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(α, β)-vague subgroups

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Abstract. In this paper, by using the concept of belonging to and quasi-coincident, the notions of $(\alpha, \beta)-$vague subgroups is introduced. We show that A is a $(\alpha, \beta)-$vague subgroup of G if and only if for any $a \in (0, 1]$, the cut set A_a of A is a 3-valued vague subgroup of G, and A is an $(\in, \in \vee q)-$vague subgroup of G if and only if for any $a \in (0, 0.5]$ the cut set of A is a 3-valued vague subgroup of G.

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1. Introduction

In most cases of judgements, evaluation is done by human beings (or by an intelligent agent) where there certainly is a limitation of knowledge or intellectual functionaries. Naturally, every decision-maker hesitates more or less, on every evaluation activity. To judge whether a patient has cancer or not, a doctor (the decision-maker) will hesitate because of the fact that a fraction of evaluation he thinks in favor of truthness, another fraction in favor of falseness and rest part remains undecided to him. This is the breaking philosophy in the notion of vague set theory introduced by Gau and Buehrer.

Since the concept of fuzzy group was introduced by Rosenfeld in 1971, the theories and approaches on different fuzzy algebraic structures developed rapidly. It is worth pointing out that Bhakat and Das [1] gave the concepts of $(\alpha, \beta)-$ fuzzy subgroups by using the belong to relation $(\in)$ and quasi-coincident with relation (q) between a fuzzy point x_a and a fuzzy set A, and introduced the concept of $(\in, \in \vee q)-$ vague. Yuan et al. Clearly, in order to answer this question, the neighborhood relations between a fuzzy point x_a and an intuitionistic fuzzy set A should be built.

In this paper, we deal with the various equivalent conditions for the vague groups. This paper will be organized as follows: in Section 2, definitions and notations are given. In Section 3, based on the concept of cut sets on intuitionistic fuzzy sets presented in [5], we establish the relationship between a fuzzy point and a vague set. In section 3, we provide the summary of $(\alpha, \beta)-$vague groups and the corresponding theorems are shown. In section 4, $(\alpha, \beta)-$vague normal subgroups and the corresponding theorems are shown.

2. Preliminaries

In this section, we present now some preliminaries of vague sets.

Let $G = \{x_1, \cdots, x_n\}$ be the universe of discourse. The membership function for fuzzy sets can take any value from the closed interval $[0, 1]$. Fuzzy set A is defined as the set of ordered pairs $A = \{(x; \mu_A(x))|x \in G\}$, where $\mu_A(x)$, is the grade of membership of element x in set A. The greater $\mu_A(x)$, the greater is the truth of the statement that the element x belongs to the setA'. But Gau and Buehrer [4] pointed out that this single value combines the evidence for x' and the evidence against x'.

It does not indicate the evidence for x' and the evidence against x', and it does not also indicate how much there is of each. Consequently, there is a genuine necessity of a different kind of fuzzy sets which could be treated as a generalization of Zadeh's fuzzy sets [7].

Definition 2.1 ([3])**.** *A vague set A in the universe of discourse G is characterized by two membership functions given by:*

(1) a truth membership function

$$t_A : G \to [0, 1]$$

and

(2) a false membership function

$$f_A : G \to [0, 1],$$

Where $t_A(x)$ is a lower bound of the grade of membership of x derived from the evidence for x, and $f_A(x)$ is a lower bound of the negation of x derived from the evidence against x and $t_A(x) + f_A(x) \leq 1$

Thus the grade of membership of x in the vague set A is bounded by a sub interval $[t_A(x), 1 - f_A(x)]$ of $[0, 1]$. This indicates that if the actual grade of membership is $\mu(x)$, then

$$t_A(x) \leq \mu(x) \leq 1 - f_A(x)$$

The vague set A is written as

$$A = \{(x, [t_A(x), 1 - f_A(x)])|x \in U\}.$$

where the interval $[t_A(x), 1 - f_A(x)]$ is called the vague value of x in A and is denoted by $V_A(x)$.

Example 2.2. Let A be a subset of R that defined by :

$$t_A(x) = \begin{cases} \frac{x-4}{4} & 4 < x < 6 \\ \frac{10-x}{8} & 6 < x < 10 \end{cases}$$

$$f_A(x) = \begin{cases} \frac{6-x}{2} & 4 < x < 6 \\ \frac{x-6}{4} & 6 < x < 10 \end{cases}$$

then A is a vague set.

Definition 2.3 ([2]). Let $(G, *)$ be a group. A vague set A of G is called vague group (VG) of G if and only if the following conditions are true:

$f_A(xy) \geq min\{f_A(x), f_A(y)\}$ and $f_A(x^{-1}) \geq f_A(x)$, for all $x, y \in G$, i.e.,
1) $t_A(xy) \geq min\{t_A(x), t_A(y)\}$ and $f_A(xy) \leq max\{f_A(x)), f_A(y)\}$ and
2) $t_A(x^{-1}) \geq t_A(x), f_A(x^{-1}) \leq f_A(x)$.

Let G be a universe discourse. A fuzzy point, denoted by x_a is a fuzzy set that satisfying

$$A(x) := \begin{cases} a & y = x \\ 0 & y \neq x \end{cases}$$

for any $x, y \in G, a \in [0, 1]$.

Definition 2.4 ([6]). Let A be a subset over G and x_a be a fuzzy point.
1) If $A(x) \geq a$, then we say x_a belongs to A, and denote $x_a \in A$;
2) If $a + A(x) > 1$, then we say x_a is quasi-coincident with A, and denote $x_a qA$;
3) $x_a \in \wedge A \Longleftrightarrow x_a \in A$ and $x_a qA$;
4) $x_a \in \vee A \Longleftrightarrow x_a \in A$ or $x_a qA$.

Definition 2.5 ([6]). Let $A = \{(x, [t_A(x), 1 - f_A(x)])|x \in G\}$ be a vague subset over G and $a \in [0, 1]$,
1) We call

$$A_a(x) = \begin{cases} 1 & t_A(x) \geq a \\ \frac{1}{2} & t_A(x) < a \leq 1 - f_A(x) \\ 0 & a > 1 - f_A(x). \end{cases}, A_{\underline{a}}(x) = \begin{cases} 1 & t_A(x) > a \\ \frac{1}{2} & t_A(x) \leq a < 1 - f_A(x) \\ 0 & a \geq 1 - f_A(x). \end{cases}$$

2) We call

$$A_{[a]}(x) = \begin{cases} 1 & a + t_A(x) \geq 1 \\ \frac{1}{2} & f_A(x) \leq a \leq 1 - t_A(x) \\ 0 & f_A(x) \geq a. \end{cases}, A_{\underline{[a]}}(x) = \begin{cases} 1 & a + t_A(x) > 1 \\ \frac{1}{2} & f_A(x) < a \leq 1 - t_A(x) \\ 0 & f_A(x) \geq a. \end{cases}$$

the a-the upper Q-cut set and a-strong upper Q-cut set of fuzzy set A, respectively.

