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Some non-continuous functions in generalized fuzzy topologies

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Abstract. The notion of extremal disconnectedness is introduced for generalized fuzzy topological spaces and some important properties of this notion are discussed. Using fuzzy unit interval, it is shown that extremally disconnected generalized fuzzy topological spaces are a rich source of functions which are not necessarily generalized fuzzy continuous but both generalized upper and lower fuzzy semi-continuous.

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1. Introduction

Generalized forms of open sets are encountered not only in topology but in various other fields of mathematics and even outside such as in logic programming etc. [2, 3, 15]. Thus, it is quite natural to investigate them and their corresponding topological structures. In the fuzzy setting, an attempt has been made in this regard in [10, 11, 12, 14]. In paper [10], a type of generalized open fuzzy sets, called γ-open fuzzy sets, are introduced using monotonic mappings on a set X. These γ-open sets on X are found to form a structure, which is a generalized form of a fuzzy topology. We call it a generalized fuzzy topology. Under different sets of conditions, these γ-open fuzzy sets represent various already existing weaker forms of open fuzzy sets such as semi-open fuzzy sets, pre-open fuzzy sets, α-open fuzzy sets, β-open fuzzy sets etc. Thus, the investigations in these papers, also provide a uniform study of the various weaker forms of open fuzzy sets existing in the literature. The neighbourhood system, separation axioms, convergence structures and notion of connectedness etc. in generalized fuzzy topological spaces have already been investigated in our earlier papers [10, 11, 12, 14]. In the present paper, we discuss another important notion, namely, extremal disconnectedness in the generalized fuzzy topological spaces. Due to its wide use and applications in various fields within and outside pure mathematics, extremal disconnectedness has been investigated and studied by several researchers in both topology and fuzzy topology. Therefore, it is important to discuss extremal disconnectedness in generalized fuzzy topological spaces also. In section 3, the basic properties of this notion are investigated. In section 4, we study another interesting relationship of extremal disconnectedness with the fuzzy unit interval. We find that extremally disconnected generalized fuzzy topological spaces are an abundant source of mappings which are both generalized fuzzy upper semi continuous and generalized fuzzy lower semi continuous. In this process, we have observed that in generalized fuzzy topological spaces, a function may be both fuzzy upper semi continuous and fuzzy lower semi continuous but yet may fail to be fuzzy continuous. This reveals a departure from fuzzy topology as well as from general topology.

2. Preliminaries

Throughout this paper, fuzzy sets are denoted by A, B, C etc and X, Y, Z etc. denote the ordinary sets. A fuzzy set A on a set X is a mapping $A : X \to [0, 1]$. The constant fuzzy sets which take each member of X to zero and to one respectively are denoted by $\underline{0}$ and $\underline{1}$ respectively. The union and intersection of a family of fuzzy sets $\{A_i\}$, denoted by $\vee A_i$ and $\wedge A_i$ respectively, are defined by

$$\big(\underset{i\in\Delta}{\vee} A_i\big)(x) = \sup\{A_i(x) : i \in \Delta\} \quad \text{and}$$
$$\big(\underset{i\in\Delta}{\wedge} A_i\big)(x) = \inf\{A_i(x) : i \in \Delta\} \quad \text{respectively.}$$

The complement of a fuzzy set A, denoted by A^c, is defined by

$$A^c(x) = 1 - A(x) \quad \text{for all } x \in X.$$

A fuzzy point x_α with support x and value α, $0 < \alpha \leq 1$ is a fuzzy set which takes value α at x and 0 at every other point of X. While $x_\alpha \leq A$ implies $\alpha \leq A(X)$; $x_\alpha \in A$ implies $\alpha < A(x)$. For fuzzy sets A and B, $A \leq B$ implies $A(x) \leq B(x)$ for each x. The dual fuzzy point of a fuzzy point x_α, where $0 < \alpha < 1$, is defined to be the fuzzy point $x_{1-\alpha}$ and is denoted by $x_{\alpha'}$. Two fuzzy sets A and B are said to overlap, denoted by AqB, if there exists x in X such that $A(x) + B(x) > 1$. For further definitions and notations, please refer to [1, 7].

Following definitions and results are taken from [10].

Definition 2.1. Let X be a non empty set and $I = [0, 1]$. A mapping $\gamma : I^X \to I^X$, is called *a monotonic mapping* if $\gamma(A) \leq \gamma(B)$, whenever $A \leq B$. The collection of all such monotonic mappings on I^X is denoted by $\Gamma(X)$.

