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Lattice and topological structures of soft sets over the power set of a universe

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Abstract. In this paper, we introduce soft sets over 2^U where 2^U is the power set of the universe U, propose some operations on soft sets over 2^U and investigate some types of soft sets over 2^U such as keeping intersection and keeping union. We obtain lattice and topological structures of soft sets over 2^U. We consider soft rough approximations and soft rough sets, and obtain structures of soft rough sets.

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1. Introduction

Most of traditional methods for formal modeling, reasoning, and computing are crisp, deterministic, and precise in character. However, Many practical problems within fields such as economics, engineering, environmental science, medical science and social sciences involve data that contain uncertainties. We cannot use traditional methods because of various types of uncertainties present in these problems.

There are several theories: probability theory, theory of fuzzy sets [21] and theory of rough sets [18], which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. For example, theory of probabilities can deal only with stochastically stable phenomena. To overcome these kinds of difficulties, Molodtsov [16] proposed a completely new approach, which is called soft set theory, for modeling uncertainty.

Recently there has been a rapid growth in soft set theory and its applications. Maji et al. [14, 15] defined several operations on soft sets, made a theoretical study on soft set theory and defined fuzzy soft sets by combining soft sets with fuzzy sets. Aktas et al. [2] introduced the concept of soft groups. Jun [8, 9, 10] applied soft set theory to BCK/BCI-algebras. Jiang et al. [11] extended soft sets with description logics. Roy et al. [19] discussed score value as the evaluation basis to finding an optimal choice object in fuzzy soft sets. Cağman et al [3, 4] presented soft matrix theory and uni-int decision making approach. Feng et al. [5, 6] and Ali [1] investigated the relationship among soft sets, rough sets and fuzzy sets. Ge et al.[7] discussed the relationship between soft sets and topological spaces. Tanay et al. [20] investigated the topological structure of fuzzy soft sets. Li et al. [12, 13] considered topological and lattice structures of intuitionistic fuzzy soft sets. Majumdaret al. [17] explored softness of soft sets.

The purpose of this paper is to investigate a soft set over 2^U and give their lattice and topological.

2. Preliminaries

Throughout this paper, U denotes initial universe, E denotes the set of all possible parameters and 2^U denotes the power set of U. For $\mathcal{A}, \mathcal{B} \subseteq 2^U$, denote

$$\mathcal{A}^* = \bigcup_{A \in \mathcal{A}} A,$$

$$\mathcal{A} \cap \mathcal{B} = \{M : M \in \mathcal{A} \text{ and } M \in \mathcal{B}\},$$

$$\mathcal{A} \cup \mathcal{B} = \{M : M \in \mathcal{A} \text{ or } M \in \mathcal{B}\},$$

$$\mathcal{A} \wedge \mathcal{B} = \{A \cap B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\},$$

$$\mathcal{A} \vee \mathcal{B} = \{A \cup B : A \in \mathcal{A} \text{ and } B \in \mathcal{B}\},$$

$$\mathcal{A} - \mathcal{B} = \{M : M \in \mathcal{A} \text{ and } M \notin \mathcal{B}\},\ \mathcal{A}^c = 2^U - \mathcal{A}.$$

In this paper, we only consider the case where U and E are both nonempty finite sets.

2.1. Rough sets.

Rough sets. Rough set theory was initiated by Pawlak [18] for dealing with vagueness and granularity in information systems.

Let R be an equivalence relation on U. The pair (U, R) is called a approximation space. The equivalence relation R is often called an indiscernibility relation. Using the indiscernibility relation R, one can define the following two rough approximations:

$$\underline{R}(X) = \{x \in U : [x]_R \subseteq X\},$$

$$\overline{R}(X) = \{x \in U : [x]_R \cap X \neq \varnothing\}.$$

$\underline{R}(X)$ and $\overline{R}(X)$ called the lower approximation and the upper approximation of X, respectively. In general, we refer to $\underline{R}(X)$ and $\overline{R}(X)$ as rough approximations of X.

The boundary region of X, defined by the difference between these rough approximations, that is $Bnd_R(X) = \overline{R}(X) - \underline{R}(X)$. It can easily be seen that $\underline{R}(X) \subseteq X \subseteq \overline{R}(X)$.

A set is rough if its boundary region is not empty; otherwise, the set is crisp. Thus, X is rough if $\underline{R}(X) \neq \overline{R}(X)$.

TABLE 1. Tabular representation of the soft set f_A

	h_1	h_2	h_3	h_4	h_5	h_6
e_1	1	0	1	0	0	0
e_2	1	0	1	0	0	1
e_3	1	0	1	1	1	0
e_4	1	1	1	0	0	0

2.2. Soft sets over U.

Definition 2.1 ([16]). Let $A \subseteq E$. A pair (f, A) is called a soft set over U, if f is a mapping given by $f : A \to 2^U$. We denote (f, A) by f_A.

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $e \in A$, $f(e)$ may be considered as the set of e-approximate elements of f_A.

To illustrate this idea, let us consider the following example.

Example 2.2. Let $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ be a set of houses under consideration, where $A = \{e_1, e_2, e_3, e_4\}$ is a set of parameters for selection of the house. Let e_1 stands for expensive houses, e_2 stands for wooden houses, e_3 stands for houses located in green surroundings, e_4 stands for houses located in the urban area, e_5 stands for the low cost houses.

We define f_A as follows:

$$f(e_1) = \{h_1, h_3\},\ \ f(e_2) = \{h_1, h_3, h_6\},\ f(e_3) = \{h_1, h_3, h_4, h_5\},\ f(e_4) = \{h_1, h_2, h_3\}.$$

f_A can be described as the following Table 1. If $h_i \in f(a_j)$, then $h_{ij} = 1$; otherwise $h_{ij} = 0$, where h_{ij} are the entries in Table 1.

3. SOFT SETS OVER 2^U

3.1. The concept of soft sets over 2^U.

Definition 3.1. Let $A \subseteq E$. A pair (σ, A) is called a soft set over 2^U, if σ is a mapping given by $\sigma : A \to 2^{2^U}$. We denote (σ, A) by σ_A.

To illustrate the background of Definition 3.1, let us consider the following example.

Example 3.2. In Example 2.2, it is easy to see from Table 1 that parameter e_i (i = 1, 2, 3, 4) induces an equivalence relation on U, and we denote it by σ_i (i = 1, 2, 3, 4). Thus, we get the equivalence classes as follows:

for σ_1 the equivalence classes are $\{h_1, h_3\}$, $\{h_2, h_4, h_5, h_6\}$,
for σ_2 the equivalence classes are $\{h_1, h_3, h_6\}$, $\{h_2, h_4, h_5\}$,
for σ_3 the equivalence classes are $\{h_1, h_3, h_4, h_5\}$, $\{h_2, h_6\}$,
for σ_4 the equivalence classes are $\{h_1, h_2, h_3\}$, $\{h_4, h_5, h_6\}$.

Put

$$\sigma(e_1) = \{\{h_1, h_3\}, \{h_2, h_4, h_5, h_6\}\}, \quad \sigma(e_2) = \{\{h_1, h_3, h_6\}, \{h_2, h_4, h_5\}\},$$
$$\sigma(e_3) = \{\{h_1, h_3, h_4, h_5\}, \{h_2, h_6\}\}, \quad \sigma(e_4) = \{\{h_1, h_2, h_3\}, \{h_4, h_5, h_6\}\}.$$

Then we obtain a soft set σ_A over 2^U. That is, a soft set f_A over U induces a soft set σ_A over 2^U.

Proposition 3.3. *Every soft set f_A over U may be considered as a soft set σ_A over 2^U.*

Proof. Let f_A be a soft set over U. For $a \in A$, put $\sigma(a) = \{f(a)\}$. Then σ_A is a soft set over 2^U. Thus f_A may be considered as a soft set σ_A over 2^U. □

Definition 3.4. Let $A, B \subseteq E$ and let σ_A and δ_B be two soft sets over 2^U.

(1) σ_A and δ_B are called soft equal, if $A = B$ and $\sigma(e) = \delta(e)$ for each $e \in A$. We write $\sigma_A = \delta_B$.

(2) σ_A is called a soft subset of δ_B, if $A \subseteq B$ and $\sigma(e) = \delta(e)$ for each $e \in A$. We write $\sigma_A \mathrel{\widetilde{\subset}} \delta_B$.

