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Characterizations of regular semigroups by interval valued fuzzy ideals with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$

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Abstract. In this paper we introduce the concepts of interval valued fuzzy ideals with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$, interval valued fuzzy interior ideals with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$, interval valued fuzzy quasi ideals with htresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ and interval valued fuzzy generalized bi-ideals with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$, which are generalization of interval valued fuzzy ideals, interval valued fuzzy interior ideals, interval valued fuzzy quasi ideals and interval valued fuzzy generalized bi-ideals, in a semigroup are introduced, and related properties are investigated. Characterizations of regular semigroups by the properties of interval valued fuzzy left (right, interior, bi, generalized bi, quasi) ideals with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ are given.

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1. Introduction

The fundamental concept of a fuzzy set was introduced by L. A. Zadeh in his definitive paper [21] of 1965, provides a natural frame-work for generalizing several basic notions of algebra. Murali [13] proposed a definition of a fuzzy point belonging to fuzzy subset under a natural equivalence on fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [15], played a vital role to generate some different types of fuzzy subgroups. Bhakat and Das [5] gave the concepts of (α, β)-fuzzy subgroups by using the "belongs to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroup. In [6] $(\in, \in \vee q)$-fuzzy subrings and ideals are defined. In [7] Davvaz define $(\in, \in \vee q)$-fuzzy subnearrings and ideals of a nearring. In [11] Jun and Song initiated the study of (α, β)-fuzzy interior ideals of a semigroup. In [4] Bhakat defined $(\in \vee q)$-level subset of a fuzzy set. In [16], Shabir et al, characterized regular semigroups by the properties of (α, β)-fuzzy ideals, bi-ideals and quasi-ideals. Abdullah et al, introduced a new type of fuzzy normal subgroups and fuzzy coset [1]. In [2], Abdullah et al defined direct product of finite fuzzy ideals of LA-semigroups. In [20], Shabir et al, characterized semigroups by the properties of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ideals, generalized bi-ideals and quasi-ideals of a semigroup. Shabir et. al. defined some types of $(\in, \in \vee q_k)$-fuzzy ideals of a semigroup and characterized regular semigroups by these ideals [18]. Davvaz introduced fuzzy R-subgroups with threshold of near-rings and implications operators [8]. In [19], Shabir and Mehmood studied $(\in, \in \vee q_k)$-fuzzy h-ideals of hemirings and characterized different classes of hemirings by the properties of $(\in, \in \vee q_k)$-fuzzy h-ideals. Hong et al. studied the concept of fuzzy interior ideal of semigroups in [9]. The notion of interval valued fuzzy sets was first introduced by Zadeh [22] as an extension of fuzzy sets in which the values of the membership degrees are intervals of numbers instead of the numbers. Thus, interval valued fuzzy sets provide a more adequate description of uncertainty than the traditional fuzzy sets. Interval valued fuzzy set theory has been shown to be a useful tool to describe situations in which the data are imprecise or vague. In [14], Narayanan and Manikantan initiated the notion of interval valued fuzzy ideals in semigroup. In 2008, Shabir and Khan introduced the concept of interval valued fuzzy ideals generated by interval valued fuzzy subset in ordered semigroup [17]. Recently, Hedayati initiated interval valued (α,β)-fuzzy bi-ideals of semigroups and obtained some fundamental results [10]. In [3], Aslam et al defined interval valued $(\overline{\alpha}, \overline{\beta})$-fuzzy ideals of LA-semigroups and charcterized regular LA-semigroup by these ideals.

In this paper we initiate the notions of interval valued fuzzy ideals with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$, interval valued fuzzy interior ideals with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$, interval valued fuzzy quasi ideals with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ and interval valued fuzzy generalized bi-ideals with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$, which are generalization of interval valued fuzzy ideals, interval valued fuzzy interior ideals, interval valued fuzzy quasi ideals and interval valued fuzzy generalized bi-ideals, in a semigroup are introduced, and related properties are investigated. We give some characterization of regular and intra-regular semigroups by the properties of interval valued fuzzy ideals with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$.

2. Preliminaries

A semigroup is an algebraic system $(S, .)$ consisting of a non-empty set S together with an associative binary operation ".". By a subsemigroup of S we mean a nonempty subset A of S such that $A^2 \subseteq A$. A nonempty subset A of S is called a left (*right*) *ideal* of S if $SA \subseteq A$ $(AS \subseteq A)$. A non-empty subset A of S is called a *two-sided ideal* or simply an ideal of S if it is both a left and a right ideal of S. A nonempty subset Q of S is called a *quasi-ideal* of S if $QS \cap SQ \subseteq Q$. A subsemigroup B of a semigroup S is called a *bi-ideal* of S if $BSB \subseteq B$. A nonempty subset B of S is called a *generalized bi-ideal* of S if $BSB \subseteq B$. A subsemigroup A of a semigroup S is called an *interior ideal* of S if $SAS \subseteq A$. Obviously every one-sided ideal of a semigroup S is a *quasi-ideal*, every *quasi-ideal* is a *bi-ideal* and every *bi-ideal* is a *generalized bi-ideal*. An element a of a semigroup S is called a regular element if there exists an element x in S such that $a = axa$. A semigroup S is called regular if every element of S is regular. It is well known that for a regular semigroup the concepts of quasi-ideal, bi-ideal and generalized bi-ideal coincide.

A fuzzy subset λ of a universe X is a function from X into the unit closed interval $[0,1]$, i.e. $\lambda : X \to [0,1]$. For any two fuzzy subsets λ and μ of S, $\lambda \leq \mu$ means that, for all $x \in S$, $\lambda(x) \leq \mu(x)$. The symbols $\lambda \wedge \mu$, and $\lambda \vee \mu$ will mean the following fuzzy subsets of S

$$(\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x)$$
$$(\lambda \vee \mu)(x) = \lambda(x) \vee \mu(x).$$

for all $x \in S$.

Let λ and μ be two fuzzy subsets of a semigroup S. The product $\lambda \circ \mu$ is defined by

$$(\lambda \circ \mu)(x) = \begin{cases} \bigvee\limits_{x=yz} \{\lambda(y) \wedge \mu(z)\}, & \text{if } \exists\ y, z \in S, \text{ such that } x = yz \\ 0 & \text{otherwise} \end{cases}$$

By an *interval number* D we mean an interval $[\alpha^-, \alpha^+]$, where $0 \leq \alpha^- \leq \alpha^+ \leq 1$. For *interval numbers* $D_1 = [\alpha^-, \alpha^+]$, $D_2 = [\beta^-, \beta^+] \in D[0,1]$, where $D[0,1]$ denotes the set of all closed subintervals of the interval $[0,1]$, we define

$$\begin{aligned}
D_1 \cap D_2 &= \min(D_1,\ D_2) = \min\left(\left[\alpha^-, \alpha^+\right],\ \left[\beta^-, \beta^+\right]\right) \\
&= \left[\min\left\{\alpha^-, \beta^-\right\},\ \min\left\{\alpha^+, \beta^+\right\}\right], \\
D_1 \cup D_2 &= \max(D_1,\ D_2) = \max\left(\left[\alpha^-, \alpha^+\right],\ \left[\beta^-, \beta^+\right]\right) \\
&= \left[\max\left\{\alpha^-, \beta^-\right\},\ \max\left\{\alpha^+, \beta^+\right\}\right], \\
D_1 &\leq D_2 \Leftrightarrow \alpha^- \leq \beta^- \text{ and } \alpha^+ \leq \beta^+, \\
D_1 &< D_2 \Leftrightarrow \alpha^- < \beta^- \text{ and } \alpha^+ < \beta^+, \\
D_1 &= D_2 \Leftrightarrow \alpha^- = \beta^- \text{ and } \alpha^+ = \beta^+, \\
mD &= m\left[\alpha^-, \alpha^+\right] = \left[m\alpha^-, m\alpha^+\right], \text{ where } 0 \leq m \leq 1, \\
D_1 + D_2 &= \left[\alpha^- + \beta^- - \alpha^-.\beta^-,\ \alpha^+ + \beta^+ - \alpha^+.\beta^+\right].
\end{aligned}$$

Let X be a given non-empty set. An interval valued fuzzy set $\widetilde{\lambda}$ on X is defined by

$$B = \left\{\left(x,\ \left[\lambda_B^-(x), \lambda_B^+(x)\right]\right) : x \in X\right\},$$

where $\lambda_B^-(x)$ and $\lambda_B^+(x)$ are fuzzy sets of X such that $\lambda_B^-(x) \leq \lambda_B^+(x)$ for all $x \in X$. Let $\widetilde{\lambda}_B(x) = \left[\lambda_B^-(x), \lambda_B^+(x)\right]$. Then,

$$B = \left\{\left(x,\ \widetilde{\lambda}_B(x)\right) : x \in X\right\},$$

where $\widetilde{\lambda}_B : X \longrightarrow D[0,1]$.

An interval valued fuzzy subset $\widetilde{\lambda}$ in a universe X of the form

$$\widetilde{\lambda}(y) = \{.\widetilde{t} \in D(0,1] \quad \text{if } y = x\widetilde{0} \qquad\qquad \text{if } y \neq x$$

is said to be an interval valued fuzzy point with support x and value $\widetilde{t}$ and is denoted by $x_{\widetilde{t}}$. An interval valued fuzzy point $x_{\widetilde{t}}$ is said to belong to (resp. quasi-coincident with) an interval valued fuzzy set $\widetilde{\lambda}$ written $x_{\widetilde{t}} \in \widetilde{\lambda}$ (resp. $x_{\widetilde{t}} q\widetilde{\lambda}$) if $\widetilde{\lambda}(x) \geq \widetilde{t}$ resp. $(\widetilde{\lambda}(x) + \widetilde{t} > \widetilde{1})$, and in this case, $x_{\widetilde{t}} \in \vee q\widetilde{\lambda}$ (resp. $x_{\widetilde{t}} \in \wedge q\widetilde{\lambda}$) means that $x_{\widetilde{t}} \in \widetilde{\lambda}$ or $x_{\widetilde{t}} q\widetilde{\lambda}$ (resp. $x_{\widetilde{t}} \in \widetilde{\lambda}$ and $x_{\widetilde{t}} q\widetilde{\lambda}$) . To say that $x_{\widetilde{t}} \overline{\alpha} \widetilde{\lambda}$ means that $x_{\widetilde{t}} \alpha \widetilde{\lambda}$ does not hold. For any two interval valued fuzzy subsets $\widetilde{\lambda}$ and $\widetilde{\mu}$ of S, $\widetilde{\lambda} \leq \widetilde{\mu}$ means that, for all $x \in S$, $\widetilde{\lambda}(x) \leq \widetilde{\mu}(x)$. The symbols $\widetilde{\lambda} \wedge \widetilde{\mu}$, and $\widetilde{\lambda} \vee \widetilde{\mu}$ will mean the following interval valued fuzzy subsets of S

$$(\widetilde{\lambda} \wedge \widetilde{\mu})(x) = \widetilde{\lambda}(x) \wedge \widetilde{\mu}(x)$$
$$(\widetilde{\lambda} \vee \widetilde{\mu})(x) = \widetilde{\lambda}(x) \vee \widetilde{\mu}(x).$$

for all $x \in S$.

Let $\widetilde{\lambda}$ and $\widetilde{\mu}$ be two interval valued fuzzy subsets of a semigroup S. The product $\widetilde{\lambda} \circ \widetilde{\mu}$ is defined by

$$\left(\widetilde{\lambda} \circ \widetilde{\mu}\right)(x) = \begin{cases} \bigvee\limits_{x=yz} \left\{\widetilde{\lambda}(y) \wedge \widetilde{\mu}(z)\right\}, & \text{if } \exists\ y,z \in S, \text{ such that } x = yz \\ \widetilde{0} & \text{otherwise} \end{cases}$$

Throughout this paper, the following notations will be used.

(1) We use $\widetilde{0}$ to denote the interval valued fuzzy empty set and $\widetilde{1}$ to denote the interval valued fuzzy whole set in a set X, and define $\widetilde{0}(x) = [0,0]$ and $\widetilde{1}(x) = [1,1]$, for all $x \in X$.

(2) We write $\widetilde{t} = [t_1, t_2]$ and $\widetilde{s} = [s_1, s_2] \in D[0,1]$.

Definition 2.1 ([12]). A fuzzy subset λ of S is called a fuzzy subsemigroup of S if for all $x, y \in S$.

$$\lambda(xy) \geq \min\{\lambda(x), \lambda(y)\}$$

Definition 2.2 ([12]). A fuzzy subset λ of S is called a fuzzy left (right) ideal of S if for all $x, y \in S$.

$$\lambda(xy) \geq \lambda(y) \qquad (\lambda(xy) \geq \lambda(x))$$

A fuzzy subset λ of S is called a fuzzy ideal of S if it is both a fuzzy left and a fuzzy right ideal of S.

Definition 2.3 ([12]). A fuzzy subset λ of S is called a fuzzy quasi-ideal of S if

$$(\lambda \circ \mathcal{S}) \wedge (\mathcal{S} \circ \lambda) \leq \lambda$$

where $\mathcal{S}$ is the fuzzy subset of S mapping every element of S on 1.

