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Bipolar fuzzy KU-subalgebras/ideals of KU-algebras

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Abstract. The notions of bipolar fuzzy KU-subalgebras and bipolar fuzzy KU-ideals in KU-algebras are introduced, and related properties are investigated. Characterizations of a bipolar fuzzy KU-subalgebra/ideal in KU-algebras are established. Relations between a bipolar fuzzy subalgebra and a bipolar fuzzy ideal are given. Conditions for a bipolar fuzzy subalgebra to be a bipolar fuzzy ideal are provided. Using a collection of KU-ideals, a bipolar fuzzy KU-ideal is established.

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1. Introduction

In the traditional fuzzy sets, the membership degrees of elements range over the interval $[0, 1]$. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set, and the membership degree 0 indicates that an element does not belong to the fuzzy set. The membership degrees on the interval $(0, 1)$ indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set (see [5, 13]). In the viewpoint of satisfaction degree, the membership degree 0 is assigned to elements which do not satisfy some property. The elements with membership degree 0 are usually regarded as having the same characteristics in the fuzzy set representation. By the way, among such elements, some have irrelevant characteristics to the property corresponding to a fuzzy set and the others have contrary characteristics to the property. The traditional fuzzy set representation cannot tell apart contrary elements from irrelevant elements. Only with the membership degrees ranged on the interval [0, 1], it is difficult to express the difference of the irrelevant elements from the contrary elements in fuzzy sets. If a set representation could express this kind of difference, it would be more informative than the traditional fuzzy set representation. Based on these observations, K. M. Lee [9] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets. Later on, K. J. Lee [8] applied the bipolar-valued fuzzy set theory to BCK/BCI-algebras, and introduced the notions of bipolar fuzzy subalgebras and bipolar fuzzy ideals of a BCK/BCI-algebra. Kim et al. [7] discussed bipolar-valued fuzzy set theory in the ideal theory of semigroups. Jun et al. [6] studied ideals and filters in CI-algebras based on bipolar-valued fuzzy set. In the sequel, Akram et al. studied bipolar-valued fuzzy set theory in graphs, Lie algebras, Lie superalgebras and K-algebras (see [1]- [4]). The notion of KU-algebras was introduced by Prabpayak and Leerawat [12]. Further, Mostafa et al. [11] studied the fuzzification of KU-ideals in KU-algebras.

In this paper, we apply the bipolar-valued fuzzy set theory to KU-algebras, and introduce the notions of bipolar fuzzy KU-subalgebras and bipolar fuzzy KU-ideals in KU-algebras. We consider characterizations of a bipolar fuzzy KU-subalgebra and a bipolar fuzzy KU-ideal in KU-algebras. We discuss relations between a bipolar fuzzy KU-subalgebra and a bipolar fuzzy KU-ideal. We provide conditions for a bipolar fuzzy KU-subalgebra to be a bipolar fuzzy KU-ideal. Using a collection of KU-ideals, we establish a bipolar fuzzy KU-ideal.

2. Preliminaries

By a *KU-algebra* (see [12]) we mean an algebra $\mathcal{X} := (X, *, 0)$ satisfying the following axioms:

(a1) $(x * y) * ((y * z) * (x * z)) = 0$,
(a2) $0 * x = x$ and $x * 0 = 0$,
(a3) $x * y = 0 = y * x$ implies $x = y$

for all $x, y, z \in X$. We define a binary relation $\leq$ on $\mathcal{X} := (X, *, 0)$ as follows:

$$(\forall x, y \in X)(x \leq y \;\Leftrightarrow\; y * x = 0).$$

Every KU-algebra $\mathcal{X} := (X, *, 0)$ satisfies the following conditions.

(b1) $(\forall x, y, z \in X)\,(x * (y * z) = y * (x * z))$.
(b2) $(\forall x \in X)\,(x * x = 0)$.