Remark 2.6. (1) It is obvious that $A_{\underline{[a]}}(x) = A_{\underline{1-a}}(x), A_{\underline{a}} \subseteq A_a$. If $a < b$, then $A_a \supseteq A_b$, $A_{\underline{a}} \supseteq A_{\underline{b}}, and A_{\underline{a}} \supseteq A_b$.

(2) Let X be a set. We call the map $A : G \to \{0, \frac{1}{2}, 1\}$ a $3-$ valued fuzzy set.

Definition 2.7 ([6]). (1) Let $[x_a \in A]$ and $[x_a qA]$ represent the grades of membership of $x_a \in A$ and $x_a qA$, respectively, then

$$[x_a \in A] := A_a(x);$$

$$[x_a qA] := A_{\underline{[a]}}(x).$$

2) Let $[x_a \in \wedge qA]$ and $[x_a \in \vee qA]$ represent the grade of membership of $x_a \in A$ and $x_a qA$, $x_a \in A$ or $x_a qA$, then

$$[x_a \in \wedge qA] := [x_a \in A] \wedge [x_a qA] = A_a(x) \wedge A_{[a]}(x)$$

$$[x_a \in \vee qA] := [x_a \in A] \vee [x_a qA] = A_a(x) \vee A_{\underline{[a]}}(x)$$

Definition 2.8 ([6]). Let x_a be a fuzzy point, $s \in [0,1]$ and $A = \{(x, [t_A(x), 1 - f_A(x)])|x \in G\}$ be vague set of U. The neighborhood relations between a fuzzy point x_a and vague set A are related as follows:

(1) $[x_a q_s A] = \begin{cases} 1 & a + t_A(x) > 2s \\ \frac{1}{2} & t_A(x) \leq 2s - a < 1 - f_A(x) \\ 0 & f_A(x) + 2s \geq a + 1. \end{cases}$

(2) $[x_a \in \wedge q_s A] \triangleq [x_a \in A] \wedge [x_a q_s A]$;

(3) $[x_a \in \vee q_s A] \triangleq [x_a \in A] \vee [x_a q_s A]$.

When $s = 0.5$, $[x_a q_s A] = [x_a qA]$.

By Definition 2.8 we know that $[x_a \alpha A] \in \{0, \frac{1}{2}, 1\}$.

3. (α, β)- vague subgroups

Definition 3.1. Let G be a group, $A = \{(x, [t_A(x), 1 - f_A(x)])|x \in G\}$ be a vague subset of G and $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$, $\alpha \neq \in \wedge q$. A is called a $(\alpha, \beta)-$vague subgroup of G if and only if for any $x, y \in G$ and $s, t \in (0,1]$:

(1) $[x_s y_t \beta A] \geq [x_s \alpha A] \wedge [y_t \alpha A]$.

(2) $[x_s^{-1} \beta A] \geq [x_s \alpha A]$.

Example 3.2. Consider the group $G = \{1, \omega, \omega^2\}$ with respect to the binary operation 'multiplication complex number ' where ω is the an imaginary cube root of unity and let, $A = \{\langle 1, [0.9, 0.1]\rangle, \langle \omega, [0.6, 0.2]\rangle, \langle \omega^2, [0.6, 0.2]\rangle\}$. Then $A = \{(x, [t_A(x), 1 - f_A(x)])|x \in G\}$ is a $(\in, \in)-$vague subgroup of the group G.

Example 3.3. Let $G = \{a, b, c, d\}$ be a group with operation $+$ as follows:

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

And let, $A = \{\langle a, [0.7, 0.2]\rangle, \langle b, [0.5, 0.3]\rangle, \langle c, [0.3, 0.4]\rangle, \langle d, [0.3, 0.4]\rangle\}$. Then

$$A = \{(x, [t_A(x), 1 - f_A(x)])|x \in U\}$$

is a $(\in, \in \vee q)-$vague subgroup of G.

In Definition 3.1, α can be chosen one of four kinds of relations, and β also can be chosen one from four kinds of relations. Next, we will discuss the properties of these kinds of $(\alpha, \beta)-$vague subgroups.

Theorem 3.4. *Let A be an (α, β)- vague subgroups of G. If $\alpha \neq \in \wedge q$, then $A_{\underline{0}}$ is a 3- value fuzzy subgroups of G, i.e., $\forall x, y \in G, A_{\underline{0}}(xy) \geq A_{\underline{0}}(x) \wedge A_{\underline{0}}(y), A_{\underline{0}}(x^{-1}) \geq A_{\underline{0}}(x)$.*

Proof. (1) We prove that $A_0(x) \wedge A_0(y) = 1$ implies $A_0(xy) = 1$. Let $A_{\underline{0}}(x) \wedge A_{\underline{0}}(y) = 1$. Denote $t = t_A(x) \wedge t_A(y)$, then there exists $s \in (0,1)$ such that $0 < 1 - s < t = t_A(x) \wedge t_A(y)$. Thus, $[x_t \in A] = A_t(x) = 1$, $[y_t \in A] = A_t(y) = 1$, $[x_s qA] = A_{\underline{[s]}}(x) = 1$ and $[y_s qA] = A_{\underline{[s]}}(y) = 1$.

(i) If $\alpha = \in$ or $\alpha = \in \vee q$, then $[x_t \alpha A] = [y_t \alpha A] = 1$. Thus, for $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, we have $1 \geq [xy\beta A] \geq [x_t \alpha A] \wedge [y_t \alpha A] = 1$, i.e., $((xy)_t \beta A) = 1$. Hence, $A_t((1-\lambda)x + \lambda y) = 1$ or $A_{\underline{t}}(xy) = 1$, which implies that $t_A(xy) \geq t$, or $t_A(xy) > 1 - t \geq 0$. Therefore, $A_{\underline{0}}(xy) = 1$.

(ii) If $\alpha = q$, so $[x_s \alpha A] = [y_s \alpha A] = 1$. Thus, for $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, we have $[(xy)_s \beta A] = 1$. Hence, $A_s(xy) = 1$ or $A_{\underline{s}}(xy) = 1$, which implies that $t_A(xy) \geq s > 0$, or $t_A(xy) > 1 - s \geq 0$. Therefore, $A_{\underline{0}}(xy) = 1$.

Second, we show that $A_{\underline{0}}(x) \wedge A_{\underline{0}}(y) = \frac{1}{2} \Longrightarrow A_{\underline{0}}(xy) \geq \frac{1}{2}$.

