Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 8, No. 3, (September 2014), pp. 393–408
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

On characteristic semigroup of state (input) independent fuzzy automata

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Received 15 December 2013; Revised 24 January 2014; Accepted 4 March 2014

Abstract. The present paper concerns with the characteristic semigroup of input independent and state independent fuzzy automaton. Apart from the relationship between input independent and state independent fuzzy automaton, various properties of characteristic semigroups of state independent fuzzy automaton are established. Also, quasi-perfect fuzzy automaton and quasi strongly connected fuzzy automaton are introduced and their properties are discussed.

2010 AMS Classification: 08A72, 68Q70, 68Q45, 03E05

Keywords: Input independent and state independent fuzzy automaton, Characteristic semigroup, Quasi-perfect fuzzy automaton, Singly generated fuzzy automaton.

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1. Introduction

Recently fuzzy automata are studied by many researchers in varied directions [5, 6, 11, 13, 14, 15, 16, 17]. Different types of fuzzy groups are discussed in [10, 12]. One way of studying fuzzy automaton is to generalize the results of classical automaton. Characteristic semigroup of classical automaton was introduced by Fleck in [8] and he discussed its properties over perfect automaton. Many researchers then discussed characteristic semigroups of various classes of classical automaton along with their properties [1, 2, 9, 19, 20].

In this paper we extend these concepts and results for fuzzy automaton. Precisely, input independent fuzzy automaton is newly introduced and its relation with state independent fuzzy automaton is discussed. Also, characteristic semigroups of right simple (right group, group) type fuzzy automaton are discussed and their various properties are established. In [7], we have discussed the cardinality of the group of weak fuzzy automaton isomorphisms. Since the group of weak fuzzy automaton isomorphisms determines the structure of fuzzy automaton, we have also investigated the relationships between them in the context of state independent fuzzy automaton.

In this paper we introduced the concept of input independent fuzzy automaton and obtained that every input independent fuzzy automaton is connected. Further, it is established that the characteristic semigroup of input independent fuzzy automaton is a right zero semigroup. Condition for input independent fuzzy automaton to be strongly connected is also derived. We have proved that the input independent fuzzy automaton is a sink if and only if it is weakly abelian. We have also characterized strict quasi-cyclic input independent fuzzy automaton.After introducing state independent fuzzy automaton, we have established its relation with input independent fuzzy automaton. Characteristic semigroups of state independent fuzzy automaton, right simple type, right group type, group type and quasi-perfect fuzzy automaton are discussed and their characterizations are established. We have established that a strongly connected fuzzy automaton is quasi-perfect if and only if it is state independent. Further, in quasi-perfect fuzzy automaton it is observed that the characteristic group, the group of its states and the group of its weak fuzzy automaton automorphisms are isomorphic. Index and period of a component of a fuzzy automaton with single input are discussed to find their various properties. It is also established that the component of state independent fuzzy automaton contains exactly one strongly connected subautomaton. Existence of weak fuzzy automaton isomorphism between any two strongly connected fuzzy subautomata of state independent fuzzy automaton is also established. Finally, it is established that the number of states of the strongly connected subautomaton of a state independent fuzzy automaton gives the cardinality of its characteristic semigroup.

2. Preliminaries

In this section, we recall preliminary concepts and notations of fuzzy automata that are needed for the rest of the paper.

Definition 2.1. Let A and B be sets. A *fuzzy relation* from A to B is a fuzzy set R of $A \times B$ i.e.$R : A \times B \to [0,1]$. The number $R(a,b)$ denotes the degree to which a is related to b.

Definition 2.2. [11] A fuzzy relation R from A to B is said to be *complete*, if for each $a \in A$, there exists $b \in B$ such that $R(a,b) > 0$. A fuzzy relation R is said to be *fuzzy function*, if for each $a \in A$, there is *unique* $b \in B$ such that $R(a,b) > 0$.

Since Supp(R) is actually a (crisp) function, the above definition resembles to that of the definition of the (crisp) function, in the sense of unique image for each element of the domain.

Definition 2.3. A *fuzzy automaton* is a triplet $A = (Q, \Sigma, \mu)$, where Q is a nonempty finite set called set of *states*, Σ is a nonempty finite set called set of *inputs* and μ is a *fuzzy function* from $Q \times \Sigma$ to Q.

If $A = (Q, \Sigma, \mu)$ is a fuzzy automaton, then Σ^* denotes the set of all strings of symbols in Σ including the empty string ϵ and the fuzzy function μ is extended to a fuzzy function μ^* from $Q \times \Sigma^*$ to Q as follows : for all $p, q \in Q, a \in \Sigma$ and $x \in \Sigma^*$ we have $\mu^*(p, ax, q) = \mu(p, a, r) \wedge \mu^*(r, x, q)$, where $r \in Q$ is such that $\mu(p, a, r) > 0$

and
$\mu^*(p,\epsilon,q) = \begin{cases} 1, & \text{if p = q;} \\ 0, & \text{otherwise.} \end{cases}$

Here onward, in this paper, we write μ for both μ and μ^* without any ambiguity.

Definition 2.4. Let $A_1 = (Q_1, \Sigma_1, \mu_1)$ and $A_2 = (Q_2, \Sigma_2, \mu_2)$ be two fuzzy automata. A pair (h,k) of maps, where $h : Q_1 \to Q_2$, $k : \Sigma_1 \to \Sigma_2$, is called *fuzzy automaton homomorphism* from A_1 to A_2, symbolically $(h,k) : A_1 \to A_2$, if for $p,q \in Q_1$ and $x \in \Sigma_1^*$,$\mu_2(h(p), k(x), h(q)) = \mu_1(p,x,q)$.
A pair of maps $(h,k) : A_1 \to A_2$ is said to be *weak fuzzy automaton homomorphism*, if for $p,q \in Q_1$ and $x \in \Sigma_1^*$,$\mu_1(p,x,q) > 0 \Rightarrow \mu_2(h(p), k(x), h(q)) > 0$

In case, if $\Sigma_1 = \Sigma_2 = \Sigma$ and k is the identity function on Σ ,then we shall denote the homomorphism simply by $h : A_1 \to A_2$.

Remark 2.5. Every fuzzy automaton homomorphism is a weak fuzzy automaton homomorphism, but not conversely.

A (weak) fuzzy automaton homomorphism (h,k) from A_1 to A_2 is said to be *(weak) fuzzy automaton isomorphism*, if both h and k are bijective functions.

We shall adopt the following notations throughout this paper
$H^F(A \to B)$: The set of all fuzzy automaton homomorphisms from A to B.
$WH^F(A \to B)$: The set of all weak fuzzy automaton homomorphisms from A to B.
$I^F(A \to B)$: The set of all fuzzy automaton isomorphisms from A to B.
$WI^F(A \to B)$: The set of all weak fuzzy automaton isomorphisms from A to B.
$E^F(A)$: The set of all fuzzy automaton endomorphisms on A.
$WE^F(A)$: The set of all weak fuzzy automaton endomorphisms on A.
$G^F(A)$: The set of all fuzzy automaton automorphisms on A.
$WG^F(A)$: The set of all weak fuzzy automaton automorphisms on A.

