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On interrelations between fuzzy congruence axioms through indicators

S. S. Desai, S. R. Chaudhari

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Abstract. The present paper studies the indicators of the fuzzy direct revelation axiom, fuzzy transitive-closure coherence axiom, fuzzy consistent-closure coherence axiom and fuzzy intermediate congruence axiom. The positions of these indicators towards the degree of the weak fuzzy congruence axiom, the strong fuzzy congruence axiom and the weak axiom of fuzzy revealed preference are established. The equivalence between the fuzzy Arrow axiom and the weak fuzzy congruence axiom is established through the indicators.

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Corresponding Author: S. S. Desai(santosh_desai21@rediffmail.com)

1. Introduction

Rational choice theory is a mathematical approach used by social scientists to study the human behaviour. To study the rationality of a consumers Samuelson [24] introduced the theory of the revealed preferences through a preference relation associated with a demand function. Uzawa [30] and Arrow [1] have developed the revealed preference theory in an abstract setting of a choice function. They assumed that the domain of the choice functions contains all non-empty finite sets of the alternatives. Sen [25, 26, 27] continued their approach and noticed that it is sufficient that the domain of the choice function contains all two-element and three-element sets. Richter [23], Hansson [19] and Suzumura [28, 29] studied the rationality of choice function without any restriction on the domain of choice function. In their approach, the domain of choice function is an arbitrary family of non-empty subsets of the universal set of alternatives. They studied the rationality of choice functions by introducing the revealed preference axioms, the congruence axioms and the consistency conditions etc.

Richter-Hansson-Suzumura theory has not characterized the rationality of the choice functions whose rationalization is not fully transitive but it possesses some weaker properties such as quasi-transitivity or acyclicity. So, there remains a gap between Arrow-Sen theory and Richter-Hansson-Suzumura theory. Bossert et al. [7] narrowed down this gap in two ways. In the first place, they defined choice functions on the domain that contains all the singletons and the two-element subsets of the universal set and characterized rational choice functions whose underlying preference relation is transitive, quasi-transitive and acyclic. In the second place, they developed necessary conditions for the choice functions defined on arbitrary domain to be full, quasi-transitive and acyclic rational.

Orlovsky [22] was the first who introduced the concept of the fuzzy preference relations and his work was continued by many researchers [3, 4, 5, 12, 13, 21]. Banerjee [2] introduced the concept of fuzzy choice functions whose domain is crisp and range is fuzzy and studied the fuzzy revealed preference theory with the help of three axioms. Later Wang [31] has proved that these axioms are dependent. Georgescu [14] generalized the fuzzy choice function by fuzzifying both of its domain and codomain. In the subsequent papers [14, 15, 16] and the monograph [17], Georgescu has further developed the fuzzy choice theory by introducing the revealed preference axioms, the fuzzy congruence axioms and the consistent axioms. Recently, Wu et al. [32] studied rationality conditions on fuzzy choice functions in the framework of Banerjee's fuzzy choice function and obtained more satisfactory results than Georgescu. In [8, 9, 10] we have defined the fuzzy choice functions on the domain consisting of all characteristic functions of all single and two-elements subsets of the universal set (i.e. base domain) and studied the rationality by introducing various congruence and revealed preference axioms. In [9, 10] we introduced four new congruence axioms namely the fuzzy direct revelation axiom (FDRA), the fuzzy transitive closure coherence axiom (FTCCA), the fuzzy consistent closure coherence axiom (FCCCA) and the fuzzy intermediate congruence axiom (FICA) to study rationality of fuzzy choice functions. However, in this paper we have introduced the indicators of various axioms defined in [9, 10]. We have also set an aim to study interrelations between FDRA, FTCCA, FCCCA, FICA, WFCA, SFCA and WAFRP.

This paper consists of 5 sections. Section 1 is introductory in nature and in the section 2 we have recalled preliminaries related to fuzzy preference relation, properties of fuzzy implications, fuzzy choice function and indicators of fuzzy congruence and revealed preference axioms. In the section 3 we have introduced indicators of FDRA, FTCCA, FCCCA and FICA and studied their interrelations and their relation with indicators of WFCA, SFCA and WAFRP. A condition for the equivalence of indicator of Fuzzy Arrow Axiom and WFCA is given in section 4.

2. Preliminaries

In this section we have recalled some properties of t-norms and its residuum as well as some basic definitions on fuzzy sets and fuzzy preference relations. The background is given by Bělohlávek [6] and Klir and Yuan [20]. Also, few basic definitions and axioms related to fuzzy choice functions theory are recalled from Chaudhari and Desai [10, 11] and Georgescu [17].

Let [0,1] be the unit interval. For any $a, a_i, b, b_i \in [0,1]$ we denote $a \vee b = max(a,b)$ and $a \wedge b = min(a,b)$. $\bigvee\limits_{i \in I} a_i = sup\{a_i : i \in I\}$ and $\bigwedge\limits_{i \in I} a_i = inf\{a_i : i \in I\}$. Clearly $\bigvee\limits_{i \in I} a_i \geq a_i$ for all $i \in I$ and $\bigwedge\limits_{i \in I} a_i \leq a_i$ for all $i \in I$.

A triangular norm (or t-norm) is a binary relation $*$ on $[0,1]$ such that (i) $a * b = b * a$; (ii) $a * (b * c) = (a * b) * c$; (iii) $a * b \leq a * c$, when $b \leq c$ and (iv) $a * 1 = a$ for all $a, b, c \in [0,1]$. If $*$ satisfies (i) - (iii) and $a * 0 = a$, for all $a \in [0,1]$ then it is called triangular co-norm (or s-norm). We say that a t-norm $*$ is continuous, if it is continuous as a function on [0, 1]. For any continuous t-norm $*$, a binary operation $\longrightarrow$ on [0,1] defined as

$$a \longrightarrow b = sup\{c \in [0,1] : a * c \leq b\}$$

is called the residuum or the fuzzy implication associated with $*$. The biresiduum operation $\longleftrightarrow$ on [0,1] is defined by

$$a \longleftrightarrow b = (a \longrightarrow b) * (b \longrightarrow a)$$

If $a * b = max(0, a + b - 1)$, then $*$ is t-norm (called Lukasiewicz t-norm) and $a \longrightarrow b = min(1, 1 - a + b)$ is the associated fuzzy implication, known as Lukasiewicz fuzzy implication. Similarly If $a * b = min(a,b)$, then $*$ is called Gödel t-norm and $a \longrightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}$ is the associated fuzzy implication. If $a * b = ab$, then $*$ is called product t-norm and $a \longrightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ b/a & \text{if } a > b \end{cases}$ is the associated fuzzy implication. For other t-norms and their associated fuzzy implications we refer to [20].