Let $A_{\underline{0}}(x) \wedge A_{\underline{0}}(y) = \frac{1}{2}$. So $f_A(x) < 1$ and $f_A(y) < 1$. Let $s, t \in (0,1)$ such that $f_A(x) \vee f_A(y) < 1 - t < s < 1$. Then $[x_t \in A] = A_t(x) \geq \frac{1}{2}$, $[y_t \in A] = A_t(y) \geq \frac{1}{2}$, $[x_s qA] = A_{\underline{[s]}}(x) \geq \frac{1}{2}$, $[y_s qA] = A_{\underline{[s]}}(y) \geq \frac{1}{2}$.

(i) If $\alpha = \in$ or $\alpha = \in \vee q$, hence $[x_t \alpha A] \wedge [y_t \alpha A] \geq \frac{1}{2}$. Thus, for $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, we have $[(xy)_t \beta A] \geq [x_t \alpha A] \wedge [y_t \alpha A] \geq \frac{1}{2}$. Hence, $A_t(xy) \geq \frac{1}{2}$ or $A_{\underline{t}}(xy) \geq \frac{1}{2}$, which implies that $f_A(xy) \leq 1 - t < 1$ or $f_A(xy) \leq t < 1$. Therefore, $A_{\underline{0}}(xy) \geq \frac{1}{2}$.

(ii) If $\alpha = q$, therefore $[x_t \alpha A] \wedge [y_t \alpha A] = \frac{1}{2}$. Thus for $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, we have $[x_s y_s \beta A] \geq \frac{1}{2}$. Hence, $A_s(xy) \geq \frac{1}{2}$ or $A_{\underline{[s]}}(xy) \geq \frac{1}{2}$, which implies that $f_A(xy) \leq 1 - s < 1$ or $f_A(xy) \leq s < 1$. Therefore, $A_{\underline{0}}(xy) \geq \frac{1}{2}$. By $A_{\underline{0}}(x), A_{\underline{0}}(y)$, $A_{\underline{0}}(xy) \in \{0, \frac{1}{2}, 1\}$ and the proof of (i) and (ii), we know that $A_{\underline{0}}(x) \wedge A_{\underline{0}}(y) \leq A_{\underline{0}}(xy)$. By similar reasoning, we have $A_{\underline{0}}(x^{-1}) \geq A_{\underline{0}}(x)$. Therefore, $A_{\underline{0}}$ is a 3- vague subgroups of G. □

Lemma 3.5. *If A is an $(\alpha, \beta)-$vague subgroups of G and $\alpha \neq \in \wedge q$, then $A_{\underline{0}}$ is a $3-$vague subgroups of G, and $f_A(e) \geq f_A(x)$, for all $x \in G$.*

Proof. For all $x \in G$, we have $xx^{-1} = e$, hence $f_A(e) = f_{A_{\underline{0}}}(xx^{-1}) \geq f_{A_0}(x) \wedge f_{A_0}(x^{-1})$ by the Theorem 3.4, $f_{A_0}(x) \wedge f_{A_0}(x^{-1}) \geq f_{A_0}(x) \wedge f_{A_0}(x) = f_{A_0}(x)$ thus $f_{A_0}(e) \geq f_{A_0}(x)$. □

If $\{x_n\}$ is a sequence in G then $f_A(x_n)$ is an interval $[t_A(x_n), 1 - f_A(x_n)]$, for any positive integer n. We define $\lim_{n\to\infty} f_A(x_n) = [\lim_{n\to\infty} t_A(x_n), 1 - \lim_{n\to\infty} f_A(x_n)]$.

Theorem 3.6. *Let A_0 be a (α, β)- vague subgroups of G. If there exists a sequence $\{x_n\}$ in G, such that $\lim_{n\to\infty} f_{A_0}(x_n) = [1,1]$ then $f_{A_0}(e) = [1,1]$.*

Proof. we have $f_{A_0}(e) \geq f_{A_0}(x)$, for $x \in G$, thus $f_{A_0}(e) \geq f_{A_0}(x_n)$ For every positive integer n. Since $t_{A_0}(e) \leq 1$ and $1 - f_{A_0}(e) \leq 1$ then we have $f_{A_0}(e) = [t_{A_0}(e), 1 - f_{A_0}(e)] \leq [1,1]$. Consider $f_{A_0}(e) \geq \lim_{n\to\infty} f_{A_0}(x_n) = [1,1]$ hence $f_{A_0}(e) = [1,1]$. □

Theorem 3.7. *If A and B are two $(\alpha, \beta)-$vague subgroup of a group G, then $A \cap B$ is also an $(\alpha, \beta)-$vague subgroup of G.*

Proof.

$$\begin{aligned} t_{A\cap B}(xy^{-1}) &= \min\{t_A(xy^{-1}), t_B(xy^{-1})\} \\ &\geq \min\{\min\{t_A(x), t_A(y)\},\ min\{t_B(x), t_B(y)\} \\ &= \min\{t_{A\cap B}(x), t_{A\cap B}(y)\}. \end{aligned}$$

□

Theorem 3.8. *If A and B are two $(\alpha, \beta)-$vague subgroup of a group G and $A \subseteq B$, then $A \cup B$ is also a $(\alpha, \beta)-$vague subgroup of G.*

Definition 3.9. An I-vague set A of a non-empty set G is a pair (t_A, f_A) where $t_A : G \to I$ and $f_A : G \to I$ with $t_A(x) \leq 1 - f_A(x)$, for all $x \in G$.

Definition 3.10. Let G be a group. An $I-$ vague set A of a group G is called an $I - (\alpha, \beta)$ vague group of G if:
(i) $f_A(xy) \geq min\{f_A(x), f_A(y)\}$,
(ii) $f_A(x^{-1}) \geq f_A(x)$,
for all $x, y \in G$.

Example 3.11. Consider the group $(Z, +)$. Let I be the unit interval [0, 1] of real numbers. Let $a \oplus b = min\{1, a + b\}$.

Define the $I-$vague set A of Z as follows:

$$f_A(x) = \begin{cases} [a_1, b_1] & x \in 4z \\ [a_2, b_2] & x \in 2z - 4z \\ [a_3, b_3] & \text{otherwise.} \end{cases}$$

$$t_A(x) = \begin{cases} [c_1, d_1] & x \in 4z \\ [c_2, d_2] & x \in 2z - 4z \\ [c_3, d_3] & \text{otherwise.} \end{cases}$$

where $[a_3, b_3] \leq [a_2, b_2] \leq [a_1, b_1]$, $[c_1, d_1] \leq [c_2, d_2] \leq [c_3, d_3]$ and $a_i, b_i, c_i, d_i \in [0, 1]$ for $i = 1, 2, 3$. Then A is an $I - (\alpha, \beta)-$vague group of G.