Let $A = (Q, \Sigma, \mu)$ be a fuzzy automaton and $M \subseteq Q$.Then the *successor of* M is the set $S(M) = \{p \in Q | \mu(q,x,p) > 0, \text{ for some } (q,x) \in M \times \Sigma^*\}$ and *x-successor of* M is the set $S_x(M) = \{p \in Q | \mu(q, x^k, p) > 0, \text{ for some } q \in M \text{ and } k \in \mathbb{N} \cup \{0\}\}$, where $x^0 = \epsilon$. We denote the successor of $\{q\}$ by $S(q)$ and the x-successor of $\{q\}$ by $S_x(q)$.

Definition 2.6 ([18])**.** A fuzzy automaton $B = (R, \Sigma, \lambda)$ is called a *subautomaton* of automaton $A = (Q, \Sigma, \mu)$, if $R \subseteq Q$, $S(R) = R$ and $\mu|_{R\times\Sigma\times R} = \lambda$. This subautomaton is called *seperated*, if $S(Q - R) \cap R = \varnothing$

Definition 2.7 ([13])**.** A fuzzy automaton $A = (Q, \Sigma, \mu)$ is said to be
(i) *connected*, if A has no proper seperated subautmaton.
(ii) *strongly connected*, if $\forall p, q \in Q$,we have $q \in S(p)$.
(iii) *abelian*, if $\mu(p, xy, q) = \mu(p, yx, q), \forall x, y \in \Sigma^*$ and $p, q \in Q$.
(iv) *weakly abelian*, if $\mu(p, xy, q) > 0 \Leftrightarrow \mu(p, yx, q) > 0$, for $x, y \in \Sigma^*$ and $p, q \in Q$.

Definition 2.8. For a fuzzy automaton $A = (Q, \Sigma, \mu)$, *the center of* A, denoted by $Z(A)$, is the set $\{x \in \Sigma^* |$ for $q, s \in Q$, we have $\mu(q, xy, s) > 0 \Leftrightarrow \mu(q, yx, s) > 0\}, \forall y \in \Sigma^*$.

Clearly, $Z(A) = \Sigma^*$ if and only if A is weakly abelian.
Let $A = (Q, \Sigma, \mu)$ be a fuzzy automaton and $q \in Q$. Then the fuzzy automaton *generated* by q is, denoted by $A(q)$, $A(q) = (S(q), \Sigma, \mu')$, where μ' is a restriction of μ to $S(q) \times \Sigma \times S(q)$.

Definition 2.9. [18] A fuzzy automaton A is said to be *singly generated*, if there exists $q \in Q$ such that $A = A(q)$.

The set of *generators* of $A(q)$ is the set $genA(q) = \{r \in S_{A(q)} | A(r) = A(q)\}$.

Definition 2.10. Let $A = (Q, \Sigma, \mu)$ be a fuzzy automaton. If $A(q)$ is strongly connected for each $q \in Q$, then A is called quasi-strongly connected .

Definition 2.11 ([7])**.** Let $A = (Q, \Sigma, \mu)$ be a fuzzy automaton, $q \in Q$ and $y, z \in \Sigma^*$. Then y is *q-fuzzy equivalent* to z , if $\mu(q, y, p) > 0$ and $\mu(q, z, p) > 0$, for some $p \in Q$. We shall denote it by $y \equiv_q^F z$.

Remark 2.12. The relation $\equiv_q^F$ is an equivalence relation of finite index on Q(i.e. on A). We shall denote the equivalence class of $x \in \Sigma^*$ with respect to this relation by $[x]_q$.

In following example, we discuss application of fuzzy automata for approximate minimization. By an approximate minimization of fuzzy automaton, we mean smallest fuzzy automaton (in the sense of number of states) having approximate languages. Two fuzzy languages L_1 and L_2 are approximate with threshold $\alpha \in [0, 1]$, if $E(L_1, L_2) = \bigwedge_x (L_1(x) \leftrightarrow L_2(x)) \geq \alpha$, where $a \leftrightarrow b = (a \to b) \wedge (b \to a)$ and $a \to b = \vee\{x \in [0, 1] | a \wedge x \leq b\}$ [3].

Example 2.13. Consider a fuzzy automaton A with $Q = \{q_0, \ldots, q_5\}, \Sigma = \{a\}$, $Q_F(q_i) = \frac{1}{i+1}$, when $i < 5$, $Q_F(q_5) = 0$ and $\mu(q_i, a, q)_{i+1} = \frac{1}{i+2}$, for $i < 5$, $\mu(q_5, a, q_5) = \frac{1}{7}$, $\mu(p, x, q) = 0$ otherwise.

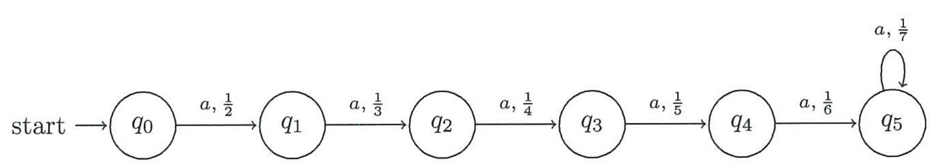

FIGURE 1.

Then $L_A(a^i) = \begin{cases} \frac{1}{i+1}, & \text{if } i < 5 \text{ ;} \\ \frac{1}{7}, & \text{if } i \geq 5. \end{cases}$
In classical sense (i.e. having equal languages), there does not exist a deterministic fuzzy automaton A' with less than 6 states having the same language that of A.
But if we consider fuzzy automaton A' given by following diagram

$$(Q = \{p_0, p_1\}, \Sigma = \{a\}, Q_F(p_0) = \tfrac{3}{4}, Q_F(p_1) = \tfrac{1}{2})$$

and μ as shown in the figure :

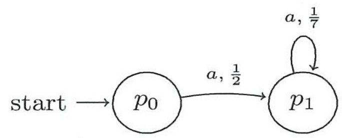

FIGURE 2. $\frac{1}{7}$-approximate minimization

Then $L_{A'}(a^i) = \begin{cases} \frac{3}{4}, & \text{if } i = 0\ ; \\ \frac{1}{2}, & \text{if } i = 1\ ; \\ \frac{1}{4}, & \text{if } i \geq 2. \end{cases}$

Then, $E(L_A, L_{A'}) \geq \frac{1}{7}$. Therefore, A' is an approximate minimization with threshold $\frac{1}{7}$ of A.

For more details on approximate minimization of fuzzy automaton, we refer to [4].

3. Input independent fuzzy automaton

This section is devoted to input independent fuzzy automaton which, we have defined first time, forms a subclass of state independent fuzzy automaton, that we are going to discuss in the next section. The characteristic semigroup of this input independent fuzzy automaton is a right zero semigroup. Let $A = (Q, \Sigma, \mu)$ be any fuzzy automaton. Then for any $x \in \Sigma^+$, where $\Sigma^+ = \Sigma^* - \{\epsilon\}$, we define the relation $\sim_x$ on Q as: $q_1 \sim_x q_2$ if and only if $\mu(q_1, x, p) \wedge \mu(q_2, x, p) > 0$, for some $p \in Q$.
Clearly "$\sim_x$" is an equivalence relation on Q and $[q]_x$ denote the equivalence class q with respect to $\sim_x$.
Note that the notation $\mu(q_1, x, p) \wedge \mu(q_2, x, p) > 0$ is used here for $\mu(q_1, x, p) > 0$ and $\mu(q_2, x, p) > 0$. This notation should not be confused with extension of μ to μ^*.