Definition 2.2. Let $\gamma \in \Gamma(X)$. A fuzzy set A on X is called γ-*open* if $A \leq \gamma(A)$. The complement of a γ-open fuzzy set is called a γ-*closed* fuzzy set.

It is observed that the γ-open fuzzy sets defined on a set X are closed under arbitrary union and $\underline{0}$ is always γ-open. These two characteristics can be used to define a generalized form of a fuzzy topological space in the following way:

Definition 2.3. Let X be a non empty set. Let ζ be a collection of fuzzy sets on X, such that

(i) $\underline{0} \in \zeta$;
(ii) For $G_i \in \zeta$, $i \in \Delta$, $\bigvee_{i\in\Delta} G_i \in \zeta$.

Then ζ is called a *generalized fuzzy topology* on X (GFT, in short) and (X, ζ) is called a generalized fuzzy topological space (GFTS,in short). Members of ζ are called *generalized open fuzzy sets* and their complements are called *generalized closed fuzzy sets* on X. Clearly, each $\gamma \in \Gamma(X)$ defines a GFT on X, as it can be verified that for a family of γ-open fuzzy sets $\{A_i\}_{i\in\wedge}$, we have union of γ-open fuzzy sets is γ-open , that is , $\bigvee_i A_i \leq \gamma(\bigvee_i A_i)$. Therefore, γ-open fuzzy sets form a GFT and every γ-open fuzzy set is a generalized open fuzzy set.

For a fuzzy set A on X, interior of A is defined to be the largest generalized open fuzzy set contained in A and closure of A is the smallest generalized closed fuzzy set containing A. For other notations and definitions, please refer to [10, 11, 12].

Here, it may be mentioned that some generalized form of fuzzy topology was also studied by some other authors[4]. However, the approach adopted and motivation in [10] is different from that of [4]. In fact the notion in [4] becomes a particular case of that of [10] for $L = [0, 1]$.

3. Extremal disconnectedness

In the following, we introduce and investigate extremal disconnectedness for the generalized fuzzy topological spaces.

Definition 3.1. A generalized fuzzy topological space X is called *extremally disconnected* if the closure of every generalized open fuzzy set is a generalized open fuzzy set in X.

Here is an example:

Example 3.2. Let X be a non empty set and let $\zeta = [\underline{0}, .\underline{4}, .\underline{5}]$. Then, (X, ζ) is an extremally disconnected $GFTS$.

Remark 3.3. Extremal disconnectedness for Chang's fuzzy topological spaces was introduced in 1994 in [8]. However, the one provided in this paper is more general in nature.

The following result for non overlapping generalized open fuzzy sets is immediate:

Theorem 3.4. *A generalized fuzzy topological space is extremally disconnected iff for every pair of non overlapping generalized open fuzzy sets U and V in X, $\mathrm{cl}(U)$ and $\mathrm{cl}(V)$ are non overlapping generalized open fuzzy sets in X.*

In the following, we provide yet another characterization of extremal disconnectedness of a generalized fuzzy topological space.

Theorem 3.5. *A generalized fuzzy topological space X is extremally disconnected iff for every generalized open fuzzy set G and generalized closed fuzzy set F with $G \leq F$, there exists a generalized open fuzzy set G_1 and a generalized closed fuzzy set F_1 such that $G \leq F_1 \leq G_1 \leq F$.*

Proof. Let X be an extremally disconnected generalized fuzzy topological space. Let G be a generalized open fuzzy set and F be a generalized closed fuzzy set such that $G \leq F$. Then $G \not{q} F^c$. Since G and F^c are both generalized open fuzzy sets, thus by Theorem 3.4, $\text{cl}(G) \not{q} \text{cl}(F^c)$. That is, $\text{cl}(G) \leq (\text{cl}(F^c))^c = ((\text{int}(F))^c)^c$. Let $\text{cl}(G) = F_1$ and $(\text{cl}(F^c))^c = G_1$. Then, $G \leq F_1 \leq G_1 \leq F$.