Definition 2.4 ([12]). A fuzzy subsemigruop λ of S is called a fuzzy bi-ideal of S if for all $x, y, z \in S$.

$$\lambda(xyz) \geq \min\{\lambda(x), \lambda(z)\}$$

Definition 2.5 ([12]). A fuzzy subset λ of S is called a fuzzy generalized bi-ideal of S if for all $x, y, z \in S$,

$$\lambda(xyz) \geq \min\{\lambda(x), \lambda(z)\}$$

Definition 2.6 ([12]). A fuzzy subsemigruop $\widetilde{\lambda}$ of S is called a fuzzy interior ideal of S if for all $x, a, y \in S$,

$$\lambda(xay) \geq \lambda(a)$$

Theorem 2.7. *For a semigroup S the following conditions are equivalent.*
(1) *S is regular.*
(2) *$R \cap L = RL$ for every right ideal R and every left ideal L of S.*
(3) *$ASA = A$ for every quasi-ideal A of S.*

3. Major section

In this section we defined interval valued fuzzy left (right, interior, generalized, quasi) ideals with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ and investigated some useful results.

Definition 3.1. An interval valued fuzzy subset $\widetilde{\lambda}$ of a semigroup S is called an interval valued fuzzy subsemigroup with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S, where $\widetilde{\alpha} < \widetilde{\beta}$ and $\widetilde{\alpha}, \widetilde{\beta} \in D[0,1]$, if it satisfies the following condition
$\max\left\{\widetilde{\lambda}(xy), \widetilde{\alpha}\right\} \geq \min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \widetilde{\beta}\right\}$.

Definition 3.2. An interval valued fuzzy subset $\widetilde{\lambda}$ of a semigroup S is called an interval valued fuzzy left (right) ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S, where $\widetilde{\alpha} < \widetilde{\beta}$ and $\widetilde{\alpha}, \widetilde{\beta} \in D[0,1]$, if it satisfies the following condition for all $x, y \in S$ $\max\left\{\widetilde{\lambda}(xy), \widetilde{\alpha}\right\} \geq \min\left\{\widetilde{\lambda}(y), \widetilde{\beta}\right\}$ $\left(\max\left\{\widetilde{\lambda}(xy), \widetilde{\alpha}\right\} \geq \min\left\{\widetilde{\lambda}(x), \widetilde{\beta}\right\}\right)$.

An interval valued fuzzy subset $\widetilde{\lambda}$ of a semigroup S is called an interval valued fuzzy ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S if it is both an interval valued fuzzy left ideal and interval valued fuzzy right ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S.

Definition 3.3. An interval valued fuzzy subset $\widetilde{\lambda}$ of a semigroup S is called an interval valued fuzzy interior ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S, where $\widetilde{\alpha} < \widetilde{\beta}$ and $\widetilde{\alpha}, \widetilde{\beta} \in D[0,1]$, if it satisfies the following conditions for all $x, a, y \in S$
(1) $\max\left\{\widetilde{\lambda}(xy), \widetilde{\alpha}\right\} \geq \min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \widetilde{\beta}\right\}$,
(2) $\max\left\{\widetilde{\lambda}(xay), \widetilde{\alpha}\right\} \geq \min\left\{\widetilde{\lambda}(a), \widetilde{\beta}\right\}$.

Definition 3.4 ([10]). An interval valued fuzzy subset $\widetilde{\lambda}$ of a semigroup S is called an interval valued fuzzy bi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S, where $\widetilde{\alpha} < \widetilde{\beta}$ and $\widetilde{\alpha}, \widetilde{\beta} \in D[0,1]$, if it satisfies the following conditions for all $x, y, z \in S$
(1) $\max\left\{\widetilde{\lambda}(xy), \widetilde{\alpha}\right\} \geq \min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \widetilde{\beta}\right\}$,
(2) $\max\left\{\widetilde{\lambda}(xyz), \widetilde{\alpha}\right\} \geq \min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \widetilde{\beta}\right\}$.

Definition 3.5. An interval valued fuzzy subset $\widetilde{\lambda}$ of a semigroup S is called an interval valued fuzzy generalized bi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S, where $\widetilde{\alpha} < \widetilde{\beta}$ and $\widetilde{\alpha}, \widetilde{\beta} \in D[0,1]$, if it satisfies the following condition for all $x, y, z \in S$,

$$\max\left\{\widetilde{\lambda}(xyz), \widetilde{\alpha}\right\} \geq \min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \widetilde{\beta}\right\}.$$

Theorem 3.6. *Let $\widetilde{\alpha}, \widetilde{\beta} \in D[0,1]$ and $\widetilde{\alpha} < \widetilde{\beta}$ and $\widetilde{\lambda}$ be a nonzero interval valued fuzzy subsemigroup with threshholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S. Then, the set $\widetilde{\lambda}_{\widetilde{\alpha}} = \left\{x \in S \mid \widetilde{\lambda}(x) > \widetilde{\alpha}\right\}$ is a subsemigroup of S.*

Proof. Let $x, y \in \widetilde{\lambda}_{\widetilde{\alpha}}$. Then, $\widetilde{\lambda}(x) > \widetilde{\alpha}$ and $\widetilde{\lambda}(y) > \widetilde{\alpha}$. Since $\widetilde{\lambda}$ be an interval valued fuzzy subsemigroup with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S, so we have $\max\left\{\widetilde{\lambda}(xy), \widetilde{\alpha}\right\} \geq \min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \widetilde{\beta}\right\} > \min\left\{\widetilde{\alpha}, \widetilde{\beta}\right\} = \widetilde{\alpha}$. This implies that $\widetilde{\lambda}(xy) > \widetilde{\alpha}$. So $xy \in \widetilde{\lambda}_{\widetilde{\alpha}}$. Thus, $\widetilde{\lambda}_{\widetilde{\alpha}}$ is a subsemigroup of S. □

Theorem 3.7. *Let $\widetilde{\alpha}, \widetilde{\beta} \in D[0,1]$ and $\widetilde{\alpha} < \widetilde{\beta}$ and $\widetilde{\lambda}$ be a nonzero interval valued fuzzy generalized bi-ideal with threshholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S. Then, the set $\widetilde{\lambda}_{\widetilde{\alpha}} = \left\{x \in S \mid \widetilde{\lambda}(x) > \widetilde{\alpha}\right\}$ is a generalized bi-ideal of S.*

Proof. Let $x, z \in \widetilde{\lambda}_{\widetilde{\alpha}}$. Then, $\widetilde{\lambda}(x) > \widetilde{\alpha}$ and $\widetilde{\lambda}(z) > \widetilde{\alpha}$. Since $\widetilde{\lambda}$ is interval valued fuzzy generalized bi-ideals with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S, so we have $\max\left\{\widetilde{\lambda}(xyz), \widetilde{\alpha}\right\} \geq \min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \widetilde{\beta}\right\} > \min\left\{\widetilde{\alpha}, \widetilde{\beta}\right\} = \widetilde{\alpha}$. Which implies that $\widetilde{\lambda}(xyz) > \widetilde{\alpha}$. So $xyz \in \widetilde{\lambda}_{\widetilde{\alpha}}$. Thus, $\widetilde{\lambda}_{\widetilde{\alpha}}$ is a generalized bi-ideal of S. □

Theorem 3.8. *Let $\widetilde{\alpha}, \widetilde{\beta} \in D[0,1]$ and $\widetilde{\alpha} < \widetilde{\beta}$ and $\widetilde{\lambda}$ be a nonzero interval valued fuzzy bi-ideal with threshholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S. Then, the set $\widetilde{\lambda}_{\widetilde{\alpha}} = \left\{x \in S \mid \widetilde{\lambda}(x) > \widetilde{\alpha}\right\}$ is a bi-ideal of S.*

Proof. Follows from Theorem 3.6 and Theorem 3.7. □

Theorem 3.9. *Let $\widetilde{\alpha}, \widetilde{\beta} \in D[0,1]$ and $\widetilde{\alpha} < \widetilde{\beta}$ and $\widetilde{\lambda}$ be a nonzero interval valued fuzzy left (resp. right) ideal with threshholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S. Then, the set $\widetilde{\lambda}_{\alpha} = \left\{x \in S \mid \widetilde{\lambda}(x) > \widetilde{\alpha}\right\}$ is a left (resp. right) ideal of S.*

Proof. Follows from Theorem 3.6 and Theorem 3.7. □

Theorem 3.10. *Let L be a left (resp. right) ideal of S and let $\widetilde{\lambda}$ be an interval valued fuzzy subset in S such that,*

$$\begin{aligned} \widetilde{\lambda}(x) &\geq \widetilde{\beta} && \text{if } x \in L \\ \widetilde{\lambda}(x) &\leq \widetilde{\alpha} && \text{if } x \in S \backslash L \end{aligned}$$

Then, $\widetilde{\lambda}$ is an interval valued fuzzy left (resp. right) ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S.

Proof. (1) On the contrary suppose that there exist $x, y \in S$ such that

$$\max\left\{\widetilde{\lambda}(xy), \widetilde{\alpha}\right\} < \min\left\{\widetilde{\lambda}(y), \widetilde{\beta}\right\}.$$

Then, $y \in L$ and $xy \notin L$. This is a contradiction to the fact that L is a left ideal of S. So, $\max\left\{\widetilde{\lambda}(xy), \widetilde{\alpha}\right\} \geq \min\left\{\widetilde{\lambda}(y), \widetilde{\beta}\right\}$. Thus, $\widetilde{\lambda}$ is an interval valued fuzzy left ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S. □

Similarly, we can prove the following Theorems.

Theorem 3.11. *Let A be a subsemigroup of S and let $\widetilde{\lambda}$ be an interval valued fuzzy subset in S such that*

$$\begin{array}{rcll} \widetilde{\lambda}(x) & \geq & \widetilde{\beta} & \textit{if } x \in A \\ \widetilde{\lambda}(x) & \leq & \widetilde{\alpha} & \textit{if } x \in S \backslash A \end{array}$$

Then, $\widetilde{\lambda}$ is an interval valued fuzzy subsemigroup with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S.

Theorem 3.12. *Let B be a generalized bi-ideal of S and let $\widetilde{\lambda}$ be an interval valued fuzzy subset in S such that*

$$\begin{array}{rcll} \widetilde{\lambda}(x) & \geq & \widetilde{\beta} & \textit{if } x \in B \\ \widetilde{\lambda}(x) & \leq & \widetilde{\alpha} & \textit{if } x \in S \backslash B \end{array}$$

Then, $\widetilde{\lambda}$ is an interval valued fuzzy generalized bi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S.

Theorem 3.13. *Let B be a bi-ideal of S and let $\widetilde{\lambda}$ be an interval valued fuzzy subset in S such that*

$$\begin{array}{rcll} \widetilde{\lambda}(x) & \geq & \widetilde{\beta} & \textit{if } x \in B \\ \widetilde{\lambda}(x) & \leq & \widetilde{\alpha} & \textit{if } x \in S \backslash B \end{array}$$

Then, $\widetilde{\lambda}$ is an interval valued fuzzy bi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S.

Definition 3.14. Let $\widetilde{\lambda}$ and $\widetilde{\mu}$ be interval valued fuzzy subsets of S. Then, define $\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}$ as

$$\left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)(x) = \begin{cases} \bigvee_{x=yz}(\left\{\widetilde{\lambda}(y) \wedge \widetilde{\mu}(z) \wedge \widetilde{\beta}\right\} \vee \widetilde{\alpha}), & \text{if } \exists\, y, z \in S, \text{ such that } x = yz \\ \widetilde{\alpha} & \text{otherwise} \end{cases}$$

Theorem 3.15. *If $\widetilde{\lambda}$ is an interval valued fuzzy left ideal and $\widetilde{\mu}$ is an interval valued fuzzy right ideal with threshholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of a semigroup S, then $\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}$ is an interval valued fuzzy two-sided ideal with threshholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S.*

Proof. Let $x, y \in S$. Then,

$$\begin{aligned}\left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)(y) \wedge \widetilde{\beta} &= [(\bigvee_{y=pq} \{\widetilde{\lambda}(p) \wedge \widetilde{\mu}(q) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha})] \wedge \widetilde{\beta} \\ &= [(\bigvee_{y=pq} \{\widetilde{\lambda}(p) \wedge \widetilde{\beta} \wedge \widetilde{\mu}(q)\} \vee \widetilde{\alpha})] \wedge \widetilde{\beta}.\end{aligned}$$

(If $y = pq$, then $xy = x(pq) = (xp)q$. Since $\widetilde{\lambda}$ is an interval valued fuzzy left ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$, so by definition $\max\{\widetilde{\lambda}(xp), \widetilde{\alpha}\} \geq \min\{\widetilde{\lambda}(p), \widetilde{\beta}\}$). Thus,

$$\begin{aligned}\left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)(y) \wedge \widetilde{\beta} &= \left[\bigvee_{y=pq} \{\widetilde{\lambda}(p) \wedge \widetilde{\beta} \wedge \widetilde{\mu}(q)\} \vee \widetilde{\alpha}) \wedge \widetilde{\beta}\right] \wedge \widetilde{\beta} \\ &\leq \left[\bigvee_{xy=xpq} \{(\widetilde{\lambda}(xp) \vee \widetilde{\alpha}) \wedge \widetilde{\beta} \wedge \widetilde{\mu}(q)\} \vee \widetilde{\alpha})\right] \wedge \widetilde{\beta} \\ &= \left[\bigvee_{xy=xpq} \{(\widetilde{\lambda}(xp) \vee \widetilde{\alpha}) \wedge (\widetilde{\beta} \vee \widetilde{\alpha}) \wedge (\widetilde{\mu}(q) \vee \widetilde{\alpha})\})\right] \wedge \widetilde{\beta} \\ &\leq \left[\bigvee_{xy=ab} \{\widetilde{\lambda}(a) \wedge \widetilde{\mu}(b) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha})\right] \wedge \widetilde{\beta} \\ &= \left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)(xy) \wedge \widetilde{\beta} \\ &\leq \left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)(xy) \\ &\leq \left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)(xy) \vee \widetilde{\alpha}.\end{aligned}$$

So,

$$\min\{\left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)(y), \widetilde{\beta}\} \leq \max\{\left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)(xy), \widetilde{\alpha}\}.$$

Similarly, we can show that $\max\{\left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)(xy), \widetilde{\alpha}\} \geq \min\{\left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)(x), \widetilde{\beta}\}$.