A nonempty subset G of a KU-algebra $\mathcal{X} := (X, *, 0)$ is called a *KU-subalgebra* of $\mathcal{X} := (X, *, 0)$ (see [12]) if $(G, *, 0)$ is a KU-algebra. Note that a nonempty subset G of a KU-algebra $\mathcal{X} := (X, *, 0)$ is a KU-subalgebra of $\mathcal{X} := (X, *, 0)$ if and only if $x * y \in G$ for all $x, y \in G$ (see [12]).

A subset G of a KU-algebra $\mathcal{X} := (X, *, 0)$ is called a *KU-ideal* of $\mathcal{X} := (X, *, 0)$ (see [12]) if it satisfies:

$$0 \in G \tag{2.1}$$

$$(\forall x, y, z \in X)(x * (y * z) \in G,\; y \in G \;\Rightarrow x * z \in G). \tag{2.2}$$

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise,} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

Let X be a universe of discourse. A *bipolar-valued fuzzy set* f in X is an object having the form

$$f = \{(x; f_n(x), f_p(x)) \mid x \in X\}$$

where $f_n : X \to [-1, 0]$ and $f_p : X \to [0, 1]$ are mappings. The positive membership degree $f_p(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $f = \{(x; f_n(x), f_p(x)) \mid x \in X\}$, and the negative membership degree $f_n(x)$ denotes the satisfaction degree of x to some implicit counter-property of $f = \{(x; f_n(x), f_p(x)) \mid x \in X\}$. If $f_p(x) \neq 0$ and $f_n(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for $f = \{(x; f_n(x), f_p(x)) \mid x \in X\}$. If $f_p(x) = 0$ and $f_n(x) \neq 0$, it is the situation that x does not satisfy the property of $f = \{(x; f_n(x), f_p(x)) \mid x \in X\}$ but somewhat satisfies the counter-property of $f = \{(x; f_n(x), f_p(x)) \mid x \in X\}$. It is possible for an element x to be $f_p(x) \neq 0$ and $f_n(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain (see [10]). For the sake of simplicity, we shall use the symbol $f = (X; f_n, f_p)$ for the bipolar-valued fuzzy set $f = \{(x; f_n(x), f_p(x)) \mid x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

3. Bipolar fuzzy KU-subalgebras of KU-algebras

In what follows let $\mathcal{X} := (X, *, 0)$ denote a KU-algebra unless otherwise specified.

Definition 3.1. A bipolar fuzzy set $f = (X; f_n, f_p)$ in $\mathcal{X} := (X, *, 0)$ is called a *bipolar fuzzy KU-subalgebra* of $\mathcal{X} := (X, *, 0)$ if it satisfies the following condition:

$$(\forall x, y \in X) \left(\begin{array}{l} f_n(x * y) \leq \bigvee \{f_n(x), f_n(y)\} \\ f_p(x * y) \geq \bigwedge \{f_p(x), f_p(y)\} \end{array} \right). \tag{3.1}$$

Example 3.2. Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	0	0	0	2
2	0	2	0	1
3	0	0	0	0

Then $\mathcal{X} := (X, *, 0)$ is a KU-algebra (see [12]). Define a bipolar fuzzy set $f = (X; f_n, f_p)$ by

$$f = \{(0; -0.7, 0.8), (1; -0.6, 0.6), (2; -0.4, 0.5), (3; -0.3, 0.2)\}.$$

It is easy to verify that $f = (X; f_n, f_p)$ is a bipolar fuzzy KU-subalgebra of X.