Lemma 2.1 ([6, 20]). *If $\longrightarrow$ is a fuzzy implication associated with a continuous t-norm $*$ on [0, 1], then*

(i) $a * (a \longrightarrow b) \leq a \wedge b$
(ii) $a * b \leq c \Longrightarrow a \leq b \longrightarrow c$
(iii) $a \leq b \Longleftrightarrow a \longrightarrow b = 1$
(iv) $1 \longrightarrow a = a$
(v) $a \leq b \Longrightarrow b \longrightarrow c \leq a \longrightarrow c$ *and* $c \longrightarrow a \leq c \longrightarrow b$
(vi) $(a \longrightarrow b) \wedge (b \longrightarrow c) \leq a \longrightarrow c$

If $*$ is a continuous t-norm, then a unary operation $\neg$ on [0, 1] defined as $\neg a = a \longrightarrow 0 = sup\{c \in [0,1] : a * c = 0\}$ is called the negation associated with a t-norm $*$ on [0, 1]. Note that $\neg a = 1 - a$ and $\neg a = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{if } a > 0 \end{cases}$ are negations associated with Lukasiewicz and Gödel t-norms respectively.

Lemma 2.2 ([6, 20]). *If $\neg$ is a negation associated with a continuous t-norm $*$, then*

(i) $a \leq \neg b \Longleftrightarrow a * b = 0$
(ii) $a * \neg a = 0$
(iii) $a \longrightarrow b \leq \neg b \longrightarrow \neg a$

(iv) $a \leq b \Longrightarrow \neg b \leq \neg a$
(v) $\neg(a \vee b) = \neg a \wedge \neg b$ and $\neg(a \wedge b) = \neg a \vee \neg b$

Let X be a non-empty set. Let us denote by $\mathcal{P}(X)$ the set of subsets of X. A fuzzy subset of X is a function $A : X \longrightarrow [0,1]$. For $x \in X$, $A(x)$ denotes the membership degree of x in A. We denote by $\mathcal{F}(X)$ the family of fuzzy subsets of X. Thus, $\mathcal{P}(X) \subseteq \mathcal{F}(X)$. A fuzzy subset A of X is called *normal*, if $A(x) = 1$, for some $x \in X$ and is called non-zero if $A(x) > 0$, for some $x \in X$.

For any $x_1, x_2, ..., x_n \in X$ we denote the *characteristic function* of $\{x_1, x_2, ..., x_n\}$ by $[x_1, x_2, ..., x_n]$. Thus,

$$[x_1, x_2, ..., x_n](y) = \begin{cases} 1 & \text{if } y \in \{x_1, x_2, ..., x_n\} \\ 0 & \text{otherwise} \end{cases}$$

For $A, B \in \mathcal{F}(\mathcal{X})$, let us denote $I(A,B) = \bigwedge_{z \in X} [A(z) \longrightarrow B(z)]$ and $E(A,B) = \bigwedge_{z \in X} [A(z) \longleftrightarrow B(z)]$. $I(A,B)$ is called the *subsethood degree* of A in B and $E(A,B)$ is called the degree of equality of A and B. We note that $I(A,B) = 1$ if and only if $A \subseteq B$ and $E(A,B) = 1$ if and only if $A = B$. $I(A,B)$ gives the truth value of the statement "A is included in B" and $E(A,B)$ gives the truth value of the statement "A and B contain the same elements".

A fuzzy subset Q of $X \times X$ is called *fuzzy binary relation* from X to Y. If $X = Y$, then Q is called fuzzy *binary relation* on X. $Q(x,y)$ denotes the degree to which x is preferred to y. Therefore, a fuzzy binary relation on X is also called *fuzzy preference relation* on X. We shall denote the *strict fuzzy preference* relation by $P(Q)$ and define it by $P(Q)(x,y) = Q(x,y) * \neg Q(y,x)$. The transitive closure $\overline{Q}$ of Q is given by

$$\overline{Q}(x,y) = Q(x,y) \vee \left\{ \bigvee_{k \in N} \bigvee_{z_1, z_2, ... z_k \in X} [Q(x,z_1) * Q(z_1,z_2) * ... * Q(z_k,y)] \right\}$$

Definition 2.3 ([10]). Let Q be a fuzzy preference relation X. The *fuzzy consistent closure* of Q is denoted by $\hat{Q}$ and is defined by

$$\hat{Q}(x,y) = Q(x,y) \vee \left(\overline{Q}(x,y) * Q(y,x)\right)$$

Lemma 2.4 ([10]). *Let Q, Q_1 and Q_2 be fuzzy preference relations on X, then*

(i) $\hat{Q} \supseteq Q$
(ii) $Q_1 \supseteq Q_2 \Longrightarrow \hat{Q}_1 \supseteq \hat{Q}_2$
(iii) $\hat{Q} \subseteq \overline{Q}$

Remark 2.5. If the fuzzy preference relation Q is transitive, then the fuzzy consistent closure $\hat{Q}$ and the fuzzy preference relation Q coincide.

The following are the few basic definitions, lemmas and theorems related to fuzzy choice functions that form the foundation for the rest of the paper.

Definition 2.6. Let X be a non-empty set and $\mathcal{B}$ is a non-empty family of non-zero fuzzy subsets of X. A fuzzy choice function (or fuzzy consumer) on $(X, \mathcal{B})$ is a function $C : \mathcal{B} \longrightarrow \mathcal{F}(X)$ such that for each $S \in \mathcal{B}$, $C(S)$ is non-zero and $C(S) \subseteq S$.

In view of terminology we shall call the elements of X alternatives and the elements of $\mathcal{B}$ fuzzy choice sets. The number $C(S)(x)$ denotes the degree of choice of an alternative x from the fuzzy choice set S and $S(x)$ the availability degree of an alternative x in the fuzzy choice set S.

