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Integral boundary value problem for fuzzy partial hyperbolic differential equations

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Abstract. This paper presents some new results on the existence and uniqueness of fuzzy solutions for some classes of fuzzy partial hyperbolic differential equations with integral boundary conditions. Our results are demonstrated in some computational examples. In this we use the same strategy as Buckley-Feuring to build fuzzy solutions from fuzzifying the deterministic solutions. Furthermore, by using the continuity of the Zadeh's extension principle combining with numerical simulations for $\alpha-$cuts of fuzzy solutions we give the representation of the surface of fuzzy solutions.

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1. Introduction

Fuzzy sets were introduced by Zadeh in [23], which is a tool that makes possible the description of vague notions and manipulations with them. Since then, there has been a increasing in study theoretical fuzzy sets as well as its applications, dramaticaly. Today, fuzzy set theory has become a fashionable theory used in many branches of real life such as dynamics systems, biological phenomena, financial forecasting, geo demographic information systems, etc (see in [2, 21, 22] for example). The concepts of fuzzy numbers, arithmetic operations and necessary calculus of fuzzy functions for developing of fuzzy analysis were first introduced and investigated by Zadeh, Chang, Dubois and Prade [11, 12, 23]. In view of the development of calculus for fuzzy functions, the investigation of fuzzy differential equations (fuzzy DEs) and fuzzy partial differential equations (fuzzy PDEs) have been initiated by Kaleva, Seikkala, Buckley and Feuring [9, 15, 20]. A survey of diversified results on the existence and uniqueness of solutions of fuzzy differential and integral equations is shown in the monograph of Lakshmikantham and Mohapatra in [16] and some references cited in this paper (see [4, 6, 7, 13, 15, 20]).

The concepts of fuzzy PDEs was first introduced by Buckley and Feuring in [9], in which they gained the existence of BF solutions and Seikkala solutions for some classes of elementary PDEs by fuzzifying crisp solutions [9]. After that, some other efforts have been done to deal with this kind of equations. And the achievements are included in some researches of Allahviranloo et al. [3], Arara et al. [5], Bertone et al. [8], Narayanamoorthy and Murugan [18]. Especially in [3] Allahviranloo et al. succeeded in applying the same strategy as Buckley and Feuring to find the exact solutions for fuzzy wave-like equations with variable coefficients. However, the theory of fuzzy PDEs is still in the initial stages and many aspects of this theory need to be explored.

In this paper, we investigate some results on the existence and uniqueness of fuzzy solutions for some class of fuzzy partial hyperbolic differential equations with integral boundary conditions. As we know, integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics, and so forth. For a detailed description of the integral boundary conditions, we refer the reader to the papers [2, 6]. These type of boundary conditions include two, three, multi-points and nonlocal boundary value problems as special cases (see [4, 7]). Concretely, We consider fuzzy PDEs with integral boundary conditions, which have the form

$$u(x,0) + \int_0^b k_1(x)u(x,y)dy = g_1(x),\ x \in J_a, \tag{1.1}$$

$$u(0,y) + \int_0^a k_2(y)u(x,y)dx = g_2(y),\ y \in J_b, \tag{1.2}$$

where $k_1 \in C(J_a,\mathbb{R}),\ k_2 \in C(J_b,\mathbb{R}),\ g_1 \in C(J_a,E^n),\ g_2 \in C(J_b,E^n)$ are given functions. By using the Banach fixed point theorem, we will prove that the fuzzy solution of the problem for partial hyperbolic differential equations exists with some conditions on databases.

The structure of the paper is organized as follows. In Section 2, we give some basic definitions and notations. In Section 3, we gain the existence and uniqueness of fuzzy solution for the partial hyperbolic differential equations with integral boundary conditions. Section 4 expands naturally the results in Section 3 for partial hyperbolic functional differential equations with integral boundary conditions. Finally, some conclusions are discussed in Section 5.

2. Preliminaries

In this section, we recall some concepts of fuzzy metric space that will be used throughout the paper. For a more thorough treatise on fuzzy analysis, we refer to monograph of Lakshmikantham and Mohapatra [16] and paper [19]. Let E^n be the space of functions u: $\mathbb{R}^n \to [0,1]$ satisfying:

i) there exists a $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$;

ii) u is fuzzy convex, that is for $x, z \in \mathbb{R}^n$ and $0 < \lambda \leq 1$,

$$u(\lambda x + (1-\lambda)z) \geq min[u(x), u(z)];$$

iii) u is semi-continuous;

iv) $[u]^0 = \overline{\{x \in \mathbb{R}^n : u(x) > 0\}}$ is a compact set in $\mathbb{R}^n$.