Thus $\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}$ is an interval valued fuzzy two-sided ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ *of* S. □

Lemma 3.16. *Intersection of interval valued fuzzy left (resp. right) ideals with thresholds* $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ *of a semigroup* S *is an interval valued fuzzy left (resp. right) ideal with thresholds* $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ *of* S.

Proof. Let $\left\{\widetilde{\lambda}_i\right\}_{i \in I}$ be a family of interval valued fuzzy left ideals with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S. Let $x, y \in S$.

Then,

$$\left(\bigwedge_{i \in I} \widetilde{\lambda}_i\right)(xy) = \bigwedge_{i \in I} \left(\widetilde{\lambda}_i(xy)\right)$$

(Since each $\widetilde{\lambda}_i$ is an interval valued fuzzy left ideal of S, so $\max\{\widetilde{\lambda}_i(xy), \widetilde{\alpha}\} \geq \min\{\widetilde{\lambda}_i(y), \widetilde{\beta}\}$ for all $i \in I$). Thus,

$$\begin{aligned}
\left(\bigwedge_{i\in I} \widetilde{\lambda}_i\right)(xy) \vee \widetilde{\alpha} &= [\bigwedge_{i\in I}\left(\widetilde{\lambda}_i(xy)\right)] \vee \widetilde{\alpha} \\
&= \bigwedge_{i\in I}\left(\widetilde{\lambda}_i(xy)\right) \vee \widetilde{\alpha}) \\
&\geq \bigwedge_{i\in I}\left(\widetilde{\lambda}_i(y) \wedge \widetilde{\beta}\right) \\
&= \left(\bigwedge_{i\in I} \widetilde{\lambda}_i(y)\right) \wedge \widetilde{\beta} \\
&= \left(\bigwedge_{i\in I} \widetilde{\lambda}_i\right)(y) \wedge \widetilde{\beta}.
\end{aligned}$$

Hence, $\bigwedge_{i\in I} \widetilde{\lambda}_i$ is an interval valued fuzzy left ideal with threshold $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S. □

Similarly, we can prove that intersection of interval valued fuzzy right ideals with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of a semigroup S is an interval valued fuzzy right ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S. Thus intersection of interval valued fuzzy two-sided ideals with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of a semigroup S is an interval valued fuzzy two-sided ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S.

Lemma 3.17. *Let $\widetilde{\lambda}$ and $\widetilde{\mu}$ be interval valued fuzzy left ideals with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S. Then, $\widetilde{\lambda} \wedge^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mu}$ is an interval valued fuzzy left ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S, where $\widetilde{\lambda} \wedge^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mu}$ is defined as*

$$\left(\widetilde{\lambda} \wedge^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mu}\right)(x) = \max\left\{\min\{\widetilde{\lambda}(x), \widetilde{\mu}(x), \widetilde{\beta}\right\}, \widetilde{\alpha}\}$$

Proof. Let $x, y \in S$. Then,

$$\begin{aligned}
\left(\widetilde{\lambda} \wedge^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mu}\right)(xy) \vee \widetilde{\alpha} &= \{\left(\widetilde{\lambda}(xy) \wedge \widetilde{\mu}(xy) \wedge \widetilde{\beta}\right) \vee \widetilde{\alpha}\} \vee \widetilde{\alpha} \\
&= \left(\widetilde{\lambda}(xy) \wedge \widetilde{\mu}(xy) \wedge \widetilde{\beta}\right) \vee \widetilde{\alpha} \\
&= \left(\widetilde{\lambda}(xy) \vee \widetilde{\alpha}) \wedge (\widetilde{\mu}(xy) \vee \widetilde{\alpha}) \wedge (\widetilde{\beta} \vee \widetilde{\alpha}\right) \\
&\geq \{(\widetilde{\lambda}(y) \wedge \widetilde{\beta}) \vee \widetilde{\alpha}\} \wedge \{(\widetilde{\mu}(y) \wedge \widetilde{\beta}) \vee \widetilde{\alpha}\} \wedge (\widetilde{\beta} \vee \widetilde{\alpha}) \\
&= \{\widetilde{\lambda}(y) \wedge \widetilde{\mu}(y) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \left(\widetilde{\lambda} \wedge^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mu}\right)(y) \\
&\geq \left(\widetilde{\lambda} \wedge^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mu}\right)(y) \wedge \widetilde{\beta}.
\end{aligned}$$

Thus, $\widetilde{\lambda} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}$ is an interval valued fuzzy left ideal with threshold $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S. □

It is clear that every interval valued fuzzy bi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of a semigroup S is an interval valued fuzzy generalized bi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S. The next example shows that the interval valued fuzzy generalized bi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S is not necessarily an interval valued fuzzy bi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S.

Example 3.18. Consider the semigroup $S = \{a, b, c, d\}$.

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Let $\widetilde{\lambda}$ be an interval valued fuzzy subset of S such that

$$\widetilde{\lambda}(a) = [0.4, 0.5], \quad \widetilde{\lambda}(b) = [0.1, 0.2], \quad \widetilde{\lambda}(c) = [0.2, 0.3], \quad \widetilde{\lambda}(d) = [0, 0]$$

Then, $\widetilde{\lambda}$ is an interval valued fuzzy generalized bi-ideal with thresholds $\left(\widetilde{\alpha} = \widetilde{0.1}, \widetilde{\beta} = \widetilde{0.5}\right)$ of S. Because $\max\{\widetilde{\lambda}(xyz), \widetilde{0.1}\} = \widetilde{\lambda}(a) \vee \widetilde{0.1} = [0.4, 0.5] \geq \widetilde{\lambda}(x) \wedge \widetilde{\lambda}(z) \wedge \widetilde{0.5}$. But $\widetilde{\lambda}$ is not an interval valued fuzzy bi-ideal with thresholds $\left(\widetilde{\alpha} = \widetilde{0.1}, \widetilde{\beta} = \widetilde{0.5}\right)$ of S. Because $\widetilde{\lambda}(cc) \vee \widetilde{0.1} = \widetilde{\lambda}(b) = [0.1, 0.2] \ngeq [0.2, 0.3] = \widetilde{\lambda}(c) \wedge \widetilde{\lambda}(c) \wedge \widetilde{0.5}$.

Lemma 3.19. *Every interval valued fuzzy generalized bi-ideal with thresholds* $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ *of a regular semigroup* S *is an interval valued fuzzy bi-ideal with thresholds* $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ *of* S.

Proof. Let $\widetilde{\lambda}$ be any interval valued fuzzy generalized bi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S and let a, b be any elements of S. Then, there exists an element $x \in S$ such that $b = bxb$. Thus, we have $\widetilde{\lambda}(ab) \vee \widetilde{\alpha} = \widetilde{\lambda}(a(bxb)) \vee \widetilde{\alpha} = \widetilde{\lambda}(a(bx)b) \vee \widetilde{\alpha} \geq \min\{\widetilde{\lambda}(a), \widetilde{\lambda}(b), \widetilde{\beta}\}$. This shows that $\widetilde{\lambda}$ is an interval valued fuzzy subsemigroup with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S and so $\widetilde{\lambda}$ is an interval valued fuzzy bi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S. □

Definition 3.20. An interval valued fuzzy subset $\widetilde{\lambda}$ of a semigroup S is called an interval valued fuzzy quasi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S, if it satisfies,

$$\max\{\widetilde{\lambda}(x), \widetilde{\alpha}\} \geq \min\{\left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mathcal{S}}\right)(x), \left(\widetilde{\mathcal{S}} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda}\right))(x)\}.$$

Where $\widetilde{\mathcal{S}}$ is an interval valued fuzzy subset of S mapping every element of S on $\widetilde{1}$.

Theorem 3.21. *Let $\widetilde{\lambda}$ be an interval valued fuzzy quasi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of a semigroup S. Then, the set $\widetilde{\lambda}_{\widetilde{\alpha}} = \left\{x \in S \mid \widetilde{\lambda}(x) > \widetilde{\alpha}\right\}$ is a quasi-ideal of S.*

Proof. In order to show that $\widetilde{\lambda}_{\widetilde{\alpha}}$ is a quasi-ideal of S, we have to show that $S\widetilde{\lambda}_{\widetilde{\alpha}} \cap \widetilde{\lambda}_{\widetilde{\alpha}}S \subseteq \widetilde{\lambda}_{\widetilde{\alpha}}$. Let $a \in S\widetilde{\lambda}_{\widetilde{\alpha}} \cap \widetilde{\lambda}_{\widetilde{\alpha}}S$. This implies that $a \in S\widetilde{\lambda}_{\widetilde{\alpha}}$ and $a \in \widetilde{\lambda}_{\widetilde{\alpha}}S$. So, $a = sx$ and $a = yt$ for some $s, t \in S$ and $x, y \in \widetilde{\lambda}_{\alpha}$. Thus $\widetilde{\lambda}(x) > \widetilde{\alpha}$ and $\widetilde{\lambda}(y) > \widetilde{\alpha}$. Now $\max\{\widetilde{\lambda}(a), \widetilde{\alpha}\} \geq \min\{\left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mathcal{S}}\right)(x), \left(\widetilde{\mathcal{S}} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda}\right))(x)\}$.

Since

$$\begin{aligned}
\left(\widetilde{\mathcal{S}} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda}\right)(a) &= \bigvee_{a=pq} \left(\left\{\widetilde{\mathcal{S}}(p) \wedge \widetilde{\lambda}(q) \wedge \widetilde{\beta}\right\} \vee \widetilde{\alpha}\right) \\
&\geq \left\{\widetilde{\mathcal{S}}(s) \wedge \widetilde{\lambda}(x) \wedge \widetilde{\beta}\right\} \vee \widetilde{\alpha} \quad \text{because } a = sx \\
&= (\widetilde{\lambda}(x) \wedge \widetilde{\beta}) \vee \widetilde{\alpha} \\
&= (\widetilde{\lambda}(x) \vee \widetilde{\alpha}) \wedge (\widetilde{\beta} \vee \widetilde{\alpha}) \\
&> (\widetilde{\alpha} \vee \widetilde{\alpha}) \wedge (\widetilde{\beta} \vee \widetilde{\alpha}) \\
&= \widetilde{\alpha} \wedge \widetilde{\beta} \\
&= \widetilde{\alpha}.
\end{aligned}$$

Similarly, $\left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mathcal{S}}\right)(a) > \widetilde{\alpha}$. Thus, we have

$$\begin{aligned}
\max\{\widetilde{\lambda}(a), \widetilde{\alpha}\} &\geq \min\{\left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mathcal{S}}\right)(x), \left(\widetilde{\mathcal{S}} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda}\right))(x)\} \\
&> \min\{\widetilde{\alpha}, \widetilde{\alpha}\} = \widetilde{\alpha}
\end{aligned}$$

Which implies that $\widetilde{\lambda}(a) > \widetilde{\alpha}$. Thus, $a \in \widetilde{\lambda}_{\widetilde{\alpha}}$. Hence, $\widetilde{\lambda}_{\widetilde{\alpha}}$ is a quasi-ideal of S. □

Remark 3.22. Every fuzzy quasi-ideal of S is a fuzzy quasi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S.

Lemma 3.23. *A non-empty subset Q of a semigroup S is a quasi-ideal of S if and only if the characteristic function $\widetilde{C}_Q$ is an interval valued fuzzy quasi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S.*

Proof. Suppose Q is a quasi-ideal of S. Let $\widetilde{C}_Q$ be the characteristic function of Q. Let $x \in S$. If $x \notin Q$, then $x \notin SQ$ or $x \notin QS$. If $x \notin SQ$, then $\left(\widetilde{\mathcal{S}} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{C}_Q\right)(x) = \widetilde{\alpha}$ and so $\min\left\{\left(\widetilde{C}_Q \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mathcal{S}}\right)(x), \left(\widetilde{\mathcal{S}} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{C}_Q\right)(x)\right\} = \widetilde{\alpha} = \widetilde{C}_Q(x) \vee \widetilde{\alpha}$. If $x \in Q$, then $\max\{\widetilde{C}_Q(x), \widetilde{\alpha}\} = \widetilde{1} \geq \min\left\{\left(\widetilde{C}_Q \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mathcal{S}}\right)(x), \left(\widetilde{\mathcal{S}} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{C}_Q\right)(x)\right\}$. Hence, $\widetilde{C}_Q$ is a fuzzy quasi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ *of* S.