Obviously, $\sigma_A = \delta_B$ if and only if $\sigma_A \mathrel{\widetilde{\subset}} \delta_B$ and $\delta_B \mathrel{\widetilde{\supset}} \sigma_A$.

3.2. **Some types of soft sets over 2^U.**

Definition 3.5. Let σ_A be a soft set over 2^U.

(1) σ_A is called full, if $\bigcup\limits_{a\in A} \sigma(a)^* = U$.

(2) σ_A is called keeping intersection, if for any $a, b \in A$, $M \in \sigma(a)$ and $N \in \sigma(b)$, there exist $c \in A$ and $Q \in \sigma(c)$ such that $M \cap N = Q$.

(3) σ_A is called keeping union, if for any $a, b \in A$, $M \in \sigma(a)$ and $N \in \sigma(b)$, there exist $c \in A$ and $Q \in \sigma(c)$ such that $M \cup N = Q$.

(4) σ_A is called uniform keeping intersection, if for any $a, b \in A$, there exists $c \in A$ such that $\sigma(a) \wedge \sigma(b) \subseteq \sigma(c)$.

(5) σ_A is called uniform keeping union, if for any $a, b \in A$, there exists $c \in A$ such that $\sigma(a) \vee \sigma(b) \subseteq \sigma(c)$.

Obviously,

$$\sigma_A \text{ is uniform keeping intersection} \Longrightarrow \sigma_A \text{ is keeping intersection},$$

$$\sigma_A \text{ is uniform keeping union} \Longrightarrow \sigma_A \text{ is keeping union}.$$

Example 3.6. Let $U = \{h_1, h_2, h_3, h_4, h_5\}$, let $A = \{a_1, a_2, a_3, a_4\}$ and let σ_A be a soft set over 2^U, defined as follows:

$\sigma(a_1) = \{\varnothing, \{h_1\}, \{h_1, h_2\}\}$, $\sigma(a_2) = \{\{h_1, h_2\}, \{h_1, h_3\}\}$,
$\sigma(a_3) = \{\{h_1, h_3\}\}$, $\sigma(a_4) = \{\{h_4, h_5\}\}$.

Then σ_A is keeping intersection.

$\sigma(a_2) \wedge \sigma(a_3) = \{\{h_1\}, \{h_1, h_3\}\} \not\subseteq \sigma(a)$ for any $a \in A$. Thus σ_A is not uniform keeping intersection.

This example illustrates that

$$\sigma_A \text{ is keeping intersection} \not\Longrightarrow \sigma_A \text{ is uniform keeping intersection}.$$

Example 3.7. Let $U = \{h_1, h_2, h_3, h_4, h_5\}$, $A = \{a_1, a_2, a_3, a_4\}$ and let σ_A be a soft set over 2^U, defined as follows:

$$\sigma(a_1) = \{\{h_1, h_3, h_4\}, \{h_1, h_2\}\},\ \ \sigma(a_2) = \{\{h_1, h_2\}, \{h_1, h_3\}\},$$

$$\sigma(a_3) = \{\{h_1, h_2, h_3\}\},\ \ \{h_3, h_4\}\},\ \ \sigma(a_4) = \{\{h_1, h_2, h_3, h_4\}\}.$$

Then σ_A is keeping union.

$\sigma(a_1) \vee \sigma(a_2) = \{\{h_1, h_2, h_3, h_4\}, \{h_1, h_3, h_4\}, \{h_1, h_2\}, \{h_1, h_2, h_3\}\} \nsubseteq \sigma(a)$ for any $a \in A$. Thus σ_A is not uniform keeping union.

This example illustrates that

$$\sigma_A \text{ is keeping union} \not\Longrightarrow \sigma_A \text{ is uniform keeping union.}$$

Definition 3.8. Let σ_A be a soft set over 2^U. σ_A is bijective, if σ_A satisfies the following conditions:

(i) For any $a \in A$, $\sigma(a)^* = U$;
(ii) If $M, N \in \sigma(a)$ and $M \neq N$ for any $a \in A$, then $M \cap N = \varnothing$.

In order to elaborate this concept, we consider the following example.

Example 3.9. Let $U = \{h_1, h_2, h_3, h_4, h_5\}$ be a universe consisting of five houses as possible alternatives, and $A = \{a_1, a_2, a_3, a_4\} \subseteq E$ be a set of parameters considered by the decision makers, where

a_1 represents the parameter "beauty", we divided it into three grades: "prettyish", "beautiful" and "wonderful";

a_2 represents the parameter "modernization", we divided it into two grades: "plain" and "modern";

a_3 represents the parameter "price", we divided it into two grades: "cheap" and "expensive";

a_4 represents the parameter "in the green surroundings", we divided it into three grades: "a little green", "green" and "much more green".

Now, we consider a bijective soft set σ_A, which describes the "attractiveness of the houses" that Mr.X is going to buy. In this case, to define the soft set σ_A means to point out classification based on the parameters beauty, modernization and so on. Consider the mapping σ given by classification based on one of the parameters $a_i \in A$. For instance, $\sigma(a_1)$ means the classification based on the parameter a_1. Let $U = \{h_1, h_2, h_3, h_4, h_5\}$, $A = \{a_1, a_2, a_3, a_4\}$ and let σ_A be a bijective soft set over 2^U, defined as follows

$$\sigma(a_1) = \{\{h_1, h_2\}, \{h_3, h_4\}, \{h_5\}\}, \quad \sigma(a_2) = \{\{h_1, h_3\}, \{h_2, h_4, h_5\}\},$$

$$\sigma(a_3) = \{\{h_1, h_2, h_4\}, \{h_3, h_5\}\}, \quad \sigma(a_4) = \{\{h_1, h_2, h_3\}, \{h_4\}, \{h_5\}\}.$$

Then a bijective soft set σ_A is described as the following Table 2.

TABLE 2. Tabular representation of the bijective soft set σ_A

	h_1	h_2	h_3	h_4	h_5
a_1	prettyish	prettyish	beautiful	beautiful	wonderful
a_2	plain	modern	plain	modern	modern
a_3	cheap	cheap	expensive	cheap	expensive
a_4	a little green	a little green	a little green	green	much more green

3.3. **Some operations on soft sets over 2^U.**

Definition 3.10. Let $A, B \subseteq E$ and let σ_A and δ_B be two soft sets over 2^U.
(1) h_C is called the intersection of σ_A and δ_B, if $C = A \cap B$ and $h(e) = \sigma(e) \cap \delta(e)$ for each $e \in C$. We write $\sigma_A \mathbin{\widetilde{\cap}} \delta_B = h_C$.
(2) h_C is called the union of σ_A and δ_B, if $C = A \cup B$ and

$$h(e) = \begin{cases} \sigma(e), & \text{if } \ e \in A - B, \\ \delta(e), & \text{if } \ e \in B - A, \\ \sigma(e) \cup \delta(e), & \text{if } \ e \in A \cap B. \end{cases}$$

We write $\sigma_A \mathbin{\widetilde{\cup}} \delta_B = h_C$.

Example 3.11. Let $U = \{h_1, h_2, h_3\}$, $A = \{a_1, a_2\}$, $B = \{a_2, a_3\}$and let σ_A and δ_B be two soft sets over 2^U, defined as follows:

$$\sigma(a_1) = \{\{h_1, h_2, h_3\}, \{h_1, h_2\}\}, \ \sigma(a_2) = \{\{h_1, h_2\}, \{h_1, h_3\}\}.$$

$$\delta(a_2) = \{\{h_2, h_3\}, \{h_1, h_2\}\}, \ \delta(a_3) = \{\{h_2, h_3\}, \varnothing\}.$$

(1) Put

$$\sigma_A \mathbin{\widetilde{\cap}} \delta_B = h_C.$$

Then $C = A \cap B = \{a_2\}$ and $h(a_2) = \sigma(a_2) \cap \delta(a_2) = \{\{h_1, h_2\}\}$.

(2) Put

$$\sigma_A \mathbin{\widetilde{\cup}} \delta_B = k_D.$$

Then $D = A \cup B = \{a_1, a_2, a_3\}$ and

$k(a_1) = \sigma(a_1) = \{\{h_1, h_2, h_3\}, \{h_1, h_2\}\}$,
$k(a_2) = \sigma(a_2) \cup \delta(a_2) = \{\{h_1, h_2\}, \{h_1, h_3\}, \{h_2, h_3\}\}$,
$k(a_3) = \delta(a_3) = \{\{h_2, h_3\}, \varnothing\}$.