Definition 3.12. Let A be an (α, β)- vague set of a G with the true-membership function t_A and false-membership function f_A. The (α, β)-cut of the (α, β)- vague set A is a crisp subset $A_{(\alpha,\beta)}$ of the set U given by $A_{(\alpha,\beta)} = \{x \in G \mid V_A(x) \geq [\alpha, \beta]\}$, where $\alpha \leq \beta$.

Theorem 3.13. *If A and B be two $(\alpha, \beta)-$vague subgroups of G, then following holds:*
(i) $A_{(\alpha,\beta)} \subseteq A_{(\delta,\theta)}$ *if* $\alpha \geq \delta$ *and* $\beta \leq \theta$
(ii) $A_{(1-\beta,\beta)} \subseteq A_{(\alpha,\beta)} \subseteq A_{(\alpha,1-\alpha)}$
(iii) $A \subseteq B$ *implies* $A_{(\alpha,\beta)} \subseteq B_{(\alpha,\beta)}$
(iv) $(A \cap B)_{(\alpha,\beta)} = A_{(\alpha,\beta)} \cap B_{(\alpha,\beta)}$
(v) $(A \cup B)_{(\alpha,\beta)} \supseteq A_{(\alpha,\beta)} \cup B_{(\alpha,\beta)}$
(vi) $(\cap A_i)_{(\alpha,\beta)} = \cap A_{i(\alpha,\beta)}$
(vii) $A_{(0,1)} = 1$

Proof. (i) Let $x \in A_{(\alpha,\beta)}$. Then $t_A(x) \geq \alpha$ and $f_A(x) \leq \beta$. Since $\delta \leq \alpha$ and $\theta \geq \beta$ implies that $t_A(x) \geq \alpha \geq \delta$ and $f_A(x) \leq \beta \leq \theta$ thus $t_A(x) \geq \delta$ and $f_A(x) \leq \theta$ therefore $x \in A_{(\delta,\theta)}$. Hence, $A_{(\alpha,\beta)} \subseteq A_{(\delta,\theta)}$.

(ii) Since, $\alpha + \beta \leq 1$ implies that $1 - \beta \geq \alpha$ and $\beta \leq \beta$. Therefore by part (i) we get

$$A_{(1-\beta,\beta)} \subseteq A_{(\alpha,\beta)} \tag{3.1}$$

$\alpha + \beta \leq 1$ implies that $\beta \leq 1 - \alpha$. Therefore by part (i) we get

$$A_{(1-\beta,\beta)} \subseteq A_{(\alpha,\beta)} \tag{3.2}$$

From (3.1) and (3.2) we get that $A_{(1-\beta,\beta)} \subseteq A_{(\alpha,\beta)} \subseteq A_{(\alpha,1-\alpha)}$.

(iii) If $x \in A_{(\alpha,\beta)}$, thus $t_A(x) \geq \alpha$ and $f_A(x) \leq \alpha$. As $A \subseteq B$ implies $t_B(x) \geq t_A(x) \geq \alpha$ and $f_B(x) \leq f_A(x) \leq \beta$ so $t_B(x) \geq \alpha$ and $f_B(x) \leq \beta$ therefore $x \in B_{(\alpha,\beta)}$. Hence $A_{(\alpha,\beta)} \subseteq B_{(\alpha,\beta)}$.

(iv) Since $A \cap B = A$ and $A \cap B \subseteq B$. Therefore by part (i) $(A \cap B)_{(\alpha,\beta)} \subseteq A_{(\alpha,\beta)}$ and $(A \cap B)_{(\alpha,\beta)} \subseteq B_{(\alpha,\beta)}$ thus

$$(A \cap B)_{(\alpha,\beta)} \subseteq A_{(\alpha,\beta)} \cap B_{(\alpha,\beta)}. \tag{3.3}$$

Let $x \in A_{(\alpha,\beta)} \cap B_{(\alpha,\beta)}$ thus $x \in A_{(\alpha,\beta)}$ and $x \in B_{(\alpha,\beta)}$ therefore $t_A(x) \geq \alpha$, $f_A(x) \leq \beta$, $t_B(x) \geq \alpha$ and $f_B(x) \leq \beta$ so $t_A(x) \wedge t_B(x) \geq \alpha$ and $f_A(x) \vee f_B(x) \leq \beta$ therefore $(t_A \cap t_B)(x) \geq \alpha$ and $(f_A \cap f_B)(x) \leq \beta$ then $x \in (A \cap B)_{(\alpha,\beta)}$. Thus

$$A_{(\alpha,\beta)} \cap B_{(\alpha,\beta)} \subseteq (A \cap B)_{(\alpha,\beta)} \tag{3.4}$$

From (3.3) and (3.4), we get that $(A \cap B)_{(\alpha,\beta)} = A_{(\alpha,\beta)} \cap B_{(\alpha,\beta)}$.

(V) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Therefore by part (i) $A_{(\alpha,\beta)} \subseteq (A \cup B)_{(\alpha,\beta)}$ and $B_{(\alpha,\beta)} \subseteq (A \cup B)_{(\alpha,\beta)}$ thus

$$A_{(\alpha,\beta)} \cup B_{(\alpha,\beta)} \subseteq (A \cup B)_{(\alpha,\beta)} \tag{3.5}$$

Now, equality is hold if $\alpha + \beta = 1$. We show that $(A \cup B)_{(\alpha,\beta)} \subseteq A_{(\alpha,\beta)} \cup B_{(\alpha,\beta)}$. Let $x \in (A \cup B)_{(\alpha,\beta)}$ then $(t_A \cup t_B)(x) \geq \alpha$ and $(f_A \cup f_B)(x) \leq \beta$ thus $t_A(x) \vee t_B(x) \geq \alpha$ and $f_A(x) \wedge f_B(x) \leq \beta$. If $t_A(x) \geq \alpha$, then $f_A(x) \leq 1 - t_A(x) \leq 1 - \alpha = \beta$. Implies that $x \in A_{(\alpha,\beta)} \cup B_{(\alpha,\beta)}$. Similarly if $t_B(x) \geq \alpha$, then $f_B(x) \leq 1 - t_B(x) \leq 1 - \alpha = \beta$. Implies that $x \in B_{(\alpha,\beta)} \subseteq A_{(\alpha,\beta)} \cup B_{(\alpha,\beta)}$. Thus we get that $x \in (A \cup B)_{(\alpha,\beta)}$ so $x \in A_{(\alpha,\beta)} \cup B_{(\alpha,\beta)}$.

$$(A \cup B)_{(\alpha,\beta)} \subseteq A_{(\alpha,\beta)} \cup B_{(\alpha,\beta)} \tag{3.6}$$

From (3.5) and (3.6), we get $(A \cup B)_{(\alpha,\beta)} = A_{(\alpha,\beta)} \cup B_{(\alpha,\beta)}$.