Conversely, assume that $\widetilde{C}_Q$ is an interval valued fuzzy quasi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S. Let $a \in QS \cap SQ$. Then, there exist $b, c \in S$ and $x, y \in Q$ such that

$a = xb$ and $a = cy$. Then, we have

$$\begin{aligned}\left(\widetilde{C}_Q \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mathcal{S}}\right)(a) &= \vee_{a=pq}(\left\{\widetilde{C}_Q(p) \wedge \widetilde{\mathcal{S}}(q) \wedge \widetilde{\beta}\right\} \vee \widetilde{\alpha}) \\ &\geq \{\widetilde{C}_Q(x) \wedge \widetilde{\mathcal{S}}(b) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\ &= \left(\widetilde{1} \wedge \widetilde{\beta}\right) \vee \widetilde{\alpha} \\ &= \widetilde{\beta}.\end{aligned}$$

So, $\left(\widetilde{C}_Q \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mathcal{S}}\right)(a) = \widetilde{\beta}$. Similarly, $\left(\widetilde{\mathcal{S}} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{C}_Q\right)(a) = \widetilde{\beta}$.

Hence $\widetilde{C}_Q(a) \vee \widetilde{\alpha} \geq \min\left\{\left(\widetilde{C}_Q \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mathcal{S}}\right)(a), \left(\widetilde{\mathcal{S}} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{C}_Q\right)(a)\right\} = \widetilde{\beta}$. Since $\widetilde{\alpha} < \widetilde{\beta}$, so $\widetilde{C}_Q(a) \geq \widetilde{\beta}$. Thus, $\widetilde{C}_Q(a) = \widetilde{1}$, which implies that $a \in Q$. Hence, $SQ \cap QS \subseteq Q$, that is Q is a quasi-ideal of S. □

Lemma 3.24. *The characteristic function $\widetilde{C}_L$ is an interval valued fuzzy left ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S if and only if L is a left ideal of S.*

Proof. Let $\widetilde{C}_L$ be an interval valued fuzzy left ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ *of* S. Let $y \in L$, then $\widetilde{C}_L(y) = \widetilde{1}$. Since $\widetilde{C}_L$ *is* an interval valued fuzzy left ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ *of* S, so $\widetilde{C}_L(xy) \vee \widetilde{\alpha} \geq \widetilde{C}_L(y) \wedge \widetilde{\beta} = \widetilde{\beta}$. Since $\widetilde{\alpha} < \widetilde{\beta}$, so $\widetilde{C}_L(xy) \geq \widetilde{\beta}$. Which implies that $\widetilde{C}_L(xy) = \widetilde{1}$. Thus, $xy \in L$.

Conversely, suppose that L is a left ideal of S. On the contrary suppose that there exist $x, y \in S$ such that $\widetilde{C}_L(xy) \vee \widetilde{\alpha} < \widetilde{C}_L(y) \wedge \widetilde{\beta}$, which implies that $\widetilde{C}_L(y) = 1$ and $\widetilde{C}_L(xy) = 0$. If $\widetilde{C}_L(y) = \widetilde{1}$, then $y \in L$. So, $xy \in L$, but $\widetilde{C}_L(xy) = \widetilde{0}$ means that $xy \notin L$, which is a contradiction. Hence, $\widetilde{C}_L(xy) \vee \widetilde{\alpha} \geq \widetilde{C}_L(y) \wedge \widetilde{\beta}$ holds. Therefore, $\widetilde{C}_L$ is a interval valued fuzzy left ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$. □

Similarly, the characteristic function $\widetilde{C}_R$ *is* an interval valued fuzzy right ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ *of* S if and only if R *is a* right ideal *of* S. Hence, it follows that characteristic function $\widetilde{C}_I$ is an interval valued fuzzy two-sided ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S if and only if I is a two-sided ideal of S.

Theorem 3.25. *Every interval valued fuzzy left ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S is an interval valued fuzzy quasi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S.*

Proof. Let $x \in S$. Then

$$\begin{aligned}
\left(\widetilde{\mathcal{S}} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\lambda}\right)(x) &= \vee_{x=yz}\left(\left\{\widetilde{\mathcal{S}}(y) \wedge \widetilde{\lambda}(z) \wedge \widetilde{\beta}\right\} \vee \widetilde{\alpha}\right) \\
&= \vee_{x=yz}(\{\widetilde{\lambda}(z) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha}) \\
&\leq (\vee_{x=yz}(\{\widetilde{\lambda}(yz) \vee \widetilde{\alpha}\} \vee \widetilde{\alpha})) \\
&= (\{\widetilde{\lambda}(x) \vee \widetilde{\alpha}\} \vee \widetilde{\alpha})) \\
&= \widetilde{\lambda}(x) \vee \widetilde{\alpha} \left(\begin{array}{c}\text{because } \widetilde{\lambda} \text{ is an interval valued fuzzy} \\ \text{left ideal with thresholds } \left(\widetilde{\alpha}, \widetilde{\beta}\right) \text{ of } S.\end{array}\right)
\end{aligned}$$

Thus, $\left(\mathcal{S} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\lambda}\right)(x) \leq \widetilde{\lambda}(x) \vee \widetilde{\alpha}$. Hence,

$$\widetilde{\lambda}(x) \vee \widetilde{\alpha} \geq \left(\widetilde{\mathcal{S}} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\lambda}\right)(x) \geq \min\left\{\left(\widetilde{\lambda} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mathcal{S}}\right)(x), \left(\widetilde{\mathcal{S}} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\lambda}\right)(x)\right\}.$$

Thus, $\widetilde{\lambda}$ is a interval valued fuzzy quasi-ideal *with thresholds* $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ *of* S. □

Similarly, we can show that every interval valued fuzzy right ideal *with thresholds* $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S is an interval valued fuzzy quasi-ideal *with thresholds* $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S.

Lemma 3.26. *Every interval valued fuzzy quasi-ideal with thresholds* $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ *of* S *is an interval valued fuzzy bi-ideal with thresholds* $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ *of* S.

Proof. Suppose that $\widetilde{\lambda}$ is an interval valued fuzzy quasi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of a semigroup S. Now,

$$\begin{aligned}
\widetilde{\lambda}(xy) \vee \widetilde{\alpha} &\geq \left(\widetilde{\lambda} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mathcal{S}}\right)(xy) \wedge \left(\widetilde{\mathcal{S}} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\lambda}\right)(xy) \\
&= \min\left\{\begin{array}{c} \bigvee_{xy=ab}(\{\widetilde{\lambda}(a) \wedge \widetilde{\mathcal{S}}(b) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha}), \\ \bigvee_{xy=pq}(\{\mathcal{S}(p) \wedge \widetilde{\lambda}(q) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha}) \end{array}\right\} \\
&\geq \min\left\{\begin{array}{c} \{\widetilde{\lambda}(x) \wedge \mathcal{S}(y) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha}, \\ \{\mathcal{S}(x) \wedge \widetilde{\lambda}(y) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \end{array}\right\} \\
&= [(\widetilde{\lambda}(x) \wedge \widetilde{\beta}) \vee \widetilde{\alpha}] \wedge [(\widetilde{\lambda}(y) \wedge \widetilde{\beta}) \vee \widetilde{\alpha}] \\
&= [(\widetilde{\lambda}(x) \wedge \widetilde{\beta}) \vee \widetilde{\alpha}] \wedge [(\widetilde{\lambda}(y) \wedge \widetilde{\beta}) \vee \widetilde{\alpha}] \\
&= \{\widetilde{\lambda}(x) \wedge \widetilde{\lambda}(y) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&\geq \min\left\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \widetilde{\beta}\right\}.
\end{aligned}$$

So, $\max\{\widetilde{\lambda}(xy), \alpha\} \geq \min\{\widetilde{\lambda}(x), \widetilde{\lambda}(y), \beta\}$. Also,

$$\begin{aligned}
\max\{\widetilde{\lambda}(xyz), \widetilde{\alpha}\} &\geq \left(\widetilde{\lambda} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mathcal{S}}\right)(xyz) \wedge \left(\widetilde{\mathcal{S}} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\lambda}\right)(xyz) \\
&= \min \left\{ \begin{array}{l} \bigvee_{xyz=ab} (\{\widetilde{\lambda}(a) \wedge \widetilde{\mathcal{S}}(b) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha}), \\ \bigvee_{xyz=pq} (\{\widetilde{\mathcal{S}}(p) \wedge \widetilde{\lambda}(q) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha}), \end{array} \right\} \\
&\geq \min \left\{ \begin{array}{l} (\widetilde{\lambda}(x) \wedge \mathcal{S}(yz) \wedge \widetilde{\beta}) \vee \widetilde{\alpha}, \\ (\mathcal{S}(xy) \wedge \widetilde{\lambda}(z) \wedge \widetilde{\beta}) \vee \widetilde{\alpha} \end{array} \right\} \\
&\geq \min \left\{ (\widetilde{\lambda}(x) \wedge \widetilde{\beta}) \vee \widetilde{\alpha}, (\widetilde{\lambda}(z) \wedge \widetilde{\beta}) \vee \widetilde{\alpha} \right\} \\
&= \min \left\{ \widetilde{\lambda}(x), \widetilde{\lambda}(z), \widetilde{\beta} \right\} \vee \widetilde{\alpha} \\
&\geq \min \left\{ \widetilde{\lambda}(x), \widetilde{\lambda}(z), \widetilde{\beta} \right\}.
\end{aligned}$$

So, $\max\{\widetilde{\lambda}(xyz), \widetilde{\alpha}\} \geq \min\{\widetilde{\lambda}(x), \widetilde{\lambda}(z), \widetilde{\beta}\}$. Thus, $\widetilde{\lambda}$ is an interval valued fuzzy bi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S. □

Lemma 3.27. *Every interval valued fuzzy two-sided ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S is an interval valued fuzzy interior ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S.*

Proof. Let $\widetilde{\lambda}$ be an interval valued fuzzy two-sided ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S. Now, $\widetilde{\lambda}(xy) \vee \widetilde{\alpha} \geq \widetilde{\lambda}(y) \wedge \widetilde{\beta} \geq \widetilde{\lambda}(x) \wedge \widetilde{\lambda}(y) \wedge \widetilde{\beta}$. So, $\widetilde{\lambda}(xy) \vee \widetilde{\alpha} \geq \widetilde{\lambda}(x) \wedge \widetilde{\lambda}(y) \wedge \widetilde{\beta}$. Also, for all $x, a, y \in S$. $\widetilde{\lambda}(xay) \vee \widetilde{\alpha} = \widetilde{\lambda}(x(ay)) \vee \widetilde{\alpha} \geq (\widetilde{\lambda}(ay) \vee \widetilde{\alpha}) \wedge \widetilde{\beta} \geq \widetilde{\lambda}(a) \wedge \widetilde{\beta}$. So $\widetilde{\lambda}(xay) \vee \widetilde{\alpha} \geq \widetilde{\lambda}(a) \wedge \widetilde{\beta}$. Hence, $\widetilde{\lambda}$ is an interval valued fuzzy interior ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S. □

The following example shows that the converse of Lemma 3.27 does not hold in general.

Example 3.28. Consider the semigroup $S = \{0, a, b, c\}$

.	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	a	b

Let $\widetilde{\lambda}$ be an interval valued fuzzy subset of S such that

$$\widetilde{\lambda}(0) = [0.7, 0.8], \ \widetilde{\lambda}(a) = [0.4, 0.5], \ \widetilde{\lambda}(b) = [0.6, 0.7], \ \widetilde{\lambda}(c) = [0, 0].$$

Then, $\widetilde{\lambda}$ is an interval valued fuzzy interior ideal with thresholds $\left(\widetilde{\alpha} = \widetilde{0.3}, \beta = \widetilde{0.5}\right)$ of S which is not an interval valued fuzzy two-sided ideal with thresholds $\left(\widetilde{\alpha} = \widetilde{0.3}, \beta = \widetilde{0.5}\right)$ of S. Since $\widetilde{\lambda}(bc) \vee \widetilde{\alpha} = \widetilde{\lambda}(a) \vee \widetilde{0.3} = [0.4, 0.5] < \widetilde{0.5} = \widetilde{\lambda}(b) \wedge \widetilde{\beta}$. So, $\widetilde{\lambda}$ is not an interval valued fuzzy right ideal with thresholds $\left(\widetilde{\alpha}=\widetilde{0.3},\beta=\widetilde{0.5}\right)$ of S, that is it is not an interval valued fuzzy two-sided ideal with thresholds $\left(\widetilde{\alpha}=\widetilde{0.3},\beta=\widetilde{0.5}\right)$ of S.