Definition 3.12. Let $A \subseteq E$ and let σ_A be an soft set over 2^U. The complement of σ_A is denoted by $(\sigma_A)^c$ and is defined by ${\sigma^c}_A$ *or* $(\sigma_A)^c = (\sigma^c, A)$, where $\sigma^c : A \to 2^{2^U}$ is a mapping given by $\sigma^c(a) = 2^U - \sigma(a)$ for each $a \in A$.

Example 3.13. In Example 3.11, we obtained ${\sigma^c}_A$ as follows:

$$\sigma^c(a_1) = \{\varnothing, \{h_1\}, \{h_2\}, \{h_3\}, \{h_2, h_3\}, \{h_1, h_3\}\},$$

$$\sigma^c(a_2) = \{\varnothing, \{h_1\}, \{h_2\}, \{h_3\}, \{h_2, h_3\}, \{h_1, h_2, h_3\}\}.$$

Definition 3.14. Let $A, B \subseteq E$ and let σ_A and δ_B be two soft sets over 2^U.
(1) h_C is called the intersection of σ_A and δ_B, if $C = A \cap B$ and $h(e) = \sigma(e) \wedge \delta(e)$ for each $e \in C$. We write $\sigma_A \mathbin{\widetilde{\wedge}} \delta_B = h_C$.
(2) h_C is called the union of σ_A and δ_B, if $C = A \cup B$ and

$$h(e) = \begin{cases} \sigma(e), & \text{if } \ e \in A - B, \\ \delta(e), & \text{if } \ e \in B - A, \\ \sigma(e) \vee \delta(e), & \text{if } \ e \in A \cap B. \end{cases}$$

We write $\sigma_A \mathbin{\widetilde{\vee}} \delta_B = h_C$.

Example 3.15. In Example 3.11, we have

(1) Put

$$\sigma_A \mathbin{\widetilde{\wedge}} \delta_B = h_C.$$

Then $C = A \cap B = \{a_2\}$ and $h(a_2) = \sigma(a_2) \wedge \delta(a_2) = \{\{h_1\}, \{h_2\}, \{h_3\}, \{h_1, h_2\}\}$.

(2) Put

$$\sigma_A \mathbin{\widetilde{\vee}} \delta_B = k_D.$$

Then $D = A \cup B = \{a_1, a_2, a_3\}$ and

$k(a_1) = \sigma(a_1) = \{\{h_1, h_2, h_3\}, \{h_1, h_2\}\}$.
$k(a_2) = \sigma(a_2) \vee \delta(a_2) = \{\{h_1, h_2\}, \{h_1, h_2, h_3\}\}$.
$k(a_3) = \delta(a_3) = \{\{h_2, h_3\}, \varnothing\}$.

Proposition 3.16. *Let $A, B, C \subseteq E$ and let σ_A, δ_B and h_C be three soft sets over 2^U. Then*

(1) $\delta_A \mathbin{\widetilde{\cup}} \delta_A = \delta_A$.
(2) $\delta_A \mathbin{\widetilde{\cup}} \delta_B = \delta_B \mathbin{\widetilde{\cup}} \delta_A$.
(3) $(\delta_A \mathbin{\widetilde{\cup}} \delta_B) \mathbin{\widetilde{\cup}} h_C = \delta_A \mathbin{\widetilde{\cup}} (\delta_B \mathbin{\widetilde{\cup}} h_C)$.
(4) $\delta_A \mathbin{\widetilde{\vee}} \delta_A = \delta_A$.
(5) $\delta_A \mathbin{\widetilde{\vee}} \delta_B = \delta_B \mathbin{\widetilde{\vee}} \delta_A$.
(6) $(\delta_A \mathbin{\widetilde{\vee}} \delta_B) \mathbin{\widetilde{\vee}} h_C = \delta_A \mathbin{\widetilde{\vee}} (\delta_B \mathbin{\widetilde{\vee}} h_C)$.

Proof. (1) and (2) are obvious.

(3) Put

$$(\sigma_A \mathbin{\widetilde{\cup}} \delta_B) \mathbin{\widetilde{\cup}} h_C = k_{A\cup B\cup C}, \quad \sigma_A \mathbin{\widetilde{\cup}} (\delta_B \mathbin{\widetilde{\cup}} h_C) = l_{A\cup B\cup C};$$
$$\sigma_A \mathbin{\widetilde{\cup}} \delta_B = s_{A\cup B}, \quad \delta_B \mathbin{\widetilde{\cup}} h_C = t_{B\cup C}.$$

For any $e \in A \cup B \cup C$, it follows that $e \in A$, or $e \in B$, or $e \in C$.

Case 1 $e \in C$.

a) If $e \notin A$ and $e \notin B$, then $k(e) = h(e) = t(e) = l(e)$.
b) If $e \notin A$ and $e \in B$, then $k(e) = s(e) \cup h(e) = \delta(e) \cup h(e) = t(e) = l(e)$.
c) If $e \in A$ and $e \notin B$, then $k(e) = s(e) \cup h(e) = \sigma(e) \cup h(e) = \sigma(e) \cup t(e) = l(e)$.
d) If $e \in A$ and $e \in B$, $k(e) = s(e) \cup h(e) = (\sigma(e) \cup \delta(e)) \cup h(e) = \sigma(e) \cup (\delta(e) \cup h(e)) = \sigma(e) \cup t(e) = l(e)$.

Case 2 $e \notin C$.

a) If $e \notin A$ and $e \in B$, then $k(e) = s(e) = \delta(e) = t(e) = l(e)$.
b) If $e \in A$ and $e \notin B$, then $k(e) = s(e) = \sigma(e) = t(e) = l(e)$.
c) If $e \in A$ and $e \in B$, then $k(e) = s(e) = \sigma(e) \cup \delta(e) = \sigma(e) \cup t(e) = l(e)$.

Thus $(\sigma_A \mathbin{\widetilde{\cup}} \delta_B) \mathbin{\widetilde{\cup}} h_C = \sigma_A \mathbin{\widetilde{\cup}} (\delta_B \mathbin{\widetilde{\cup}} h_C)$.

(4) and (5) are obvious.

(6) Put

$$(\sigma_A \mathbin{\widetilde{\vee}} \delta_B) \mathbin{\widetilde{\vee}} h_C = k'_{A\cup B\cup C}, \quad \sigma_A \mathbin{\widetilde{\vee}} (\delta_B \mathbin{\widetilde{\vee}} h_C) = l'_{A\cup B\cup C};$$
$$\sigma_A \mathbin{\widetilde{\vee}} \delta_B = s'_{A\cup B}, \quad \delta_B \mathbin{\widetilde{\vee}} h_C = t'_{B\cup C}.$$

For any $e \in A \cup B \cup C$, it follows that $e \in A$, or $e \in B$, or $e \in C$.

Case 1 $e \in C$.

a) If $e \notin A$ and $e \notin B$, then $k'(e) = h(e) = t'(e) = l'(e)$.
b) If $e \notin A$ and $e \in B$, then $k'(e) = s'(e) \vee h(e) = \delta(e) \vee h(e) = t'(e) = l'(e)$.
c) If $e \in A$ and $e \notin B$, then $k'(e) = s'(e) \vee h(e) = \sigma(e) \vee h(e) = \sigma(e) \vee t'(e) = l'(e)$.
d) If $e \in A$ and $e \in B$, then $k'(e) = s'(e) \vee h(e) = (\sigma(e) \vee \delta(e)) \vee h(e) = \sigma(e) \vee (\delta(e) \vee h(e)) = \sigma(e) \vee t'(e) = l'(e)$.