Denote $[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}$ for $0 < \alpha \leq 1$. Then from (i) to (iv), it follows that $[u]^\alpha$ is a nonempty compact, convex subset of $\mathbb{R}^n$.

In the following $CC(\mathbb{R}^n)$ denotes the set of all nonempty compact, convex subsets of $\mathbb{R}^n$. Let A and B be in $CC(\mathbb{R}^n)$. The distance between A and B is defined by the Hausdorff metric

$$d_H(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B}||a-b||, \sup_{b\in B}\inf_{a\in A}||a-b||\right\},$$

here $||.||$ is usual Euclidean norm in $\mathbb{R}^n$. It is easy to see that the Hausdorff metric has some following properties

i) $H_d(tA, tB) = |t|H_d(A, B)$,

ii) $H_d(A + A', B + B') \leq H_d(A, B) + H_d(A', B')$;

iii) $H_d(A + C, B + C) = H_d(A, B)$,

where $A, B, C, A', B' \in CC(\mathbb{R}^n)$ and $t \in \mathbb{R}$. Moreover, $(CC(\mathbb{R}^n), H_d)$ is a complete metric space.

The supremum metric d_∞ on E^n is defined by

$$d_\infty(u, v) = \sup_{0<\alpha\leq 1} H_d\left([u]^\alpha, [v]^\alpha\right)$$

for all $u, v \in E^n$. It is obviously that (E^n, d_∞) is a complete metric space.

If g is a function from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$, then according to Zadeh's extension principle we can extend g to $E^n \times E^n \to E^n$ by the function defined by

$$g(u, v)(z) = \sup_{z=g(x,\overline{z})} \min\{u(x), v(\overline{z})\}.$$

If g is continuous then

$$[g(u, v)]^\alpha = g\left([u]^\alpha, [v]^\alpha\right)$$

for all $u, v \in E^n$, $0 \leq \alpha \leq 1$.

Let J is a rectangular of $\mathbb{R}^2$. A map $f : J \to E^n$ is called continuous at $(t_0, s_0) \in J \subset \mathbb{R}^2$ if the multi-valued map $f_\alpha(t, s) = [f(t, s)]^\alpha$ is continuous at $(t, s) = (t_0, s_0)$ with respect to the Hausdorff metric H_d for all $\alpha \in [0, 1]$. $C(J, E^n)$ is denoted a space of all continuous functions $f : J \to E^n$ with the supremum metric H_1 defined by

$$H_1(f, g) = \sup_{(s,t)\in J} d_\infty(f(s, t), g(s, t)).$$

It can be shown that $(C(J, E^n), H_1)$ is also a complete metric space.

A map $f : J \times E^n \to E^n$ is called continuous at point $(t_0, s_0, u_0) \in J \times E^n$ provided, for any fixed $\alpha \in [0, 1]$ and arbitrary $\epsilon > 0$, there exists $\delta(\epsilon, \alpha) > 0$ such that

$$H_d\left([f(t, s, u)]^\alpha, [f(t_0, s_0, u_0)]^\alpha\right) < \epsilon$$

whenever $\max\{|t - t_0|, |s - s_0|\} < \delta(\epsilon, \alpha)$ and $H_d\left([u]^\alpha, [u_0]^\alpha\right) < \delta(\epsilon, \alpha)$ for all $(t, s, u) \in J \times E^n$.

Let $f : J = [x_1, y_1] \times [x_2, y_2] \to E^n$. The integral of f over J, denoted by $\int_{x_1}^{y_1}\int_{x_2}^{y_2} f(t,s)\,dsdt$ is defined by

$$\left[\int_{x_1}^{y_1}\int_{x_2}^{y_2} f(t,s)\,dsdt\right]^{\alpha} = \int_{x_1}^{y_1}\int_{x_2}^{y_2} f_{\alpha}(t,s)\,dsdt$$
$$= \{\int_{x_1}^{y_1}\int_{x_2}^{y_2} v(t,s)\,dsdt | v : J \to \mathbb{R}^n \text{is a measurable selection for } f_{\alpha}\}$$

for all $\alpha \in (0,1]$. A function $f : J \to E^n$ is integrable on J if $\int_{x_1}^{y_1}\int_{x_2}^{y_2} f(t,s)\,dsdt$ is in E^n.

Definition 2.1. Given $u, v \in E^n$, if there exists $w \in E^n$ such that $u = v + w$, we call $w = u - v$ the Hukuhara difference of u and v.