(vi) If $x \in (\cap A_i)_{(\alpha,\beta)}$, then $(\cap t_i)(x) \geq \alpha$ and $(\cap f_i)(x) \leq \beta \wedge t_i(x) \geq \alpha$ and $\vee f_i)(x) \leq \beta$ therefore $x \in A_{i(\alpha,\beta)}$ for all $i \in I$, thus $x \in \cap A_{i(\alpha,\beta)}$ hence $\cap A_{i(\alpha,\beta)} \subseteq \cap B_{i(\alpha,\beta)}$.

(vii) Follows from definition. □

Theorem 3.14. *Let A be (α, β)- vague subset of a group G. Then, A is (α, β)- vague subgroup of G if and only if $A_{(\alpha,\beta)}$ is a subgroup of group G for all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$, where $t_A(e) \geqslant \alpha, f_A(e) \leq \beta$ and e is the identity element of G.*

Proof. ($\Longrightarrow$) Let A be (α, β)- vague subgroup of group G. Then the proof follows by Definition 3.11.

($\Longleftarrow$) Let A is vague subset of a group G such that $A_{(\alpha,\beta)}$ is a subgroup of group G for all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. To show that A be (α, β)- vague subgroup of group G. For this we show that

(i) $t_A(xy) \geq t_A(x) \wedge t_A(y), f_A(xy) \geq f_A(x) \vee f_A(y)$, for all $x, y \in G$

(ii) $t_A(x^{-1}) = t_A(x), f_A(x^{-1}) = f_A(x)$.

For (i) Let $x, y \in G$ and let $\alpha = t_A(x) \wedge t_A(y)$ and $\beta = f_A(x) \vee f_A(y)$. Then, $t_A(x), t_A(y) \geq \alpha$ and $f_A(x), f_A(y) \leq \beta$. Thus, $t_A(x) \geq \alpha, f_A(x) \leq \beta$, and $t_A(y) \geq \alpha, f_A(y) \leq \beta$. Therefore $x, y \in A_{(\alpha,\beta)}$ and because $A_{(\alpha,\beta)}$ is group thus $xy \in A_{(\alpha,\beta)}$. Therefore $t_A(xy) \geq \alpha = t_A(x) \wedge t_A(y), f_A(xy) \leq \beta = f_A(x) \vee f_A(y)$ thus $t_A(xy) \geq t_A(x) \wedge t_A(y), f_A(xy) \geq f_A(x)$.

For (ii) Let $x \in G$ and $t_A(x) = \alpha, f_A(x) = \beta$. Then, $t_A(x) \geq \alpha, f_A(x) \leq \beta$ is true, so $x \in A_{(\alpha,\beta)}$. As $x \in A_{(\alpha,\beta)}$ is a subgroup of G, therefore we have $x^{-1} \in A_{(\alpha,\beta)}$ thus $t_A(x^{-1}) \geq \alpha, f_A(x^{-1}) \leq \beta$. Thus $t_A(x^{-1}) \geq \alpha = t_A(x)$ and $f_A(x^{-1}) \leq \beta = f_A(x)$. Thus $t_A(x) = t_A((x^{-1})^{-1}) \geq t_A(x^{-1}) \geq t_A(x)$ implies that $t_A(x^{-1}) = t_A(x)$ and $f_A(x) = f_A((x^{-1})^{-1}) \leq f_A(x^{-1}) \geq f_A(x)$ implies that $f_A(x^{-1}) = f_A(x)$. Hence, A is (α, β)- vague subgroup of group G. □

Definition 3.15. Let A and B be any two $(\alpha, \beta)-$vague subgroup of G and H respectively. Then the cartesian product of A and B is denoted by $A \times B$ and is defined as;

$A \times B = \{ < (x, y), t_{A\times B}(x, y), f_{A\times B}(x, y) \ >: x \in G$ and $y \in H\}$

where $t_{A\times B}(x, y) = \min\{t_A(x), t_B(y)\}$ and $f_{A\times B}(x, y) = \max\{f_A(x), f_B(y)\}$

Example 3.16. Let $G = \{e_1, a\}$, where $a^2 = e_1$ and let $H = \{e_2, x, y, xy\}$, where $x^2 = y^2 = e_2$ and $xy = yx$.

$G_1 \times H = \{(e_1, e_2), (e_1, x), (e_1, y), (e_1, xy), (a, e_2), (a, x), (a, y), (a, xy)\}$.

Let $B = \{\langle e_1, 0.6, 0.2\rangle, \langle a, 0.5, 0.2\rangle\}$, and

$$C = \{\langle e_2, 0.9, 0.1\rangle, \langle x, 1, 0\rangle, \langle y, 0.7, 0.1\rangle, \langle xy, 0.6, 0.1\rangle\},$$

be $(\alpha, \beta)-$vague subgroup of G_1 and H respectively. Then

$B \times C = \{\langle (e_1, e_2), 0.6, 0.2\rangle, \langle (e_1, x), 0.6, 0.2\rangle, \langle (e_1, y), 0.6, 0.2\rangle, \langle (e_1, xy), 0.6, 0.2\rangle,$
$\langle (a, e_2), 0.5, 0.2\rangle, \langle (a, x), 0.5, 0.2\rangle, \langle (a, y), 0.5, 0.2\rangle, \langle (a, xy), 0.5, 0.2\rangle\}$.

Here $B \times C$ is also $(\alpha, \beta)-$vague subgroup of group $G \times H$.

Theorem 3.17. *If A and B be two $(\alpha, \beta)-$vague subgroup of G and H respectively. Then $(A \times B)_{(\alpha,\beta)} = A_{(\alpha,\beta)} \times B_{(\alpha,\beta)}$, for all $\alpha, \beta \in [0, 1]$ with $0 \leq \alpha + \beta \leq 1$*

Proof. Let $x, y \in (A \times B)_{(\alpha,\beta)}$ if and only if $t_{A\times B}(x, y) \geq \alpha$ and $f_{A\times B}(x, y) \leq \beta$ and only if $\min\{t_A(x), t_B(y)\} \geq \alpha$ and $\max\{f_A(x), f_B(y)\} \leq \beta$ if and only if $t_A(x) \geq \alpha, t_B(y) \geq \alpha$ and $f_A(x) \leq \beta, f_B(y) \leq \beta$ if and only if $t_A(x) \geq \alpha, f_A(x) \leq \beta$ and $t_B(y) \geq \alpha, f_B(y) \leq \beta$ if and only if $x \in A_{(\alpha,\beta)}$ and $y \in B_{(\alpha,\beta)}$ if and only if $x, y \in A_{(\alpha,\beta)} \times B_{(\alpha,\beta)}$. Hence $(A \times B)_{(\alpha,\beta)} = A_{(\alpha,\beta)} \times B_{(\alpha,\beta)}$. □

Theorem 3.18. *(1) A is an $(\in, \in)-$vague subgroup of G if and only if for any $a \in [0, 1]$, A_a is a 3-valued fuzzy subgroup of G.*

(2) A is an $(\in, \in \vee q)-$ vague subgroup of G if and only if for any $a \in [0, 0.5]$, A_a is a 3-valued fuzzy subgroup of G.