In classical choice theory it is always assumed that every available set S must have at least one choice i.e $C(S)$ is non-empty, for every S. To meet this requirement, throughout this paper, we assume that every fuzzy choice set of C must be normal, i.e. for every $S \in \mathcal{B}$, $C(S)(x) = 1$, for some $x \in X$.

Definition 2.7 ([8, 9, 10])**.** Let $C : \mathcal{B} \longrightarrow \mathcal{F}(X)$ be a fuzzy choice function on $(X, \mathcal{B})$. Then define the fuzzy revealed preference relations on X as

$$\begin{aligned} R(x,y) &= \bigvee_{S\in\mathcal{B}} [C(S)(x) * S(y)]; \\ I(x,y) &= R(x,y) * R(y,x); \\ P(x,y) &= R(x,y) * \neg R(y,x) \text{ and} \\ \tilde{P}(x,y) &= \bigvee_{S\in\mathcal{B}} [C(S)(x) * S(y) * \neg C(S)(y)] \end{aligned}$$

We call R the fuzzy revealed preference relation generated by C; I the indifference fuzzy revealed preference relation generated by C; P the strict fuzzy revealed preference relation generated by C and $\tilde{P}$ the strong fuzzy revealed preference relation generated by C.

Recall that a choice function C satisfies

i) Direct-revelation coherence: For all $S \in \mathcal{B}$ and $x \in X$, if $(x, y) \in R$ for all $y \in S$ then $x \in C(S)$

ii) Transitive-closure coherence: For all $S \in \mathcal{B}$ and $x \in X$, if $(x, y) \in \bar{R}$ for all $y \in S$ then $x \in C(S)$

iii) Consistent-closure coherence: For all $S \in \mathcal{B}$ and $x \in X$, if $(x, y) \in \hat{R}$ for all $y \in S$ then $x \in C(S)$

iv) Intermediate congruence: For all $S \in \mathcal{B}$ and $x, y \in X$, if $(x, y) \in \hat{R}$, $y \in C(S)$ and $x \in S$ then $x \in C(S)$.

We have fuzzified them and characterized rationality of fuzzy choice functions in different ways in [9, 10] and established interrelations between them in [11]. Here, we recall few results.

Definition 2.8 ([10, 11])**.** A fuzzy choice function C is said to satisfy

i) Fuzzy Direct Revelation Axiom (FDRA), if for any $S \in B$ and $x \in X$ we have $S(x) * \bigwedge_{z\in X} [S(z) \to R(x,z)] \leq C(S)(x)$

ii) Fuzzy Fuzzy Transitive-closure coherence Axiom (FTCCA) if for any $S \in B$ and $x \in X$, we have $S(x) * \bigwedge_{z\in X} [S(z) \to \overline{R}(x,z)] \leq C(S)(x)$

iii) Fuzzy Consistent-closure coherence Axiom (FCCCA) if for any $S \in B$ and $x \in X$, we have $S(x) * \bigwedge_{z\in X} [S(z) \to \hat{R}(x,z)] \leq C(S)(x)$

iv) Fuzzy Intermediate Congruence Axiom (FICA)if for any $S \in B$ and $x, y \in X$, we have $\hat{R}(x,y) * C(S)(y) * S(x) \leq C(S)(x)$

In the following theorems we have established the interrelations between the above axioms and their relations with WFCA, SFCA and WAFRP.

Theorem 2.9 ([11]). *Let $C : B \longrightarrow F(X)$ be a fuzzy choice function with an arbitrary non-empty domain B. Then*

(a) FTCCA implies FICA
(b) FICA implies FCCCA
(c) FCCCA implies FDRA

Theorem 2.10 ([11]). *Let $C : B \longrightarrow F(X)$ be a fuzzy choice function with an arbitrary non-empty domain B. Then*

(a) FTCCA and SFCA are equivalent
(b) FICA implies WFCA
(c) WFCA implies FDRA
(d) FICA implies WAFRP
(e) SFCA implies FICA

Recall that the indicators of weak fuzzy congruence axiom (WFCA), strong fuzzy congruence axiom (SFCA), weak axiom of fuzzy revealed preference (WAFRP) are defined by Georgescu in [17] as follows:

Definition 2.11 ([17]). For a fuzzy choice function C on $(X, \mathcal{B})$, the indicators of the axioms WFCA, SFCA and WAFRP are respectively given as:

$$WFCA(C) = \bigwedge_{x,y \in X} \bigwedge_{S \in \mathcal{B}} [R(x,y) * C(S)(y) * S(x) \longrightarrow C(S)(x)],$$

$$SFCA(C) = \bigwedge_{x,y \in X} \bigwedge_{S \in \mathcal{B}} \left[\overline{R}(x,y) * C(S)(y) * S(x) \longrightarrow C(S)(x)\right] \text{ and}$$

$$WAFRP(C) = \bigwedge_{x,y \in X} \left[\tilde{P}(x,y) \longrightarrow \neg R(y,x)\right]$$

Intuitively, $WFCA(C)$ gives the extent to which the statement "C satisfies the weak fuzzy congruence axiom" is true. Similar interpretation are given to $SFCA(C)$ and $WAFRP(C)$.

Remark 2.12 ([17]). For a fuzzy choice function C the following hold

(i) $WFCA(C) = 1$ if and only if C satisfies WFCA
(ii) $SFCA(C) = 1$ if and only if C satisfies SFCA
(iii) $WAFRP(C) = 1$ if and only if C satisfies WAFRP

3. Indicators of fuzzy congruence axioms

In [9, 10] we have studied the rationality of fuzzy choice function by introducing various fuzzy axioms namely fuzzy direct revelation axiom (FDRA), fuzzy transitive closure coherence axiom (FTCCA), fuzzy consistent closure coherence axiom (FCCCA) and fuzzy intermediate congruence axiom (FICA).

We believe that the further study of fuzzy choice functions will be influenced, if one discusses the interrelation between the fuzzy axioms introduced so far. Therefore, in this section we will introduce the indicators of FDRA, FTCCA, FCCCA and FICA and their interrelations will be studied. Also, their positions towards the degree of WFCA, SFCA and WAFRP will be determined.