Definition 3.29. Let $\widetilde{\lambda}$ be an interval valued fuzzy subset of a semigroup S. We define the $\widetilde{\lambda}_{\widetilde{\beta}}^{\widetilde{\alpha}}$ as $\widetilde{\lambda}_{\widetilde{\beta}}^{\widetilde{\alpha}}(x)=\{\widetilde{\lambda}(x)\wedge\widetilde{\beta}\}\vee\widetilde{\alpha}$.

Lemma 3.30. *Let $\widetilde{\lambda}$ and $\widetilde{\mu}$ be interval valued fuzzy subsets of a semigruop S. Then the following holds.*

(1) $\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)_{\widetilde{\beta}}^{\widetilde{\alpha}}=\left(\widetilde{\lambda}_{\widetilde{\beta}}^{\widetilde{\alpha}}\wedge\widetilde{\mu}_{\widetilde{\beta}}^{\widetilde{\alpha}}\right)$

(2) $\left(\widetilde{\lambda}\vee\widetilde{\mu}\right)_{\widetilde{\beta}}^{\widetilde{\alpha}}=\left(\widetilde{\lambda}_{\widetilde{\beta}}^{\widetilde{\alpha}}\vee\widetilde{\mu}_{\widetilde{\beta}}^{\widetilde{\alpha}}\right)$

(3) $\left(\widetilde{\lambda}\circ\widetilde{\mu}\right)_{\widetilde{\beta}}^{\widetilde{\alpha}}\geq\left(\widetilde{\lambda}_{\widetilde{\beta}}^{\widetilde{\alpha}}\circ\widetilde{\mu}_{\widetilde{\beta}}^{\widetilde{\alpha}}\right)$

If every element x of S is expressible as $x=bc$, then $\left(\widetilde{\lambda}\circ\widetilde{\mu}\right)_{\widetilde{\beta}}^{\widetilde{\alpha}}=\left(\widetilde{\lambda}_{\widetilde{\beta}}^{\widetilde{\alpha}}\circ\widetilde{\mu}_{\widetilde{\beta}}^{\widetilde{\alpha}}\right)$.

Proof. For all $a\in S$.
(1)

$$\begin{aligned}\left(\widetilde{\lambda}\wedge\widetilde{\mu}\right)_{\widetilde{\beta}}^{\widetilde{\alpha}}(a) &= \left(\widetilde{\lambda}(a)\wedge\widetilde{\mu}(a)\wedge\widetilde{\beta}\right)\vee\widetilde{\alpha}\\ &= \left\{\left(\widetilde{\lambda}(a)\wedge\widetilde{\beta}\right)\wedge\left(\widetilde{\mu}(a)\wedge\widetilde{\beta}\right)\right\}\vee\widetilde{\alpha}\\ &= \left\{\left(\widetilde{\lambda}(a)\wedge\widetilde{\beta}\right)\vee\widetilde{\alpha}\right\}\wedge\left\{\left(\widetilde{\mu}(a)\wedge\widetilde{\beta}\right)\vee\widetilde{\alpha}\right\}\\ &= \widetilde{\lambda}_{\widetilde{\beta}}^{\widetilde{\alpha}}(a)\wedge\widetilde{\mu}_{\widetilde{\beta}}^{\widetilde{\alpha}}(a).\\ &= \left(\widetilde{\lambda}_{\widetilde{\beta}}^{\widetilde{\alpha}}\wedge\widetilde{\mu}_{\widetilde{\beta}}^{\widetilde{\alpha}}\right)(a).\end{aligned}$$

(2)

$$\begin{aligned}\left(\widetilde{\lambda}\vee\widetilde{\mu}\right)_{\widetilde{\beta}}^{\widetilde{\alpha}}(a) &= \{\left(\widetilde{\lambda}\vee\widetilde{\mu}\right)(a)\wedge\widetilde{\beta}\}\vee\widetilde{\alpha}\\ &= \{\left(\widetilde{\lambda}(a)\vee\widetilde{\mu}(a)\right)\wedge\widetilde{\beta}\}\vee\widetilde{\alpha}\\ &= \{(\widetilde{\lambda}(a)\wedge\widetilde{\beta})\vee(\widetilde{\mu}(a)\wedge\widetilde{\beta})\}\vee\widetilde{\alpha}\\ &= \{(\widetilde{\lambda}(a)\wedge\widetilde{\beta})\vee\widetilde{\alpha}\}\vee\{(\widetilde{\mu}(a)\wedge\widetilde{\beta})\vee\widetilde{\alpha}\}\\ &= \widetilde{\lambda}_{\widetilde{\beta}}^{\widetilde{\alpha}}(a)\vee\widetilde{\mu}_{\widetilde{\beta}}^{\widetilde{\alpha}}(a)\\ &= \left(\widetilde{\lambda}_{\widetilde{\beta}}^{\widetilde{\alpha}}\vee\widetilde{\mu}_{\widetilde{\beta}}^{\widetilde{\alpha}}\right)(a).\end{aligned}$$

(3) If a is not expressible as $a=bc$ for some $b,c\in S$, then $(\widetilde{\lambda}\circ\widetilde{\mu})(a)=0$. Thus, $\left(\widetilde{\lambda}\circ\widetilde{\mu}\right)_{\widetilde{\beta}}^{\widetilde{\alpha}}(a)=(\left(\widetilde{\lambda}\circ\widetilde{\mu}\right)(a)\wedge\widetilde{\beta})\vee\widetilde{\alpha}=\widetilde{\alpha}$. Since a is not expressible as $a=bc$, so $\left(\widetilde{\lambda}_{\widetilde{\beta}}^{\widetilde{\alpha}}\circ\widetilde{\mu}_{\widetilde{\beta}}^{\widetilde{\alpha}}\right)(a)=0$. Thus, in this case $\left(\widetilde{\lambda}\circ\widetilde{\mu}\right)_{\widetilde{\beta}}^{\widetilde{\alpha}}\geq\left(\widetilde{\lambda}_{\widetilde{\beta}}^{\widetilde{\alpha}}\circ\widetilde{\mu}_{\widetilde{\beta}}^{\widetilde{\alpha}}\right)$. And if a is expressible

$a = xy$ for some $x, y \in S$, then

$$\begin{aligned}
(\widetilde{\lambda} \circ \widetilde{\mu})_{\widetilde{\beta}}^{\widetilde{\alpha}}(a) &= (\left(\widetilde{\lambda} \circ \widetilde{\mu}\right)(a) \wedge \widetilde{\beta}) \vee \widetilde{\alpha} \\
&= [(\bigvee_{a=xy} \left\{\widetilde{\lambda}(x) \wedge \widetilde{\mu}(y)\right\}) \wedge \widetilde{\beta}] \vee \widetilde{\alpha} \\
&= [\bigvee_{a=xy} (\left\{\widetilde{\lambda}(x) \wedge \widetilde{\mu}(y)\right\} \wedge \widetilde{\beta})] \vee \widetilde{\alpha} \\
&= [\bigvee_{a=xy} \left\{\widetilde{\lambda}(x) \wedge \widetilde{\mu}(y) \wedge \widetilde{\beta}\right\}] \vee \widetilde{\alpha} \\
&= \bigvee_{a=xy} [\left\{\widetilde{\lambda}(x) \wedge \widetilde{\beta} \wedge \widetilde{\mu}(y) \wedge \widetilde{\beta}\right\} \vee \widetilde{\alpha}] \\
&= \bigvee_{a=xy} \left\{[(\widetilde{\lambda}(x) \wedge \widetilde{\beta}) \vee \widetilde{\alpha}] \wedge [(\widetilde{\mu}(y) \wedge \widetilde{\beta}) \vee \widetilde{\alpha}]\right\} \\
&= \bigvee_{a=xy} \left\{\widetilde{\lambda}_{\widetilde{\beta}}^{\widetilde{\alpha}}(x) \wedge \widetilde{\mu}_{\widetilde{\beta}}^{\widetilde{\alpha}}(y)\right\} \\
&= (\widetilde{\lambda}_{\widetilde{\beta}}^{\widetilde{\alpha}} \circ \widetilde{\mu}_{\widetilde{\beta}}^{\widetilde{\alpha}})(a).
\end{aligned}$$

Hence, $(\widetilde{\lambda} \circ \widetilde{\mu})_{\widetilde{\beta}}^{\widetilde{\alpha}}(a) = (\widetilde{\lambda}_{\widetilde{\beta}}^{\widetilde{\alpha}} \circ \widetilde{\mu}_{\widetilde{\beta}}^{\widetilde{\alpha}})(a)$. □

Let A be a nonempty subset of a semigroup S. Then, define $C_{\widetilde{\beta}}^{\widetilde{\alpha}}$ the interval valued characteristic function is,

$$\widetilde{C}_{\widetilde{\beta}}^{\widetilde{\alpha}}(a) = \{.\widetilde{\beta} \quad \text{if } a \in A\widetilde{\alpha} \quad \text{if } a \notin A$$

Lemma 3.31. *Let A and B be non-empty subsets of a semigroup S. Then, the followoing properties hold.*

(1) $A \subseteq B$ if and only if $(\widetilde{C}_A)_{\widetilde{\beta}}^{\widetilde{\alpha}} \leq (\widetilde{C}_B)_{\widetilde{\beta}}^{\widetilde{\alpha}}$.
(2) $(\widetilde{C}_A \wedge \widetilde{C}_B)_{\widetilde{\beta}}^{\widetilde{\alpha}} = (\widetilde{C}_{A\cap B})_{\widetilde{\beta}}^{\widetilde{\alpha}}$
(3) $(\widetilde{C}_A \vee \widetilde{C}_B)_{\widetilde{\beta}}^{\widetilde{\alpha}} = (\widetilde{C}_{A\cup B})_{\widetilde{\beta}}^{\widetilde{\alpha}}$
(4) $(\widetilde{C}_A \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} C_B) = (\widetilde{C}_{AB})_{\widetilde{\beta}}^{\widetilde{\alpha}}$

Proof. (1) Suppose $A, B \subseteq S$ and $(\widetilde{C}_A)_{\widetilde{\beta}}^{\widetilde{\alpha}} \leq (\widetilde{C}_B)_{\widetilde{\beta}}^{\widetilde{\alpha}}$. Let $a \in A$. Then, $(\widetilde{C}_A)_{\widetilde{\beta}}^{\widetilde{\alpha}}(a) = \widetilde{\beta}$. Since $(\widetilde{C}_A)_{\widetilde{\beta}}^{\widetilde{\alpha}} \leq (\widetilde{C}_B)_{\widetilde{\beta}}^{\widetilde{\alpha}}$, so $\widetilde{\beta} = (\widetilde{C}_A)_{\widetilde{\beta}}^{\widetilde{\alpha}} \leq (\widetilde{C}_B)_{\widetilde{\beta}}^{\widetilde{\alpha}}$. Which implies that $(\widetilde{C}_B)_{\widetilde{\beta}}^{\widetilde{\alpha}}(a) = \widetilde{\beta}$, so $a \in B$. Thus, $A \subseteq B$.

Conversely, $A, B \subseteq S$ such that $A \subseteq B$. On the contrary suppose that there exist $x \in S$ such that $(\widetilde{C}_A)_{\widetilde{\beta}}^{\widetilde{\alpha}}(x) > (\widetilde{C}_B)_{\widetilde{\beta}}^{\widetilde{\alpha}}(x)$. Then, $(\widetilde{C}_A)_{\widetilde{\beta}}^{\widetilde{\alpha}}(x) = \widetilde{\beta}$ and $(\widetilde{C}_B)_{\widetilde{\beta}}^{\widetilde{\alpha}}(x) = \widetilde{\alpha}$, which implies that $x \in A$ and $x \notin B$. So, $A \nsubseteq B$, which is a contradiction. Thus, $(\widetilde{C}_A)_{\widetilde{\beta}}^{\widetilde{\alpha}} \leq (\widetilde{C}_B)_{\widetilde{\beta}}^{\widetilde{\alpha}}$.