For a bipolar fuzzy set $f = (X; f_n, f_p)$ in X and $(s, t) \in [-1, 0] \times [0, 1]$, we define

$$\begin{aligned} N(f_n; s) &= \{x \in X \mid f_n(x) \leq s\}, \\ P(f_p; t) &= \{x \in X \mid f_p(x) \geq t\} \end{aligned} \tag{3.2}$$

which are called the *negative s-cut* of $f = (X; f_n, f_p)$ and the *positive t-cut* of $f = (X; f_n, f_p)$, respectively (see [6]). The set

$$C(f; (s,t)) := N(f_n; s) \cap P(f_p; t)$$

is called the *(s,t)-cut* of $f = (X; f_n, f_p)$. For every $k \in [0,1]$, if $(s,t) = (-k, k)$ then the set

$$C(f; k) := N(f_n; -k) \cap P(f_p; k)$$

is called the *k-cut* of $f = (X; f_n, f_p)$ (see [6]).

Theorem 3.3. *Let $f = (X; f_n, f_p)$ be a bipolar fuzzy KU-subalgebra of $\mathcal{X} := (X, *, 0)$. Then for any $(s,t) \in [-1,0] \times [0,1]$, the nonempty negative s-cut and positive t-cut of $f = (X; f_n, f_p)$ are KU-subalgebras of $\mathcal{X} := (X, *, 0)$.*

Proof. (1) Let $s \in [-1, 0]$ be such that $N(f_n; s) \neq \varnothing$. If $x, y \in N(f_n; s)$, then $f_n(x) \leq s$ and $f_n(y) \leq s$. It follows from (3.1) that

$$f_n(x * y) \leq \bigvee \{f_n(x), f_n(y)\} \leq s$$

so that $x * y \in N(f_n; s)$. Hence $N(f_n; s)$ is a KU-subalgebra of X.

(2) Let $t \in [0,1]$ be such that $P(f_p; t) \neq \varnothing$. Let $x, y \in P(f_p; t)$. Then $f_p(x) \geq t$ and $f_p(y) \geq t$, which imply from (3.1) that

$$f_p(x * y) \geq \bigwedge \{f_p(x), f_p(y)\} \geq t.$$

Thus $x * y \in P(f_p; t)$, and so $P(f_p; t)$ is a KU-subalgebra of X. $\square$

Corollary 3.4. *If $f = (X; f_n, f_p)$ is a bipolar fuzzy KU-subalgebra of X, then the sets $N(f_n; f_n(0))$ and $P(f_p; f_p(0))$ are KU-subalgebras of X.*

Now we consider the converse of Theorem 3.3.

Theorem 3.5. *For a bipolar fuzzy set $f = (X; f_n, f_p)$ in $\mathcal{X} := (X, *, 0)$, if the nonempty negative s-cut and positive t-cut of $f = (X; f_n, f_p)$ are KU-subalgebras of $\mathcal{X} := (X, *, 0)$ for all $(s,t) \in [-1,0] \times [0,1]$, then $f = (X; f_n, f_p)$ is a bipolar fuzzy KU-subalgebra of $\mathcal{X} := (X, *, 0)$.*

Proof. Assume that the nonempty negative s-cut $N(f_n; s)$ and positive t-cut $P(f_p; t)$ of $f = (X; f_n, f_p)$ are KU-subalgebras of $\mathcal{X} := (X, *, 0)$ for all $(s,t) \in [-1,0] \times [0,1]$. If there exist $a, b \in X$ such that $f_n(a * b) > \bigvee \{f_n(a), f_n(b)\}$, then

$$f_n(a * b) > s_0 \geq \bigvee \{f_n(a), f_n(b)\}$$

for some $s_0 \in [-1, 0]$. It follows that $a, b \in N(f_n; s_0)$ but $a * b \notin N(f_n; s_0)$, a contradiction. Therefore $f_n(x * y) \leq \bigvee \{f_n(x), f_n(y)\}$ for all $x, y \in X$. Suppose that there exist $a, b \in X$ such that $f_p(a * b) < \bigwedge \{f_p(a), f_p(b)\}$. Then

$$f_p(a * b) < t_0 \leq \bigwedge \{f_p(a), f_p(b)\},$$

which implies that $a, b \in P(f_p; t_0)$ but $a * b \notin P(f_p; t_0)$. This is impossible, and thus $f_p(x * y) \geq \bigwedge \{f_p(x), f_p(y)\}$ for all $x, y \in X$. Consequently, $f = (X; f_n, f_p)$ is a bipolar fuzzy KU-subalgebra of $\mathcal{X} := (X, *, 0)$. $\square$