Case 2 $e \notin C$.
a) If $e \notin A$ and $e \in B$, then $k'(e) = s'(e) = \delta(e) = t'(e) = l'(e)$.
b) If $e \in A$ and $e \notin B$, then $k'(e) = s'(e) = \sigma(e) = t'(e) = l'(e)$.
c) If $e \in A$ and $e \in B$, then $k'(e) = s'(e) = \sigma(e) \vee \delta(e) = \sigma(e) \vee t'(e) = l'(e)$.
Thus $(\sigma_A \mathbin{\widetilde{\vee}} \delta_B) \mathbin{\widetilde{\vee}} h_C = \sigma_A \mathbin{\widetilde{\vee}} (\delta_B \mathbin{\widetilde{\vee}} h_C)$. □

Proposition 3.17. *Let $A, B, C \subseteq E$ and let σ_A, δ_B and h_C be three soft sets over 2^U. Then*

(1) $\sigma_A \mathbin{\widetilde{\cap}} \sigma_A = \sigma_A$.
(2) $\sigma_A \mathbin{\widetilde{\cap}} \delta_B = \delta_B \mathbin{\widetilde{\cap}} \sigma_A$.
(3) $(\sigma_A \mathbin{\widetilde{\cap}} \delta_B) \mathbin{\widetilde{\cap}} h_C = \sigma_A \mathbin{\widetilde{\cap}} (\delta_B \mathbin{\widetilde{\cap}} h_C)$.
(4) $\sigma_A \mathbin{\widetilde{\wedge}} \sigma_A = \sigma_A$.
(5) $\sigma_A \mathbin{\widetilde{\wedge}} \delta_B = \delta_B \mathbin{\widetilde{\wedge}} \sigma_A$.
(6) $(\sigma_A \mathbin{\widetilde{\wedge}} \delta_B) \mathbin{\widetilde{\wedge}} h_C = \sigma_A \mathbin{\widetilde{\wedge}} (\delta_B \mathbin{\widetilde{\wedge}} h_C)$.

Proof. (1) and (2) are obvious.

(3) Put

$$(\sigma_A \mathbin{\widetilde{\cap}} \delta_B) \mathbin{\widetilde{\cap}} h_C = k_{A\cap B\cap C},\ \sigma_A \mathbin{\widetilde{\cap}} (\delta_B \mathbin{\widetilde{\cap}} h_C)) = l_{A\cap B\cap C}.$$

For any $e \in A \cap B \cap C$, it follows that $e \in A$, $e \in B$ and $e \in C$. $k(e) = (\sigma(e) \cap \delta(e)) \cap h(e) = \sigma(e) \cap (\delta(e) \cap h(e)) = l(e)$, then $(\sigma_A \mathbin{\widetilde{\cap}} \delta_B) \mathbin{\widetilde{\cap}} h_C = \sigma_A \mathbin{\widetilde{\cap}} (\delta_B \mathbin{\widetilde{\cap}} h_C)$.

(4) and (5) are obvious.

(6) Put

$$(\sigma_A \mathbin{\widetilde{\wedge}} \delta_B) \mathbin{\widetilde{\wedge}} h_C = k'_{A\cap B\cap C},\ \sigma_A \mathbin{\widetilde{\wedge}} (\delta_B \mathbin{\widetilde{\wedge}} h_C)) = l'_{A\cap B\cap C}.$$

For any $e \in A\cap B\cap C$, it follows that $e \in A$, $e \in B$ and $e \in C$. Then $k'(e) = (\sigma(e) \wedge \delta(e)) \wedge h(e) = \sigma(e) \wedge (\delta(e) \wedge h(e)) = l'(e)$. So $(\sigma_A \mathbin{\widetilde{\wedge}} \delta_B) \mathbin{\widetilde{\wedge}} h_C = \sigma_A \mathbin{\widetilde{\wedge}} (\delta_B \mathbin{\widetilde{\wedge}} h_C)$. □

Proposition 3.18. *Let $A, B, C \subseteq E$ and let σ_A, δ_B and h_C be three soft sets over 2^U. Then*

(1) $(\sigma_A \mathbin{\widetilde{\cup}} \delta_B) \mathbin{\widetilde{\cap}} h_C = (\sigma_A \mathbin{\widetilde{\cap}} h_C) \mathbin{\widetilde{\cup}} (\delta_B \mathbin{\widetilde{\cap}} h_C)$.
(2) $(\sigma_A \mathbin{\widetilde{\cap}} \delta_B) \mathbin{\widetilde{\cup}} h_C = (\sigma_A \mathbin{\widetilde{\cup}} h_C) \mathbin{\widetilde{\cap}} (\delta_B \mathbin{\widetilde{\cup}} h_C)$.
(3) $(\sigma_A \mathbin{\widetilde{\vee}} \delta_B) \mathbin{\widetilde{\wedge}} h_C \mathbin{\widetilde{\subseteq}} (\sigma_A \mathbin{\widetilde{\wedge}} h_C) \mathbin{\widetilde{\vee}} (\delta_B \mathbin{\widetilde{\wedge}} h_C)$.
(4) $(\sigma_A \mathbin{\widetilde{\wedge}} \delta_B) \mathbin{\widetilde{\vee}} h_C \mathbin{\widetilde{\subseteq}} (\sigma_A \mathbin{\widetilde{\vee}} h_C) \mathbin{\widetilde{\wedge}} (\delta_B \mathbin{\widetilde{\vee}} h_C)$.

Proof. (1) Put

$$(\sigma_A \mathbin{\widetilde{\cup}} \delta_B) \mathbin{\widetilde{\cap}} h_C = k'_{(A\cup B)\cap C},\ (\sigma_A \mathbin{\widetilde{\cap}} h_C) \cup (\delta_B \mathbin{\widetilde{\cap}} h_C) = l'_{(A\cap C)\cup(B\cap C)}.$$

Obviously, $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. For any $e \in (A \cup B) \cap C$, it follows that $e \in A \cap C$, or $e \in B \cap C$.

1) If $e \notin A \cap C$ and $e \in B \cap C$, then $e \notin A, e \in B$ *and* $e \in C$. So $k'(e) = \delta(e) \cup h(e) = l'(e)$.

2) If $e \in A \cap C$ and $e \notin B \cap C$, then $e \in A, e \notin B$ *and* $e \in C$. So $k'(e) = \sigma(e) \cup h(e) = l'(e)$.
3) If $e \in A \cap C$ and $e \in B \cap C$, then $e \in A, e \in B$ *and* $e \in C$. $k'(e) = (\sigma(e) \cup \delta(e)) \cap h(e) = (\sigma(e) \cap h(e)) \cup (\delta(e) \cap h(e)) = l'(e)$.
Thus

$$(\sigma_A \,\widetilde{\cup}\, \delta_B) \,\widetilde{\cap}\, h_C = (\sigma_A \,\widetilde{\cap}\, h_C) \widetilde{\cup}\, (\delta_B \,\widetilde{\cap}\, h_C).$$

(2) This is similar to the proof of (1).

(3) Put

$$(\sigma_A \,\widetilde{\vee}\, \delta_B) \,\widetilde{\wedge}\, h_C = k_{(A \cup B) \cap C},\ (\sigma_A \,\widetilde{\wedge}\, h_C) \vee (\delta_B \,\widetilde{\wedge}\, h_C) = l_{(A \cap C) \cup (B \cap C)}.$$

Obviously, $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. For any $e \in (A \cup B) \cap C$, it follows that $e \in A \cap C$, or $e \in B \cap C$.
1) If $e \notin A \cap C$ and $e \in B \cap C$, then $e \notin A, e \in B$ *and* $e \in C$. So $k(e) = \delta(e) \vee h(e) = l(e)$.
2) If $e \in A \cap C$ and $e \notin B \cap C$, then $e \in A, e \notin B$ *and* $e \in C$. So $k(e) = \sigma(e) \vee h(e) = l(e)$.
3) If $e \in A \cap C$ and $e \in B \cap C$, then $e \in A, e \in B$ *and* $e \in C$. So $k(e) = (\sigma(e) \vee \delta(e)) \wedge h(e) \subseteq (\sigma(e) \wedge h(e)) \vee (\delta(e) \wedge h(e)) = l(e)$.
Thus

$$(\sigma_A \,\widetilde{\vee}\, \delta_B) \,\widetilde{\wedge}\, h_C \,\widetilde{\subset}\, (\sigma_A \,\widetilde{\wedge}\, h_C) \,\widetilde{\vee}\, (\delta_B \,\widetilde{\wedge}\, h_C).$$