Definition 2.2. Given mapping $f : J \to E^n$, we say that f is Hukuhara partial differentiable with respect to x at $(x_0, y_0) \in J$ if for each $h > 0$ the Hukuhara-difference $f(x_0+\Delta t, y) - f(x_0, y)$ and $f(x_0, y) - f(x_0 - \Delta t, y_0)$ exists in E^n for every $0 < \Delta t < h$ and if it exists $\dfrac{\partial f(x_0, y_0)}{\partial x} \in E^n$ such that

$$\lim_{h\to 0^+} d_{\infty}\left(\frac{f(x_0+\Delta t, y_0) - f(x_0, y_0)}{h}, \frac{\partial f(x_0, y_0)}{\partial x}\right) = 0$$

and

$$\lim_{h\to 0^+} d_{\infty}\left(\frac{f(x_0, y_0) - f(x_0 - \Delta t, y_0)}{h}, \frac{\partial f(x_0, y_0)}{\partial x}\right) = 0.$$

In this case, $\dfrac{\partial f(x_0, y_0)}{\partial x} \in E^n$ is called the Hukuhara partial derivative of f at (x_0, y_0).

The fuzzy partial derivative of f with respect to y and higher order of fuzzy partial derivative of f at the point $(x_0, y_0) \in J$ are defined similarly.

3. The fuzzy solutions of the partial hyperbolic differential equations

Denote $J_a = [0, a];\ J_b = [0, b],\ a, b \in (0, 1]$. We consider the hyperbolic partial differential equation

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = f(x, y, u(x,y)),\ (x,y) \in J_a \times J_b, \tag{3.1}$$

where $f : J_a \times J_b \times E^n \to E^n$ is a given function.

In this paper, we are concerned with the existence of fuzzy solutions for partial hyperbolic differential equations with integral boundary conditions having the form 1.1 and 1.2

Definition 3.1. A function $u \in C(J_a \times J_b, E^n)$ is called a solution of the problem (3.1) with integral boundary conditions 1.1 and 1.2 if u satisfies the following integral equation

$$\begin{aligned} u(x,y) =& q(x,y) - \int_0^b k_1(x)u(x,y)dy - \int_0^a k_2(y)u(x,y)dx \\ &- k_1(0)\int_0^b \int_0^a k_2(y)u(x,y)dxdy + \int_0^x \int_0^y f\left(t,s,u(t,s)\right)dsdt, \end{aligned}$$

where

$$q(x,y) = g_1(x) + g_2(y) - g_1(0) + k_1(0)\int_0^b g_2(s)ds$$

for all $(x,y) \in J_a \times J_b$.

Set $k_1 = \sup_{t\in J_a} |k_1(t)|,\ k_2 = \sup_{s\in J_b} |k_2(s)|$. By applying the fixed point theorem, we prove the following result.

Theorem 3.2. *Suppose that there exists a positive number K such that*

(1) *$H_d([f(t,s,u)]^\alpha, [f(t,s,v)]^\alpha) \le KH_d([u]^\alpha,[v]^\alpha)$ holds for all $(t,s) \in J_a \times J_b$ and $u,v \in E^n$*
(2) *and $k_1 + k_2 + k_1k_2 + K < 1$.*

Then the fuzzy PDEs (3.1) with integral conditions (1.1) and (1.2) have a unique fuzzy solution in $C(J_a \times J_b, E^n)$.

Proof. Integrating both sides of the equation (3.1) on $[0,x]\times[0,y]$ and substituting the boundary conditions (1.1) and (1.2) leads to integral equation

$$\begin{aligned} u(x,y) =& q(x,y) - \int_0^b k_1(t)u(x,s)ds - \int_0^a k_2(s)u(t,y)dt \\ &- k_1(0)\int_0^b \int_0^a k_2(s)u(t,s)dtds + \int_0^x \int_0^y f\left(t,s,u(t,s)\right)dsdt \end{aligned}$$

where $q(x,y) = g_1(x) + g_2(y) - g_1(0) + k_1(0)\int_0^b g_2(s)ds$. The fuzzy solution of the problem (3.1), (1.1), (1.2) (if it exists) is a fixed point of the operator $N : C(J_a \times J_b, E^n) \to C(J_a \times J_b, E^n)$ defined as follows

$$\begin{aligned} N(u(x,y)) =& q(x,y) - \int_0^b k_1(t)u(x,s)ds - \int_0^a k_2(s)u(t,y)dt \\ &- k_1(0)\int_0^b \int_0^a k_2(s)u(t,s)dtds + \int_0^x \int_0^y f\left(t,s,u(t,s)\right)dsdt. \end{aligned}$$