Definition 3.1. A fuzzy automaton A is said to be *input independent*, if $[q]_x = Q$, for all $x \in \Sigma^+$ and for some $q \in Q$.

A fuzzy automaton A', given in Figure 2 is actually an input independent fuzzy automaton. From this example we conclude that for each fuzzy automaton which is not input independent, there exists an input independent α-approximate minimization. Hence, the theory developed in this paper can be applied.

Lemma 3.2. *If $A = (Q, \Sigma, \mu)$ is an input independent fuzzy automaton, then for any $x \in \Sigma^+$, we have $[x]_q = [x]_s$ for all $q, s \in Q$.*

Therefore, for input independent fuzzy automaton, here onward we shall write the equivalence class $[x]_q$ by $[x]$ without any ambiguity and the collection of all such classes by $[\Sigma^+]$.

Remark 3.3. For input independent fuzzy automaton $A = (Q, \Sigma, \mu)$, we have
(i) $[z] = \{xz | x \in \Sigma^+\}$, for any $z \in \Sigma^+$
(ii) The operation $[x][y] = [y]$ is well defined on $[\Sigma^+]$ and $[\Sigma^+]$ is semigroup under this operation.
(iii) $S(p) = S(q)$, for all $p, q \in Q$.
(iv) $|S(q)| = |[\Sigma^+]|$, for all $q \in Q$.

The semigroup $[\Sigma^+]$ is called the characteristic semigroup of A.

Lemma 3.4. *If A is an input independent fuzzy automaton, then its characteristic semigroup $[\Sigma^+]$, with operation $[x][y] = [y]$ is a right zero semigroup.*

Theorem 3.5. *Every input independent fuzzy automaton is connected.*

Proof. Let C_1 and C_2 be two components of A and $q_1 \in C_1$, $q_2 \in C_2$. Choose $a \in \Sigma$ be such that $\mu(q_1, a, p) > 0$, for some $p \in C_1$. Since A is input independent fuzzy automaton, we have $\mu(q_2, a, p) > 0$. This leads to $p \in C_1 \cap C_2$, which is a contradiction. □

Theorem 3.6. *Let A be an input independent fuzzy automaton. If A is strongly connected, then $|Q| \leq |\Sigma|$.*

Proof. Since A is strongly connected, for each $q \in Q$, there exists $x_q \in \Sigma$ such that $\mu(p, x_q, q) > 0$, for some $p \in Q$. Again since A is input independent fuzzy automaton, for $q_1 \neq q_2$, we have $x_{q_1} \neq x_{q_2}$. this proves that $|Q| \leq |\Sigma|$. □

The converse of the above theorem is not true in general. This can be seen in the following example.

Example 3.7. Let $Q = \{q_1, q_2, q_3\}$ and $I = \{a, b, c\}$. Let μ be defined by the following diagram.

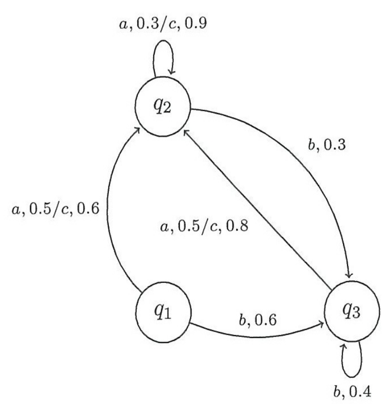

FIGURE 3.

Definition 3.8. An input independent fuzzy automaton with trivial characteristic semigroup, i.e.$|[\Sigma^+]| = 1$, is called as *sink*.

Theorem 3.9. *An input independent fuzzy automaton is a sink if and only if it is weakly abelian.*

Proof. Let $A = (Q, \Sigma, \mu)$ be an input independent fuzzy automaton. Suppose $x, y \in \Sigma^+$ such that $\mu(q_1, xy, q_2) > 0$. Since A is input independent fuzzy automaton, we have $[x][y] = [y]$ and $[y][x] = [x]$. Also, A is sink implies that $[x] = [y]$. Therefore $[x][y] = [y][x]$. Hence, $\mu(q_1, yx, q_2) > 0$. Conversely, suppose that A is a not sink. Let $[x], [y] \in [\Sigma^+]$ with $[x] \neq [y]$. Then $[x][y] = [y] \neq [x] = [y][x]$. Hence, A is not weakly abelian. □

Theorem 3.10. *Every singly generated input independent fuzzy automaton is strongly connected.*

Proof. Consider $A = A(q)$. Let $q_1, q_2 \in Q$. Then there exist $x, y \in \Sigma^+$ such that $\mu(q, x, q_1) > 0$ and $\mu(q, y, q_2) > 0$. Since A is an input independent fuzzy automaton, we have $\mu(q_1, y, q_2) > 0$. □

Corollary 3.11. *Every input independent fuzzy automaton is quasi-strongly connected.*

Theorem 3.12. *An input independent fuzzy automaton A is strongly connected if and only if for every $q \in Q$, there exists $a \in \Sigma$ such that $\mu(q, a, q) > 0$.*

Proof. Suppose A is strongly connected. Let $q \in Q$. Then there exists $x \in \Sigma^*$ such that $\mu(q, x, q) > 0$. Thus, there is $a \in \Sigma$ and $p \in Q$ such that $\mu(p, a, q) > 0$. Since A is input independent fuzzy automaton, $\mu(q, a, q) > 0$. Conversely, let $q_1, q_2 \in Q$. By hypothesis there exist $a, b \in \Sigma$ such that $\mu(q_1, a, q_1) > 0$ and $\mu(q_2, b, q_2) > 0$. Then $\mu(q_1, b, q_2) > 0$. This proves that A is strongly connected. □

Definition 3.13. A fuzzy automaton A is said to be *quasi-cyclic*, if there exists $q \in Q$ such that $p \in S(q)$, for all $p \in Q - \{q\}$. In this case q is called a quasi-generator of A.

A fuzzy automaton A is strict quasi-cyclic, if there exists $q \in Q$ such that $p \in S(q)$ if and only if $p \in Q - \{q\}$.
In a fuzzy automaton $A = (Q, \Sigma, \mu)$ a state q is said to have a self-loop if $\mu(q, a, q) > 0$, for some $a \in \Sigma$.

Theorem 3.14. *If A is a strict quasi-cyclic input independent fuzzy automaton, then it has unique quasi-generator. Further, any state which does not have a self-loop is the quasi-generator of A.*

Proof. Suppose q_1 and q_2 are quasi-generator of A. There exist $a, b \in \Sigma$ such that $\mu(q_1, a, q_2) > 0$ and $\mu(q_2, b, q_1) > 0$. Since A is input independent fuzzy automaton, we have $\mu(q_2, a, q_2) > 0$. This is impossible, as A is strict quasi-cyclic and q_2 is quasi-generator. Now suppose that for some $q \in Q$, we have $\mu(q, a, q) = 0$, for all $a \in \Sigma^+$. If $p(\neq q)$ is a quasi-generator of A, then there exists $b \in \Sigma$ such that $\mu(p, b, q) > 0$. Then $\mu(q, b, q) > 0$, which is a contradiction. Therefore, quasi-cyclicity of A proves that q is the quasi-generator of A. □

Theorem 3.15. *Let A be an input independent fuzzy automaton. Then A is strict quasi-cyclic if and only if there are exactly $|Q| - 1$ elements in Q with $\mu(q, a, q) > 0$, for some $a \in \Sigma$.*

Proof. Immediate by above theorem. □

Definition 3.16. A fuzzy automaton $A = (Q, \Sigma, \mu)$ is said to be a *cycle* of length n if $|Q| = n$ and there is $q \in Q$,$x \in \Sigma^*$ with $|x| = n$ such that $\mu(q, x, q) > 0$.