Definition 3.1. For a fuzzy choice function C on $(X, \mathcal{B})$, we define the indicators of the congruence axioms FDRA, FCCCA, FTCCA and FICA respectively as follows

i) $FDRA(C) = \bigwedge_{S\in\mathcal{B}} \bigwedge_{x\in X} \left[S(x) * \bigwedge_{z\in X} [S(z) \longrightarrow R(x,z)] \longrightarrow C(S)(x)\right]$

ii) $FCCCA(C) = \bigwedge_{S\in\mathcal{B}} \bigwedge_{x\in X} \left[S(x) * \bigwedge_{z\in X} \left[S(z) \longrightarrow \hat{R}(x,z)\right] \longrightarrow C(S)(x)\right]$

iii) $FTCCA(C) = \bigwedge_{S\in\mathcal{B}} \bigwedge_{x\in X} \left[S(x) * \bigwedge_{z\in X} \left[S(z) \longrightarrow \overline{R}(x,z)\right] \longrightarrow C(S)(x)\right]$

iv) $FICA(C) = \bigwedge_{S\in\mathcal{B}} \bigwedge_{x,y\in X} \left[\hat{R}(x,y) * C(S)(y) * S(x) \longrightarrow C(S)(x)\right]$

The notion $FDRA(C)$ is called the indicator of the fuzzy direct revelation axiom and it gives the extent to which the fuzzy choice function C satisfies the fuzzy direct revelation axiom. Similar interpretation are given to $FCCCA(C)$, $FTCCA(C)$ and $FICA(C)$.

Remark 3.2. For a fuzzy choice function C the following hold

i) $FDRA(C) = 1$ if and only if C satisfies FDRA
ii) $FCCCA(C) = 1$ if and only if C satisfies FCCCA
iii) $FTCCA(C) = 1$ if and only if C satisfies FTCCA
iv) $FICA(C) = 1$ if and only if C satisfies FICA

The following theorem establishes the positions of the above defined indicators towards the degree of other indicators.

Theorem 3.3. *If $C : \mathcal{B} \longrightarrow \mathcal{F}(X)$ is a fuzzy choice function defined on domain $\mathcal{B}$, then*

(i) $FTCCA(C) \leq FICA(C)$
(ii) $FICA(C) \leq FCCCA(C)$
(iii) $FCCCA(C) \leq FDRA(C)$

Proof. **(i)** Let $S \in \mathcal{B}$ and $x, y \in X$. Then for any $z \in X$, by Lemma 2.4-(iii), we have

$$\begin{aligned}\hat{R}(x,y) * C(S)(y) * S(z) &\leq \overline{R}(x,y) * C(S)(y) * S(z)\\ &\leq \overline{R}(x,y) * R(y,z)\\ &\leq \overline{R}(x,y) * \overline{R}(y,z)\\ &\leq \overline{R}(x,z)\end{aligned}$$

By Lemma 2.1-(ii), we get

$$\hat{R}(x,y) * C(S)(y) \leq S(z) \longrightarrow \overline{R}(x,z)$$

This last inequality holds for any $z \in X$, therefore

$$\hat{R}(x,y) * C(S)(y) * S(x) \leq S(x) * \bigwedge_{z\in X} \left[S(z) \longrightarrow \overline{R}(x,z)\right]$$

Applying Lemma 2.1-(v) to the above inequality we get

$$S(x) * \bigwedge_{z\in X} [S(z) \longrightarrow \overline{R}(x,z)] \longrightarrow C(S)(x) \leq \hat{R}(x,y) * C(S)(y) * S(x) \longrightarrow C(S)(x)$$

The above inequality is true for all $S \in \mathcal{B}$ and $x, y \in X$. Therefore $FTCCA(C) \leq FICA(C)$.

(ii). Let $S \in \mathcal{B}$ and $x, y \in X$. Since the fuzzy choice function C is normal, for every $S \in \mathcal{B}$ there exists $t \in X$ such that $C(S)(t) = 1$. Hence $S(t) = 1$.
Then, for any $S \in \mathcal{B}$ and $x \in X$ by Lemma 2.1-(iv) if follows that

$$\begin{aligned} S(x) * \bigwedge_{z\in X} \left[S(z) \longrightarrow \hat{R}(x,z)\right] &\leq S(x) * \left[S(t) \longrightarrow \hat{R}(x,t)\right] \\ &= S(x) * \hat{R}(x,t) \\ &= \hat{R}(x,t) * C(S)(t) * S(x) \end{aligned} \tag{3.1}$$

By Lemma 2.1-(v) it follows that

$$\hat{R}(x,t) * C(S)(t) * S(x) \longrightarrow C(S)(x) \leq S(x) * \bigwedge_{z\in X} \left[S(z) \longrightarrow \hat{R}(x,z)\right] \longrightarrow C(S)(x)$$

The above inequality holds for all $S \in \mathcal{B}$, $x \in X$ and a particular $t \in X$. Therefore $FICA(C) \leq FCCCA(C)$
(iii). Since $R \subseteq \hat{R}$, by Lemma 2.1-(v), we have

$$S(y) \longrightarrow R(x,y) \leq S(y) \longrightarrow \hat{R}(x,y), \text{ for all } S \in \mathcal{B} \text{ and } x, y \in X$$

Therefore,

$$S(x) * \bigwedge_{y\in X} [S(y) \longrightarrow R(x,y)] \leq S(x) * \bigwedge_{y\in X} \left[S(y) \longrightarrow \hat{R}(x,y)\right]$$

Now, for any $S \in \mathcal{B}$ and $x \in X$, we have

$$\begin{aligned} & FCCCA(C) * S(x) * \bigwedge_{y\in X} [S(y) \longrightarrow R(x,y)] \\ &\leq FCCCA(C) * S(x) * \bigwedge_{y\in X} \left[S(y) \longrightarrow \hat{R}(x,y)\right] \\ &\leq S(x) * \bigwedge_{y\in X} \left[S(y) \to \hat{R}(x,y)\right] * \left[\left(S(x) * \bigwedge_{y\in X} \left[S(y) \to \hat{R}(x,y)\right]\right) \to C(S)(x)\right] \\ &\leq \left(S(x) * \bigwedge_{y\in X} \left[S(y) \longrightarrow \hat{R}(x,y)\right]\right) \wedge C(S)(x) \\ &\leq C(S)(x) \end{aligned}$$