For all $u,v \in C(J_a \times J_b, E^n)$ and $\alpha \in (0,1]$, one gets

$$\begin{aligned} N(u(x,y)) =& q(x,y) - \int_0^b k_1(t)u(x,s)ds - \int_0^a k_2(s)u(t,y)dt \\ &- k_1(0)\int_0^b \int_0^a k_2(s)u(t,s)dtds + \int_0^x \int_0^y f\left(t,s,u(t,s)\right)dsdt \end{aligned}$$

and

$$N(v(x,y)) = q(x,y) - \int_0^b k_1(t)v(x,s)ds - \int_0^a k_2(s)v(t,y)dt \\ - k_1(0)\int_0^b \int_0^a k_2(s)v(t,s)dtds + \int_0^x \int_0^y f\left(t,s,v(t,s)\right)dsdt.$$

We have

$$\begin{aligned}
& H_d([N(u(x,y))]^\alpha, [N(v(x,y))]^\alpha) \\
& \leq H_d([\int_0^b k_1(t)u(x,s)ds]^\alpha, [\int_0^b k_1(t)v(x,s)ds]^\alpha) \\
& + H_d([\int_0^a k_2(s)u(t,y)dt]^\alpha, [\int_0^a k_2(s)v(t,y)dt]^\alpha) \\
& + H_d([k_1(0)\int_0^b \int_0^a k_2(s)u(t,s)dtds]^\alpha, [k_1(0)\int_0^b \int_0^a k_2(s)v(t,s)dtds]^\alpha) \\
& + H_d([\int_0^x \int_0^y f\left(t,s,u(t,s)\right)dsdt]^\alpha, [\int_0^x \int_0^y f\left(t,s,v(t,s)\right)dsdt]^\alpha) \\
& \leq k_1 \int_0^b H_d([u(x,s)]^\alpha, [v(x,s)]^\alpha)ds + k_2 \int_0^a H_d([u(t,y)]^\alpha, [v(t,y)]^\alpha)dt \\
& + |k_1(0)| \sup_{s \in J_b} |k_2(s)| \int_0^b \int_0^a H_d([u(t,s)]^\alpha, [v(t,s)]^\alpha)dtds \\
& + \int_0^x \int_0^y H_d([f\left(t,s,u(t,s)\right)]^\alpha, [f\left(t,s,v(t,s)\right)]^\alpha)dsdt \\
& \leq (k_1 b + k_2 a + k_1 k_2 ab) d_\infty(u(t,s), v(t,s)) + \int_0^x \int_0^y K H_d([u(t,s)]^\alpha, [v(t,s)]^\alpha)dsdt \\
& \leq (k_1 b + k_2 a + k_1 k_2 ab) H_1(u,v) + K \int_0^a \int_0^b d_\infty(u(t,s), v(t,s))dsdt \\
& \leq (k_1 + k_2 + k_1 k_2 + K) H_1(u,v).
\end{aligned}$$

Hence

$$\begin{aligned}
H_1(N(u), N(v)) &= \sup_{(x,y)\in J_a \times J_b} d_\infty(N(u\left(x,y\right)), N(v\left(x,y\right))) \\
&= \sup_{(x,y)\in J_a \times J_b} \left(\sup_{0<\alpha\leq 1} H_d\left([N(u(x,y))]^\alpha, [N(v(x,y))]^\alpha\right)\right) \\
&\leq (k_1 + k_2 + k_1 k_2 + K) H_1(u,v).
\end{aligned}$$

Because $k_1 + k_2 + k_1k_2 + K < 1$, we imply that N is a contraction operator on complete metric space $C(J_a \times J_b, E^n)$. So N has a unique fixed point (see [16]). That is the fuzzy solution of the problem (3.1), (1.1), (1.2). The theorem is proved completely. □

The following example will demonstrate the existence of fuzzy solution for fuzzy hyperbolic PDEs, we will use the same strategy as Buckley-Feuring to build fuzzy solution from fuzzifying the deterministic solution (see [9, 10]).

Example 3.3. Consider the following fuzzy hyperbolic equation

$$\frac{U(x,y)}{\partial x \partial y} = Ce^{x+y} = F(x,y,C) \tag{3.2}$$

where $C > 0$ is a triangular fuzzy number in $I = [0, M]$, $M > 0$, $(x, y) \in [0,1] \times [0,1]$. And the integral boundary conditions are

$$U(x,0) + \int_0^1 U(x,y)dy = Ce^{x+1}, \tag{3.3}$$

$$U(0,y) + \int_0^1 u(x,y)dx = Ce^{y+1}. \tag{3.4}$$

It is clear that the hypothesis (H_1) is satisfied with an positive number $K = \frac{1}{8}$ and $k_1 = k_2 = 1$, $a = b = 1$. That follows all the conditions in the Theorem 3.1 hold. Therefore there exists a unique fuzzy solution of this problem. We will find a fuzzy solution of this problem by using Buckley-Feuring's method in [9, 10]. The deterministic solution of the crisp hyperbolic equation