Theorem 3.17. *If A is an input independent fuzzy automaton with $|[\Sigma]| = n$, then there is a cycle of length n. In fact any state of this cycle is a self-loop state.*

Proof. Let $q \in Q$ and $[\Sigma] = \{[a_1], [a_2], \ldots, [a_n]\}$. Then we must have $\{q_1, q_2, \ldots, q_n\}$ such that $\mu(q, a_i, q_i) > 0$, for $i = 1, 2, \ldots, n$. Then $\mu(q_{i+1}, a_i, q_i) > 0$, for $i = 1, 2, \ldots, n-1$ and $\mu(q_1, a_n, q_n) > 0$. Therefore, $\{q_1, q_2, \ldots, q_n\}$ is cycle of length n. Suppose $\{q'_1, q'_2, \ldots, q'_n\}$ is another cycle of length n. Then $\mu(q, a_1, q_i) > 0$ and $\mu(q, a_1, q'_i) > 0$. This implies that $q_i = q'_i$. □

Theorem 3.18. *Let $A_1 = (Q_1, \Sigma, \mu_1)$, $A_2 = (Q_2, \Sigma, \mu_2)$ and $h \in WH^F(A_1 \to A_2)$ with h onto. If A_1 is a sink, then A_2 is also a sink.*

Proof. Let $q_i \in Q_2$, for $i = 1, 2, 3, 4$ and $x \in \Sigma^+$ be such that $\mu_2(q_1, x, q_2) \wedge \mu_2(q_3, x, q_4) > 0$. Since h is onto, there exist $p_i \in Q_1$ with $h(p_i) = q_i$, for $i = 1, 2, 3, 4$. Then $\mu_1(p_1, x, p_2) \wedge \mu_1(p_3, x, p_4) > 0$. This implies that $p_2 = p_4$, as A_1 is input independent fuzzy automaton. Therefore, $q_2 = q_4$. This prove that A_2 is also input independent. Let $[x], [y] \in [\Sigma^+]$ and $q \in Q_2$. Then there exists $p \in Q_1$ such that $h(p) = q$. Since A_1 is a sink, we have $x \equiv_p^F y$. Thus, $\mu_1(p, x, p_1) \wedge \mu_1(p, y, p_1) > 0$, for some $p_1 \in Q_1$. Therefore, $\mu_2(q, x, q_1) \wedge \mu_2(q, y, q_1) > 0$, where $h(p_1) = q_1$. This implies that, $[x]_q = [y]_q$. Hence, A_2 is also a sink. □

4. State independent fuzzy automaton

In this section we introduce state independent fuzzy automaton and its types namely right simple, right group and group type fuzzy automaton. Apart from various properties of these types of state independent fuzzy automaton, we will establish that the characteristic semigroup of state independent fuzzy automaton is the semigroup of successor of any state with a suitable binary operation. We begin this section with the definition of state independent fuzzy automaton.

Definition 4.1. A is said to be a *state independent fuzzy automaton*, if for all $q, s \in Q$ and $x \in \Sigma^*$, $[x]_q = [x]_s$. i.e.$\forall q, s \in Q$ and $x, y \in \Sigma^*$, one has $\mu(q, x, p) > 0$ and $\mu(q, y, p) > 0$, for some $p \in Q$ if and only if $\mu(s, x, t) > 0$ and $\mu(s, y, t) > 0$, for some $t \in Q$.

Therefore, for state independent fuzzy automaton, here onward we shall write the equivalence class $[x]_q$ by $[x]$ without any ambiguity and the collection of all such classes by $[\Sigma^*]$. Under the natural operation, $[\Sigma^*]$ is the characteristic semigroup of A. In fact

Lemma 4.2. *For state independent fuzzy automaton $A = (Q, \Sigma, \mu)$,the $[\Sigma^*]$ forms a monoid with respect to the operation $[x][y] = [xy]$.*

The equivalence class of the empty string is the identity of this monoid.

Lemma 4.3. *If $A = (Q, \Sigma, \mu)$ is a state independent fuzzy automaton, then for $x, y \in \Sigma^*$, we have $[x] = [y] \Rightarrow [x^k] = [y^k]$, for all $k \in \mathbb{N}$.*

Lemma 4.4. *If $A = (Q, \Sigma, \mu)$ is a state independent fuzzy automaton, then the map $x \to [x]$ is an onto homomorphism from Σ^* to $[\Sigma^*]$.*

If the condition of state independence holds for all $x \in \Sigma^+$, then A is called as partially state independent fuzzy automaton.

Theorem 4.5. *Every input independent fuzzy automaton is partially state independent fuzzy automaton.*

Proof. Suppose $x \equiv_q^F y$, for some $x, y \in \Sigma^+$ and $q \in Q$. Since A is input independent fuzzy automaton, we have $x \equiv_t^F y$, for all $t \in Q$. Therefore, A is partially state independent fuzzy automaton. □

Recall that a semigroup S is right simple, if for any $a, b \in S$, there exists $c \in S$ such that $ac = b$. It is a right group, if S is a right simple semigroup with a left identity[2].

Definition 4.6. A fuzzy automaton $A = (Q, \Sigma, \mu)$ is said to be a *right simple (right group,group)-type fuzzy automaton*, if A is state independent fuzzy automaton and the semigroup $[\Sigma^*]$ is a right simple (right group,group respectively).

Definition 4.7. Fuzzy automaton $A = (Q, \Sigma, \mu)$ is said to be *invertible*, if $\mu(p, x, q) > 0$ for some $p, q \in Q$ and $x \in \Sigma^*$ then there exists $y \in \Sigma^*$ such that $\mu(q, y, p) > 0$.

Theorem 4.8. *Every group type fuzzy automaton is invertible.*

Proof. Let A be a group type fuzzy automaton and $\mu(p, x, q) > 0$, for some $p, q \in Q$ and $x \in \Sigma^*$. Then $[x] \in [\Sigma^*]$ and $[\Sigma^*]$ is a group. Thus, there exists $[y] \in [\Sigma^*]$ such that $[x][y] = [\epsilon]$. Therefore, $\mu(p, xy, p) > 0$. This implies that $\mu(p, x, q) \wedge \mu(q, y, p) > 0$. Hence, $\mu(q, y, p) > 0$. □

The following example shows that a group-type fuzzy automaton is not strongly connected.

Example 4.9. Let $Q = \{q_1, q_2, q_3, q_4, q_5, q_6\}$ and $\Sigma = \{a\}$. Let μ be defined by the following diagram.

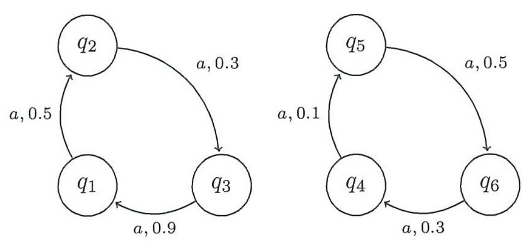

FIGURE 4.