(2) Let a be any element of S. Suppose $a \in A \cap B$. It implies that $a \in A$ and $a \in B$. So

$$\begin{aligned}
(\widetilde{C}_A \wedge \widetilde{C}_B)^{\widetilde{\alpha}}_{\widetilde{\beta}}(a) &= \{(\widetilde{C}_A \wedge \widetilde{C}_B)(a) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \{(\widetilde{C}_A(a) \wedge \widetilde{C}_B(a)) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \{(\widetilde{C}_A(a) \wedge \widetilde{C}_B(a) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= (\widetilde{1} \wedge \widetilde{1} \wedge \widetilde{\beta}) \vee \widetilde{\alpha} \\
&= \widetilde{\beta} \vee \widetilde{\alpha} \\
&= \widetilde{\beta} \\
&= (\widetilde{C}_{A\cap B})^{\widetilde{\alpha}}_{\widetilde{\beta}}(a).
\end{aligned}$$

If $a \notin A \cap B$, then it implies $a \notin A$ or $a \notin B$. So,

$$\begin{aligned}
(\widetilde{C}_A \wedge \widetilde{C}_B)^{\widetilde{\alpha}}_{\widetilde{\beta}}(a) &= \{(\widetilde{C}_A \wedge \widetilde{C}_B)(a) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \{(\widetilde{C}_A(a) \wedge \widetilde{C}_B(a)) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \left\{\widetilde{0} \wedge \widetilde{\beta}\right\} \vee \widetilde{\alpha} \\
&= \widetilde{0} \vee \widetilde{\alpha} \\
&= \widetilde{\alpha} \\
&= (\widetilde{C}_{A\cap B})^{\widetilde{\alpha}}_{\widetilde{\beta}}(a).
\end{aligned}$$

(3) Let $a \in A \cup B$, it implies that $a \in A$ or $a \in B$. So,

$$\begin{aligned}
(\widetilde{C}_A \vee \widetilde{C}_B)^{\widetilde{\alpha}}_{\widetilde{\beta}}(a) &= \{(\widetilde{C}_A \vee \widetilde{C}_B)(a) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \{(\widetilde{C}_A(a) \vee \widetilde{C}_B(a)) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \{\widetilde{1} \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \widetilde{\beta} \vee \widetilde{\alpha} \\
&= \widetilde{\beta} \\
&= (\widetilde{C}_{A\cup B})^{\widetilde{\alpha}}_{\widetilde{\beta}}(a).
\end{aligned}$$

If $a \notin A \cup B$, then it implies $a \notin A$ and $a \notin B$. So,

$$\begin{aligned}
(\widetilde{C}_A \vee \widetilde{C}_B)^{\widetilde{\alpha}}_{\widetilde{\beta}}(a) &= \{(\widetilde{C}_A \vee \widetilde{C}_B)(a) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \{(\widetilde{C}_A(a) \vee \widetilde{C}_B(a)) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \left\{\widetilde{0} \wedge \widetilde{\beta}\right\} \vee \widetilde{\alpha} \\
&= \widetilde{0} \vee \widetilde{\alpha} \\
&= \widetilde{\alpha} \\
&= (\widetilde{C}_{A\cup B})^{\widetilde{\alpha}}_{\widetilde{\beta}}(a).
\end{aligned}$$

(4) Let a be any element of S. Suppose $a \in AB$. Then, $a = xy$ for some $x \in A$ and $y \in B$. Thus, we have

$$\begin{aligned}
(\widetilde{C}_A \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{C}_B)(a) &= \bigvee_{a=uv} (\{\widetilde{C}_A(u) \wedge \widetilde{C}_B(v) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha}) \\
&\geq \{\widetilde{C}_A(x) \wedge \widetilde{C}_B(y) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \{\widetilde{1} \wedge \widetilde{1} \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \widetilde{\beta} \vee \widetilde{\alpha} \\
&= \widetilde{\beta}
\end{aligned}$$

and so $(\widetilde{C}_A \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{C}_B)(a) = \widetilde{\beta}$.

Since $a \in AB$, $(\widetilde{C}_{AB})^{\widetilde{\alpha}}_{\widetilde{\beta}}(a) = \widetilde{\beta}$. So, $(\widetilde{C}_A \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{C}_B)(a) = (\widetilde{C}_{AB})^{\widetilde{\alpha}}_{\widetilde{\beta}}(a)$. Now, if $a \notin AB$, then $a \neq xy$ for all $x \in A$ and $y \in B$. If $a = uv$ for some $u, v \in S$, then we have

$$\begin{aligned}
(\widetilde{C}_A \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{C}_B)(a) &= \bigvee_{a=uv} (\{\widetilde{C}_A(u) \wedge \widetilde{C}_B(v) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha}) \\
&= \alpha \\
&= (C_{AB})^{\widetilde{\alpha}}_{\widetilde{\beta}}(a).
\end{aligned}$$

Thus, $(\widetilde{C}_A \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{C}_B) = (\widetilde{C}_{AB})^{\widetilde{\alpha}}_{\widetilde{\beta}}$. □

Lemma 3.32. *The lower part of interval valued characteristic function $(\widetilde{C}_L)^{\widetilde{\alpha}}_{\widetilde{\beta}}$ is an interval valued fuzzy left ideal with thesholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S if and only if L is a left ideal of S.*

Proof. Let L be a left ideal of S. Then, by Theorem 3.10 $(\widetilde{C}_L)^{\widetilde{\alpha}}_{\widetilde{\beta}}$ is an interval valued fuzzy left ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S.

Conversely, assume that $(\widetilde{C}_L)^{\widetilde{\alpha}}_{\widetilde{\beta}}$ is an interval valued fuzzy left ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S. Let $y \in L$. Then, $(\widetilde{C}_L)^{\widetilde{\alpha}}_{\widetilde{\beta}}(y) = \widetilde{\beta}$. Since $(\widetilde{C}_L)^{\widetilde{\alpha}}_{\widetilde{\beta}}$ is an interval valued fuzzy left ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S, so $(\widetilde{C}_L)^{\widetilde{\alpha}}_{\widetilde{\beta}}(xy) \vee \widetilde{\alpha} \geq (\widetilde{C}_L)^{\widetilde{\alpha}}_{\widetilde{\beta}}(y) \wedge \widetilde{\beta} = \widetilde{\beta}$. Since $\widetilde{\alpha} < \widetilde{\beta}$, so $(\widetilde{C}_L)^{\widetilde{\alpha}}_{\widetilde{\beta}}(xy) \geq \widetilde{\beta}$, which imlpies that $(\widetilde{C}_L)^{\widetilde{\alpha}}_{\widetilde{\beta}}(xy) = \widetilde{\beta}$. Hence, $xy \in (\widetilde{C}_L)^{\widetilde{\alpha}}_{\widetilde{\beta}}$. Thus, L is a left ideal of S. □

Similarly, we can prove that the interval valued characteristic function $(\widetilde{C}_R)^{\widetilde{\alpha}}_{\widetilde{\beta}}$ is an interval valued fuzzy right ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S if and only if R is a right ideal of S. Thus, an interval valued characteristic function $(\widetilde{C}_I)^{\widetilde{\alpha}}_{\widetilde{\beta}}$ is an interval valued fuzzy two-sided ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S if and only if I is a two-sided ideal of S.

Lemma 3.33. *Let Q be a non-empty subset of a semigroup S. Then, Q is a quasi-ideal of S if and only if an interval valued characteristic function $(\widetilde{C}_Q)^{\widetilde{\alpha}}_{\widetilde{\beta}}$ is an interval valued fuzzy quasi-ideal with thresholds $\left(\widetilde{\alpha},\widetilde{\beta}\right)$ of S.*

Proof. Suppose Q is a quasi-ideal of S. Let $(\widetilde{C}_Q)^{\widetilde{\alpha}}_{\widetilde{\beta}}$ be an interval valued characteristic function of Q. Let $x \in S$. If $x \notin Q$, then $x \notin SQ$ or $x \notin QS$. If $x \notin SQ$, then $\left(\widetilde{\mathcal{S}} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} (\widetilde{C}_Q)^{\widetilde{\alpha}}_{\widetilde{\beta}}\right)(x) = \widetilde{\alpha}$ and so $\max\left\{\left((\widetilde{C}_Q)^{\widetilde{\alpha}}_{\widetilde{\beta}} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mathcal{S}}\right)(x), \left(\widetilde{\mathcal{S}} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} (\widetilde{C}_Q)^{\widetilde{\alpha}}_{\widetilde{\beta}}\right)(x)\right\} = \widetilde{\alpha} = (\widetilde{C}_Q)^{\widetilde{\alpha}}_{\widetilde{\beta}}(x) \vee \widetilde{\alpha}$. If $x \in Q$, then

$$\max\{(\widetilde{C}_Q)^{\widetilde{\alpha}}_{\widetilde{\beta}}(x), \widetilde{\alpha}\} = \widetilde{\beta} \geq \min\left\{\left((\widetilde{C}_Q)^{\widetilde{\alpha}}_{\widetilde{\beta}} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mathcal{S}}\right)(x), \left(\widetilde{\mathcal{S}} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} (\widetilde{C}_Q)^{\widetilde{\alpha}}_{\widetilde{\beta}}\right)(x)\right\}.$$

Hence, $(\widetilde{C}_Q)^{\widetilde{\alpha}}_{\widetilde{\beta}}$ is an interval valued fuzzy quasi-ideal with thresholds $\left(\widetilde{\alpha},\widetilde{\beta}\right)$ of S.

Conversely, assume that $(\widetilde{C}_Q)^{\widetilde{\alpha}}_{\widetilde{\beta}}$ is an interval valued fuzzy quasi-ideal thresholds $\left(\widetilde{\alpha},\widetilde{\beta}\right)$ of S. Let $a \in QS \cap SQ$. Then, there exist $b, c \in S$ and $x, y \in Q$ such that $a = xb$ and $a = cy$. Then,

$$\begin{aligned}
\left((\widetilde{C}_Q)^{\widetilde{\alpha}}_{\widetilde{\beta}} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mathcal{S}}\right)(a) &= \vee_{a=pq}\left(\left\{(\widetilde{C}_Q)^{\widetilde{\alpha}}_{\widetilde{\beta}}(p) \wedge \widetilde{\mathcal{S}}(q) \wedge \widetilde{\beta}\right\} \vee \widetilde{\alpha}\right) \\
&\geq \{(\widetilde{C}_Q)^{\widetilde{\alpha}}_{\widetilde{\beta}}(x) \wedge \widetilde{\mathcal{S}}(b) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \left(\widetilde{\beta} \wedge \widetilde{\beta}\right) \vee \widetilde{\alpha} \\
&= \widetilde{\beta}.
\end{aligned}$$

So, $\left((\widetilde{C}_Q)^{\widetilde{\alpha}}_{\widetilde{\beta}} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mathcal{S}}\right)(a) = \widetilde{\beta}$. Similarly, $\left(\widetilde{\mathcal{S}} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} (\widetilde{C}_Q)^{\widetilde{\alpha}}_{\widetilde{\beta}}\right)(a) = \widetilde{\beta}$.

Hence $(C_Q)^{\widetilde{\alpha}}_{\widetilde{\beta}}(a) \vee \widetilde{\alpha} \geq \min\left\{\left((\widetilde{C}_Q)^{\widetilde{\alpha}}_{\widetilde{\beta}} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mathcal{S}}\right)(a), \left(\widetilde{\mathcal{S}} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} (\widetilde{C}_Q)^{\widetilde{\alpha}}_{\widetilde{\beta}}\right)(a)\right\} = \widetilde{\beta}$. Since $\widetilde{\alpha} < \widetilde{\beta}$, so $(\widetilde{C}_Q)^{\widetilde{\alpha}}_{\widetilde{\beta}}(a) \geq \widetilde{\beta}$. Thus, $(\widetilde{C}_Q)^{\widetilde{\alpha}}_{\widetilde{\beta}}(a) = \widetilde{\beta}$, which implies that $a \in Q$. Hence, $SQ \cap QS \subseteq Q$, thus Q is a quasi-ideal of S. □

Theorem 3.34. *For a semigroup S the following conditions are equivalent.*

(1) *S is regular.*

(2) $(\widetilde{\lambda} \wedge^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mu}) = (\widetilde{\lambda} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mu})$ *for every interval valued fuzzy right ideal $\widetilde{\lambda}$ and every interval valued fuzzy left ideal $\widetilde{\mu}$ with thresholds $\left(\widetilde{\alpha},\widetilde{\beta}\right)$ of S.*

Proof. First assume that (1) holds. Let $\widetilde{\lambda}$ be an interval valued fuzzy right ideal and $\widetilde{\mu}$ an interval valued fuzzy left ideal with thresholds $\left(\widetilde{\alpha},\widetilde{\beta}\right)$ of S. Let $a \in S$, we have

$$
\begin{aligned}
(\widetilde{\lambda}\circ^{\widetilde{\alpha}}_{\widetilde{\beta}}\widetilde{\mu})(a) &= \bigvee_{a=yz}(\{\widetilde{\lambda}(y)\wedge\widetilde{\mu}(z)\wedge\widetilde{\beta}\}\vee\widetilde{\alpha}) \\
&= \bigvee_{a=yz}(\{(\widetilde{\lambda}(y)\wedge\widetilde{\beta})\wedge(\widetilde{\mu}(z)\wedge\widetilde{\beta})\}\vee\widetilde{\alpha}) \\
&\leq \bigvee_{a=yz}(\{[(\widetilde{\lambda}(yz)\vee\widetilde{\alpha})\wedge\widetilde{\beta}]\wedge[(\widetilde{\mu}(yz)\vee\widetilde{\alpha})\wedge\widetilde{\beta}]\}\vee\widetilde{\alpha}) \\
&= \{[(\widetilde{\lambda}(yz)\vee\widetilde{\alpha})\wedge\widetilde{\beta}]\wedge[(\widetilde{\mu}(yz)\vee\widetilde{\alpha})\wedge\widetilde{\beta}]\}\vee\widetilde{\alpha} \\
&= \left(\{\widetilde{\lambda}(a)\wedge\widetilde{\mu}(a)\wedge\widetilde{\beta}\}\vee\widetilde{\alpha}\right)\vee\widetilde{\alpha} \\
&= \{\widetilde{\lambda}(a)\wedge\widetilde{\mu}(a)\wedge\widetilde{\beta}\}\vee\widetilde{\alpha} \\
&= (\widetilde{\lambda}\wedge^{\widetilde{\alpha}}_{\widetilde{\beta}}\widetilde{\mu})(a).
\end{aligned}
$$

So, $(\widetilde{\lambda}\wedge^{\widetilde{\alpha}}_{\widetilde{\beta}}\widetilde{\mu})\leq(\widetilde{\lambda}\circ^{\widetilde{\alpha}}_{\widetilde{\beta}}\widetilde{\mu})$.