Conversely, let U and V be two non overlapping generalized open fuzzy sets. Then $U \leq V^c$ and V^c is closed. Thus, there exists, a generalized open fuzzy set G and a generalized closed fuzzy set F such that $U \leq F \leq G \subseteq V^c$. This implies, $U \leq \text{cl}(U) \leq F \leq G \leq \text{int}\, V^c \leq V^c$. Thus, $\text{cl}(U) \leq \text{int}\, V^c$. This implies $\text{cl}(U) \not{q} (\text{int}\, V^c)^c$, where $(\text{int}\, V^c)^c = \text{cl}(V)$. Thus, X is an extremally disconnected generalized fuzzy topological space. □

In the following, we investigate the interrelationship between semi-closure and closure of a fuzzy set in the light of extremal disconnectedness. It may be recalled that a fuzzy set A in a GFT is called *semi-open* if there exists an open fuzzy set A_0 such that $A_0 \leq A \leq \text{cl}(A_0)$. Equivalently, A is *semi-open* iff $A \leq \text{cl}(\text{int}\, A)$. The complement of a semi-open fuzzy set is called *semi-closed* fuzzy set. The intersection of all semi-closed fuzzy sets containing a fuzzy set A is called the *semi-closure* of A and is denoted by $s\text{-cl}(A)$.

Remark 3.6. It may be mentioned here that the collection of semi-open fuzzy sets of a given topology on a set X is an example of a generalized fuzzy topology, which is not a fuzzy topology.

Theorem 3.7. *Let A be a fuzzy set in a GFTS X. Then $x_\alpha \leq s\text{-cl}(A)$ iff every semi-open fuzzy set overlapping with x_α overlaps with A.*

For a GFT, $\text{cl}(A)$ and $s\text{-cl}(A)$ may be different but in an extremally disconnected GFT both are same for a semi-open fuzzy set. In the following, we prove the same.

Theorem 3.8. *In an extremally disconnected generalized fuzzy topological space X, $\text{cl}(A) = s\text{-cl}(A)$ for every semi-open fuzzy set A in X.*

Proof. $s\text{-cl}(A) \leq \text{cl}(A)$ is obvious. It is sufficient to prove the reverse inclusion. Let x_α be a fuzzy point in X such that $x_\alpha \not\leq s\text{-cl}(A)$. By Theorem 3.7, there exists a semi-open fuzzy set B in X such that $x_\alpha q B$ and $B \not{q} A$. This implies that $x_\alpha q B$ and $\text{int}(B) \not{q} \text{int}(A)$. Since X is extremally disconnected, by Theorem 3.4, we have $\text{cl}(\text{int}(A)) \not{q} \text{cl}(\text{int}(B))$. Also B is semi-open, so $B \leq \text{cl}(\text{int}(B))$. Therefore, $\text{cl}(\text{int}(A)) \not{q} B$. Again as A is semi-open there exists an open fuzzy set A_0 such that $A_0 \leq A \leq \text{cl}(A_0)$. Therefore, $A_0 \leq \text{int}\, A \leq A \leq \text{cl}(A) \leq \text{cl}(A_0)$. This implies that $\text{cl}(A_0) \leq \text{cl}(\text{int}(A)) \leq \text{cl}(A) \leq \text{cl}(\text{cl}(A)) \leq \text{cl}(\text{cl}(A_0))$. Hence $\text{cl}(A) = \text{cl}(\text{int}(A))$. Thus it follows that $\text{cl}(A) \not{q} B$. Also $x_\alpha q B$. Therefore, $\text{cl}(A)(x) + B(x) \leq 1$ and $\alpha + B(x) > 1$, so that $\alpha > \text{cl}(A)(x)$. Hence $x_\alpha \not\leq \text{cl}(A)$. Thus $\text{cl}(A) \leq s\text{-cl}(A)$. □

4. Extremal disconnectedness and fuzzy unit interval

In 1974, Hutton constructed the L-fuzzy unit interval $L(I)$ [6]. Later, Gantner, Steinlage and Warren, by developing Hutton's idea introduced L-fuzzy real line $R(L)$ [5]. In what follows, we first discuss, fuzzy real line and fuzzy unit interval for

$L = I = [0, 1]$ and also discuss the relationship of extremal disconnectedness with fuzzy unit interval.