(3) A is an $(\in \wedge q, \in)-$ vague subgroup of G if and only if for any $a \in [0.5, 1]$, A_a is a 3-valued fuzzy subgroup of G.

Proof. (1) $(\Longrightarrow)$ Since A is an $(\in, \in)-$vague subgroup, then for any $a \in [0, 1]$ and $x \in G$, $[x_a y_a \in A] \geq [x_a \in A] \wedge [y_a \in A], [x_a^{-1} \in A] \geq [x_a \in A]$, hence $A_a(xy) \geq A_a(x) \wedge A_a(y)$ and $A_a(x^{-1}) \geq A_a(x)$. So A is a 3$-$valued fuzzy subgroup of G.

$(\Longleftarrow)$ for any $x, y \in G$ and $s, t \in [0, 1]$, $[x_s y_t \in A] = A_{s\wedge t}(xy) \geq A_{s\wedge t}(x) \wedge A_{s\wedge t}(y) \geq A_s(x) \wedge A_t(y) = [x_s \in A] \wedge [y_t \in A]$ and $[x_s^{-1} \in A] = A_s(x^{-1}) \geq A_s(x) = [x_s \in A]$. Therefore, A is an $(\in, \in)-$ vague subgroup of G.

(2) $(\Longrightarrow)$ since A is an $(\in, \in \vee q)-$ vague subgroup, for any $a \in [0, 0.5]$ and $x \in G$, we have $[x_a y_a \in \vee q A] \geq [x_a \in A] \wedge [y_a \in A]$. So $A_a(xy) \vee A_{\underline{[a]}}(xy) \geq A_a(xy) \geq A_a(x) \wedge A_a(y)$. By $0 < a \leq 0.5$, we have that $a \leq 0 \leq 1 - a$. Thus

$A_{\underline{[a]}}(xy) = A_{\underline{1-a}}(xy) \leq A_{\underline{a}}(xy) \leq A_a(xy)$. Hence $A_a(xy) \geq A_a(x) \wedge A_a(y)$. Similarly, we have $A_a(x^{-1}) \geq A_a(x)$. So A_a is a 3− valued fuzzy subgroup of G.

($\Longleftarrow$) Let $s, t \in (0, 1]$. If $s \wedge t \leq 0.5$, then $1 - s \wedge t \geq 0.5 \geq s \wedge t$. Thus $A_{\underline{[s\wedge t]}}(xy) \leq A_{s\wedge t}(xy)$. So $[x_s y_t \in \vee qA] = A_{s\wedge t}(xy) \vee A_{\underline{[s\wedge t]}}(xy) = A_{s\wedge t}(xy) \geq A_{s\wedge t}(x) \wedge A_{s\wedge t}(y) \geq A_s(x) \wedge A_t(y) = [x_s \in A] \wedge [y_t \in A]$. If $s \wedge t > 0.5$, then let $a \in (0, 1)$ such that $1 - s \wedge t \geq 0.5 \geq s \wedge t$. Thus $A_{s\wedge t}(xy) \leq A_{\underline{[s\wedge t]}}(xy)$ and $A_{\underline{[s\wedge t]}}(xy) \geq A_a(xy)$. Hence $[xy \in \vee qA] = A_{s\wedge t}(xy) \vee A_{\underline{[s\wedge t]}}(xy) = A_{\underline{[s\wedge t]}}(xy) \geq A_a(xy) \geq A_a(x) \wedge A_a(y) \geq A_s(x) \wedge A_t(y) = [x_s \in A] \wedge [y_t \in A]$. Hence, $[x_s y_t \in \vee qA] \geq [x_s \in A] \wedge [y_t \in A]$. Similarly, we have $[x_s^{-1} \in \vee qA] \geq [x_s \in A]$

(3) ($\Longrightarrow$) Let $\alpha \in (0, 0.5]$ and $x \in G$. Then $A_{\underline{a}}(x) \geq A_a(x)$. Thus $A_a(xy) = [x_a y_a) \in A] \geq [x_a \in \wedge qA] \wedge [y_a \in \wedge qA] \geq A_a(x) \wedge A_{\underline{[a]}}(x) \wedge A_a(y) \wedge A_{\underline{[a]}}(y) \geq A_a(x) \wedge A_a(y)$. Similarly, we have $A_a(x^{-1}) \geq A_a(x)$. So A_a is a 3− valued fuzzy subgroup of G.

($\Longleftarrow$) For any $x, y \in U$ and $s, t \in (0, 1]$, let $a = [x_s \in \wedge qA] \wedge [y_t \in \wedge qA]$.

Case 1. $a = 1$. Then $t_A(x) \geq s, t_A(x) \geq 1 - s, t_A(y) \geq t$ and $t_A(y) \geq 1 - t$. Thus $t_A(x) > 0.5, t_A(y) > 0.5$. So $t_A(xy) \geq t_A(x) \wedge t_A(y) \geq s \wedge t$, i.e., $[x_s y_t \in A] = 1$.

Case 2. $a = \frac{1}{2}$. Then $1 - f_A(x) \geq s \geq f_A(x)$ and $1 - f_A(y) \geq t \geq f_A(y)$. Therefore $f_A(x) < 0.5, f_A(y) < 0.5$. So $f_A(xy) \leq f_A(x) \wedge f_A(y)$ and $1 - f_A(xy) \geq (1 - f_A(x)) \wedge (1 - f_A(y)) \geq s \wedge t$. Hence $[x_s y_t \in A] \geq \frac{1}{2}$. So $[x_s y_t \in A] \geq [x_s \in \wedge qA] \wedge [y_t \in \wedge qA]$. Similarly, we have $[x_s^{-1} \in A] \geq [x_s \in \wedge qA]$. Therefore, A is an $(\in \wedge q, \in)-$vague subgroup of G. □

Theorem 3.19. *A is a $(\in, \in \vee q)$- vague subgroup of G if and only if for any $x, y \in G$*

$$t_A(xy) \geq t_A(x) \wedge t_A(y) \wedge \tfrac{1}{2}, \quad t_A(x^{-1}) \geq t_A(x) \wedge 0.5$$

$$f_A(xy) \leq f_A(x) \vee f_A(y) \vee \tfrac{1}{2}, \quad f_A(x^{-1}) \leq f_A(x) \vee 0.5$$

Proof. ($\Longrightarrow$) Suppose that $t = t_A(x) \wedge t_A(y) \wedge \frac{1}{2}$, so $[x_t y_t \in \vee qA] \geq [x_t \in A] \wedge [y_t \in A] = 1$, which implies that $t_A(xy) \geq t$ or $t_A(xy) > 1 - t \geq \frac{1}{2} \geq t$, thus $t_A(xy) \geq t_A(x) \wedge t_A(y) \wedge \frac{1}{2}$. Similarly, we have $t_A(x^{-1}) \geq t_A(x) \wedge 0.5$ and $f_A(x^{-1}) \leq t_A(x) \vee 0.5$.