$$u_{xy} = ce^{x+y} = f(x,y,c) \tag{3.5}$$

corresponding to (3.2)-(3.4) is

$$u(x,y) = g(x,y,c) = ce^{x+y}.$$

We now fuzzify this crisp solution to find fuzzy solution of fuzzy equations (3.2)-(3.4). We apply the fuzzification in c, and supposed that the parametric form of corresponding fuzzy number C is

$$[C]^\alpha = [C_1(\alpha), C_2(\alpha)]$$

where the sufficient conditions are
(a) $C_1(\alpha)$ is a bounded left continuous non-decreasing function with respect to α.
(b) $C_2(\alpha)$ is a bounded left continuous non-increasing function with respect to α.
(c) $C_1(\alpha) \le C_2(\alpha)$, for all $\alpha \in [0,1]$.

By using the Zadeh's extension principle we compute $F(x,y,C)$ from $f(x,y,c)$

$$\begin{aligned}[F]^\alpha =& [F_1(x,y,\alpha), F_2(x,y,\alpha)]\\ =& [\min\{f(x,y,c)|c \in C[\alpha]\}, \max\{f(x,y,c)|c \in C[\alpha]\}]\\ =& [C_1(\alpha)e^{x+y}, C_2(\alpha)e^{x+y}] = e^{x+y}[C_1(\alpha), C_2(\alpha)],\end{aligned}$$

satisfied conditions $(a)-(c)$, so $[F]^\alpha$ are the $\alpha-$cuts of fuzzy number Ce^{x+y}. Similarly, we compute $Y(x,y)$ from $g(x,y,c)$, we have

$$[Y]^\alpha = [Y_1(x,y,\alpha), Y_2(x,y,\alpha)] = [C_1(\alpha)e^{x+y}, C_2(\alpha)e^{x+y}].$$

In order to see if $Y(x,y)$ is differentiable, we consider fuzzy differential operator

$$\varphi(D_x, D_y)U(x,y) = \frac{U(x,y)}{\partial x \partial y}$$

and compute

$$S(x,y,\alpha) = [\varphi(D_x,D_y)Y_1(x,y,\alpha), \varphi(D_x,D_y)Y_2(x,y,\alpha)]$$

which equals to

$$[C_1(\alpha)e^{x+y}, C_2(\alpha)e^{x+y}]$$

which are the $\alpha-$cuts of fuzzy number Ce^{x+y}. Hence, $Y(x,y)$ is differentiable in the sense of Buckley and Feuring (see [9]). Because all partials of F and G with respect to C are all positive, $Y(x,y)$ is a fuzzy solution (see [10]). The integral boundary conditions are

$$Y_1(x,0,\alpha) + \int_0^1 Y_1(x,y,\alpha)dy = C_1(\alpha)e^{x+1},$$

$$Y_2(x,0,\alpha) + \int_0^1 Y_1(x,y,\alpha)dy = C_2(\alpha)e^{x+1},$$

$$Y_1(0,y,\alpha) + \int_0^1 Y_1(x,y,\alpha)dx = C_1(\alpha)e^{y+1}$$

$$Y_2(0,y,\alpha) + \int_0^1 Y_2(x,y,\alpha)dx = C_2(\alpha)e^{x+1},$$

which are all true. Therefore, $Y(x,y)$ is a fuzzy solution which also satisfies the boundary conditions (3.3), (3.4). This solution may be written

$$Y(x,y) = Ce^{x+y}.$$

4. The fuzzy solutions of partial hyperbolic functional differential equations

Functional DEs with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last year; see for instance [14] and the references therein. However, the literatures related to functional PDEs with state-dependent delay are limited; see for instance [1]. Motivated by these mentions, in this paper we investigate the existence of fuzzy solutions of hyperbolic functional PDEs with state-dependent delay. Our results may be interpreted as extensions of previous results of Arara et al. [6] for fuzzy hyperbolic PDEs with local and nonlocal initial conditions and Bertone et al. [8] with linear type of hyperbolic equations.

For any positive real number $r > 0$, we denote $J_r = [-r,a] \times [-r,b],\ a,b \in (0,1]$, $\tilde{J}_r = J_r \backslash (0,a] \times (0,b]$ and $J_0 = [-r,0] \times [-r,0]$. For each $(x,y) \in J_a \times J_b$, the state-dependent delays $u_{(x,y)}(t,s)$ is defined by

$$u_{(x,y)}(t,s) = u(x+t, y+s),\ (t,s) \in \tilde{J}_r,$$

here $u_{(x,y)}(.,.)$ represents the history of the state from time $(x-r, y-r)$ up to the present time (x,y).