The fuzzy real line $R(I)$ [9], is the set of all equivalence classes $[\lambda]$, where $\lambda : R \to [0, 1]$ is monotonically decreasing mapping such that $\vee\{\lambda(t) : t \in R\} = 1$ and $\wedge\{\lambda(t) : t \in R\} = 0$; $\mu \in [\lambda]$ iff $\wedge\{\lambda(r) : r < t\} = \wedge\{\mu(r) : r < t\}$ and $\vee\{\lambda(r) : r > t\} = \vee\{\mu(r) : r > t\}$ for each $t \in R$. The fuzzy topology on $R(I)$ is generated from the subbase $\{L_t, R_t : t \in R\}$ where $L_t(\lambda) = (\lambda(t-))^c = (\underset{s<t}{\wedge}(\lambda(s)))^c$ and $R_t(\lambda) = (\lambda(t+)) = (\underset{s>t}{\vee}(\lambda(s)))$. The fuzzy unit interval is the subset $I(I)$ of $R(I)$ in which for each $[\lambda]$, $\lambda(t) = 1$ if $t < 0$ and $\lambda(t) = 0$ if $t > 1$ equipped with the subspace topology.

In the following, we study the interrelationship between extremal disconnectedness and generalized fuzzy upper and lower semi continuous mappings. We also discuss the relationship of extremal disconnectedness with the fuzzy unit interval.

The upper and lower semi continuous mappings for a generalized fuzzy topology may be defined in the following way:

Definition 4.1. Let (X, ζ) be a generalized fuzzy topological space and $R(I)$ be the fuzzy real line. A mapping $f : X \to R(I)$ is said to be *generalized fuzzy upper semi-continuous* (g.f.u.s.c, in brief) if for each $t \in R$, $f^{-1}(R_t) \in \zeta$ *(generalized fuzzy lower semi-continuous* (g.f.l.s.c, in brief) if for each $t \in R$, $f^{-1}(L_t) \in \zeta$ respectively).

The following example shows that unlike in fuzzy topology, a mapping may be both g.f.u.s.c and g.f.l.s.c, yet may fail to be fuzzy continuous in generalized fuzzy topologies.

Example 4.2. Let $X = I(I)$. Let $\zeta=$ union of members of $\{L_t, R_t : t \in I\}$. Then (X, ζ) is a generalized fuzzy topological space. Let $Y = I(I)$ be the fuzzy unit interval under the subspace topology of fuzzy real line. Consider the identity mapping $i : X \to Y$. Then i is both g.f.l.s.c and g.f.u.s.c but is not generalized fuzzy continuous as $i^{-1}(R_a \wedge L_b) \notin \zeta$ for $a < b, a, b \in (0, 1)$.

In the following, we show that extremally disconnected generalized fuzzy topological spaces are rich sources of g.f.l.s.c and g.f.u.s.c mappings. Here, we prove a result which resembles Hutton Uryshon lemma for normal fuzzy spaces.

Theorem 4.3. *Let X be an extremally disconnected generalized fuzzy topological space. Then for each generalized open fuzzy set G and generalized closed fuzzy set F such that $G \subseteq F$, there exists a mapping $f : X \to I(I)$ such that*

(i) *for every $x \in X, G(x) \le f(x)(1-) \le f(x)(0+) \le F(x)$.*
(ii) *f is both g.f.u.s.c and g.f.l.s.c.*

Proof. Let $t \in (0, 1)$. Since $G \le F$, by Theorem 3.5, there exist, a generalized open fuzzy set, say G_t and a generalized closed fuzzy set, say F_t such that $G \le F_t \le G_t \le F$. Again, let $s < t < s_1$ where $s_1 \in (0, 1)$. By the same reasoning, there exists generalized open fuzzy sets G_s, G_{s_1} and generalized closed fuzzy sets F_s, F_{s_1} such that $G \le F_{s_1} \le G_{s_1} \le F_t$ and $G_t \le F_s \le G_s \le F$ as $G \le F_t$ and $G_t \le F$. Continuing this process, for each $t \in (0, 1)$, there exists a fuzzy set V_t such that $G \le V_t \le F$ and for $s > t$, we have $G \le V_s \le \mathrm{cl}(V_s) \le \mathrm{int}(V_t) \le V_t \le F$.