Let $1 - s = f_A(x) \vee f_A(y) \vee \frac{1}{2}$. Then $[x_s y_s \in \vee qA] \geq [x_s \in A] \wedge [y_s \in A] \geq \frac{1}{2}$, which implies that $s \leq 1 - f_A(xy)$ or $f_A(xy) < s \leq 1 - s$. Furthermore, $f_A(xy) \leq 1 - s = f_A(x) \vee f_A(y) \vee \frac{1}{2}$.

($\Longleftarrow$) For any $x, y \in G$ and $s, t \in [0, 1]$, put $a = [x_s \in A] \wedge [y_t \in A]$ in the case $a = 1$, suppose that $[x_s y_t \in \vee qA] \leq \frac{1}{2}$, therefore $t_A(x) \geq s, t_A(y) \geq t$, $t_A(xy) \leq s \wedge t$ and $t_A(xy) \leq 1 - s \wedge t$, thus $\frac{1}{2} \geq t_A(xy) \geq t_A(x) \wedge t_A(y) \wedge \frac{1}{2}$. So $t_A(xy) \geq t_A(x) \wedge t_A(y) \geq s \wedge t$. This is a contradiction with $t_A(xy) < s \wedge t$. Therefore we have $[x_t y_t \in \vee qA] = 1$. In the case $a = \frac{1}{2}$, we have $1 - f_A(x) \geq s$, $1 - f_A(y) \geq t$ and $1 - f_A(x) \vee f_A(y) \geq s \wedge t$. Suppose that $[x_s y_t \in \vee qA] = 0$, then $s \wedge t > 1 - f_A(xy)$ and $f_A(xy) \geq s \wedge t$, thus $f_A(xy) > \frac{1}{2}$, consequently $f_A(xy) \leq f_A(x) \vee f_A(y)$ and $1 - f_A(xy) \geq 1 - f_A(x) \vee f_A(y) \geq s \wedge t$. This is a contradiction with $1 - f_A(xy) < s \wedge t$. Therefore, we have $[x_s y_t \in \vee qA] \geq \frac{1}{2}$. Ultimately, $[x_s y_t \in \vee qA] \geq [x_s \in A] \wedge [y_t \in A]$. Similarly, we have $[x_s^{-1} \in \vee qA] \geq [x_s \in A]$. This shows that A is an $(\in, \in \vee q)$- vague subgroup of G. □

Corollary 3.20. *A is an* $(\in\wedge q, \in)$*- vague subgroup of* G *if and only if*

$$t_A(xy) \vee \tfrac{1}{2} \geq t_A(x) \wedge t_A(y), \quad t_A(x^{-1}) \vee \tfrac{1}{2}$$

$$f_A(xy) \wedge \tfrac{1}{2} \leq f_A(x) \vee f_A(y), \quad f_A(x^{-1}) \wedge \tfrac{1}{2} \leq f_A(x)$$

Theorem 3.21. *A is an* $(\in\wedge q, \in\vee q)$*- vague subgroup of* G *if and only if for any* $x, y \in G$

(1) $f_A(xy) \leq f_A(x) \vee f_A(y) \vee \frac{1}{2}$ *0r* $f_A(xy) \wedge \frac{1}{2} \leq f_A(x) \vee f_A(y)$,

(2) $f_A(x^{-1}) \leq f_A(x) \vee \frac{1}{2}, f_A(x^{-1}) \wedge \frac{1}{2} \leq f_A(x)$,

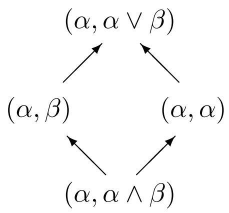
(3) $t_A(xy) \geq t_A(x) \wedge t_A(y) \wedge \frac{1}{2}$ *or* $t_A(xy) \vee \frac{1}{2} \geq t_A(x) \wedge t_A(y)$,

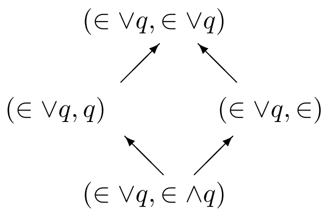
(4) $t_A(x^{-1}) \geq t_A(x) \wedge \frac{1}{2}$ *or* $t_A(x^{-1}) \vee \frac{1}{2} \geq t_A(x)$.

Proof. ($\Longrightarrow$) (1)Suppose that $t_A(xy) \vee \frac{1}{2} < t = t_A(x) \wedge t_A(y)$, so $t_A(x) \geq t > \frac{1}{2}$, $t_A(y) \geq t > \frac{1}{2}$. Thus $[x_{0.5}y_{0.5} \in \vee qA] \geq [x_{0.5} \in \wedge qA] \wedge [y_{0.5} \in \wedge qA] = 1$, which implies that $t_A(xy) \geq \frac{1}{2}$ or $t_A(xy) + \frac{1}{2} > 1$. Consequently, $t_A(xy) \geq \frac{1}{2} \geq t_A(x) \wedge t_A(y) \wedge \frac{1}{2}$. (3) Suppose that $f_A(xy) \wedge \frac{1}{2} > t = 1 - s = f_A(x) \vee f_A(y)$, so $s \leq 1 - f_A(x)$, $s \leq 1 - f_A(y)$ and $s > \frac{1}{2}$. Therefore $[x_{0.5}y_{0.5} \in \vee qA] \geq [x_{0.5} \in \wedge qA] \wedge [y_{0.5} \in \wedge qA] \geq \frac{1}{2}$ which implies that $\frac{1}{2} \leq 1 - f_A(xy)$ or $f_A(xy) < \frac{1}{2}$. Hence $f_A(xy) \leq \frac{1}{2} \leq f_A(x) \vee f_A(y) \vee \frac{1}{2}$. $(2, 4)$ can be proved similarly.