(4) This is similar to the proof of (3). □

Example 3.19. Let $U = \{h_1, h_2, h_3\}$, $A = \{a_1, a_2\}$, $B = \{a_2, a_3\}$, $C = \{a_2, a_4\}$ and let σ_A, δ_B and h_C be three soft sets over 2^U, defined as follows:

$$\sigma(a_1) = \{\{h_1, h_2, h_3\}, \{h_1, h_2\}\},\ \sigma(a_2) = \{\{h_1\}\}.$$
$$\delta(a_2) = \{\varnothing, \{h_2, h_3\}\},\ \delta(a_3) = \{\{h_2, h_3\}, \varnothing\}.$$
$$h(a_2) = \{\{h_1\}, \{h_3\}\},\ h(a_4) = \{\{h_1, h_2\}\}.$$

(1) Put

$$(\sigma_A \,\widetilde{\vee}\, \delta_B) \,\widetilde{\wedge}\, h_C = k_D,\ (\sigma_A \,\widetilde{\wedge}\, h_C) \,\widetilde{\vee}\, (\ \delta_B \,\widetilde{\wedge}\, h_C) = l_N.$$

Then $D = (A \cup B) \cap C = (A \cap C) \cup (B \cap C) = N = \{a_2\}$.

$$\begin{aligned} &k(a_2) = (\sigma(a_2) \vee \delta(a_2)) \wedge h(a_2) = \{\{h_1\}, \varnothing, \{h_3\}\}, \\ &l(a_2) = (\sigma(a_2) \wedge h(a_2)) \vee (\delta(a_2) \wedge h(a_2)) = \{\{h_1\}, \{h_1, h_3\}, \{h_3\}, \varnothing\} \\ &\quad \neq k(a_2). \end{aligned}$$

Thus

$$(\sigma_A \,\widetilde{\vee}\, \delta_B) \,\widetilde{\wedge}\, h_C \ \neq\ (\sigma_A \,\widetilde{\wedge}\, h_C) \,\widetilde{\vee}\, (\ \delta_B \,\widetilde{\wedge}\, h_C).$$

(2) Put

$$(\sigma_A \,\widetilde{\wedge}\, \delta_B) \,\widetilde{\vee}\, h_C = k'_{D'},\ (\sigma_A \,\widetilde{\vee}\, h_C) \,\widetilde{\wedge}\, (\ \delta_B \,\widetilde{\vee}\, h_C) = l'_{N'}.$$

Then $D' = (A \cap B) \cup C = (A \cup C) \cap (B \cup C) = N' = \{a_2, a_4\}$.

$$\begin{aligned} &k'(a_2) = (\sigma(a_2) \wedge \delta(a_2)) \vee h(a_2) = \{\{h_1\}, \{h_3\}\}, \\ &l'(a_2) = (\sigma(a_2) \vee h(a_2)) \wedge (\delta(a_2) \vee h(a_2)) = \{\{h_1\}, \{h_1, h_3\}, \{h_3\}, \varnothing\} \\ &\quad \neq k'(a_2). \end{aligned}$$

$k'(a_4) = h(a_4) = \{\{h_1, h_2\}\}$,
$l'(a_4) = h(a_4) \wedge h(a_4) = h(a_4) = \{\{h_1, h_2\}\} = k'(a_4)$.

Thus
$$(\sigma_A \mathbin{\widetilde{\wedge}} \delta_B) \mathbin{\widetilde{\vee}} h_C \neq (\sigma_A \mathbin{\widetilde{\vee}} h_C) \mathbin{\widetilde{\wedge}} (\delta_B \mathbin{\widetilde{\vee}} h_C).$$

Proposition 3.20. *Let $A \subseteq E$ and let σ_A, δ_A be two soft sets over 2^U. Then*

(1) $((\sigma_A)^c)^c = \sigma_A$.
(2) $\sigma_A \mathbin{\widetilde{\cup}} (\sigma_A)^c = U_A$.
(3) $\sigma_A \mathbin{\widetilde{\cap}} (\sigma_A)^c = \varnothing_A$.
(4) $(\sigma_A \mathbin{\widetilde{\cup}} \delta_A)^c = (\sigma_A)^c \mathbin{\widetilde{\cap}} (\delta_A)^c$.
(5) $(\sigma_A \mathbin{\widetilde{\cap}} \delta_A)^c = (\sigma_A)^c \mathbin{\widetilde{\cup}} (\delta_A)^c$.
(6) $(\sigma_A \mathbin{\widetilde{\vee}} \delta_A)^c \mathbin{\widetilde{\supseteq}} (\sigma_A)^c \mathbin{\widetilde{\wedge}} (\delta_A)^c$.
(7) $(\sigma_A \mathbin{\widetilde{\wedge}} \delta_A)^c \mathbin{\widetilde{\supseteq}} (\sigma_A)^c \mathbin{\widetilde{\vee}} (\delta_A)^c$.

Proof. (1) For any $e \in A$, $(\sigma(e)^c)^c = \sigma(e)$. That is, $((\sigma_A)^c)^c = \sigma_A$.

(2) Put
$$\sigma_A \mathbin{\widetilde{\cup}} (\sigma_A)^c = h_A.$$
For any $e \in A$, $h(e) = \sigma(e) \cup f^c(e) = 2^U$. That is, $\sigma_A \mathbin{\widetilde{\cup}} (\sigma_A)^c = U_A$.

(3) This is similar to the proof of (2).

(4) Put
$$(\sigma_A \mathbin{\widetilde{\cup}} \delta_A)^c = h_A, \ (\sigma_A)^c \mathbin{\widetilde{\cap}} (\delta_A)^c = l_A.$$
For any $e \in A$, $h(e) = (\sigma(e) \cup \delta(e))^c = \sigma(e)^c \cap \delta(e)^c = l(e)$.
That is,
$$(\sigma_A \mathbin{\widetilde{\cup}} \delta_A)^c = (\sigma_A)^c \mathbin{\widetilde{\cap}} (\delta_A)^c.$$

(5) This is similar to the proof of (4).

(6) Put
$$(\sigma_A \mathbin{\widetilde{\vee}} \delta_A)^c = h'_A, \ (\sigma_A)^c \mathbin{\widetilde{\wedge}} (\delta_A)^c = l'_A.$$
For any $e \in A$, $h'(e) = (\sigma(e) \vee \delta(e))^c$, $l'(e) = \sigma(e)^c \wedge \delta(e)^c$.
Then $h'(e) = (\sigma(e) \vee \delta(e))^c \supseteq \sigma(e)^c \wedge \delta(e)^c = l'(e)$.
That is,
$$(\sigma_A \mathbin{\widetilde{\vee}} \delta_A)^c \mathbin{\widetilde{\supseteq}} (\sigma_A)^c \mathbin{\widetilde{\wedge}} (\delta_A)^c.$$

(7) This is similar to the proof of (6). □

Example 3.21. Let $U = \{h_1, h_2, h_3\}$, $A = \{a_1, a_2\}$ and let σ_A and δ_A be two soft sets over 2^U, defined as follows:
$$\sigma(a_1) = \{\{h_1, h_2, h_3\}, \{h_1, h_2\}\}, \ \sigma(a_2) = \{\{h_1, h_2\}, \{h_1, h_3\}\}.$$
$$\delta(a_1) = \{\{h_1, h_3\}, \{h_1, h_2\}\}, \ \delta(a_2) = \{\{h_2, h_3\}, \varnothing\}.$$

(1)
$$\sigma(a_1)^c = \{\varnothing, \{h_1\}, \{h_2\}, \{h_3\}, \{h_2, h_3\}, \{h_1, h_3\}\},$$
$$\sigma(a_2)^c = \{\varnothing, \{h_1\}, \{h_2\}, \{h_3\}, \{h_2, h_3\}, \{h_1, h_2, h_3\}\},$$

so
$$\sigma(a_1) \vee \sigma(a_1)^c = \{\{h_1, h_2\}, \{h_1, h_2, h_3\}\} \neq 2^U,$$

$$\sigma(a_2) \vee \sigma(a_2)^c = \{\{h_1, h_2\}, \{h_1, h_3\}, \{h_1, h_2, h_3\}\} \neq 2^U.$$