In this part of the paper we give an existence and uniqueness result for the hyperbolic problem in the following form

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \frac{\partial(p(x,y)u(x,y))}{\partial y} + f(x,y,u_{(x,y)}), (x,y) \in J_a \times J_b, \tag{4.1}$$

with integral boundary conditions

$$u(x,0) + \int_0^b k_1(x)u(x,y)dy = g_1(x),\ x \in J_a, \tag{4.2}$$

$$u(0,y)+\int_0^a k_2(y)u(x,y)dx=g_2(y),\ \ y\in J_b \tag{4.3}$$

and initial condition

$$u(x,y)=\varphi(x,y),\ \ (x,y)\in \tilde{J}_r, \tag{4.4}$$

where $f: J_{ab}\times C(J_0,E^n)\to E^n,\ p\in C(J_a\times J_b,\mathbb{R}),\ g_1\in C(J_a,E^n),\ g_2\in C(J_b,E^n)$, $k_1\in C([0,a],\mathbb{R}),\ k_2\in C([0,b],\mathbb{R})$ are given functions and $\varphi\in C(\tilde{J}_r,E^n)$.

Definition 4.1. A function $u\in C(J_r,E^n)$ is called a fuzzy solution of the problem (4.1)-(4.4) if u satisfies the following integral equation

$$\begin{aligned}u(x,y)&=q(x,y)-\int_0^b k_1(x)u(x,y)dy-\int_0^a k_2(y)u(x,y)dx+\int_0^x p(t,y)u(t,y)dt\\&-k_1(0)\int_0^b\int_0^a k_2(y)u(x,y)dxdy+\int_0^x\int_0^b p(t,0)k_1(t)u(t,y)dydt\\&+\int_0^x\int_0^y f\left(t,s,u_{(t,s)}\right)dsdt\end{aligned}$$

if $(x,y)\in J_a\times J_b$ and $u(x,y)=\varphi(x,y)$ if $(x,y)\in\tilde{J}_r$, where

$$q(x,y)=g_1(x)+g_2(y)-g_1(0)+k_1(0)\int_0^b g_2(y)dy-\int_0^x p(t,0)g_1(t)dt.$$

Let $k_1=\sup_{t\in J_a}|k_1(t)|,\ k_2=\sup_{s\in J_b}|k_2(s)|$ and $\sup_{(t,s)\in J_r}|p(t,s)|=p$.

Theorem 4.2. *Suppose that there exists a positive number K satisfied*

(1) $d_\infty(f(x,y,u_{(x,y)}),f(x,y,v_{(x,y)}))\le Kd_\infty(u(x+w,y+\theta),v_(x+w,y+\theta))$ *hold for all* $(w,\theta)\in J_0,\ u,v\in C(J_r,E^n)$ *and*
(2) $(k_1+1)(k_2+p+1)+K<2$.

Then the problem (4.1)-(4.4) has a unique fuzzy solution in $C(J_r,E^n)$.

Proof. The fuzzy solution of the problem (4.1)-(4.4) (if it exists) is a fixed point of the operator $N: C(J_r,E^n)\to C(J_r,E^n)$ defined as follows

$$N(u(x,y))=\begin{cases}\Phi(x,y,u) & \text{if } (x,y)\in J_a\times J_b,\\ \varphi(x,y) & \text{if } (x,y)\in\tilde{J}_r\end{cases}$$

where

$$\begin{aligned}\Phi(x,y,u)=q(x,y)&-\int_0^b k_1(x)u(x,y)dy-\int_0^a k_2(y)u(x,y)dx\\&+\int_0^x p(t,y)u(t,y)dt-k_1(0)\int_0^b\int_0^a k_2(y)u(x,y)dxdy\\&+\int_0^x\int_0^b p(t,0)k_1(t)u(t,y)dydt+\int_0^x\int_0^y f\left(t,s,u_{(t,s)}\right)dsdt\end{aligned}$$