($\Longleftarrow$) For any $x, y \in G$ and $s, t \in [0, 1]$, put $a = [x_s \in \wedge qA] \wedge [y_t \in \wedge qA]$. In the case $a = 1$, hence $t_A(x) \geq s, t_A(x) \geq 1-s, t_A(y) \geq t, t_A(y) \geq 1-t$, so, $t_A(x) \wedge t_A(y) > \frac{1}{2}$. Suppose that $[x_s y_t \in \vee qA] \leq \frac{1}{2}$, then $t_A(xy) < s \wedge t$ and $t_A(xy) \leq 1 - s \wedge t$, therefore $t(xy) < \frac{1}{2} < t_A(x) \wedge t_A(y)$. Furthermore, $t_A(xy) < t_A(x) \wedge t_A(y) \wedge \frac{1}{2}$ and $t_A(xy) \vee \frac{1}{2} < t_A(x) \wedge t_A(y)$, which is a contradiction. Consequently, $[x_s y_t \in \vee qA] = 1$ in the case $a = \frac{1}{2}$, then $1 - f_A(x) \geq s > f_A(x)$ or $1 - f_A(y) \geq t > f_A(y)$, thus $f_A(x) \vee f_A(y) < \frac{1}{2}$. Suppose that $[x_s y_t) \in \vee qA] = 0$, so $f_A(xy) \geq s \wedge t > 1 - f_A(xy)$, thus $f_A(xy) > \frac{1}{2}$. Furthermore, $f_A(xy) \wedge \frac{1}{2} = \frac{1}{2} > f_A(x) \vee f_A(y)$ and $f_A(xy) > f_A(x) \vee f_A(y) \vee \frac{1}{2}$, which is a contradiction. Hence, $[x_s y_t \in \vee qA] \geq \frac{1}{2}$. From the above we have $[x_s y_t \in \vee qA)] \geq [x_s \in \wedge qA] \wedge [y_t \in \wedge qA]$, similarly, we have $[x_s^{-1} \in \vee qA] \geq [x_s \in \wedge qA]$. This show that A is an $(\in\wedge qA, \in\vee qA)$- vague subgroup of G. □

Theorem 3.22. *Let A be a* $(\alpha, \beta)-$*vague subgroup of* G*. Then the left diagram shows the relationship between* $(\alpha, \beta)-$ *vague subgroup of* G*, where* α, β *are one of* $\in$ *and* q*. Also we have the right diagram.*





Theorem 3.23. *If A is a* $(\in\vee qA, \in\vee qA)$*-vague subgroup of* G*, then A is a* $(\in, \in\vee qA)$*-vague subgroup of* G*.*

4. $(\alpha, \beta)-$VAGUE NORMAL SUBGROUP

Definition 4.1. A $(\alpha, \beta)-$vague subgroup A of group G is $(\alpha, \beta)-$vague normal subgroup of G if:
(i) $t_A(xy) = t_A(yx)$
(ii) $f_A(xy) = f_A(yx)$, for all $x, y \in G$.

Example 4.2. Let A be a subgroup of group G that defined by:
$t_A(x) = t_A(e), f_A(x) = f_A(e)$.
Then A is a $(\alpha, \beta)-$vague subgroup normal of group G.

Remark 4.3. Let A is $(\alpha, \beta)-$vague subgroup of group G. Then A is normal if and only if:
(i) $t_A(g^{-1}xg) = t_A(x)$.
(ii)$f_A(g^{-1}xg) = f_A(x)$, for all $x \in A$ and $g \in G$.

Theorem 4.4. *Let A be $(\alpha, \beta)-$vague normal subgroup of a group G. Then $A_{(\alpha,\beta)}$ is a normal subgroup of group G, where $t_A(e) \geqslant \alpha, f_A(e) \leq \beta$ and e is the identity element of G.*

Proof. Let $x \in A_{(\alpha,\beta)}$ and $g \in G$. Then $t_A(x) \geq \alpha, f_A(x) \leq \beta$. Also, A be $(\alpha, \beta)-$vague normal subgroup of a group G. Therefore, $t_A(g^{-1}xg) = t_A(x)$ and $f_A(g^{-1}xg) = f_A(x)$, for all $x \in A$ and $g \in G$. Therefore $t_A(g^{-1}xg) = t_A(x) \geq \alpha$ and $f_A(g^{-1}xg) = f_A(x) \leq \beta$ implies that $t_A(g^{-1}xg) \geq \alpha$ and $f_A(g^{-1}xg) \leq \beta$ therefore $g^{-1}xg \in A_{(\alpha,\beta)}$. Hence, $A_{(\alpha,\beta)}$ is normal subgroup of G. □

Theorem 4.5. *Let A and B be $(\alpha, \beta)-$vague normal subgroup of group G_1 and G_2 respectively. Then $A \times B$ is also $(\alpha, \beta)-$vague subgroup of group $G_1 \times G_2$.*

Proof. Let A and B be $(\alpha, \beta)-$vague normal subgroup of group G_1 and G_2 respectively. Then by Theorem 4.4, $A_{(\alpha,\beta)}, B_{(\alpha,\beta)}$ are normal subgroup of G_1, G_2 respectively. Thus, $A_{(\alpha,\beta)}, B_{(\alpha,\beta)}$ is subgroup of group $G_1 \times G_2$. By Theorem 3.19 $A \times B$ is $(\alpha, \beta)-$vague subgroup of group $G_1 \times G_2$. □

5. CONCLUSIONS

In this paper we have presented basic concept of vague groups. We have also the notion of (α, β)- vague subgroup and studied their properties are investigate.

We obtained the following results:

1. Among 16 kinds of $(\alpha, \beta)-$vague sets, the significant ones are the $(\in, \in)-$vague subgroup, the $(\in, \in \vee q)-$vague subgroup and the $(\in \wedge q, \in)-$vague subgroup.

2. A is a $(\alpha, \beta)-$vague subgroup of G if and only if, for any $a \in (0, 1]$, the cut set A_a of A is a 3-valued vague subgroup of G and A is an $(\in, \in \vee q)-$vague subgroup (or $(\in \wedge q, \in)-$ vague subgroup) of G if and only if for any $a \in (0, 0.5]$ (or $a \in (0.5, 1]$), the cut set of A is a 3-valued vague set of G.

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References

[1] S. K. Bhakat and P. Das, $(\in, \in \vee q)$-fuzzy subgroup, Fuzzy Sets and Systems 80 (1996) 359–368.
[2] R. Biswas, Vague group, International Journal Computational 4(2) (2006) 21–22.
[3] A. Borumand Saeid, Vague BCK/BCI-algebra, Opuscula Math. 29(2) (2009) 177–186.
[4] W. L. Gau and D. J. Buehrer, Vague sets, IEEE Ttransactions on Systems, Man and Cybernetic 23 (1993) 610–614.
[5] X. H. Yuan, X. H. Li and K. B. Sun, The cut sets, decomposition theorems on the intuitionistic fuzzy sets and interval-valued fuzzy sets, Inform. Sci. 39(9) (2009) 933–942.
[6] X. H. Yuan, C. Zhang and Y. H. Ren, Generalized fuzzy groups and many-valued implications, Fuzzy Sets and Systems 138 (2003) 206–211.
[7] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

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