By Lemma 2.1-(ii) we have

$$FCCCA(C) \leq \left[\left(S(x) * \bigwedge_{y\in X} [S(y) \longrightarrow R(x,y)]\right) \longrightarrow C(S)(x)\right]$$

The above inequality is true for all $S \in \mathcal{B}$ and $x \in X$, therefore

$$FCCCA(C) \leq \bigwedge_{S\in\mathcal{B}} \bigwedge_{x\in X} \left[\left(S(x) * \bigwedge_{y\in X} [S(y) \longrightarrow R(x,y)] \right) \longrightarrow C(S)(x) \right]$$

i.e. $FCCCA(C) \leq FDRA(C)$ □

The following theorem shows that the degree to which the fuzzy choice function satisfies the FTCCA is equal to the degree to which it satisfies the SFCA.

Theorem 3.4. *Let C be a fuzzy choice function with arbitrary domain $\mathcal{B}$. Then $FTCCA(C) = SFCA(C)$.*

Proof. First we shall prove $FTCCA(C) \leq SFCA(C)$.
For this, let $S \in \mathcal{B}$ and $x, y \in X$. Then for any $z \in X$, we have

$$\begin{aligned} \overline{R}(x,y) * C(S)(y) * S(z) &\leq \overline{R}(x,y) * R(y,z) \\ &\leq \overline{R}(x,y) * \overline{R}(y,z) \\ &\leq \overline{R}(x,z) \end{aligned}$$

By Lemma 2.1-(ii) we have

$$\overline{R}(x,y) * C(S)(y) \leq S(z) \longrightarrow \overline{R}(x,z)$$

The above inequality is true for $z \in X$, therefore

$$\overline{R}(x,y) * C(S)(y) * S(x) \leq S(x) * \bigwedge_{z\in X} \left[S(z) \longrightarrow \overline{R}(x,z)\right]$$

By Lemma (2.1)-(v) it follows that

$$S(x) * \bigwedge_{z\in X} \left[S(z) \longrightarrow \overline{R}(x,z)\right] \longrightarrow C(S)(x) \leq \overline{R}(x,y) * C(S)(y) * S(x) \longrightarrow C(S)(x)$$

The above inequality is true for all $S \in \mathcal{B}$ and $x, y \in X$. Thus $FTCCA(C) \leq SFCA(C)$.

Next, we will prove that $SFCA(C) \leq FTCCA(C)$.
Since the fuzzy choice function C is normal, for every $S \in \mathcal{B}$ there exists $t \in X$ such that $C(S)(t) = 1$ and hence $S(t) = 1$.
Then, for any $x, y \in X$ by Lemma 2.1-(i) it follows that

$$\begin{aligned} &SFCA(C) * S(x) * \bigwedge_{z\in X} \left[S(z) \longrightarrow \overline{R}(x,z)\right] \\ &\leq SFCA(C) * S(x) * \left[S(t) \longrightarrow \overline{R}(x,t)\right] \\ &= SFCA(C) * S(x) * \overline{R}(x,t) \\ &\leq \overline{R}(x,t) * S(x) * \left[\overline{R}(x,t) * S(x) * C(S)(t) \longrightarrow C(S)(x)\right] \\ &= \overline{R}(x,t) * S(x) * \left[\overline{R}(x,t) * S(x) \longrightarrow C(S)(x)\right] \\ &\leq (\overline{R}(x,t) * S(x)) \wedge C(S)(x) \\ &\leq C(S)(x) \end{aligned}$$

Thus,

$$SFCA(C) * S(x) * \bigwedge_{z \in X} \left[S(z) \longrightarrow \overline{R}(x,z)\right] \longrightarrow C(S)(x)$$

The last inequality is true for all $S \in \mathcal{B}$ and $x \in X$, therefore

$$SFCA(C) \leq \bigwedge_{S \in \mathcal{B}} \bigwedge_{x \in X} \left[S(x) * \bigwedge_{z \in X} \left[S(z) \longrightarrow \overline{R}(x,z)\right] \longrightarrow C(S)(x)\right]$$

i.e. $SFCA(C) \leq FTCCA(C)$. □

The following theorem gives the position of the degrees of FICA and FDRA towards the degree of WFCA, SFCA and WAFRP.

Theorem 3.5. *Let C be a fuzzy choice function with arbitrary domain $\mathcal{B}$. Then*

(i) $FICA(C) \leq WFCA(C)$
(ii) $WFCA(C) \leq FDRA(C)$
(iii) $FICA(C) \leq WAFRP(C)$
(iv) $SFCA(C) \leq FICA(C)$

Proof. **(i)** Let $S \in \mathcal{B}$ and $x, y \in X$. Then, by Lemmas 2.4-(iii) and 2.1-(i)

$$\begin{aligned} FICA(C) * R(x,y) * C(S)(y) * S(x) &= R(x,y) * C(S)(y) * S(x) * FICA(C) \\ &\leq R(x,y) * C(S)(y) * S(x) * \Big(\hat{R}(x,y) * C(S)(y) * S(x) \longrightarrow C(S)(x)\Big) \\ &\leq \hat{R}(x,y) * C(S)(y) * S(x) * \Big(\hat{R}(x,y) * C(S)(y) * S(x) \longrightarrow C(S)(x)\Big) \\ &\leq (\hat{R}(x,y) * C(S)(y) * S(x)) \wedge C(S)(x) \\ &\leq C(S)(x) \end{aligned}$$

By Lemma 2.1-(ii), we have

$$FICA(C) \leq R(x,y) * C(S)(y) * S(x) \longrightarrow C(S)(x)$$

The last inequality holds for all $S \in \mathcal{B}$ and $x, y \in X$, therefore

$$FICA(C) \leq \bigwedge_{x,y \in X} \bigwedge_{S \in \mathcal{B}} [R(x,y) * C(S)(y) * S(x) \longrightarrow C(S)(x)]$$

i.e. $FICA(C) \leq WFCA(C)$

(ii) Let $S \in \mathcal{B}$ and $x \in X$. Since the fuzzy choice function C is normal, for every $S \in \mathcal{B}$ there exists $t \in X$ such that $C(S)(t) = 1$ and hence $S(t) = 1$. Then

$$\begin{aligned} S(x) * \bigwedge_{z \in X} [S(z) \longrightarrow R(x,z)] &\leq S(x) * [S(t) \longrightarrow R(x,t)] \\ &= S(x) * R(x,t) \end{aligned}$$