4. Bipolar fuzzy KU-ideals of KU-algebras

Definition 4.1. A bipolar fuzzy set $f = (X; f_n, f_p)$ in $\mathcal{X} := (X, *, 0)$ is called a *bipolar fuzzy KU-ideal* of $\mathcal{X} := (X, *, 0)$ if it satisfies the following condition:

$$(\forall x \in X)\,(f_n(0) \leq f_n(x),\ f_p(0) \geq f_p(x))\,, \tag{4.1}$$

$$(\forall x, y, z \in X) \begin{pmatrix} f_n(x * z) \leq \bigvee \{f_n(x * (y * z)), f_n(y)\} \\ f_p(x * z) \geq \bigwedge \{f_p(x * (y * z)), f_p(y)\} \end{pmatrix}. \tag{4.2}$$

Example 4.2. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the following Cayley table:

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	3
2	0	1	0	1	4
3	0	0	0	0	3
4	0	0	0	0	0

Then $\mathcal{X} := (X, *, 0)$ is a KU-algebra (see [11]). Define a bipolar fuzzy set $f = (X; f_n, f_p)$ by

$$f = \{(0; -0.8, 0.9), (1; -0.5, 0.7), (2; -0.5, 0.7), (3; -0.3, 0.4), (4; -0.3, 0.4)\}.$$

It is easy to verify that $f = (X; f_n, f_p)$ is a bipolar fuzzy KU-ideal of X.

Proposition 4.3. *Every bipolar fuzzy KU-ideal of $\mathcal{X} := (X, *, 0)$ satisfies the following condition.*

$$(\forall x, y \in X)\,(f_n(x * y) \leq f_n(y),\ f_p(x * y) \geq f_p(y))\,. \tag{4.3}$$

Proof. Let $f = (X; f_n, f_p)$ be a bipolar fuzzy KU-ideal of $\mathcal{X} := (X, *, 0)$. If we put $y = z$ in (4.2), then

$$\begin{aligned} f_n(x * y) &\leq \bigvee \{f_n(x * (y * y)), f_n(y)\} \\ &= \bigvee \{f_n(x * 0), f_n(y)\} \\ &= \bigvee \{f_n(0), f_n(y)\} \\ &= f_n(y) \end{aligned}$$

and

$$\begin{aligned} f_p(x * y) &\geq \bigwedge \{f_p(x * (y * y)), f_p(y)\} \\ &= \bigwedge \{f_p(x * 0), f_p(y)\} \\ &= \bigwedge \{f_p(0), f_p(y)\} \\ &= f_p(y) \end{aligned}$$

for all $x, y \in X$ by using (a2), (b1) and (b2). □

Theorem 4.4. *Every bipolar fuzzy KU-ideal of $\mathcal{X} := (X, *, 0)$ is a bipolar fuzzy KU-subalgebra of $\mathcal{X} := (X, *, 0)$.*

Proof. Let $f = (X; f_n, f_p)$ be a bipolar fuzzy KU-ideal of $\mathcal{X} := (X, *, 0)$. Since $f_n(x * y) \leq f_n(y)$ and $f_p(x * y) \geq f_p(y)$ for all $x, y \in X$ by Proposition 4.3, we have

$$f_n(x * y) \leq \bigvee \{f_n(x), f_n(y)\}, \; f_p(x * y) \geq \bigwedge \{f_p(x), f_p(y)\}$$

for all $x, y \in X$. Therefore $f = (X; f_n, f_p)$ be a bipolar fuzzy KU-subalgebra of $\mathcal{X} := (X, *, 0)$. □

The following example shows that the converse of Theorem 4.4 is not true in general.

