \((X, T)\) is a fuzzy second category set in \((X, T)\).

Theorem 3.19 ([13]). If \((X, T)\) is a totally fuzzy second category, fuzzy regular space, then \((X, T)\) is a fuzzy Baire space.

Proposition 3.20. If \((X, T)\) is a totally fuzzy second category, fuzzy regular and fuzzy strongly irresolvable space, then \((X, T)\) is a fuzzy Volterra space.

Proof. Let \(\lambda_i\)'s \((i = 1 \text{ to } N)\) be fuzzy dense and fuzzy \(G_\delta\)-sets in \((X, T)\). Since \((X, T)\) is a totally fuzzy second category, fuzzy regular space, by Theorem 3.19, \((X, T)\) is a fuzzy Baire space. Then \((X, T)\) is a fuzzy strongly irresolvable Baire space. By Proposition 3.11, \((X, T)\) is a fuzzy Volterra space. \(\Box\)

Theorem 3.21 ([11]). If \(\lambda\) is a fuzzy dense and fuzzy open set in \((X, T)\), then \(1 - \lambda\) is a fuzzy nowhere dense set in \((X, T)\).

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Theorem 3.22 ([11]). Let \((X, T)\) be a fuzzy topological space. Then the following are equivalent:

1. \((X, T)\) is a fuzzy Baire space.
2. \(\text{int}(\lambda) = 0\) for every fuzzy first category set \(\lambda\) in \((X, T)\).
3. \(\text{cl}(\mu) = 1\) for every fuzzy residual set \(\mu\) in \((X, T)\).

Proposition 3.23. If \((X, T)\) is a totally fuzzy second category, fuzzy regular and fuzzy P-space, then \((X, T)\) is a fuzzy Volterra space.

Proof. Let \((X, T)\) be a totally fuzzy second category, fuzzy regular and fuzzy P-space. Let \(\lambda_i\)’s \((i = 1 \text{ to } \infty)\) be fuzzy dense and fuzzy G_δ-sets in \((X, T)\). Since \((X, T)\) is a P-space, \(\lambda_i\)’s are fuzzy G_δ-sets in \((X, T)\). By Theorem 3.21, \((1 - \lambda_i)\)’s are fuzzy nowhere dense sets in \((X, T)\).

Now the fuzzy set \(\lambda = \bigvee_{i=1}^{\infty} (1 - \lambda_i)\) is a fuzzy first category set in \((X, T)\). Since \((X, T)\) is a totally fuzzy second category, fuzzy regular space, by Theorem 3.19, \((X, T)\) is a fuzzy Baire space. Then by Theorem 3.22, \(\text{int}(\lambda) = 0\) in \((X, T)\). This implies that \(\text{int}(\bigwedge_{i=1}^{\infty} (1 - \lambda_i)) = 0\). Then \(\text{int}(1 - \bigwedge_{i=1}^{\infty} (\lambda_i)) = 1 - \text{cl}(1 - \bigwedge_{i=1}^{\infty} (\lambda_i))\) implies that \(1 - \text{cl}(\bigwedge_{i=1}^{\infty} (\lambda_i)) = 0\). That is, \(\text{cl}(\bigwedge_{i=1}^{\infty} (\lambda_i)) = 1\). Then we have \(\text{cl}(\bigwedge_{i=1}^{\infty} (\lambda_i)) = 1\). (since \(\text{cl}(\bigwedge_{i=1}^{\infty} (\lambda_i)) \leq \text{cl}(\bigwedge_{i=1}^{N} (\lambda_i))\)). Therefore \((X, T)\) is a fuzzy Volterra space. □

4. Conclusions

In this paper we studied the conditions under which a fuzzy topological space becomes a fuzzy Volterra space.

REFERENCES


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