In fact, we conclude the following

Lemma 4.10. *Every singly generated group-type fuzzy automaton is strongly connected.*

Proof. Consider a singly generated group type fuzzy automaton $A = A(q)$. Let $q_1, q_2 \in Q$. Then there exist $x, y \in \Sigma^*$ such that $\mu(q, x, q_1) \wedge \mu(q, y, q_2) > 0$. Since $[\Sigma^*]$ is a group, we have $[x^{-1}] \in [\Sigma^*]$ such that $[x][x^{-1}] = [\epsilon]$. Therefore, $\mu(q_1, x^{-1}, q) > 0$. Hence, $\mu(q_1, x^{-1}y, q_2) > 0$. □

If we designate a singly generated group-type fuzzy automaton as *quasi-perfect fuzzy automaton*, then due to Lemma 4.10, we have

Theorem 4.11. *Every quasi-perfect fuzzy automaton is strongly connected.*

Let $A = (Q, \Sigma, \mu)$ be a state independent fuzzy automaton. Define a binary operation $*$ on $S(q)$ as $p*r = s$, where $\mu(q, x, p) > 0, \mu(q, y, r) > 0$ and $\mu(q, xy, s) > 0$, for some $x, y \in \Sigma^*$. Then $(S(q), *)$ is a semigroup.

Theorem 4.12. *Let $A = (Q, \Sigma, \mu)$ be a state independent fuzzy automaton. Then, $[\Sigma^*]$ is isomorphic to $(S(q), *)$, for each $q \in Q$.*

Proof. Define $h : S(q) \to [\Sigma^*]$ by $h(p) = [x]$, where $\mu(q, x, p) > 0$. Since A is state independent, h is well defined. Now, let $p, r, s \in Q$ with $\mu(q, x, p) > 0, \mu(q, y, r) > 0$ and $\mu(q, xy, s) > 0$. Then $h(p * r) = h(s) = [xy] = [x][y] = h(p)h(r)$. i.e. h is a homomorphism. Let $h(p) = h(r)$. Then $[x] = [y]$. In particular $[x]_q = [y]_q$. This shows that h is 1-1. Now, for $[x] \in [\Sigma^*]$ we choose $p \in Q$ such that $\mu(q, x, p) > 0$. Then $p \in S(q)$ and $h(p) = [x]$. Hence h is an isomorphism. □

Corollary 4.13. *If $A = (Q, \Sigma, \mu)$ is a quasi-perfect fuzzy automaton, then $(Q, *)$ is a group and $[\Sigma^*]$ is isomorphic to $(Q, *)$.*

Theorem 4.14. *Every right simple type fuzzy automaton is quasi-strongly connected.*

Proof. Let $q \in Q$ and $q_1, q_2 \in S(q)$. Then there exists $x_1, x_2 \in \Sigma^*$ such that $\mu(q, x_1, q_1) > 0$ and $\mu(q, x_2, q_2) > 0$. Since A is a right simple type fuzzy automaton, we have $[z] \in [\Sigma^*]$ such that $[x_1][z] = [x_2]$. Therefore $\mu(q, x_2, q_2) = \mu(q, x_1 z, q_2) = \mu(q, x_1, q_1) \wedge \mu(q_1, z, q_2) > 0$. This gives us $\mu(q_1, z, q_2) > 0$. Therefore, $A(q)$ is strongly connected for any $q \in Q$. Hence, A is quasi-strongly connected. □

The above theorem is the fuzzy counterpart of the theorem for classical deterministic automata theory given in [20].

Corollary 4.15. *Every connected right simple type fuzzy automaton is strongly connected.*

Theorem 4.16. *Let $A = (Q, \Sigma, \mu)$ be a strongly connected fuzzy automaton. Then A is a quasi-perfect fuzzy automaton if and only if A is state independent.*

Proof. Clearly, quasi-perfect fuzzy automaton is a state independent fuzzy automaton. Conversely, we have strongly connectedness of A implies A is singly generated. Also $[\Sigma^*]$ is always monoid with $[\epsilon]$ as the identity. Let $[x] \in [\Sigma^*]$. Since A is strongly connected for $p, q \in Q$, there exists $y \in \Sigma^*$ such that $\mu(q, x, p) > 0$ and $\mu(p, y, q) > 0$. Now, $\mu(q, xy, q) = \mu(q, x, p) \wedge \mu(p, y, q) > 0$ implies that $[xy] = [x][y] = [\epsilon]$, as A is state independent. Similarly, we have $[y][x] = [\epsilon]$. Therefore, $[y] = [x]^{-1}$ and $[\Sigma^*]$ is a group. Hence, A is a quasi-perfect fuzzy automaton. □

Theorem 4.17. *Let $A = (Q, \Sigma, \mu)$ be a quasi-perfect fuzzy automaton and $x_0 \in \Sigma^*$ be fixed element. Let the map $h : Q \to Q$ defined by $h(q) = p$, where $\mu(q, x_0, p) > 0$. Then $h \in WG^F(A)$ if and only if $x_0 \in Z(A)$.*

Proof. Suppose $x_0 \in Z(A)$. We first prove that h is one one. Let $h(q_1) = h(q_2)$, for $q_1, q_2 \in Q$. Then $\mu(q_1, x_0, p) > 0$ and $\mu(q_2, x_0, p) > 0$, for some $p \in Q$. Since A is strongly connected, there exists $z \in \Sigma^*$ such that $\mu(q_1, z, q_2) > 0$. Then, $\mu(q_1, zx_0, p) = \mu(q_1, z, q_2) \wedge \mu(q_2, x_0, p) > 0$. Thus, $[zx_0] = [z][x_0] = [x_0] = [x_0][z]$, since $\mu(q_1, x_0, p) > 0$ and $x_0 \in Z(A)$. Then, $[z]$ is an identity of $[\Sigma^*]$. Thus, $q_1 = q_2$.

Hence h is 1-1. Since Q is finite we have h is onto. To prove that $h \in WG^F(A)$, let $q, r \in Q$, $z \in \Sigma^*$ and $\mu(q, z, r) > 0$. Let $[y] \in [\Sigma^*]$ be such that $[z] = [y][x_0]$. Then, $\mu(q, yx_0, r) = \mu(q, x_0y, r) = \mu(q, x_0, p) \wedge \mu(p, y, r) > 0$. Thus, $\mu(h(q), z, h(r)) = \mu(p, yx_0, h(r)) = \mu(p, y, r) \wedge \mu(r, x_0, h(r))$. Therefore, $\mu(h(q), z, h(r)) > 0$. Hence $h \in WG^F(A)$. Conversely, suppose that $h \in WG^F(A)$. Let $y \in \Sigma^*$. For $q, r \in Q$, let $\mu(q, x_0y, r) > 0$. Then $\mu(q, x_0y, r) = \mu(q, x_0, p) \wedge \mu(p, y, r) > 0$, for some $p \in Q$. Now, $\mu(q, yx_0, t) = \mu(q, y, s) \wedge \mu(s, x_0, t) > 0$, for some $s \in Q$. Then $h(s) = t$. Therefore, $\mu(h(q), y, h(s)) > 0$ i.e. $\mu(p, y, t) > 0$. This implies that $t = r$. Thus, $\mu(q, x_0y, r) > 0 \Rightarrow \mu(q, yx_0, r) > 0$. □

A permutation group G on a set Q is said to be transitive on Q, if given any ordered pair (q, q') of elements of Q, there exists $g \in G$ such that $g(q) = q'$