Example 4.5. Consider the KU-algebra $\mathcal{X} := (X, *, 0)$ which is given in Example 3.2. Let $f = (X; f_n, f_p)$ be a bipolar fuzzy set in $\mathcal{X} := (X, *, 0)$ defined by

$$f = \{(0; -0.9, 0.8), (1; -0.7, 0.8), (2; -0.6, 0.5), (3; -0.3, 0.4)\}.$$

Then $f = (X; f_n, f_p)$ is a bipolar fuzzy KU-subalgebra of $\mathcal{X} := (X, *, 0)$, but it is not a bipolar fuzzy KU-ideal of $\mathcal{X} := (X, *, 0)$ since $f_n(2 * 1) = -0.6 > -0.7 = \bigvee\{f_n(2 * (1 * 1)), f_n(1)\}$ and/or $f_p(2 * 1) = 0.5 < 0.8 = \bigwedge\{f_p(2 * (1 * 1)), f_p(1)\}$.

Proposition 4.6. *Every bipolar fuzzy KU-ideal $f = (X; f_n, f_p)$ of $\mathcal{X} := (X, *, 0)$ satisfies the following condition:*

$$y * z \leq x \;\Rightarrow\; f_n(z) \leq \bigvee\{f_n(x), f_n(y)\}, \; f_p(z) \geq \bigwedge\{f_p(x), f_p(y)\} \tag{4.4}$$

for all $x, y, z \in X$.

Proof. Let $x, y, z \in X$ be such that $y * z \leq x$. Then $x * (y * z) = 0$, which implies from (4.1), (4.2) and (b1) that

$$\begin{aligned} f_n(y * z) &\leq \bigvee\{f_n(y * (x * z)), f_n(x)\} \\ &= \bigvee\{f_n(x * (y * z)), f_n(x)\} \\ &= \bigvee\{f_n(0), f_n(x)\} \\ &= f_n(x), \end{aligned}$$

$$\begin{aligned} f_p(y * z) &\geq \bigwedge\{f_p(y * (x * z)), f_p(x)\} \\ &= \bigwedge\{f_p(x * (y * z)), f_p(x)\} \\ &= \bigwedge\{f_p(0), f_p(x)\} \\ &= f_p(x). \end{aligned}$$

Hence, if we take $x = 0$ in (4.2) and use (a2) then

$$\begin{aligned} f_n(z) = f_n(0 * z) &\leq \bigvee\{f_n(0 * (y * z)), f_n(y)\} \\ &= \bigvee\{f_n(y * z), f_n(y)\} \\ &\leq \bigvee\{f_n(x), f_n(y)\}, \end{aligned}$$

$$\begin{aligned} f_p(z) = f_p(0 * z) &\geq \bigwedge\{f_p(0 * (y * z)), f_p(y)\} \\ &= \bigwedge\{f_p(y * z), f_p(y)\} \\ &\geq \bigwedge\{f_p(x), f_p(y)\}. \end{aligned}$$

This completes the proof. □

We now provide a condition for a bipolar fuzzy KU-subalgebra to be a bipolar fuzzy KU-ideal.

Theorem 4.7. *If a bipolar fuzzy KU-subalgebra $f = (X; f_n, f_p)$ of $\mathcal{X} := (X, *, 0)$ satisfies the condition (4.4), then $f = (X; f_n, f_p)$ is a bipolar fuzzy KU-ideal of $\mathcal{X} := (X, *, 0)$.*

Proof. Let $f = (X; f_n, f_p)$ be a KU-subalgebra that satisfies the condition (4.4). Using (b2) and (3.1), we have $f_n(0) \leq f_n(x)$ and $f_p(0) \geq f_p(x)$ for all $x \in X$. Since $(x * (y * z)) * (x * z) \leq y$ for all $x, y, z \in X$, it follows from (4.4) that

$$f_n(x * z) \leq \bigvee\{f_n(x * (y * z)), f_n(y)\} \text{ and } f_p(x * z) \geq \bigwedge\{f_p(x * (y * z)), f_p(y)\}$$

for all $x, y, z \in X$. Therefore $f = (X; f_n, f_p)$ is a bipolar fuzzy KU-ideal of $\mathcal{X} := (X, *, 0)$. □