We can see that

$$d_\infty(N(u(x,y)),N(v(x,y)))=0$$

if $(x, y) \in \tilde{J}_r$. On the other way, we have on $J_a \times J_b$

$$\begin{aligned}
&d_\infty(N(u(x,y)), N(v(x,y))) \leq d_\infty(\int_0^b k_1(x)u(x,y)dy, \int_0^b k_1(x)v(x,y)dy)\\
&+ d_\infty(k_1(0)\int_0^b\int_0^a k_2(y)u(x,y)dxdy, k_1(0)\int_0^b\int_0^a k_2(y)v(x,y)dxdy)\\
&+ d_\infty(\int_0^a k_2(y)u(x,y)dx, \int_0^a k_2(y)v(x,y)dx)\\
&+ d_\infty(\int_0^x p(t,y)u(t,y)dt, \int_0^x p(t,y)v(t,y)dt)\\
&+ d_\infty(\int_0^x\int_0^b p(t,0)k_1(t)u(t,y)dydt, \int_0^x\int_0^b p(t,0)k_1(t)v(t,y)dydt)\\
&+ d_\infty(\int_0^x\int_0^y f\left(t,s,u_{(t,s)}\right)dsdt, \int_0^x\int_0^y f\left(t,s,v_{(t,s)}\right)dsdt\\
&\leq \sup_{x\in J_a}|k_1(x)|\int_0^b d_\infty(u(x,y), v(x,y))dy\\
&+ |k_1(0)|\sup_{y\in J_b}|k_2(y)|\int_0^b\int_0^a d_\infty(u(x,y), v(x,y))dxdy\\
&+ \sup_{y\in J_b}|k_2(y)|\int_0^a d_\infty(u(x,y), v(x,y))dx\\
&+ \sup_{(t,s)\in J_r}|p(t,s)|\int_0^x d_\infty(u(t,y), v(t,y))dt\\
&+ \sup_{x\in J_a}|k_1(x)|\sup_{(t,s)\in J_r}|p(t,s)|\int_0^x\int_0^b d_\infty(u(t,y), v(t,y))dydt\\
&+ \int_0^x\int_0^y d_\infty(f\left(t,s,u_{(t,s)}\right), f\left(t,s,v_{(t,s)}\right))dsdt\\
&\leq (k_1b + k_2a + ap + k_1k_2ab + k_1pab)d_\infty(u(x,y), v(x,y))\\
&+ \int_0^x\int_0^y Kd_\infty(u(t+w, s+\theta), v(t+w, s+\theta))dsdt\\
&= (k_1b + k_2a + pa + k_1k_2ab + k_1pab + Kab)d_\infty(u(x,y), v(x,y))\\
&\leq (k_1 + k_2 + p + k_1k_2 + k_1p + K)H_1(u,v).
\end{aligned}$$

Hence we have on J_r

$$\begin{aligned}
H_1(N(u), N(v)) &= \sup_{(x,y)\in J_r} d_\infty(N(u\,(x,y)), N(v((x,y)))\\
&\leq [(k_1+1)(k_2+p+1) + K - 1]H_1(u,v).
\end{aligned}$$

Since $(k_1+1)(k_2+p+1)+K < 2$ then $[(k_1+1)(k_2+p+1)+K-1] < 1$. It implies that N is a contraction operator. By applying the Banach fixed point theorem, N has a unique fixed point, that is the fuzzy solution of the problem (4.1) – (4.4). The theorem is proved completely. □

In this example, we use the continuity of Zadeh's extension principle and numerical simulation to show some graphical representations of the fuzzy solutions.

Example 4.3. We consider a fuzzy hyperbolic functional PDEs as follows

$$\frac{\partial^2 U(x,y)}{\partial x \partial y} = \frac{\partial(2U(x,y))}{\partial y} - e^{-(h+r)}U_{(x,y)}(h,r) + C_2 e^{x-r} + C_3(1-2e^h)e^{y-h},$$

where $(x,y) \in J_{ab} = [0, \frac{1}{9}] \times [0, \frac{1}{9}]$, $(h,r) \in J_0 = [-\frac{1}{10}, 0] \times [-\frac{1}{10}, 0]$.

The boundary conditions

$$U(x,0) + \int_0^{\frac{1}{9}} U(x,y)dy = C_1 e^{\frac{1}{9}+x} + \frac{10C_2}{9}e^x + C_3 e^{\frac{1}{9}},$$

$$U(0,y) + \int_0^{\frac{1}{9}} U(x,y)dx = C_1 e^{\frac{1}{9}+y} + \frac{10C_3}{9}e^y + C_2 e^{\frac{1}{9}},$$

and initial condition

$$U(x,y) = C_1 + C_2 + C_3, (x,y) \in [-\frac{1}{10}, \frac{1}{9}]^2 \backslash (0, \frac{1}{9}]^2,$$

where $C_i = (a_i, c_i, b_i) (i = 1,2,3)$ are three triangular fuzzy numbers in $J_i = [0, k_i]$, $k_i > 0$ (see in [17]).