Theorem 4.18. *Let $A = (Q, \Sigma, \mu)$ be a fuzzy automaton. Then A is quasi-perfect if and only if A is strongly connected and $WG^F(A)$ is transitive on Q.*

Proof. Suppose that A is a quasi-perfect automaton. Let $q_1, q_2 \in Q$ and $x \in \Sigma^*$ such that $\mu(q_1, x, q_2) > 0$. Define a map h as (i) $h(q_1) = q_2$ and (ii) $h(p) = r$, whenever $\mu(q_1, z, p) \wedge \mu(q_2, z, r) > 0$, for some $z \in \Sigma^*$. Since A is state independent fuzzy automaton, h is well defined and 1-1. Also A is strongly connected implies that h is onto. Let $p_1, p_2 \in Q$ and $\mu(p_1, z, p_2) > 0$, for some $z \in \Sigma^*$. Suppose $\mu(q_1, y, p_1) > 0$, for some $y \in \Sigma^*$. Then $\mu(q_1, yz, p_2) > 0$ implies that $\mu(q_2, yz, h(p_2)) > 0$, by the definition of h. Then, $\mu(q_2, y, h(p_1)) \wedge \mu(h(p_1), z, h(p_2)) > 0$. Therefore, $\mu(h(p_1), z, h(p_2)) > 0$. Hence, $h \in WG^F(A)$ and $WG^F(A)$ is transitive on Q. Conversely suppose A is strongly connected and $WG^F(A)$ is transitive on Q. Let $p, q \in Q$ and $g, h \in WG^F(A)$ such that $g(p) = q$ and $h(q) = p$. Let $y \in [x]_p$, for some $x \in \Sigma^*$. Then $\mu(p, x, r) \wedge \mu(p, y, r) > 0$, for some $r \in Q$. Therefore, $\mu(q, x, g(r)) \wedge \mu(q, y, g(r)) > 0$. Thus, $y \in [x]_q$. Hence $[x]_p \subseteq [x]_q$. Similarly $[x]_q \subseteq [x]_p$. Thus, $[x]_p = [x]_q$, for all $p, q \in Q$. i.e. A is state independent. Theorem 4.16 proves that A is quasi-perfect fuzzy automaton. □

Lemma 4.19. *If $A = (Q, \Sigma, \mu)$ is a strongly connected fuzzy automaton and $h_1, h_2 \in WE^F(A)$ with $h_1(q_0) = h_2(q_0)$, for some $q_0 \in Q$, then $h_1 = h_2$.*

Theorem 4.20. *Let $A = (Q, \Sigma, \mu)$ be a quasi-perfect fuzzy automaton and $h : Q \to Q$. Then $h \in WG^F(A)$ if and only if for a fixed $q_0 \in Q$, $h(p) = r$, whenever $\mu(q_0, x, p) \wedge \mu(h(q_0), x, r) > 0$, $x \in \Sigma^*$.*

Proof. Let $h \in WG^F(A)$ and for fixed $q_0 \in Q$. Let $h(q_0) = q$ for some $q \in Q$. Since A is quasi-perfect, A is strongly connected. Therefore there exists $y \in \Sigma^*$ such that $\mu(q_0, y, q) > 0$. Define g by $g(p) = r$, if $\mu(p, y, r) > 0$. Then g is a weak fuzzy automaton homomorphism and $\mu(q_0, x, p) \wedge \mu(q, x, r) > 0$. Since $h(q_0) = g(q_0)$, by Lemma 4.19, $h = g$. Converse follows by the above theorem. □

Theorem 4.21. *Let $A = (Q, \Sigma, \mu)$ be a quasi-perfect fuzzy automaton. Then $[\Sigma^*]$ is isomorphic to $WG^F(A)$.*

Proof. Let $h \in WG^F(A)$. Then by above theorem, $h(p) = r$, when $\mu(q_0, x, p) \wedge \mu(h(q_0), x, r) > 0$, for some $x \in \Sigma^*$. Let $x_h \in \Sigma^*$ be such that $\mu(q_0, x_h, h(q_0)) > 0$. Define $f : WG^F(A) \to [\Sigma^*]$ by $f(h) = [x_h]$. If $h_1 = h_2$, then $h_1(q_0) = h_2(q_0)$.

Therefore, $\mu(q_0, x_{h_1}, h_1(q_0)) > 0$ and $\mu(q_0, x_{h_2}, h_2(q_0)) > 0$. i.e. $x_{h_1} \equiv^F_{q_0} x_{h_2}$. Thus, $[x_{h_1}]_{q_0} = [x_{h_2}]_{q_0}$. As A is state independent fuzzy automaton, $[x_{h_1}] = [x_{h_2}]$. This proves that f is well defined. If $f(h_1) = f(h_2)$, then $[x_{h_1}] = [x_{h_2}]$. Thus, $[x_{h_1}]_{q_0} = [x_{h_2}]_{q_0}$. Hence by Lemma 4.19, $h_1 = h_2$. Therefore f is 1-1. Let $[x] \in [\Sigma^*]$ and $p \in Q$ such that $\mu(q_0, x, p) > 0$. Define $h(q_0) = p$. Then, by above theorem, $h \in WG^F(A)$. Clearly, $f(h) = [x]$ proves f is onto. $\square$

Due to Corollary 4.13, we have

Corollary 4.22. *If $A = (Q, \Sigma, \mu)$ is a quasi-perfect fuzzy automaton, then $WG^F(A)$ is isomorphic to $(Q, *)$. In particular $|WG^F(A)| = |Q|$.*

Theorem 4.23. *Let $A = (Q, \Sigma, \mu)$ be a strongly connected fuzzy automaton. If $|WG^F(A)| = |Q|$, then A is a quasi-perfect fuzzy automaton.*

Proof. Since A is strongly connected, in light of Theorem 4.18, it is sufficient to prove that $WG^F(A)$ is transitive on Q. Let $Q = \{q_1, q_2, \ldots, q_n\}$ and $WG^F(A) = \{h_1, h_2, \ldots, h_n\}$. If $WG^F(A)$ is not transitive on Q, then there exist $q_1, q_2 \in Q$ such that $q_2 \notin \{h_1(q_1), h_2(q_1), \ldots, h_n(q_1)\}$. This gives $h_i(q_1) = h_j(q_1)$, for some $i \neq j$. By Lemma 4.19, $h_i = h_j$. This contradicts to the fact that $|WG^F(A)| = |Q|$. $\square$

Theorems 4.12, 4.17, 4.18, 4.21, 4.23 reduces to their classical deterministic case when we restrict range of μ to $\{0, 1\}$, these can be found in [19].

5. Properties of state independent fuzzy automaton

The purpose of this section is to count the number of elements in the characteristic semigroup of a state independent fuzzy automaton with the help of concepts like index and period of the state independent fuzzy automaton.

Definition 5.1. A maximal connected fuzzy subautomaton $A' = (Q', \Sigma', \mu')$ of a fuzzy automaton $A = (Q, \Sigma, \mu)$ is called a *block* of A. We shall denote Q' as a *component* of Q (or A).

Definition 5.2. Let $A = (Q, \{x\}, \mu)$ be a fuzzy automaton. Let C be a component of A. For each $q \in C$, there exist two non-negative integers m, n such that $\mu(q, x^m, p) > 0$ and $\mu(q, x^{m+n}, p) > 0$,where x^0 is the identity element. The smallest non-negative integers m and n with this property are respectively called the *index* and the *period* of q with respect to x. (or the x - index and x - period of q.) We shall symbolically denote them respectively as I^x_q and P^x_q.