Since S is regular and $a\in S$, so there exists an element $x\in S$ such that $a=axa$. So,

$$
\begin{aligned}
(\widetilde{\lambda}\circ^{\widetilde{\alpha}}_{\widetilde{\beta}}\widetilde{\mu})(a) &= \bigvee_{a=yz}(\{\widetilde{\lambda}(y)\wedge\widetilde{\mu}(z)\wedge\widetilde{\beta}\}\vee\widetilde{\alpha}) \\
&\geq \{\widetilde{\lambda}(ax)\wedge\widetilde{\mu}(a)\wedge\widetilde{\beta}\}\vee\widetilde{\alpha} \\
&= \{(\widetilde{\lambda}(ax)\wedge\widetilde{\beta})\wedge(\widetilde{\mu}(a)\wedge\widetilde{\beta})\}\vee\widetilde{\alpha} \\
&\geq \{[(\widetilde{\lambda}(a)\vee\widetilde{\alpha})\wedge\widetilde{\beta}]\wedge[(\widetilde{\mu}(a)\vee\widetilde{\alpha})\wedge\widetilde{\beta})]\}\vee\widetilde{\alpha} \\
&\geq \{[\widetilde{\lambda}(a)\wedge\widetilde{\beta}]\wedge[\widetilde{\mu}(a)\wedge\widetilde{\beta}]\}\vee\widetilde{\alpha} \\
&= \left\{\widetilde{\lambda}(a)\wedge\widetilde{\mu}(a)\wedge\widetilde{\beta}\right\}\vee\widetilde{\alpha} \\
&= (\widetilde{\lambda}\wedge^{\widetilde{\alpha}}_{\widetilde{\beta}}\widetilde{\mu})(a).
\end{aligned}
$$

So, $(\widetilde{\lambda}\circ^{\widetilde{\alpha}}_{\widetilde{\beta}}\widetilde{\mu})\geq(\widetilde{\lambda}\circ^{\widetilde{\alpha}}_{\widetilde{\beta}}\widetilde{\mu})$. Thus, $(\widetilde{\lambda}\circ^{\widetilde{\alpha}}_{\widetilde{\beta}}\widetilde{\mu})=(\widetilde{\lambda}\circ^{\widetilde{\alpha}}_{\widetilde{\beta}}\widetilde{\mu})$ and so (1) implies (2).

(2) implies (1) : assume that (2) holds. Let R and L be right ideal and left ideal of S, respectevily. In order to see that $R\cap L=RL$ holds. Let a be any element of $R\cap L$. Then, by Lemma 3.32, interval valued characteristic functions $(\widetilde{C}_R)^{\widetilde{\alpha}}_{\widetilde{\beta}}$ and $(\widetilde{C}_L)^{\widetilde{\alpha}}_{\widetilde{\beta}}$ of R and L are interval valued fuzzy right ideal and interval valued fuzzy left ideal with thresholds $\left(\widetilde{\alpha},\widetilde{\beta}\right)$ of S, respectevily. Thus, we have

$$
\begin{aligned}
(\widetilde{C}_{RL})^{\widetilde{\alpha}}_{\widetilde{\beta}}(a) &= (\widetilde{C}_R\circ^{\widetilde{\alpha}}_{\widetilde{\beta}}\widetilde{C}_L)(a) && \text{by Lemma 3.31} \\
&= (\widetilde{C}_R\wedge^{\widetilde{\alpha}}_{\widetilde{\beta}}\widetilde{C}_L)(a) && \text{by (1)} \\
&= (\widetilde{C}_{R\cap L})^{\widetilde{\alpha}}_{\widetilde{\beta}}(a) && \text{by Lemma 3.31} \\
&= \widetilde{\beta}.
\end{aligned}
$$

So, $a\in RL$, which implies that $R\cap L\subseteq RL$. Thus, $R\cap L=RL$. Hence, it follows from Theorem 2.7 that S is regular and so (2) implies (1). □

Theorem 3.35. *For a semigroup S, the following conditions are equivalent.*

(1) *S is regular.*

(2) $\left(\widetilde{\delta} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right) \leq \left(\widetilde{\delta} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)$ *for every interval valued fuzzy right ideal $\widetilde{\delta}$, every interval valued fuzzy generalized bi-ideal $\widetilde{\lambda}$ and every interval valued fuzzy left ideal $\widetilde{\mu}$ with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S.*

(3) $\left(\widetilde{\delta} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right) \leq \left(\widetilde{\delta} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)$ *for every interval valued fuzzy right ideal $\widetilde{\delta}$, every interval valued fuzzy bi-ideal $\widetilde{\lambda}$, and every interval valued fuzzy left ideal $\widetilde{\mu}$ with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S.*

(4) $\left(\widetilde{\delta} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right) \leq \left(\widetilde{\delta} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)$ *for every interval valued fuzzy right ideal $\widetilde{\delta}$, every interval valued fuzzy quasi-ideal $\widetilde{\lambda}$, and every interval valued fuzzy left ideal $\widetilde{\mu}$ with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S.*

Proof. $(1) \Rightarrow (2)$: Let $\widetilde{\delta}$, $\widetilde{\lambda}$ and $\widetilde{\mu}$ be interval valued fuzzy right ideal, interval valued fuzzy generalized bi-ideal, and any interval valued fuzzy left ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S**,** respectevily. Let a be any element of S**.** Since S is regular, so there exists an element $x \in S$ such that $a = axa$. Hence we have

$$
\begin{aligned}
\left(\widetilde{\delta} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)(a) &= \bigvee_{a=yz} \left(\{\widetilde{\delta}(y) \wedge \left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)(z) \wedge \widetilde{\beta}\}\right) \vee \widetilde{\alpha} \\
&\geq \{\widetilde{\delta}(ax) \wedge \left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)(a) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \left\{\widetilde{\delta}(ax) \wedge \widetilde{\beta} \wedge \left[\bigvee_{a=pq} \left(\{\widetilde{\lambda}(p) \wedge \widetilde{\mu}(q) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha}\right)\right]\right\} \vee \widetilde{\alpha} \\
&\geq \left\{\left(\widetilde{\delta}(a) \vee \widetilde{\alpha}\right) \wedge \left[\{\widetilde{\lambda}(a) \wedge \widetilde{\mu}(xa) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha})\right]\right\} \vee \widetilde{\alpha} \\
&\geq \left\{\left(\widetilde{\delta}(a) \vee \widetilde{\alpha}\right) \wedge \left[\{\widetilde{\lambda}(a) \wedge \widetilde{\mu}(a) \vee \widetilde{\alpha}\} \vee \widetilde{\alpha})\right]\right\} \vee \widetilde{\alpha} \\
&\geq \{\widetilde{\delta}(a) \wedge \widetilde{\lambda}(a) \wedge \widetilde{\mu}(a)\} \vee \widetilde{\alpha} \\
&\geq \{\widetilde{\delta}(a) \wedge \widetilde{\lambda}(a) \wedge \widetilde{\mu}(a) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \left(\widetilde{\delta} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)(a).
\end{aligned}
$$

So, (1) implies (2). $(2) \Rightarrow (3) \Rightarrow (4)$ straightforward.

$(4) \Rightarrow (1)$: Let $\widetilde{\delta}$ and $\widetilde{\mu}$ be any interval valued fuzzy right ideal and any interval valued fuzzy left ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S, respectively. Since $\widetilde{\mathcal{S}}$ is an interval valued fuzzy quasi-ideal with thresholds $\left(\widetilde{\alpha},\widetilde{\beta}\right)$ of S, so by the assumption, we have

$$\begin{aligned}
\left(\widetilde{\delta}\wedge_{\widetilde{\beta}}^{\widetilde{\alpha}}\widetilde{\mu}\right)(a) &= \{(\widetilde{\delta}\wedge\widetilde{\mu})(a)\wedge\widetilde{\beta}\}\vee\widetilde{\alpha} \\
&= \{\left(\widetilde{\delta}\wedge\mathcal{S}\wedge\widetilde{\mu}\right)(a)\wedge\widetilde{\beta}\}\vee\widetilde{\alpha} \\
&= \left(\widetilde{\delta}\wedge_{\widetilde{\beta}}^{\widetilde{\alpha}}\mathcal{S}\wedge_{\widetilde{\beta}}^{\widetilde{\alpha}}\widetilde{\mu}\right)(a) \\
&\leq \left(\widetilde{\delta}\circ_{\widetilde{\beta}}^{\widetilde{\alpha}}\mathcal{S}\circ_{\widetilde{\beta}}^{\widetilde{\alpha}}\widetilde{\mu}\right)(a) \\
&= \bigvee_{a=bc}(\{\left(\widetilde{\delta}\circ_{\widetilde{\beta}}^{\widetilde{\alpha}}\mathcal{S}\right)(b)\wedge\widetilde{\mu}(c)\wedge\widetilde{\beta}\}\vee\widetilde{\alpha}) \\
&= \bigvee_{a=bc}\left[\left(\left\{\bigvee_{b=pq}(\{\widetilde{\delta}(p)\wedge\mathcal{S}(q)\wedge\widetilde{\beta}\}\vee\widetilde{\alpha})\wedge\widetilde{\mu}(c)\wedge\widetilde{\beta}\right\}\vee\widetilde{\alpha}\right)\right] \\
&= \bigvee_{a=bc}\left[\left(\left\{\bigvee_{b=pq}(\{\widetilde{\delta}(p)\wedge\widetilde{\beta}\}\vee\widetilde{\alpha})\wedge\widetilde{\mu}(c)\wedge\widetilde{\beta}\right\}\vee\widetilde{\alpha}\right)\right] \\
&\leq \bigvee_{a=bc}\left[\left(\left\{\bigvee_{b=pq}(\{\widetilde{\delta}(pq)\}\vee\widetilde{\alpha})\wedge\widetilde{\mu}(c)\wedge\widetilde{\beta}\right\}\vee\widetilde{\alpha}\right)\right] \\
&= \bigvee_{a=bc}\left(\left\{(\widetilde{\delta}(b)\vee\widetilde{\alpha})\wedge\widetilde{\mu}(c)\wedge\widetilde{\beta}\right\}\vee\widetilde{\alpha}\right) \\
&= \bigvee_{a=bc}\left\{(\widetilde{\delta}(b)\vee\widetilde{\alpha})\wedge(\widetilde{\mu}(c)\vee\widetilde{\alpha})\wedge(\widetilde{\beta}\vee\widetilde{\alpha})\right\} \\
&= \bigvee_{a=bc}(\{\widetilde{\delta}(b)\wedge\widetilde{\mu}(c)\wedge\widetilde{\beta}\}\vee\widetilde{\alpha}) \\
&= \left(\widetilde{\delta}\circ_{\widetilde{\beta}}^{\widetilde{\alpha}}\widetilde{\mu}\right)(a).
\end{aligned}$$

Thus, it follows that $\left(\widetilde{\delta}\wedge_{\widetilde{\beta}}^{\widetilde{\alpha}}\widetilde{\mu}\right)\leq\left(\widetilde{\delta}\circ_{\widetilde{\beta}}^{\widetilde{\alpha}}\widetilde{\mu}\right)$ for every interval valued fuzzy right ideal $\widetilde{\delta}$ and every interval valued fuzzy left ideal $\widetilde{\mu}$ with thresholds $\left(\widetilde{\alpha},\widetilde{\beta}\right)$ of S. But $\left(\widetilde{\delta}\wedge_{\widetilde{\beta}}^{\widetilde{\alpha}}\widetilde{\mu}\right)\leq\left(\widetilde{\delta}\circ_{\widetilde{\beta}}^{\widetilde{\alpha}}\widetilde{\mu}\right)$ always. So, $\left(\widetilde{\delta}\wedge_{\widetilde{\beta}}^{\widetilde{\alpha}}\widetilde{\mu}\right)=\left(\widetilde{\delta}\circ_{\widetilde{\beta}}^{\widetilde{\alpha}}\widetilde{\mu}\right)$. Hence it is follows from Theorem 3.34 that S is regular. □

Theorem 3.36. *For a semigroup S, the following conditions are equivalent.*

(1) *S is regular.*

(2) $\widetilde{\lambda}_{\widetilde{\beta}}^{\widetilde{\alpha}}\leq(\widetilde{\lambda}\circ_{\widetilde{\beta}}^{\widetilde{\alpha}}\widetilde{\mathcal{S}}\circ_{\widetilde{\beta}}^{\widetilde{\alpha}}\widetilde{\lambda})$ *for every interval valued fuzzy generalized bi-ideal $\widetilde{\lambda}$ with thresholds $\left(\widetilde{\alpha},\widetilde{\beta}\right)$ of S.*

(3) $\widetilde{\lambda}_{\widetilde{\beta}}^{\widetilde{\alpha}}\leq(\widetilde{\lambda}\circ_{\widetilde{\beta}}^{\widetilde{\alpha}}\widetilde{\mathcal{S}}\circ_{\widetilde{\beta}}^{\widetilde{\alpha}}\widetilde{\lambda})$ *for every interval valued fuzzy bi-ideal $\widetilde{\lambda}$ with thresholds $\left(\widetilde{\alpha},\widetilde{\beta}\right)$ of S.*

(4) $\widetilde{\lambda}_{\widetilde{\beta}}^{\widetilde{\alpha}} \leq (\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mathcal{S}} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda})$ *for every interval valued fuzzy quasi-ideal* $\widetilde{\lambda}$ *with thresholds* $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ *of* S.