Since fuzzy function

$$F(x,y,U_{(x,y)}) = -e^{-(h+r)}U_{(x,y)}(h,r) + C_2 e^{x-r} + C_3(1-2e^h)e^{y-h}$$

satisfies

$$d_\infty(F(x,y,U_{(x,y)}), F(x,y,\overline{U}_{(x,y)})) \le \frac{1}{e^{(h+r)}} d_\infty(U(x+h,y+r), \overline{U}(x+h,y+r)).$$

We recognize that the hypothesis of Theorem 4.1 is satisfied with $K = \sqrt[20]{e}$ and $p = 2, k_1 = k_2 = 1$. Therefore, there exists a unique fuzzy solution of this problem. Conduct similar to example above we find a fuzzy solution of this problem in the following form

$$U(x,y,C) = C_1 e^{x+y} + C_2 e^x + C_3 e^y.$$

The membership functions of fuzzy numbers $C_i = (a_i, c_i, b_i)$ are

$$C_i(t) = \begin{cases} \frac{t-a_i}{c_i-a_i} & \text{if } a_i \le t \le c_i \\ \frac{b_i-t}{b_i-c_i} & \text{if } c_i \le t \le b_i \\ 0 & \text{otherwise.} \end{cases}$$

This functions can be represented in the form of order pair, as follows

$$[a_i + (c_i - a_i)\alpha, b_i - (b_i - c_i)\alpha].$$

The continuity of Zadeh's extension principle states that the $\alpha-$cuts of fuzzy solution $U(x,y,C)$ are

$$\Big[e^{x+y}[a_1 + (c_1 - a_1)\alpha] + e^x[a_2 + (c_2 - a_2)\alpha] + e^y[a_3 + (c_3 - a_3)\alpha],$$

$$e^{x+y}[b_1 - (b_1 - c_1)\alpha] + e^x[b_2 - (b_2 - c_2)\alpha] + e^y[b_3 - (b_3 - c_3)\alpha]\Big].$$

If we convert this interval valued function into single valued function, we receive the membership function of $U(x, y)$ that is

$$U(x, y, C)(t) = \big(a_1 e^{x+y} + a_2 e^x + a_3 e^y, c_1 e^{x+y} + c_2 e^x + c_3 e^y, b_1 e^{x+y} + b_2 e^x + b_3 e^y\big).$$

Figure 1 shows the membership functions of $C_1 = (0, 0.5, 1)$, $C_2 = (0.5, 1, 2)$, $C_3 = (0.7, 1.5, 2)$ and the membership function of fuzzy solution $U(x, y, C)$ at point $(0, 0)$.

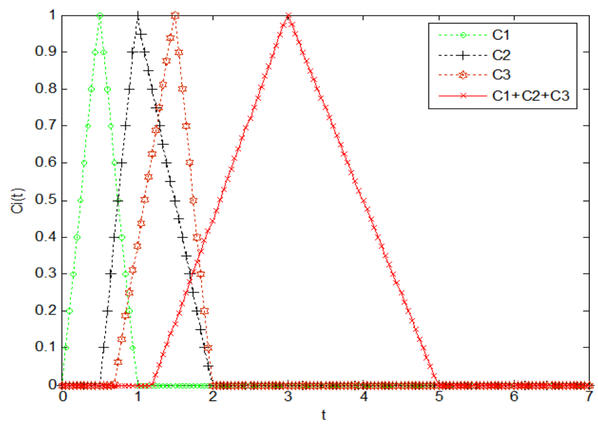

FIGURE 1. The membership functions of triangular fuzzy numbers

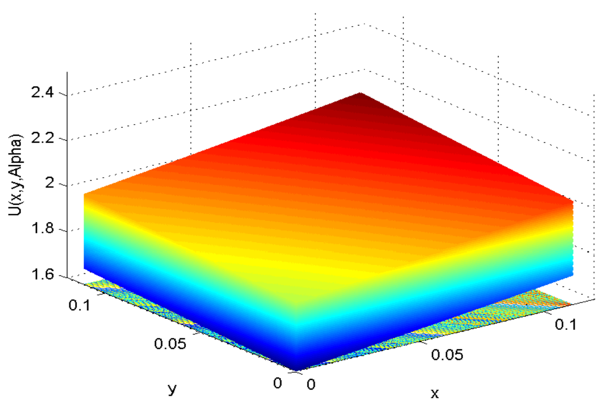

FIGURE 2. The surface of fuzzy solution $U(x, y, C)$

By using numerical simulations by Matlab 7.9.0, we present the surface of fuzzy solution in Figure 2 with there triangular fuzzy numbers $C_1 = (0.1, 0.15, 2)$, $C_2 =$

$(0.5, 0.55, 6)$ and $C_3 = (1, 1.05, 1.1)$. Obviously, the deterministic solution is the preferred solution $[U(x, y)]^1$, which means that it has membership degree 1.

5. Conclusions

This article investigates the existence and uniqueness of the fuzzy solution of some class of partial hyperbolic differential equations with integral boundary conditions. We have achieved these goals by using Banach fixed point theorem. These results are illustrated by some computational examples.

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