Thus

$$\sigma_A \ \widetilde{\vee}\ (\sigma_A)^c \neq U_A.$$

(2)

$$\sigma(a_1) \wedge \sigma(a_1)^c = \{\varnothing, \{h_1\}, \{h_2\}, \{h_3\}, \{h_2, h_3\}, \{h_1, h_3\}\} \neq \{\varnothing\},$$
$$\sigma(a_2) \wedge \sigma(a_2)^c = \{\varnothing, \{h_1\}, \{h_2\}, \{h_3\}, \{h_1, h_2\}, \{h_1, h_3\}\} \neq \{\varnothing\}.$$

Thus

$$\sigma_A \ \widetilde{\wedge}\ (\sigma_A)^c \neq \varnothing_A.$$

(3) Put

$$(\sigma_A \ \widetilde{\vee}\ \delta_A)^c = h_A, \quad (\sigma_A)^c \ \widetilde{\wedge}\ (\delta_A)^c = l_A.$$
$$h(a_1) = \{\varnothing, \{h_1\}, \{h_2\}, \{h_3\}, \{h_1, h_3\}, \{h_2, h_3\}\},$$
$$h(a_2) = \{\varnothing, \{h_1\}, \{h_2\}, \{h_3\}, \{h_2, h_3\}\}.$$
$$l(a_1) = \{\varnothing, \{h_1\}, \{h_2\}, \{h_3\}, \{h_1, h_2\}, \{h_2, h_3\}\} \neq h(a_1),$$
$$l(a_2) = \{\varnothing, \{h_1\}, \{h_2\}, \{h_3\}, \{h_1, h_2\}, \{h_1, h_3\}, \{h_2, h_3\}, \{h_1, h_2, h_3\}\} \neq h(a_2).$$

Thus

$$(\sigma_A \ \widetilde{\vee}\ \delta_A)^c \neq\ (\sigma_A)^c \ \widetilde{\wedge}\ (\delta_A)^c.$$

(4) Put

$$(\sigma_A \ \widetilde{\wedge}\ \delta_A)^c = k_A, \quad (\sigma_A)^c \ \widetilde{\vee}\ (\delta_A)^c = t_A.$$

Then

$$k(a_1) = \{\varnothing, \{h_2\}, \{h_3\}, \{h_2, h_3\}, \{h_1, h_2, h_3\}\},$$
$$k(a_2) = \{\{h_1\}, \{h_1, h_2\}, \{h_1, h_3\}, \{h_2, h_3\}, \{h_1, h_2, h_3\}\}.$$
$$t(a_1) = \{\varnothing, \{h_1\}, \{h_2\}, \{h_3\}, \{h_1, h_2\}, \{h_1, h_3\}, \{h_2, h_3\}, \{h_1, h_2, h_3\}\} \neq k(a_1),$$
$$t(a_2) = \{\{h_1\}, \{h_2\}, \{h_3\}, \{h_1, h_2\}, \{h_1, h_3\}, \{h_2, h_3\}, \{h_1, h_2, h_3\}\} \neq k(a_2).$$

Thus

$$(\sigma_A \ \widetilde{\vee}\ \delta_A)^c \neq\ (\sigma_A)^c \ \widetilde{\wedge}\ (\delta_A)^c.$$

4. Lattice structures of soft sets over 2^U

We denote

$$S(U, E) = \{\sigma_A : A \subseteq E \text{ and } \sigma_A \text{ is a soft set over } 2^U\},$$
$$S_1(U, E) = \{\sigma_E :\ \sigma_E \text{ is a soft set over } 2^U\}.$$

Obviously,

$$S_1(U, E) \subseteq S(U, E).$$

Theorem 4.1. *For any $\sigma_A, \delta_B \in S(U, E)$, define*

$$\sigma_A \leq \delta_B \ \Leftrightarrow\ \sigma_A \ \widetilde{\subset}\ \delta_B,$$

$$\sigma_A \vee \delta_B =\ \sigma_A \ \widetilde{\cup}\ \delta_B,$$

$$\sigma_A \wedge \delta_B =\ \sigma_A \ \widetilde{\cap}\ \delta_B.$$

Then $(S(U, E),\ \widetilde{\cup},\ \widetilde{\cap})$ *is a lattice.*

Proof. This is obvious. □

Theorem 4.2. *For any $\sigma_A, \delta_B \in S(U,E)$, define*

$$\sigma_A \le \delta_B \;\Leftrightarrow\; \sigma_A \mathrel{\widetilde{\subset}} \delta_B,$$

$$\sigma_A \vee \delta_B = \;\sigma_A \mathbin{\widetilde{\vee}} \delta_B,$$

$$\sigma_A \wedge \delta_B = \;\sigma_A \mathbin{\widetilde{\wedge}} \delta_B.$$

Then $(S(U,E),\ \widetilde{\vee},\ \widetilde{\wedge})$ *is a lattice.*

Proof. This is obvious. □

Theorem 4.3. *For any $\sigma_E, g_E \in S_1(U,E)$, define*

$$\sigma_E \le g_E \;\Leftrightarrow\; \sigma_E \mathrel{\widetilde{\subset}} g_E,$$

$$\sigma_E \vee g_E = \;\sigma_E \mathbin{\widetilde{\cup}} g_E,$$

$$\sigma_E \wedge g_E = \;\sigma_E \mathbin{\widetilde{\cap}} g_E.$$

Then $(S_1(U,E),\ \widetilde{\cup},\ \widetilde{\cap})$ *is a Boolean lattice.*

Proof. Denote $\sum_1 = S_1(U,E)$. It is easily proved that

$$0_{\sum_1} = \varnothing_E \textit{ and } 1_{\sum_1} = U_E.$$

By Proposition 3.18, $S_1(U,E)$ is a distributive lattice with $1_{\sum_1}$ and $0_{\sum_1}$.
For any $\sigma_E \in S_1(U,E)$, $(\sigma_E)' = f_E^c$.
Hence $(S_1(U,E),\ \widetilde{\cup},\ \widetilde{\cap})$ is a Boolean lattice. □

Theorem 4.4. *For any $\sigma_E, \delta_E \in S_1(U,E)$, define*

$$\sigma_E \le \delta_E \;\Leftrightarrow\; \sigma_E \mathrel{\widetilde{\subset}} \delta_E,$$

$$\sigma_E \vee \delta_E = \;\sigma_E \mathbin{\widetilde{\vee}} \delta_E,$$

$$\sigma_E \wedge \delta_E = \;\sigma_E \mathbin{\widetilde{\wedge}} \delta_E.$$

Then $(S_1(U,E), \widetilde{\vee}, \widetilde{\wedge})$ *is a lattice.*

Proof. This is obvious. □

Corollary 4.5. (1) $(S_1(U,E),\ \widetilde{\cup},\ \widetilde{\cap})$ *is a sublattice of* $(S(U,E),\ \widetilde{\cup},\ \widetilde{\cap})$
(2) $(S_1(U,E),\ \widetilde{\vee},\ \widetilde{\wedge})$ *is a sublattice of* $(S(U,E),\ \widetilde{\vee},\ \widetilde{\wedge})$.

5. Soft rough approximations and soft rough sets

5.1. Soft rough approximations.

Definition 5.1. Let σ_A be a soft set over 2^U. Then the pair $P = (U, \sigma_A)$ is called a soft approximation space. Based on the soft approximation space P, we define a pair of operations $\underline{apr}_P$, $\overline{apr}_P$: $2^U \longrightarrow 2^U$ as follows:

$$\underline{apr}_P(X) = \{x \in U : \exists\ a \in A,\ M \in \sigma(a)\ s.t.\ x \in M \in \sigma(a)\ and\ M \subseteq X\},$$

$$\overline{apr}_P(X) = \{x \in U : \exists\ a \in A,\ M \in \sigma(a)\ s.t.\ x \in M \in \sigma(a)\ and\ M \cap X \neq \varnothing\}.$$

$\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ are called the soft P-lower approximation and the soft P-upper approximation of X, respectively.

In general, we refer to $\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ as soft rough approximations of X with respect to P.