Proof. (1) $\Rightarrow$ (2) : Let $\widetilde{\lambda}$ be an interval valued fuzzy generalized bi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S and let a be any element of S**.** Since S is regular, so there exists an element $x \in S$ such that $a = axa$. Hence, we have

$$\begin{aligned}
(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mathcal{S}} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda})(a) &= \bigvee_{a=yz} \left(\{\left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mathcal{S}}\right)(y) \wedge \widetilde{\lambda}(z) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha}\right) \\
&\geq \{\left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mathcal{S}}\right)(ax) \wedge \widetilde{\lambda}(a) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \left\{\bigvee_{ax=pq} \left(\widetilde{\lambda}(p) \wedge \widetilde{\mathcal{S}}(q) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha}\right) \wedge \widetilde{\lambda}(a) \wedge \widetilde{\beta}\right\} \vee \widetilde{\alpha} \\
&\geq \{\left(\widetilde{\lambda}(a) \wedge \widetilde{\mathcal{S}}(x) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha}\right) \wedge \widetilde{\lambda}(a) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \{\left(\{\widetilde{\lambda}(a) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha}\right) \wedge \widetilde{\lambda}(a) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&\geq \{\{\widetilde{\lambda}(a) \wedge \widetilde{\beta}\} \wedge \widetilde{\lambda}(a) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \{\widetilde{\lambda}(a) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \widetilde{\lambda}_{\widetilde{\beta}}^{\widetilde{\alpha}}.
\end{aligned}$$

Thus, $(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mathcal{S}} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda}) \geq \widetilde{\lambda}_{\widetilde{\beta}}^{\widetilde{\alpha}}$.

Now (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (1) : Let Q be any quasi-ideal of S. Then we have $QSQ \subseteq Q(SS) \cap (SS)Q \subseteq QS \cap SQ \subseteq Q$. Let a be any element of Q. Since by Lemma 3.23 $\widetilde{C}_Q$ is an interval valued fuzzy quasi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S. Now, by assumption and Lemma 3.31, we have

$$(\widetilde{C}_Q)_{\widetilde{\beta}}^{\widetilde{\alpha}} \leq \widetilde{C}_Q \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mathcal{S}} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} C_Q = (\widetilde{C}_{QSQ})_{\widetilde{\beta}}^{\widetilde{\alpha}}.$$

Then, it follows from Lemma 3.31 that $Q \subseteq QSQ$. Therefore, $QSQ = Q$. Thus, S is regular by Lemma 2.7. □

Theorem 3.37. *For a semigruop* S, *the following conditions are equivalent.*

(1) S *is regular.*

(2) $\left(\widetilde{\lambda} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right) \leq \left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda}\right)$ *for every interval valued fuzzy quasi-ideal* $\widetilde{\lambda}$ *and every interval valued fuzzy two-sided ideal* $\widetilde{\mu}$ *with thresholds* $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ *of* S.

(3) $\left(\widetilde{\lambda} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right) \leq \left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda}\right)$ *for every interval valued fuzzy quasi-ideal* $\widetilde{\lambda}$ *and every interval valued fuzzy interior ideal* $\widetilde{\mu}$ *with thresholds* $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ *of* S.

(4) $\left(\widetilde{\lambda} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right) \leq \left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda}\right)$ *for every interval valued fuzzy bi-ideal* $\widetilde{\lambda}$ *and every interval valued fuzzy two-sided ideal* $\widetilde{\mu}$ *with thresholds* $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ *of* S.

(5) $\left(\widetilde{\lambda} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right) \leq \left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda}\right)$ *for every interval valued fuzzy bi-ideal* $\widetilde{\lambda}$ *and every interval valued fuzzy interior ideal* $\widetilde{\mu}$ *with thresholds* $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ *of* S.

(6) $\left(\widetilde{\lambda} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right) \leq \left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda}\right)$ *for every interval valued fuzzy generalized bi-ideal* $\widetilde{\lambda}$ *and every interval valued fuzzy two-sided ideal* $\widetilde{\mu}$ *with thresholds* $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ *of* S.

(7) $\left(\widetilde{\lambda} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right) \leq \left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda}\right)$ *for every interval valued fuzzy generalized bi-ideal* $\widetilde{\lambda}$ *and every interval valued fuzzy interior ideal* $\widetilde{\mu}$ *with thresholds* $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ *of* S.

Proof. $(1) \Rightarrow (7)$: Let $\widetilde{\lambda}$ and $\widetilde{\mu}$ be any interval valued fuzzy generalized bi-ideal and any interval valued fuzzy interior ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S, respectively. Now, let a be any element of S**.** Then**,** since S is regular, there exists an element $x \in S$ such that $a = axa\,(= axaxa)$. Since $\widetilde{\mu}$ is an interval valued fuzzy interior ideal of S**,** we have

$$\begin{aligned}
(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda})(a) &= \bigvee_{a=yz} \left(\{\widetilde{\lambda}(y) \wedge \left(\widetilde{\mu} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda}\right)(z) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha}\right) \\
&\geq \{\widetilde{\lambda}(a) \wedge \left(\widetilde{\mu} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda}\right)(xaxa) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \left[\widetilde{\lambda}(a) \wedge \left(\bigvee_{xaxa=pq} \left(\{\widetilde{\mu}(p) \wedge \widetilde{\lambda}(q) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha}\right)\right) \wedge \widetilde{\beta}\right] \vee \widetilde{\alpha} \\
&\geq \left[\widetilde{\lambda}(a) \wedge \left(\{\widetilde{\mu}(xax) \wedge \widetilde{\lambda}(a) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha}\right) \wedge \widetilde{\beta}\right] \vee \widetilde{\alpha} \\
&= \left[\widetilde{\lambda}(a) \wedge \{(\widetilde{\mu}(xax) \vee \widetilde{\alpha}) \wedge ((\widetilde{\lambda}(a) \wedge \widetilde{\beta}) \vee \widetilde{\alpha})\} \wedge \widetilde{\beta}\right] \vee \widetilde{\alpha} \\
&\geq \left[\widetilde{\lambda}(a) \wedge \{(\widetilde{\mu}(a) \wedge \widetilde{\beta}) \wedge ((\widetilde{\lambda}(a) \wedge \widetilde{\beta}) \vee \widetilde{\alpha})\} \wedge \widetilde{\beta}\right] \vee \widetilde{\alpha} \\
&= \{\widetilde{\lambda}(a) \vee \widetilde{\alpha}\} \wedge \{(\widetilde{\mu}(a) \wedge \widetilde{\beta}) \vee \widetilde{\alpha}\} \wedge ((\widetilde{\lambda}(a) \wedge \widetilde{\beta}) \vee \widetilde{\alpha})\} \wedge \{\widetilde{\beta} \vee \widetilde{\alpha}\} \\
&= \left\{\widetilde{\lambda}(a) \wedge \widetilde{\mu}(a) \wedge \widetilde{\beta}\right\} \vee \widetilde{\alpha} \\
&= \left(\widetilde{\lambda} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)(a).
\end{aligned}$$

So, $(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda}) \geq \left(\widetilde{\lambda} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)$. $(7) \Rightarrow (5) \Rightarrow (3) \Rightarrow (2)$ and $(7) \Rightarrow (6) \Rightarrow (4) \Rightarrow (2)$ are clear.

$(2) \Rightarrow (1)$: Let $\widetilde{\lambda}$ be any interval valued fuzzy quasi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S. Then, since $\widetilde{\mathcal{S}}$ itself is a interval valued fuzzy two-sided with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S, so we have

$$\begin{aligned}
\widetilde{\lambda}_{\widetilde{\beta}}^{\widetilde{\alpha}}(a) &= \{\widetilde{\lambda}(a) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \{\widetilde{\lambda}(a) \wedge \widetilde{\mathcal{S}}(a) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \left(\widetilde{\lambda} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mathcal{S}}\right)(a) \\
&\leq (\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mathcal{S}} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\lambda})(a).
\end{aligned}$$

Thus, it follows from Theorem 3.36 that S is regular. □

Theorem 3.38. *For a semigroup S, the following conditions are equivalent.*

(1) *S is regular.*

(2) $\left(\widetilde{\lambda} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right) \leq \left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)$ *for every interval valued fuzzy quasi-ideal $\widetilde{\lambda}$ and every interval valued fuzzy left ideal $\widetilde{\mu}$ with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S.*

(3) $\left(\widetilde{\lambda} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right) \leq \left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)$ *for every interval valued fuzzy bi-ideal $\widetilde{\lambda}$ and every interval valued fuzzy left ideal $\widetilde{\mu}$ with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S.*

(4) $\left(\widetilde{\lambda} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right) \leq \left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)$ *for every interval valued fuzzy generalized bi-ideal $\widetilde{\lambda}$ and every interval valued fuzzy left ideal $\widetilde{\mu}$ with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S.*

Proof. (1) ⇒ (4) : Let $\widetilde{\lambda}$ and $\widetilde{\mu}$ be an interval valued fuzzy generalized bi-ideal and an interval valued fuzzy left ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S, respectively. Let a be any element of S. Then, there exists an element $x \in S$ such that $a = axa$. Thus we have

$$\begin{aligned}
\left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)(a) &= \bigvee_{a=yz} \left(\{\widetilde{\lambda}(y) \wedge \widetilde{\mu}(z) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha}\right) \\
&\geq \{\widetilde{\lambda}(a) \wedge \widetilde{\mu}(xa) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \{\widetilde{\lambda}(a) \wedge (\widetilde{\mu}(a) \vee \widetilde{\alpha}) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&\geq \{\widetilde{\lambda}(a) \wedge \widetilde{\mu}(a) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha} \\
&= \left(\widetilde{\lambda} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)(a).
\end{aligned}$$

So, $\left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right) \geq \left(\widetilde{\lambda} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right)$.

Now, (4) ⇒ (3) ⇒ (2) are obvious.

(2) ⇒ (1) : Let $\widetilde{\lambda}$ be an interval valued fuzzy right ideal and $\widetilde{\mu}$ an interval valued fuzzy left ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S. Since every interval valued fuzzy right ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S is an interval valued fuzzy quasi-ideal with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S, so $\left(\widetilde{\lambda} \circ_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu}\right) \geq (\widetilde{\lambda} \wedge_{\widetilde{\beta}}^{\widetilde{\alpha}} \widetilde{\mu})$.

Now,

$$\begin{aligned}
(\widetilde{\lambda} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mu})(a) &= \bigvee_{a=yz} (\{\widetilde{\lambda}(y) \wedge \widetilde{\mu}(z) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha}) \\
&= \bigvee_{a=yz} (\{(\widetilde{\lambda}(y) \wedge \widetilde{\beta}) \wedge (\widetilde{\mu}(z) \wedge \widetilde{\beta}) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha}) \\
&\leq \bigvee_{a=yz} (\{(\widetilde{\lambda}(yz) \vee \widetilde{\alpha}) \wedge (\widetilde{\mu}(yz) \vee \widetilde{\alpha}) \wedge \widetilde{\beta}\} \vee \widetilde{\alpha}) \\
&= (\widetilde{\lambda}(a) \vee \widetilde{\alpha}) \wedge (\widetilde{\mu}(a) \vee \widetilde{\alpha}) \wedge (\widetilde{\beta} \vee \widetilde{\alpha}) \\
&= \left\{\widetilde{\lambda}\,(a) \wedge \widetilde{\mu}\,(a) \wedge \widetilde{\beta}\right\} \vee \widetilde{\alpha} \\
&= (\widetilde{\lambda} \wedge^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mu})\,(a)\,.
\end{aligned}$$

So, $(\widetilde{\lambda} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mu}) \leq (\widetilde{\lambda} \wedge^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mu})$. Hence, $(\widetilde{\lambda} \circ^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mu}) = (\widetilde{\lambda} \wedge^{\widetilde{\alpha}}_{\widetilde{\beta}} \widetilde{\mu})$ for every interval valued fuzzy right ideal $\widetilde{\lambda}$ with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S, and every interval valued fuzzy left ideal $\widetilde{\mu}$ with thresholds $\left(\widetilde{\alpha}, \widetilde{\beta}\right)$ of S. Thus, by Theorem 3.34 that S is regular. □

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