The largest x - index of all the elements of C is the *index of the component* C with respect to x (or the x - index of C), in symbol I^x_c. The x - period of any element of C is the *period of the component* C with respect to x (or the x - period of C), in symbol P^x_c.

Definition 5.3. Let $A = (Q, \Sigma, \mu)$ be any fuzzy automaton, $q \in Q$, $x \in \Sigma^*$. The *x-path of* q is the subautomaton $O_x(q) = (S_x(q), \{x\}, \mu')$, where μ' is the restriction of μ to $S_x(q) \times \{x\} \times S_x(q)$.

Definition 5.4. The *x circle of q* is the subautomaton $C_x(q) = (S^c_x(q), \{x\}, \mu')$, where $S^c_x(q) = \{t \in S_x(q) : \mu(q, x^k, t) > 0$ and $\mu(q, x^m, t) > 0$, for some integer $m > k\}$ and μ' is the restriction of μ to $S^c_x(q) \times \{x\} \times S^c_x(q)$.

The x - path of q, $O_x(q)$ is said to be *circular*, if $O_x(q) = C_x(q)$.

Lemma 5.5. *Let $A = (Q, \Sigma, \mu)$ be a fuzzy automaton and $x \in \Sigma^*$. If C is a component of $(Q, \{x\}, \mu')$, then for any $q \in C$, $P_q^x = |C_x(q)|$.*

Lemma 5.6. *Let $A = (Q, \Sigma, \mu)$ be a fuzzy automaton and $x \in \Sigma^*$. If C is a component of $(Q, \{x\}, \mu')$, then for any $q \in C$, $I_q^x = |O_x(q)| - |C_x(q)|$.*

Lemma 5.7. *Let $A = (Q, \Sigma, \mu)$ be a state independent fuzzy automaton and $x \in \Sigma^*$. If C is a component of $(Q, \{x\}, \mu')$, then for any $q \in C$, we have $I_q^x \leq 1$. Further, $I_C^x \leq 1$.*

Proof. Let $q_0 \in C$ be such that $I_{q_0}^x = m$. Then, $\mu(q_0, x^m, q_m) > 0$ and $\mu(q_0, x^{m+n}, q_m) > 0$. Let $\mu(q_0, x^j, q_j) > 0$ for $j = 0, 1, 2, \ldots, m$. Then

$$\mu(q_{m-1}, x, q_m) > 0 \text{ and } \mu(q_{m-1}, x^{1+n}, q_m) > 0.$$

Thus, $[x] = [x^{1+n}]$. This gives us $\mu(q_0, x, q_m) > 0$ and $\mu(q_0, x^{1+n}, q_m) > 0$. Hence, $m \leq 1$. □

Theorem 5.8. *Let $A = (Q, \Sigma, \mu)$ be a state independent fuzzy automaton and $x \in \Sigma^*$. Then any two components of $(Q, \{x\}, \mu')$ have same x - period.*

Proof. Let C_1 and C_2 be two components of $(Q, \{x\}, \mu')$. Let the elements of C_1 are $q_0, q_1, \ldots, q_{m_1+n_1}$ and of C_2 are $q'_0, q'_1, \ldots, q'_{m_2+n_2}$. Then $\mu(q_0, x^{m_1}, q_{m_1}) > 0$ and $\mu(q_0, x^{m_1+n_1}, q_{m_1}) > 0$. Let $\mu(q_0, x^j, q_j) > 0$, for $j = 0, 1, 2, \ldots, m_1$. Then $\mu(q_{m_1}, x^0, q_{m_1}) > 0$ and $\mu(q_{m_1}, x^{0+n_1}, q_{m_1}) > 0$. Thus, $[x^0] = [x^{n_1}]$. This gives us $\mu(q'_{m_2}, x^0, q'_{m_2}) > 0$ and $\mu(q'_{m_2}, x^{0+n_1}, q'_{m_2}) > 0$. Therefore, $n_2 \leq n_1$. Similarly, $n_1 \leq n_2$. □

Theorem 5.9. *Let $A = (Q, \Sigma, \mu)$ be a state independent fuzzy automaton and C be a component of A. Let $q \in C$ and $x \in \Sigma^*$. If $p \in Q$ is such that $\mu(q, x, p) > 0$, then p is a state of strongly connected fuzzy subautomaton of A.*

Proof. Let q'' be a state in a strongly connected subautomaton A'' of A. Let $s, r \in Q''$ and $y \in \Sigma^*$ be such that $\mu(q'', x, s) > 0$ and $\mu(s, y, r) > 0$. Since A'' is strongly connected, there exists $z \in \Sigma^*$ such that $\mu(r, z, s) > 0$. Then $\mu(q'', xyz, s) > 0$. This implies that $[x] = [xyz]$. Therefore, $\mu(t, z, p) > 0$, where $\mu(p, y, t) > 0$. Therefore, p can be reached from any of its successor. Hence, $A(p)$ is strongly connected. □

Lemma 5.10. *A component of a fuzzy automaton has distinct strongly connected subautomata if and only if it has a singly generated subautomaton with distinct strongly connected subautomata.*

Proof. Let C be a component of a fuzzy automaton A. Let B_1, B_2 be distinct strongly connected subautomata of C. For any $q_1 \in B_1$ and $q_2 \in B_2$, $\mu(q_1, x, q_2) = 0$ for all $x \in \Sigma^*$. Since C is a component, there exists $q \in C - (B_1 \cup B_2)$ such that $\mu(q, x, q_1) > 0$ and $\mu(q, y, q_2) > 0$, for some $x, y \in \Sigma^*$, $q_1 \in B_1$ and $q_2 \in B_2$. Then $A(q)$ is a singly generated subautomaton of C and B_1, B_2 are distinct strongly connected subautomata of $A(q)$. Converse is immediate. □

Theorem 5.11. *Let $A = (Q, \Sigma, \mu)$ be a state independent fuzzy automaton and C be a component of A. Then C contains exactly one strongly connected fuzzy subautomaton of A.*

Proof. Consider a singly generated fuzzy subautomaton $A(q)$, $q \in C$, of a state independent fuzzy automaton A. If $A(q)$ has two strongly connected fuzzy subautomata $A_1 = (Q_1, \Sigma, \mu_1)$ and $A_2 = (Q_2, \Sigma, \mu_2)$, then for some $x, y \in \Sigma^*$ and $q_1 \in Q_1, q_2 \in Q_2$, we have $\mu(q, x, q_1) > 0$ and $\mu(q, y, q_2) > 0$. But there is $z \in \Sigma^*$ such that $\mu(q_1, yz, t) > 0$ and $\mu(q_1, x, t) > 0$. Therefore, $[yz] = [x]$. Since A is state independent fuzzy automaton,we have$[yz]_q = [x]_q$. Then $q_1 \in A_2$. Hence $A_1 = A_2$. □

Theorem 5.12. *Let $A' = (Q', \Sigma', \mu')$ be a connected state independent fuzzy automaton and $q_1, q_2 \in Q'$. Let $x \in \Sigma'^*$ be such that $\mu'(q_1, x, p) \wedge \mu'(q_2, x, p) > 0$, for some $p \in Q'$. Then for all $y \in \Sigma'^*$, $\mu'(q_1, y, r) \wedge \mu'(q_2, y, r) > 0$, for some $r \in Q'$.*