Proposition 5.2. *Let σ_A be a soft set over 2^U and let $P = (U, \sigma_A)$ be a soft approximation space. Then for any $X, Y \in 2^U$,*

(1) $\underline{apr}_P(X) = \bigcup\{M : a \in A, M \in \sigma(a)\ and\ M \subseteq X\} \subseteq X;$

$\overline{apr}_P(X) = \bigcup\{M : a \in A, M \in \sigma(a)\ and\ M \cap X \neq \varnothing\}.$

(2) $\underline{apr}_P(\varnothing) = \overline{apr}_P(\varnothing) = \varnothing;\quad \underline{apr}_P(U) = \overline{apr}_P(U) = \bigcup\limits_{a \in A} \sigma(a)^*.$

(3) $X \subseteq Y \Rightarrow \underline{apr}_P(X) \subseteq \underline{apr}_P(Y);\quad X \subseteq Y \Rightarrow \overline{apr}_P(X) \subseteq \overline{apr}_P(Y).$

(4) $\overline{apr}_P(X \cup Y) = \overline{apr}_P(X) \cup \overline{apr}_P(Y).$

(5) $\underline{apr}_P(\underline{apr}_P(X)) = \underline{apr}_P(X);\quad \underline{apr}_P(\overline{apr}_P(X)) = \overline{apr}_P(X).$

Proof. (1) and (2) are obvious.

(3) (i) Suppose that $\underline{apr}_P(X) - \underline{apr}_P(Y) \neq \varnothing$. Pick

$$x \in \underline{apr}_P(X) - \underline{apr}_P(Y).$$

Then there exist $a \in A$ and $M \in \sigma(a)$ such that $x \in M \in \sigma(a)$ and $M \subseteq X$. Since $X \subseteq Y$, we have $M \subseteq Y$ and $x \in \underline{apr}_P(Y)$. This is a contradiction.

Thus

$$\underline{apr}_P(X) \subseteq \underline{apr}_P(Y).$$

(ii) Suppose that $\overline{apr}_P(X) - \overline{apr}_P(Y) \neq \varnothing$. Pick

$$x \in \overline{apr}_P(X) - \overline{apr}_P(Y).$$

Then there exist $a \in A$ and $M \in \sigma(a)$ such that $x \in M \in \sigma(a)$ and $M \cap X \neq \varnothing$. Since $X \subseteq Y$, $M \cap Y \neq \varnothing$. This implies that $x \in \overline{apr}_P(Y)$. This is a contradiction.

Thus

$$\overline{apr}_P(X) \subseteq \overline{apr}_P(Y).$$

(4) By (3),

$$\overline{apr}_P(X \cup Y) \supseteq \overline{apr}_P(X) \cup \overline{apr}_P(Y).$$

Suppose that $\overline{apr}_P(X \cup Y) - (\overline{apr}_P(X) \cup \overline{apr}_P(Y)) \neq \varnothing$. Pick $x \in \overline{apr}_P(X \cup Y) - (\overline{apr}_P(X) \cup \overline{apr}_P(Y))$. Then there exist $a \in A$ and $M \in \sigma(a)$ such that $x \in M \in \sigma(a)$

and $M \cap (X \cup Y) \neq \varnothing$. This implies that $M \cap X \neq \varnothing$ or $M \cap Y \neq \varnothing$. So $x \in \overline{apr}_P(X)$ or $x \in \overline{apr}_P(Y)$, then $x \in \overline{apr}_P(X) \cup \overline{apr}_P(Y)$. This is a contradiction. Thus

$$\overline{apr}_P(X \cup Y) \subseteq \overline{apr}_P(X) \cup \overline{apr}_P(Y).$$

Hence

$$\overline{apr}_P(X \cup Y) = \overline{apr}_P(X) \cup \overline{apr}_P(Y).$$

(5) (i) By (1), $\underline{apr}_P(X) \subseteq X$. By (3),

$$\underline{apr}_P(\underline{apr}_P(X)) \subseteq \underline{apr}_P(X).$$

Suppose that $x \in \underline{apr}_P(X)$. Then there exist $a \in A$ and $N \in \sigma(a)$ such that $x \in N \in \sigma(a)$ and $N \subseteq X$. Since $\underline{apr}_P(X) = \bigcup\{M : a \in A, M \in \sigma(a) \textit{ and } M \subseteq X\}$, we have $x \in N \subseteq \underline{apr}_P(X)$. This implies that $x \in \underline{apr}_P(\underline{apr}_P(X))$. Thus

$$\underline{apr}_P(\underline{apr}_P(X)) \supseteq \underline{apr}_P(X).$$

Hence

$$\underline{apr}_P(\underline{apr}_P(X)) = \underline{apr}_P(X).$$

(ii) Suppose that $x \in \overline{apr}_P(X)$, then there exist $a \in A$ and $N \in \sigma(a)$ such that $x \in N \in \sigma(a)$ and $N \cap X \neq \varnothing$. Since $\overline{apr}_P(X) = \bigcup\{M : a \in A, M \in \sigma(a) \textit{ and } M \cap X \neq \varnothing\}$, we have $x \in N \subseteq \overline{apr}_P(X)$. This implies that $x \in \underline{apr}_P(\overline{apr}_P(X))$. Thus

$$\underline{apr}_P(\overline{apr}_P(X)) \supseteq \overline{apr}_P(X).$$

By (3),

$$\underline{apr}_P(\overline{apr}_P(X)) \subseteq \overline{apr}_P(X).$$

Hence

$$\underline{apr}_P(\underline{apr}_P(X)) = \underline{apr}_P(X).$$

□

Proposition 5.3. *Let σ_A be a soft set over 2^U and let $P = (U, \sigma_A)$ be a soft approximation space. Then*

(1) *If σ_A is full, then $\underline{apr}_P(X) \subseteq X \subseteq \overline{apr}_P(X)$ for any $X \in 2^U$.*

(2) *If σ_A is full, then $\underline{apr}_P(U) = \overline{apr}_P(U) = U$.*

(3) *If σ_A is keeping intersection, then $\underline{apr}_P(X \cap Y) = \underline{apr}_P(X) \cap \underline{apr}_P(Y)$ for any $X, Y \in 2^U$.*

(4) *If σ_A is full and keeping union, then $\overline{apr}_P(X) = U$ for any $X \in 2^U \backslash \varnothing$.*

Proof. (1) By Proposition 5.2, $\underline{apr}_P(X) \subseteq X$. Suppose that $X - \overline{apr}_P(X) \neq \varnothing$. Pick

$$x \in X - \overline{apr}_P(X) \neq \varnothing.$$

Since σ_A is full, $\bigcup\limits_{a \in A} \sigma(a)^* = U$. So $x \in M \in \sigma(a)$ for some $a \in A$. Note that $x \in X$. Then $M \cap X \neq \varnothing$. Thus $x \in \overline{apr}_P(X) \neq \varnothing$. This is a contradiction. Hence

$$X \subseteq \overline{apr}_P(X).$$

(2) This holds by Proposition 5.2.

(3) By Proposition 5.2, $\underline{apr}_P(X \cap Y) \subseteq \overline{apr}_P(X) \cap \overline{apr}_P(Y)$. Suppose that $\underline{apr}_P(X) \cap \underline{apr}_P(Y) - \underline{apr}_P(X \cap Y) \neq \varnothing$. Pick

$$x \in \underline{apr}_P(X) \cap \underline{apr}_P(Y) - \underline{apr}_P(X \cap Y).$$

Then there exist $a, b \in A$, $M \in \sigma(a)$ and $N \in f(b)$ such that $M \subseteq X$, $N \subseteq Y$. Since σ_A is keeping intersection, $M \cap N = Q$ for some $c \in A$ *and* $Q \in f(c)$. This implies that $x \in Q \in f(c)$ *and* $Q \subseteq X \cap Y$. Thus $x \in \underline{apr}_P(X \cap Y)$. This is a contradiction. Thus

$$\underline{apr}_P(X \cap Y) \supseteq \underline{apr}_P(X) \cap \underline{apr}_P(Y).$$

Hence

$$\underline{apr}_P(X \cap Y) = \underline{apr}_P(X) \cap \underline{apr}_P(Y).$$

(4) Suppose that $X \in 2^U \backslash \varnothing$. Obviously, $\overline{apr}_P(X) \subseteq U$.

Since σ_A is full and keeping union, we claim that there exist $a \in A$ and $N \in \sigma(a)$ such that $N = U$.

Otherwise. Suppose that there is not $a \in A$ such that exists $N \in \sigma(a)$ and $N = U$. Since σ_A is full, there exists M_i($i \in \tau$) such that $\cup_{i\in\tau} M_i = U$. But σ_A is keeping union. This is a contradiction.

Since $\overline{apr}_P(X) = \bigcup\{M : a \in A, M \in \sigma(a) \text{ and } M \cap X \neq \varnothing\}$, $U \subseteq \overline{apr}_P(X)$.