Thus,

$$S(x) * \bigwedge_{z \in X} [S(z) \longrightarrow R(x,z)] \leq R(x,t) * C(S)(t) * S(x)$$

Now by Lemma 2.1-(v) one can write

$$R(x,t) * C(S)(t) * S(x) \longrightarrow C(S)(x) \leq S(x) * \bigwedge_{z\in X} [S(z) \longrightarrow R(x,z)] \longrightarrow C(S)(x)$$

The above inequality is true for all $S \in \mathcal{B}$, $x \in X$ and a particular $t \in X$. Therefore $WFCA(C) \leq FDRA(C)$.

(iii) For any $x, y \in X$, we have

$$\begin{aligned} & FICA(C) * \tilde{P}(x,y) * R(y,x) \leq FICA(C) * \tilde{P}(x,y) * \hat{R}(y,x) \\ & = FICA(C) * \bigvee_{S\in\mathcal{B}} [C(S)(x) * S(y) * \neg C(S)(y)] * \hat{R}(y,x) \\ & = \bigvee_{S\in\mathcal{B}} \left[FICA(C) * C(S)(x) * S(y) * \neg C(S)(y) * \hat{R}(y,x)\right] \end{aligned} \tag{3.2}$$

Now, for any $S \in \mathcal{B}$, we have

$$\begin{aligned} & FICA(C) * C(S)(x) * S(y) * \neg C(S)(y) * \hat{R}(y,x) \\ & \quad = \hat{R}(y,x) * C(S)(x) * S(y) * FICA(C) * \neg C(S)(y) \\ & \quad \leq \hat{R}(y,x) * C(S)(x) * S(y) * \left[\hat{R}(y,x) * C(S)(x) * S(y) \longrightarrow C(S)(y)\right] * \neg C(S)(y) \\ & \quad \leq [\hat{R}(y,x) * C(S)(x) * S(y) \wedge C(S)(y)] * \neg C(S)(y) \\ & \quad \leq C(S)(y) * \neg C(S)(y) \\ & \quad = 0 \end{aligned}$$

Therefore, equation (3.2) reduces to

$$FICA(C) * \tilde{P}(x,y) * R(y,x) = 0$$

Thus by Lemma 2.2-(i), we have

$$FICA(C) * \tilde{P}(x,y) \leq \neg R(y,x)$$

Then, by Lemma 2.1-(ii) we have

$$FICA(C) \leq \tilde{P}(x,y) \longrightarrow \neg R(y,x)$$

The above inequality is true for all $x, y \in X$, therefore

$$FICA(C) \leq \bigwedge_{x,y\in X} \left[\tilde{P}(x,y) \longrightarrow \neg R(y,x)\right]$$

(iv). It follows from Theorems 3.4 and 3.3-(i) □

4. Equivalence between FAA(C) and WFCA(C)

In [17] Georgescu argued that the fuzzy Arrow axiom and weak fuzzy congruence axioms are equivalent in the presence of the following hypotheses (H1) and (H2)

(H1) Every $S \in \mathcal{B}$ and $C(S)$ are normal fuzzy subsets of X.

(H2) $\mathcal{B}$ includes all fuzzy sets $[x_1, x_2, \ldots, x_n]$, $n \geq 1$ and $x_1, x_2, \ldots, x_n \in X$.

In the following theorem we will prove an equivalence between the fuzzy Arrow axiom and the weak fuzzy congruence axiom on arbitrary domain in the presence of (H1) only. Here we note that the results of this section are valid only for the Gödel t-norm.

Definition 4.1 ([18]). Let C be a fuzzy choice functions on $(X, \mathcal{B})$ satisfying (H1). Then the indicator of the fuzzy Arrow axiom, $FAA(C)$, is given by

$$FAA(C) = \bigwedge_{S_1, S_2 \in \mathcal{B}} \bigwedge_{x \in X} \left[I(S_1, S_2) \wedge S_1(x) \wedge C(S_2)(x) \longrightarrow E(S_1 \cap C(S_2), C(S_1)) \right]$$

Note that $FAA(C) = 1$ if and only if C satisfies FAA. The number $FAA(C)$ gives the degree to which the fuzzy choice function C satisfies the fuzzy Arrow axiom.

In the following example first we show $FAA(C)$ and $WFCA(C)$ are distinct on arbitrary domain. We note that $WFCA(C) \leq FAA(C)$ on any domain.

Example 4.2. Let $X = \{a, b, c, d\}$. Define a fuzzy choice function C on $B = \{[a, b, c], [a, b, d], [a, c, b], [b, d]\}$ as follows
$C([a, b, c])(a) = 0.8, C([a, b, c])(b) = 1, C([a, b, c])(c) = 1, C([a, b, d])(a) = 1, C([a, b, d])(b) = 0.7, C([a, b, d])(d) = 0.6, C([a, c, d])(a) = 1, C([a, c, d])(c) = 1, C([a, c, d])(d) = 1, C([b, d])(b) = 0.9, C([b, d])(d) = 1$
Then the fuzzy revealed preference relation R is given by

$$R = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0.9 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Here C satisfies the fuzzy Arrow axiom. Therefore, $FAA(C) = 1$. But $WFCA(C) \leq R(a, b) \wedge C([a, b, c])(b) \wedge [a, b, c](a) \longrightarrow C([a, b, c])(a) = 0.8$. This shows that $FAA(C)$ and $WFCA(C)$ are distinct on arbitrary domain.

The following theorem shows that the degree to which the fuzzy choice function C satisfies the fuzzy Arrow axiom is equal to the degree to which it satisfies the WFCA on an arbitrary domain.