Proposition 4.8. *Every bipolar fuzzy KU-ideal $f = (X; f_n, f_p)$ of $\mathcal{X} := (X, *, 0)$ satisfies the following condition:*

(1) $(\forall x, y \in X)\,(x \leq y \;\Rightarrow\; f_n(x) \leq f_n(y),\; f_p(x) \geq f_p(y))$.
(2) $(\forall x, y \in X)\,(f_n(x * (x * y)) \leq f_n(y),\; f_p(x * (x * y)) \geq f_p(y))$.

Proof. (1) Let $x, y \in X$ be such that $x \leq y$. Then $y * x = 0$, and so

$$\begin{aligned} f_n(x) = f_n(0 * x) &\leq \bigvee\{f_n(0 * (y * x)), f_n(y)\} \\ &= \bigvee\{f_n(0), f_n(y)\} = f_n(y), \end{aligned}$$

$$\begin{aligned} f_p(x) = f_p(0 * x) &\geq \bigwedge\{f_p(0 * (y * x)), f_p(y)\} \\ &= \bigwedge\{f_p(0), f_p(y)\} = f_p(y) \end{aligned}$$

by (a2) and (4.2).

(2) If we put $z = x * y$ in (4.2), then

$$\begin{aligned} f_n(x * (x * y)) &\leq \bigvee\{f_n(x * (y * (x * y))), f_n(y)\} \\ &= \bigvee\{f_n(x * (x * (y * y))), f_n(y)\} \\ &= \bigvee\{f_n(x * (x * 0)), f_n(y)\} \\ &= \bigvee\{f_n(0), f_n(y)\} \\ &= f_n(y), \end{aligned}$$

$$\begin{aligned} f_p(x*(x*y)) &\geq \bigwedge\{f_p(x*(y*(x*y))), f_p(y)\} \\ &= \bigwedge\{f_p(x*(x*(y*y))), f_p(y)\} \\ &= \bigwedge\{f_p(x*(x*0)), f_p(y)\} \\ &= \bigwedge\{f_p(0), f_p(y)\} \\ &= f_p(y) \end{aligned}$$

by using (a2), (b1), (b2) and (4.1). □

Theorem 4.9. *For a bipolar fuzzy set* $f = (X; f_n, f_p)$ *in* $\mathcal{X} := (X, *, 0)$*, the following are equivalent:*

(1) $f = (X; f_n, f_p)$ *is a bipolar fuzzy KU-ideal of* $\mathcal{X} := (X, *, 0)$.

(2) $f = (X; f_n, f_p)$ *satisfies the following assertions:*

  (i) $(\forall s \in [-1, 0])\,(N(f_n; s) \neq \varnothing \;\Rightarrow\; N(f_n; s)$ is a KU-ideal of $X)$.

  (ii) $(\forall t \in [0, 1])\,(P(f_p; t) \neq \varnothing \;\Rightarrow\; P(f_p; t)$ is a KU-ideal of $X)$.

Proof. (1) ⇒ (2). Let $s \in [-1, 0]$ be such that $N(f_n; s) \neq \varnothing$. Then there exists $y \in N(f_n; s)$, and so $f_n(y) \leq s$. It follows from (4.1) that $f_n(0) \leq f_n(y) \leq s$ so that $0 \in N(f_n; s)$. Let $x, y, z \in X$ such that $x*(y*z) \in N(f_n; s)$ and $y \in N(f_n; s)$. Then $f_n(x*(y*z)) \leq s$ and $f_n(y) \leq s$. Using (4.2), we have