Hence $\overline{apr}_P(X) = U$. □

5.2. **Structures of soft rough sets.**

Definition 5.4. Let σ_A be a soft set over 2^U and let $P = (U, \sigma_A)$ be a soft approximation space. $X \in 2^U$ is called a soft P-definable set if $\underline{apr}_P(X) = \overline{apr}_P(X)$; X is called a soft P-rough set if $\underline{apr}_P(X) \neq \overline{apr}_P(X)$.

Denote

$$\mathcal{R} = \{X \in 2^U : X \text{ is a soft } P\text{-rough set }\},$$
$$\mathcal{D} = \{X \in 2^U : X \text{ is a soft } P\text{-definable set }\},$$
$$\tau_f = \{X \in 2^U : \underline{apr}_P(X) = X\},$$
$$\sigma_f = \{X \in 2^U : \overline{apr}_P(X) = X\}.$$

Proposition 5.5. *Let σ_A be a soft set over U and let $P = (U, \sigma_A)$ be a soft approximation space. Then for each $X \in 2^U$,*

$$X \in \mathcal{D} \Longleftrightarrow \overline{apr}_P(X) \subseteq X.$$

Proof. "$\Longrightarrow$". This is obvious.

"$\Longleftarrow$". Obviously, $\underline{apr}_P(X) \subseteq \overline{apr}_P(X)$.

Suppose that $x \in \overline{apr}_P(X)$. Then there exist $a \in A$ and $N \in \sigma(a)$ such that $x \in N \in \sigma(a)$ and $N \cap X \neq \varnothing$. $\overline{apr}_P(X) = \bigcup\{M : a \in A, M \in \sigma(a) \text{ and } M \cap X \neq \varnothing$ and $\overline{apr}_P(X) \subseteq X$ imply $N \subseteq X$. So $x \in \underline{apr}_P(X)$. Thus $\underline{apr}_P(X) \supseteq \overline{apr}_P(X)$.

Hence $\underline{apr}_P(X) = \overline{apr}_P(X)$ and $X \in \mathcal{D}$.

□

Corollary 5.6. *Let σ_A be a soft set over U and let $P = (U, \sigma_A)$ be a soft approximation space. Then for each $X \in 2^U$,*

$$X \in \mathcal{R} \iff \overline{apr}_P(X) \nsubseteq X.$$

The following theorem gives structures of soft rough sets.

Theorem 5.7. *Let σ_A be a soft set over U and let $P = (U, \sigma_A)$ be a soft approximation space.*

(1)

$$\mathcal{R} \cup \mathcal{D} = 2^U,\ \mathcal{R} \cap \mathcal{D} = \varnothing \text{ and } \sigma_f \subseteq \mathcal{D}.$$

(2) *If σ_A is full, then*

$$\mathcal{R} = 2^U - \sigma_f \text{ and } \mathcal{D} = \sigma_f \subseteq \tau_f.$$

(3) *If σ_A is full and keeping union, then*

$$\mathcal{R} = 2^U - \{\varnothing, U\} \text{ and } \mathcal{D} = \{\varnothing, U\} = \sigma_f \subseteq \tau_f.$$

Proof. This holds by Propositions 5.2, 5.3 and 5.5. □

6. Topological structures of soft sets over 2^U

Theorem 6.1. *Let σ_A be a soft set over 2^U and let $P = (U, \sigma_A)$ be a soft approximation space. If σ_A is full and keeping intersection or bijective, then τ_f is a topology on U.*

Proof. This holds by Propositions 5.2 and 5.3. □

Definition 6.2. Let σ_A be a full and keeping intersection soft set over 2^U and let $P = (U, \sigma_A)$ be a soft approximation space. Then τ_f is called the topology induced by σ_A on U.

Theorem 6.3. *Let σ_A be a full and keeping intersection over 2^U, let $P = (U, \sigma_A)$ be a soft approximation space and let τ_f be the topology induced by σ_A on U. Then*

(1)

$$\{\overline{apr}_P(X) : X \in 2^U\} \subseteq \tau_f = \{\underline{apr}_P(X) : X \in 2^U\}.$$

(2)

$$\sigma(a) \subseteq \tau_f \text{ for any } a \in A.$$

(3) $\underline{apr}_P$ *is an interior operator of* τ_f.

Proof. (1) By Proposition 5.2, $\{\overline{apr}_P(X) : X \in 2^U\} \subseteq \tau_f$.

Obviously,

$$\tau_f \subseteq \{\underline{apr}_P(X) : X \subseteq U\}.$$

Let $Y \in \{\underline{apr}_P(X) : X \in 2^U\}$. Then $Y = \underline{apr}_P(X)$ for some $X \in 2^U$. By Proposition 5.2, $\underline{apr}_P(\underline{apr}_P(X)) = \underline{apr}_P(X)$. This implies that $Y \in \tau_f$. Thus

$$\tau_f \supseteq \{\underline{apr}_P(X) : X \in 2^U\}.$$

Hence

$$\{\overline{apr}_P(X) : X \in 2^U\} \subseteq \tau_f = \{\underline{apr}_P(X) : X \in 2^U\}.$$

(2) Let $a \in A$. Suppose that $M \in \sigma(a)$. Obviously, $\underline{apr}_P(M) \subseteq M$.

Let $x \in M$. Then $M \in \sigma(a)$ and $x \in M \subseteq M$, This implies that $x \in \underline{apr}_P(M)$. So $\underline{apr}_P(M) \supseteq M$. Thus $\underline{apr}_P(M) = M$.

Hence $\sigma(a) \subseteq \tau_f$ *for any* $a \in A$.

(3) It suffices to show that

$$\underline{apr}_P(X) = int(X) \text{ for each } X \in 2^U.$$

By (1), $\underline{apr}_P(X) \in \tau_f$. By Proposition 5.2, $\underline{apr}_P(X) \subseteq X$. Thus

$$\underline{apr}_P(X) \subseteq int(X).$$

Conversely, for each $Y \in \tau_f$ with $Y \subseteq X$, by Proposition 5.2, $Y = \underline{apr}_P(Y) \subseteq \underline{apr}_P(X)$. Then

$$int(X) = \bigcup\{Y : Y \in \tau_f \text{ and } Y \subseteq X\} \subseteq \underline{apr}_P(X).$$

Thus $\underline{apr}_P(X) = int(X)$. □

7. An application in decision making problems

In this section, we illustrate an application of soft sets over 2^U in decision making problems by Example 7.1.

Example 7.1. In Example 3.9, if the house is "wonderful", "modern", "cheap" and "much more green surroundings", then it is said to be satisfied. Let the score of satisfied houses be 1, the weight of any $a \in A$ be 0.25. And

if the house is prettyish, then for a_1, the score of it is 0.15;
if the house is beautiful, then for a_1, the score of it is 0.20;
if the house is wonderful, then for a_1, the score of it is 0.25;

if the house is plain, then for a_2, the score of it is 0.15;
if the house is modern, then for a_2, the score of it is 0.25;

if the house is expensive, then for a_3, the score of it is 0.15;
if the house is cheap, then for a_3, the score of it is 0.25;

if the house is in a little green surroundings, then for a_4, the score of it is 0.15;
if the house is in green surroundings, then for a_4, the score of it is 0.20;
if the house is in much more green surroundings, then for a_4, the score of it is 0.25.

Since

$$\sigma(a_1) \wedge \sigma(a_2) \wedge \sigma(a_3) \wedge \sigma(a_4) = \{\{h_1\}, \{h_2\}, \{h_3\}, \{h_4\}, \{h_5\}\},$$

we have

$$D = \{\frac{\{h_1\}}{0.70}, \frac{\{h_2\}}{0.75}, \frac{\{h_3\}}{0.65}, \frac{\{h_4\}}{0.90}, \frac{\{h_5\}}{0.90}\},$$

where $\frac{X}{\cdot}$ represents the score of houses in X.

Thus, we can conclude that h_4 or h_5 is the best choice for Mr.X.

8. Conclusions

In this paper, we considered soft sets over 2^U and obtained their lattice and topological structures. Moreover, We introduced soft rough approximations and soft rough sets, and gave structures of soft rough sets. We will study applications of soft sets over 2^U in future papers.

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