Theorem 4.3. *Let C be a fuzzy choice function with arbitrary domain $\mathcal{B}$. If $\mathcal{B}$ is closed under intersection, then $FAA(C) = WFCA(C)$.*

Proof. First we shall prove $WFCA(C) \leq FAA(C)$. To do this we establish

(a) $WFCA(C) \wedge I(S, T) \wedge S(x) \wedge C(T)(x) \leq C(S)(z) \longrightarrow S(z) \wedge C(T)(z)$
(b) $WFCA(C) \wedge I(S, T) \wedge S(x) \wedge C(T)(x) \leq S(z) \wedge C(T)(z) \longrightarrow C(S)(z)$

Let $S, T \in \mathcal{B}$ and $x \in X$. Then by the definition of $I(S, T)$ and Lemma 2.1-(i), we have $I(S, T) \wedge S(z) \leq T(z)$.

(a) For any $z \in X$, we have

$$\begin{aligned}
&WFCA(C) \wedge I(S,T) \wedge S(x) \wedge C(T)(x) \wedge C(S)(z) \\
&\quad = WFCA(C) \wedge I(S,T) \wedge S(z) \wedge S(x) \wedge C(T)(x) \wedge C(S)(z) \wedge S(z) \\
&\quad \leq WFCA(C) \wedge S(z) \wedge (S(z) \longrightarrow T(z)) \wedge S(x) \wedge C(T)(x) \wedge C(S)(z) \wedge S(z) \\
&\quad = WFCA(C) \wedge S(z) \wedge T(z) \wedge S(x) \wedge C(T)(x) \wedge C(S)(z) \wedge S(z) \\
&\quad \leq WFCA(C) \wedge T(z) \wedge S(x) \wedge C(T)(x) \wedge C(S)(z) \wedge S(z) \\
&\quad = C(S)(z) \wedge S(x) \wedge C(T)(x) \wedge T(z) \wedge WFCA(C) \wedge S(z) \\
&\quad \leq R(z,x) \wedge C(T)(x) \wedge T(z) \wedge WFCA(C) \wedge S(z) \\
&\quad \leq R(z,x) \wedge C(T)(x) \wedge T(z) \wedge (R(z,x) \wedge C(T)(x) \wedge T(z) \longrightarrow C(T)(z)) \wedge S(z) \\
&\quad = R(z,x) \wedge C(T)(x) \wedge T(z) \wedge C(T)(z) \wedge S(z) \\
&\quad \leq C(T)(z) \wedge S(z)
\end{aligned}$$

By Lemma 2.1-(ii) we have

$$WFCA(C) \wedge I(S,T) \wedge S(x) \wedge C(T)(x) \leq C(S)(z) \longrightarrow C(T)(z) \wedge S(z) \tag{4.1}$$

Next, the normality of C implies that for any $S \in \mathcal{B}$ there exists $t \in X$ such that $C(S)(t) = 1$ and hence $S(t) = 1$.
Then,

$$\begin{aligned}
&WFCA(C) \wedge I(S,T) \wedge S(x) \wedge C(T)(x) \wedge S(z) \wedge C(T)(z) \\
&\quad = C(T)(z) \wedge I(S,T) \wedge S(t) \wedge S(x) \wedge C(T)(x) \wedge S(z) \wedge WFCA(C) \\
&\quad \leq C(T)(z) \wedge S(t) \wedge (S(t) \longrightarrow T(t)) \wedge S(x) \wedge C(T)(x) \wedge S(z) \wedge WFCA(C) \\
&\quad = C(T)(z) \wedge S(t) \wedge T(t) \wedge S(x) \wedge C(T)(x) \wedge S(z) \wedge WFCA(C) \\
&\quad = C(T)(z) \wedge T(t) \wedge S(x) \wedge C(T)(x) \wedge S(z) \wedge WFCA(C) \\
&\quad \leq R(z,t) \wedge S(x) \wedge C(T)(x) \wedge S(z) \wedge WFCA(C) \\
&\quad \leq R(z,t) \wedge S(z) \wedge WFCA(C) \\
&\quad \leq R(z,t) \wedge S(z) \wedge (R(z,t) \wedge C(S)(t) \wedge S(z) \longrightarrow C(S)(z)) \\
&\quad = R(z,t) \wedge S(z) \wedge (R(z,t) \wedge S(z) \longrightarrow C(S)(z)) \\
&\quad = R(z,t) \wedge S(z) \wedge C(S)(z) \\
&\quad \leq C(S)(z)
\end{aligned}$$

Again by Lemma 2.1-(ii) we have

$$WFCA(C) \wedge I(S,T) \wedge S(x) \wedge C(T)(x) \leq S(z) \wedge C(T)(z) \longrightarrow C(S)(z) \tag{4.2}$$

By the idempotency of the Gödel t-norm and equation (4.1), we have

$$WFCA(C) \wedge I(S,T) \wedge S(x) \wedge C(T)(x) \leq S(z) \wedge C(T)(z) \longleftrightarrow C(S)(z)$$

The above inequality is true for any $z \in X$, therefore

$$WFCA(C) \wedge I(S,T) \wedge S(x) \wedge C(T)(x) \leq E(S \cap C(T), C(S))$$

By Lemma 2.1-(ii), we have

$$WFCA(C) \leq I(S,T) \wedge S(x) \wedge C(T)(x) \longrightarrow E(S \cap C(T) \longleftrightarrow C(S))$$

Thus,

$$WFCA(C) \leq FAA(C) \tag{4.3}$$

Now, we prove that $FAA(C) \leq WFCA(C)$. For any $S \in \mathcal{B}$ and $x \in X$, we have

$$\begin{aligned} FAA(C) \wedge R(x,y) \wedge & C(S)(y) \wedge S(x) \\ &= FAA(C) \wedge \bigvee_{T\in\mathcal{B}} [C(T)(x) \wedge T(y)] \wedge C(S)(y) \wedge S(x) \\ &= \bigvee_{T\in\mathcal{B}} [FAA(C) \wedge C(T)(x) \wedge T(y) \wedge C(S)(y) \wedge S(x)] \end{aligned} \tag{4.4}$$