$$f_n(x*z) \leq \bigvee\{f_n(x*(y*z)), f_n(y)\} \leq s$$

which implies that $x*z \in N(f_n; s)$. Therefore $N(f_n; s)$ is a KU-ideal of X. Assume that $P(f_p; t) \neq \varnothing$ for $t \in [0, 1]$, and let $a \in P(f_p; t)$. Then $f_p(a) \geq t$, and so $f_p(0) \geq f_p(a) \geq t$ by (4.1). Thus $0 \in P(f_p; t)$. Let $x, y, z \in X$ such that $x*(y*z) \in P(f_p; t)$ and $y \in P(f_p; t)$. Then $f_p(x*(y*z)) \geq t$ and $f_p(y) \geq t$. It follows from (4.2) that

$$f_p(x*z) \geq \bigwedge\{f_p(x*(y*z)), f_p(y)\} \geq t$$

so that $x*z \in P(f_p; t)$. Hence $P(f_p; t)$ is a KU-ideal of X.

(2) ⇒ (1). Assume that there exist $a \in X$ such that $f_n(0) > f_n(a)$. Taking

$$s_0 := \tfrac{1}{2}\,(f_n(0) + f_n(a))$$

implies $f_n(a) < s_0 < f_n(0)$. This is a contradiction, and thus $f_n(0) \leq f_n(y)$ for all $y \in X$. Suppose that

$$f_n(x*z) > \bigvee\{f_n(x*(y*z)), f_n(y)\}$$

for some $x, y, z \in X$ and let

$$s_1 := \tfrac{1}{2}\left(f_n(x*z) + \bigvee\{f_n(x*(y*z)), f_n(y)\}\right).$$

Then $\bigvee\{f_n(x*(y*z)), f_n(y)\} < s_1 < f_n(x*z)$, which is a contradiction. Therefore

$$f_n(x*z) \leq \bigvee\{f_n(x*(y*z)), f_n(y)\}$$

for all $x, y, z \in X$. Now, if $f_p(0) < f_p(y)$ for some $y \in X$, then $f_p(0) < t_0 \leq f_p(y)$ for some $t_0 \in (0, 1]$. This is a contradiction. Thus $f_p(0) \geq f_p(y)$ for all $y \in X$. If

$$f_p(x*z) < \bigwedge\{f_n(x*(y*z)), f_n(y)\}$$

for some $x, y, z \in X$, then there exists $t_1 \in (0, 1]$ such that

$$f_p(x * z) < t_1 \leq \bigwedge \{f_n(x * (y * z)), f_n(y)\}.$$

It follows that $x * (y * z) \in P(f_p; t_1)$ and $y \in P(f_p; t_1)$ but $x * z \notin P(f_p; t_0)$, a contradiction. Consequently, $f_p(x*z) \geq \bigwedge \{f_n(x * (y * z)), f_n(y)\}$ for all $x, y, z \in X$. Therefore $f = (X; f_n, f_p)$ is a bipolar fuzzy KU-ideal of X. □

Let Λ^P and Λ^N be nonempty subsets of $[0, 1]$ and $[-1, 0]$, respectively.

Theorem 4.10. *Let $\{G_k \mid k \in \Lambda^P \cup \Lambda^N\}$ be a finite collection of KU-ideals of X such that*

(1) $X = (\cup\{G_t \mid t \in \Lambda^P\}) \bigcup (\cup\{G_s \mid s \in \Lambda^N\})$,
(2) $(\forall m, n \in \Lambda^P \cup \Lambda^N)$ $(m > n \iff G_m \subset G_n)$.

Then a bipolar fuzzy set $f = (X; f_n, f_p)$ in $\mathcal{X} := (X, *, 0)$ defined by

$$f_p(x) = \bigvee\{t \in \Lambda^P \mid x \in G_t\},\ f_n(x) = \bigwedge\{s \in \Lambda^N \mid x \in G_s\} \tag{4.5}$$

for all $x \in X$ is a bipolar fuzzy KU-ideal of X.