Proof. Let $q_1, q_2 \in Q'$ and $x \in \Sigma'^*$ be such that $\mu'(q_1, x, p) \wedge \mu'(q_2, x, p) > 0$, for some $p \in Q'$. Let $y \in \Sigma'^*$ and $\mu'(q_1, y, r) > 0$. By Theorem 5.9, p and r are states of strongly connected fuzzy subautomaton of A' (by Theorem 5.11, A' has a unique strongly connected fuzzy subautomaton). Thus, there exists $z \in \Sigma'^*$ such that $\mu'(p, z, r) > 0$. Then, $\mu'(q_1, xz, r) > 0$ and $\mu'(q_1, y, r) > 0$. Therefore, $[xz] = [y]$. Thus, $\mu'(q_2, xz, s) > 0$ and $\mu'(q_2, y, s) > 0$. But $r = s$, implies that $\mu'(q_1, y, r) \wedge \mu'(q_2, y, r) > 0$. □

Lemma 5.13. *Let A_1 and A_2 be strongly connected subautomata of fuzzy automaton A. Then $WI^F(A_1 \to A_2) \neq \phi$ if and only if there are $q_1 \in Q_1$ and $q_2 \in Q_2$ such that for all $x, y \in \Sigma^*$, we have $x \equiv^F_{q_1} y \Leftrightarrow x \equiv^F_{q_2} y$.*

Proof. Let $f \in WI^F(A_1 \to A_2)$ and for $x, y \in \Sigma^*, x \equiv^F_{q_1} y$. Then, $\mu_1(q_1, x, p) \wedge \mu_1(q_1, y, p) > 0$, for some $p \in Q_1$. Thus, $\mu_2(f(q_1), x, f(p)) \wedge \mu_2(f(q_1), y, f(p)) > 0$. i.e. $\mu_2(q_2, x, f(p)) \wedge \mu_2(q_2, y, f(p)) > 0$, where $f(q_1) = q_2$. Therefore, $x \equiv^F_{q_2} y$. Similarly we have $x \equiv^F_{q_2} y \Rightarrow x \equiv^F_{q_1} y$. Conversely, let $q_1 \in Q_1$ and $q_2 \in Q_2$ such that for all $x, y \in \Sigma^*$,$x \equiv^F_{q_1} y \Leftrightarrow x \equiv^F_{q_2} y$. Define $h : Q_1 \to Q_2$ by $h(q_1) = q_2$ and $h(p) = r$, whenever $\mu_1(q_1, z, p) > 0$ and $\mu_2(q_2, z, r) > 0$. Suppose $p_1 = p_2$. Then there exists $x \in \Sigma^*$ such that $\mu(q_1, x, p_1) > 0$. Let $h(p_1) = r_1$ and $h(p_2) = r_2$. Then $\mu_1(q_1, x, p_1) > 0$, $\mu_2(q_2, x, r_1) > 0$ and $\mu_1(q_1, y, p_2) > 0$, $\mu_2(q_2, y, r_2) > 0$. Since $p_1 = p_2$, $x \equiv^F_{q_1} y$, we have $x \equiv^F_{q_2} y$. Therefore, $r_1 = r_2$. This proves that h is well defined. Similarly, one can prove that h is 1-1. Now by hypothesis for any $q_2 \in Q_2$, there exists $q_1 \in Q_1$ such that $x, y \in \Sigma^*$,$x \equiv^F_{q_1} y \Leftrightarrow x \equiv^F_{q_2} y$. Thus, $h(q_1) = q_2$. Hence, h is onto. □

Theorem 5.14. *Let $A = (Q, \Sigma, \mu)$ be a state independent fuzzy automaton. If A_1 and A_2 are strongly connected fuzzy subautomata of A, then $WI^F(A_1 \to A_2) \neq \phi$.*

Proof. Let $q_1 \in Q_1$, $q_2 \in Q_2$ and $x, y \in \Sigma^*$. For some $p \in Q_1$, if $\mu(q_1, x, p) > 0$ and $\mu(q_1, y, p) > 0$, then $[x] = [y]$ (as A is state independent fuzzy automaton). Therefore $\mu(q_2, x, r) > 0$ and $\mu(q_2, y, r) > 0$, for some $r \in Q_2$. Then, by using Lemma 5.13, one can complete the proof. □

We are now in position to talk about the number of elements of the characteristic semigroup of the state independent fuzzy automaton.

Lemma 5.15. *Let $A' = (Q', \Sigma', \mu')$ be a connected fuzzy automaton and $A'' = (Q'', \Sigma'', \mu'')$ be a strongly connected state independent fuzzy subautomaton of A'.Let $q' \in Q' - Q''$ be such that A'' be a subautomaton of $A(q')$. Then $A(q')$ is state independent if and only if there exists $q'' \in Q''$ such that for all $x \in \Sigma^*, \mu'(q', x, p) > 0$ and $\mu''(q'', x, p) > 0$, for some $p \in Q''$.*

Proof. Let $A(q')$ be a state independent fuzzy automaton. Suppose that no q'' exists satisfying required condition. Let $x \in \Sigma'^*$ be such that $\mu(q', x, p) > 0$, for some $p \in Q''$ and $\mu(s, x, p) \not> 0$, for any $s \in Q''$. This implies that the index of q' is greater than 1. This contradicts to the lemma 5.7. Then by Theorem 5.12, there exists $q'' \in Q''$ such that , $\forall x \in \Sigma^*, \mu'(q', x, p) > 0$ and $\mu''(q'', x, p) > 0$ for some $p \in Q''$. The converse is obvious. □

Theorem 5.16. *Let $A = (Q, \Sigma, \mu)$ be a state independent fuzzy automaton and $A'' = (Q'', \Sigma'', \mu'')$ be a strongly connected fuzzy subautomaton of A. Then $|[\Sigma^*]| = |[\Sigma'']^*| = |Q''|$.*

Proof. One can prove this using Corollary 4.13, Theorems 4.16 and 5.14. □

Restricting range of μ to $\{0, 1\}$, theorems 5.8, 5.9, 5.11, 5.14, 5.16 reduces to their classical deterministic case [9].

6. Conclusion

Fuzzy automaton and semigroup associated with it influence the study of each other. In this paper we have introduced the concept of input independent fuzzy automaton and introduced characteristic semigroup of an input independent as well as state independent fuzzy automaton. We have precisely established the following properties of these input and state independent fuzzy automaton in this paper.

(1) Input independent fuzzy automaton is connected. Further, if it is singly generated, then it is strongly connected.
(2) Characteristic semigroup of an input independent fuzzy automaton is right zero semigroup.
(3) Characteristic semigroup of a state independent fuzzy automaton is the semigroup of successors of any state of that fuzzy automaton with suitable binary operation.
(4) Characteristic semigroup of a quasi-perfect fuzzy automaton is the semigroup of its state set under suitable binary operation, as well as it is the group of its weak fuzzy automaton isomorphisms.
(5) Any state independent fuzzy automaton contains a unique strongly connected subautomaton up to weak fuzzy automaton isomorphism. Further, the cardinality of the characteristic semigroup of a state independent fuzzy automaton is equal to the number of states of this strongly connected subautomaton.

Acknowledgements. We are thankful to anonymous referees for their valuable suggestions. The first author thanks to the University Grant Commission, New Delhi, India for granting him teacher fellowship under the FIP scheme.

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