Define $f : X \to I(I)$ by $f(x)(t) = V_t(x)$.
Then,

$$f(x)(1-) = \underset{s<1}{\wedge}(f(x))(s) = \underset{s<1}{\wedge} V_s(x) \geq G(x)$$

and

$$f(x)(0+) = \underset{s>0}{\vee}(f(x)(s)) = \underset{s>0}{\vee} V_s(x) \leq F(x).$$

Thus, $G(x) \leq f(x)(1-) \leq f(x)(0+) \leq F(x)$.
Consider,

$$f^{-1}(R_t)(x) = R_t(f(x)) = f(x)(t+) = \underset{s>t}{\vee} f(x)(s) = \underset{s>t}{\vee} V_s(x).$$

We first show that $\underset{s>t}{\vee} V_s(x) = \underset{s>t}{\vee} int\ (V_s(x))$ so that $\underset{s>t}{\vee} V_s(x)$ is open.

This will establish that $f^{-1}(R_t)$ is open. Let $y_\alpha \leq \underset{s>t}{\vee} V_s(x)$.

(i) $y_\alpha \leq V_s$ for some $s > t$.
(ii) $y_\alpha \not\leq V_s$ for any $s > t$.

In first case, we proceed as follows:
If $y_\alpha \leq V_s$ for some $s > t$. Now, there exists s_0 such that $s > s_0 > t$, so that $V_s \leq \text{cl}(V_s) \leq \text{int}(V_{s_0}) \leq V_{s_0}$.

Thus, $y_\alpha \leq V_s$ implies $y_\alpha \leq \text{int}(V_{s_0})$ for some $s_0 > s$. Then, we have $\underset{s>t}{\vee} V_s(x) \leq \underset{s>t}{\vee} \text{int}(V_s(x))$, whence $\underset{s>t}{\vee} V_s(x) = \underset{s>t}{\vee} \text{int}(V_s(x))$.

In case, $y_\alpha \not\leq V_s(x)$, for any $s > t$, then we proceed as follows:
for $n \in N$,

$$y_{\alpha-\frac{1}{n}} < \underset{s>t}{\vee} V_s(x), (\text{assume,} \alpha - 1/n > 0, \text{without any loss of generality}).$$

$$\Rightarrow \quad y_{\alpha-\frac{1}{n}} < V_{s_n}(x) \quad \text{for some} \quad s_n > t$$

$$\Rightarrow \quad y_{\alpha-\frac{1}{n}} \leq \text{int}(V_{s_n^0}) \quad \text{for some} \quad s_n > s_n^0$$

$$\Rightarrow \quad \vee y_{\alpha-\frac{1}{n}} \leq \underset{s>t}{\vee} \text{int}(V_s(x))$$

$$\Rightarrow \quad y_\alpha \leq \underset{s>t}{\vee} \text{int}(V_s(x)).$$

Thus, we have $f^{-1}(R_t)(x) = \underset{x>t}{\vee} \text{int}(V_s)$ and hence is a generalized open fuzzy set.

Moreover, similarly, we can show

$$f^{-1}(L_t^c)(x) = L_t^c(f(x)) = f(x)(t-) = \underset{s<t}{\wedge} f(x)(s) = \underset{s<t}{\wedge} V_s(x)$$

We first show that $\underset{s<t}{\wedge} V_s(x) = \underset{s<t}{\wedge} \text{cl}(V_s)(x)$, so that $\underset{s<t}{\wedge} \text{cl}(V_s)(x)$ is closed. This will establish $f^{-1}(L_t^c)(x)$ is closed.

Now, $\underset{s<t}{\wedge} V_s(x) \leq \text{cl}(V_s)(x)$, is obvious.

Let $y_\alpha \leq \wedge\, \text{cl}(V_s)(x)$ for all $s > t$ but $y_\alpha \not\leq V_{s_0}(x)$ for some $s_0 < t$.
This implies that $y_\alpha \leq \text{cl}(V_s)$ for all $s < t$ but $y_\alpha \not\leq V_{s_0}$ for some $s_0 < t$.
Choose some t_0 such that $s_0 < t_0 < t$, then
$V_{t_0} \leq \text{cl}(V_{t_0}) \leq \text{int}(V_{s_0}) \leq V_{s_0}$
Then, $y_\alpha \leq \text{cl}(V_{t_0})$ but $y_\alpha \not\leq V_{s_0}$, which is a contradiction.

Therefore, $f^{-1}(L_t^c)(x) = \bigwedge_{s<t} \mathrm{cl}(V_s)(x)$ and hence is a generalized closed fuzzy set. □

Remark 4.4. As a particular case of the above result one may realize Theorem 2.6 of [13] for crisp topological spaces.

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