Proof. Let $(s, t) \in [-1, 0] \times [0, 1]$ be such that $P(f_p; t)$ and $N(f_n; s)$ are non-empty. We claim that $P(f_p; t)$ and $N(f_n; s)$ are KU-ideals of X. We consider the following two cases:

$$\text{(i)}\ t = \bigvee\{r \in \Lambda^P \mid r < t\} \ \text{ and } \ \text{(ii)}\ t \neq \bigvee\{r \in \Lambda^P \mid r < t\}.$$

First case implies that

$$x \in P(f_p; t) \iff x \in G_r \text{ for all } r < t \iff x \in \cap\{G_r \mid r < t\}, \tag{4.6}$$

so that $P(f_p; t) = \cap\{G_r \mid r < t\}$, which is a KU-ideal of X. For the second case, we claim that $P(f_p; t) = \cup\{G_r \mid r \geq t\}$. If $x \in \cup\{G_r \mid r \geq t\}$, then $x \in G_r$ for some $r \geq t$. It follows that $f_p(x) \geq r \geq t$ so that $x \in P(f_p; t)$. If $x \notin \cup\{G_r \mid r \geq t\}$, then $x \notin G_r$ for all $r \geq t$. Since $t \neq \bigvee\{r \in \Lambda^P \mid r < t\}$, there exists $\varepsilon > 0$ such that $(t - \varepsilon, t) \cap \Lambda^P = \varnothing$. Hence $x \notin G_r$ for all $r > t - \varepsilon$, which means that if $x \in G_r$ then $r \leq t - \varepsilon$. Thus $f_p(x) \leq t - \varepsilon < t$, and so $x \notin P(f_p; t)$. Therefore $P(f_p; t) = \cup\{G_r \mid r \geq t\}$ which is a KU-ideal of X since $\{G_k\}$ forms a chain. Next we show that $N(f_n; s)$ is a KU-ideal of X. We also consider the following two cases:

$$\text{(iii)}\ s = \bigwedge\{q \in \Lambda^N \mid s < q\} \ \text{ and } \ \text{(iv)}\ s \neq \bigwedge\{q \in \Lambda^N \mid s < q\}.$$

For the case (iii), we get

$$x \in N(f_n; s) \iff x \in G_q \text{ for all } q > s \iff x \in \cap\{G_q \mid s < q\}, \tag{4.7}$$

and so $N(f_n; s) = \cap\{G_q \mid s < q\}$, which is a KU-ideal of X. For the case (iv), we prove that $N(f_n; s) = \cup\{G_q \mid q \leq s\}$. If $x \in \cup\{G_q \mid q \leq s\}$, then $x \in G_q$ for some $q \leq s$. It follows that $f_n(x) \leq q \leq s$ so that $x \in N(f_n; s)$. Hence

$$\cup\{G_q \mid q \leq s\} \subseteq N(f_n; s).$$

Conversely, if $x \notin \cup\{G_q \mid q \leq s\}$ then $x \notin G_q$ for all $q \leq s$. Since

$$s \neq \bigwedge\{q \in \Lambda^N \mid s < q\},$$

there exists $\varepsilon > 0$ such that $(s, s + \varepsilon) \cap \Lambda^N = \varnothing$, which implies that $x \notin G_q$ for all $q < s + \varepsilon$. This says that if $x \in G_q$ then $q \geq s + \varepsilon$. Thus $f_n(x) \geq s + \varepsilon > s$, and so $x \notin N(f_n; s)$. Therefore $N(f_n; s) \subseteq \cup\{G_q \mid q \leq s\}$, and consequently

$$N(f_n; s) = \cup\{G_q \mid q \leq s\}$$

which is a KU-ideal of X. Therefore, using Theorem 4.9, $f = (X; f_n, f_p)$ is a bipolar fuzzy KU-ideal of X. □

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