Since the domain $\mathcal{B}$ of the fuzzy choice function C is closed under intersection, we have for $S, T \in \mathcal{B}$, $S \cap T \in \mathcal{B}$. Since $S \cap T \subseteq S, S \cap T \subseteq T$, we have, $I(S \cap T, S) = 1$ and $I(S \cap T, T) = 1$.
Now, for any $T \in \mathcal{B}$, we have

$$\begin{aligned} & FAA(C) \wedge C(T)(x) \wedge T(y) \wedge C(S)(y) \wedge S(x) \\ &= C(T)(x) \wedge T(y) \wedge C(S)(y) \wedge S(x) \wedge T(x) \wedge S(y) \wedge FAA(C) \\ &= (S(y) \wedge T(y) \wedge C(S)(y) \wedge FAA(C)) \wedge (S(x) \wedge T(x) \wedge C(T)(x) \wedge FAA(C)) \end{aligned}$$

Thus,

$$FAA(C) \wedge C(T)(x) \wedge T(y) \wedge C(S)(y) \wedge S(x) \tag{4.5}$$
$$= (S(y) \wedge T(y) \wedge C(S)(y) \wedge FAA(C)) \wedge (S(x) \wedge T(x) \wedge C(T)(x) \wedge FAA(C)) \tag{4.6}$$

Next,

$$\begin{aligned} & S(y) \wedge T(y) \wedge C(S)(y) \wedge FAA(C) \\ & \leq S(y) \wedge T(y) \wedge C(S)(y) \wedge [I(S \cap T, S) \wedge S(y) \wedge T(y) \wedge C(S)(y) \longrightarrow \\ & \quad E(C(S \cap T), S \cap T \cap C(S))] \\ & = S(y) \wedge T(y) \wedge C(S)(y) \wedge [S(y) \wedge T(y) \wedge C(S)(y) \longrightarrow \\ & \quad E(C(S \cap T), S \cap T \cap C(S))] \\ & = S(y) \wedge T(y) \wedge C(S)(y) \wedge E\left(C(S \cap T), S \cap T \cap C(S)\right) \\ & \leq E\left(C(S \cap T), S \cap T \cap C(S)\right) \end{aligned}$$

Thus,

$$S(y) \wedge T(y) \wedge C(S)(y) \wedge FAA(C) \leq E\left(C(S \cap T), S \cap T \cap C(S)\right)$$

Similarly, one has

$$S(x) \wedge T(x) \wedge C(T)(x) \wedge FAA(C) \leq E\left(C(S \cap T), S \cap T \cap C(T)\right)$$

Therefore,

$$
\begin{aligned}
FAA(C) \wedge C(T)(x) \wedge T(y) \wedge C(S)(y) \wedge S(x) & \\
&\leq E\left(C(S\cap T), S\cap T\cap C(S)\right) \wedge E\left(C(S\cap T), S\cap T\cap C(T)\right)\\
&= E\left(C(S\cap T), T\cap C(S)\right) \wedge E\left(C(S\cap T), S\cap C(T)\right)\\
&= E\left(S\cap C(T), C(S\cap T)\right) \wedge E\left(C(S\cap T), T\cap C(S)\right)\\
&\leq ((S\cap C(T))(x) \longrightarrow C(S\cap T)(x)) \wedge (C(S\cap T)(x)\\
&\qquad \longrightarrow (T\cap C(S))(x))\\
&\leq S(x)\wedge C(T)(x) \longrightarrow T(x)\wedge C(S)(x), \text{by Lemma 2.1-(vi)}
\end{aligned}
$$

Therefore, by Lemma 2.1-(i) and the idempotent property of the Gödel t-norm, we have

$$
\begin{aligned}
FAA(C) \wedge C(T)(x) \wedge T(y) \wedge C(S)(y) \wedge S(x) &\leq T(x) \wedge C(S)(x)\\
&\leq C(S)(x)
\end{aligned}
$$

Thus,

$$FAA(C) \wedge R(x,y) \wedge C(S)(y) \wedge S(x) \leq C(S)(x)$$

By Lemma 2.1-(ii), the above inequality reduces to

$$FAA(C) \leq R(x,y) \wedge C(S)(y) \wedge S(x) \longrightarrow C(S)(x)$$

Therefore $FAA(C) \leq WFCA(C)$. □

5. Conclusions

In classical choice theory the interrelations between the direct revelation axiom, consistent-closure coherence axiom, transitive-closure coherence axiom and intermediate congruence axiom are established. Also, their relation with the weak fuzzy congruence axiom, strong fuzzy congruence axiom and weak axiom of fuzzy revealed preference relation is established. It is expected that such theorems correspond to results that express relation between indicators. In this paper, we have introduced the indicators of the fuzzy direct revelation axiom, fuzzy transitive-closure coherence axiom, fuzzy consistent-closure coherence axiom and fuzzy intermediate congruence axiom. These indicators express the degree to which the fuzzy choice function C satisfies the direct revelation axiom, consistent-closure coherence axiom, transitive-closure coherence axiom and intermediate congruence axiom. The Theorem 3.3 generalizes the theorem that establishes the interrelations between the direct revelation axiom, consistent-closure coherence axiom, transitive-closure coherence axiom and intermediate congruence axiom. The equality $FTCCA(C) = SFCA(C)$ generalizes the theorem that establishes the equivalence between the transitive-closure coherence axiom and strong congruence axiom and the Theorem 3.5 gives the position of the degrees of FICA and FDRA towards the degree of WFCA, SFCA and WAFRP. Lastly, the equivalence between FAA and WFCA on arbitrary domain is given.

The idempotent property of the Gödel t-norm appears essentially in the proof of Theorem 4.3. An open problem is to check whether this theorem still holds for other continuous t-norms. Wu et al. [32] follow Banerjee's approach and generalize some results of Georgescu. However, in this paper we are concerned with a more general fuzzy choice function defined by Georgescu. As an open problem we propose to follow Banerjee's approach and prove the results Section 4 for an arbitrary continuous t-norm.

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S. S. Desai (santosh_desai21@rediffmail.com)
Department of Mathematics, Tatyasaheb Kore Institute of Engineering and Technology, Warananagar, Kolhapur, Pin: 416113 (M.S.), India

S. R. Chaudhari (shrikant_chaudhari@yahoo.com)
Department of Mathematics, School of Mathematical Sciences, North Maharashtra University, Jalgaon (M.S.)-425001, India